FOUNDATIONS OF p-ADIC HODGE THEORY

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CONTENTS

0. INTRODUCTION

The aim of this project is to provide a complete – and as self-contained as possible – proof of the so-called "almost purity" theorem, originally proved by Faltings in [34]. In a second stage, we plan to apply almost purity to establish general comparison theorems between the étale and the de Rham cohomologies of a scheme defined over the field of fractions of a rank one valuation ring K^+ of mixed characteristic.

In our formulation, the theorem states that a certain pair (X, Z), consisting of an affine K^+-scheme X and a closed subset $Z \subset X$ is *almost pure* (see definition 8.2.25), in analogy with the notion that is found in [44].

The proof follows the general strategy pioneered by Faltings, but our theorem is stronger than his, since we allow non-discrete valuation rings K^+. The price to pay for this extra generality is that one has to extend to this non-noetherian context a certain number of standard tools and results from commutative algebra and algebraic geometry. Especially, chapter 5 gives a rather thorough treatment of local cohomology, for rings and schemes that are not necessarily noetherian.

The complete proof is found in chapter 9. The case where $R := \mathscr{O}_X(X)$ has Krull dimension one had already been dealt with in our previous work [36]. Almost purity in dimension two is theorem 9.1.31. Theorem 9.2.16 takes care of the case of a smooth K^+-scheme of Krull dimension strictly greater than three; the main features of this case are the use of the Frobenius endomorphism of the ring R/pR, and a notion of normalized length for arbitrary R-modules.

In truth, one does not really need to consider separately the case of dimension > 3, since the argument given in section 9.4 works uniformly for every dimension ≥ 3. However, the case of dimension > 3 is considerably easier, and at the same time illuminates the more difficult case of dimension 3. For the latter, one constructs a ring $\mathbf{A}(R)^+$ with a surjection onto the p-adic completion $R^\wedge$ of R, and an endomorphism $\boldsymbol{\sigma}_R$ that lifts the Frobenius endomorphism of R/pR. This construction was originally found by Fontaine, who exploited it in case R is the ring of integers of a local field of characteristic zero.

1. CATEGORIES

1.1. **Basic category theory.** The purpose of this section is to fix some notation that shall be used throughout this work, and to collect, for ease of reference, a few well known generalities on categories and functors, which are frequently used. Our main reference on general nonsense is the treatise [10], and another good reference is the more recent [51]. For homological algebra, we mostly refer to [75].

Sooner or later, any honest discussion of categories and topoi gets tangled up with some foundational issues revolving around the manipulation of large sets. For this reason, to be able to move on solid ground, it is essential to select from the outset a definite set-theoretical framework (among the several currently available), and stick to it unwaveringly.

Thus, *throughout this work we will accept the so-called Zermelo-Fraenkel system of axioms for set theory*. (In this version of set theory, everything is a set, and there is no primitive notion of class, in contrast to other axiomatisations.)

Additionally, following [3, Exp.I, §0], we shall assume that, for every set S, there exists a *universe* V such that $S \in \mathsf{V}$. (For the notion of universe, the reader may also see [10, §1.1].)

Throughout this section, we fix some universe U. A set S is U-*small* (resp. *essentially* U-*small*), if $S \in \mathsf{U}$ (resp. if S has the cardinality of a U-small set). If the context is not ambiguous, we shall just write small, instead of U-small.

1.1.1. Recall that a *category* $\mathscr{C}$ is the datum of a set $\mathrm{Ob}(\mathscr{C})$ of *objects* and, for every $A, B \in \mathrm{Ob}(\mathscr{C})$, a set of *morphisms* from A to B, denoted :

$$\mathrm{Hom}_{\mathscr{C}}(A, B).$$

This datum must fulfill a list of standard axioms, which we omit. For any $A \in \mathrm{Ob}(\mathscr{C})$, we write $\mathbf{1}_A$ for the identity morphism of A. We also often use the notation :

$$\mathrm{End}_{\mathscr{C}}(A) := \mathrm{Hom}_{\mathscr{C}}(A, A).$$

Likewise, $\mathrm{Aut}_{\mathscr{C}}(A) \subset \mathrm{End}_{\mathscr{C}}(A)$ is the group of automorphisms of the object A. Furthermore, we denote by $\mathrm{Morph}(\mathscr{C})$ the set of all morphisms in $\mathscr{C}$.

The category of all small sets shall be denoted U-$\mathbf{Set}$ or just $\mathbf{Set}$, if there is no need to emphasize the chosen universe.

We say that the category $\mathscr{C}$ is *small*, if both $\mathrm{Ob}(\mathscr{C})$ and $\mathrm{Morph}(\mathscr{C})$ are small sets. Somewhat weaker is the condition that $\mathscr{C}$ *has small* Hom*-sets, i.e.* $\mathrm{Hom}_{\mathscr{C}}(A, B) \in \mathsf{U}$ for every $A, B \in \mathrm{Ob}(\mathscr{C})$. The collection of all small categories, together with the functors between them, forms a category U-$\mathbf{Cat}$. Unless we have to deal with more than one universe, we shall usually omit the prefix U, and write just $\mathbf{Cat}$. If $\mathscr{A}$ and $\mathscr{B}$ are any two categories, we denote by

$$\mathrm{Fun}(\mathscr{A}, \mathscr{B})$$

the set of all functors $\mathscr{A} \to \mathscr{B}$. If $\mathscr{A}$ is small and $\mathscr{B}$ has small Hom-sets, then $\mathrm{Fun}(\mathscr{A}, \mathscr{B}) \in \mathsf{U}$; especially, $\mathbf{Cat}$ is a category with small Hom-sets. Moreover, there is a natural fully faithful imbedding :

$$\mathbf{Set} \to \mathbf{Cat}.$$

Indeed, to any set S one may assign its *discrete category* also denoted S, *i.e.* the unique category such that $\mathrm{Ob}(S) = S$ and $\mathrm{Morph}(S) = \{\mathbf{1}_s \mid s \in S\}$. If S and S' are two discrete categories, the datum of a functor $S \to S'$ is clearly the same as a map of sets $\mathrm{Ob}(S) \to \mathrm{Ob}(S')$.

1.1.2. The *opposite category* $\mathscr{C}^o$ is the category with $\mathrm{Ob}(\mathscr{C}^o) = \mathrm{Ob}(\mathscr{C})$, and such that :

$$\mathrm{Hom}_{\mathscr{C}^o}(A, B) := \mathrm{Hom}_{\mathscr{C}}(B, A) \qquad \text{for every } A, B \in \mathrm{Ob}(\mathscr{C}).$$

Given an object A of $\mathscr{C}$, sometimes we denote, for emphasis, by A^o the same object, viewed as an element of $\mathrm{Ob}(\mathscr{C}^o)$; likewise, given a morphism $f : A \to B$ in $\mathscr{C}$, we write f^o for the corresponding morphism $B^o \to A^o$ in $\mathscr{C}^o$. Furthermore, any functor $F : \mathscr{A} \to \mathscr{B}$ induces a functor $F^o : \mathscr{A}^o \to \mathscr{B}^o$, in the obvious way.

1.1.3. A morphism $f : A \to B$ in $\mathscr{C}$ is said to be a *monomorphism* if the induced map :

$$\mathrm{Hom}_{\mathscr{C}}(X, f) : \mathrm{Hom}_{\mathscr{C}}(X, A) \to \mathrm{Hom}_{\mathscr{C}}(X, B) \quad : \quad g \mapsto f \circ g$$

is injective, for every $X \in \mathrm{Ob}(\mathscr{C})$. Dually, we say that f is an *epimorphism* if f^o is a monomorphism in $\mathscr{C}^o$. Obviously, an isomorphism is both a monomorphism and an epimorphism. The converse does not necessarily hold, in an arbitrary category.

Two monomorphisms $f : A \to B$ and $f' : A' \to B$ are *equivalent*, if there exists an isomorphism $h : A \to A'$ such that $f = f' \circ h$. A *subobject* of B is defined as an equivalence class of monomorphisms $A \to B$. Dually, a *quotient* of B is a subobject of B^o in $\mathscr{C}^o$.

One says that $\mathscr{C}$ is *well-powered* if, for every $A \in \mathrm{Ob}(\mathscr{C})$, the set :

$$\mathrm{Sub}(A)$$

of all subobjects of A is essentially small. Dually, $\mathscr{C}$ is *co-well-powered*, if $\mathscr{C}^o$ is well-powered.

Definition 1.1.4. Let $F : \mathscr{A} \to \mathscr{B}$ be a functor.

(i) We say that F is *faithful* (resp. *full*, resp. *fully faithful*), if it induces injective (resp. surjective, resp. bijective) maps :

$$\mathrm{Hom}_{\mathscr{A}}(A, A') \to \mathrm{Hom}_{\mathscr{B}}(FA, FA') \quad : \quad f \mapsto Ff$$

for every $A, A' \in \mathrm{Ob}(\mathscr{A})$.

(ii) We say that F *reflects monomorphisms* (resp. *reflects epimorphisms*, resp. *is conservative*) if the following holds. For every morphism $f : A \to A'$ in $\mathscr{A}$, if the morphism Ff of $\mathscr{B}$ is a monomorphism (resp. epimorphism, resp. isomorphism), then the same holds for f.

(iii) We say that F is *essentially surjective* if every object of $\mathscr{B}$ is isomorphic to an object of the form FA, for some $A \in \mathrm{Ob}(\mathscr{A})$.

(iv) We say that F is an *equivalence*, if it is fully faithful and essentially surjective.

1.1.5. Let $\mathscr{A}$, $\mathscr{B}$ be two categories, $F, G : \mathscr{A} \to \mathscr{B}$ two functors. A *natural transformation*

$$\alpha : F \Rightarrow G \tag{1.1.6}$$

from F *to* G is a family of morphisms $(\alpha_A : FA \to GA \mid A \in \mathrm{Ob}(\mathscr{A}))$ such that, for every morphism $f : A \to B$ in $\mathscr{A}$, the diagram :

$$\begin{tikzcd} FA \arrow[r, "\alpha_A"] \arrow[d, "Ff"'] & GA \arrow[d, "Gf"] \\ FB \arrow[r, "\alpha_B"] & GB \end{tikzcd}$$

commutes. If α_A is an isomorphism for every $A \in \mathscr{A}$, we say that α is a *natural isomorphism* of functors. For instance, the rule that assigns to any object A the identity morphism $\mathbf{1}_{FA} : FA \to FA$, defines a natural isomorphism $\mathbf{1}_F : F \Rightarrow F$. A natural transformation (1.1.6) is also indicated by a diagram of the type :

$$\begin{tikzcd} \mathscr{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathscr{B}. \end{tikzcd}$$

1.1.7. The natural transformations between functors $\mathscr{A} \to \mathscr{B}$ can be composed; namely, if $\alpha : F \Rightarrow G$ and $\beta : G \Rightarrow H$ are two such transformations, we obtain a new natural transformation

$$\beta \circ \alpha : F \Rightarrow H \qquad \text{by the rule :} \qquad A \mapsto \beta_A \circ \alpha_A \qquad \text{for every } A \in \mathrm{Ob}(\mathscr{A}).$$

With this composition, $\mathrm{Fun}(\mathscr{A}, \mathscr{B})$ is the set of objects of a category which we shall denote

$$\mathbf{Fun}(\mathscr{A}, \mathscr{B}).$$

There is also a second composition law for natural transformations : if $\mathscr{C}$ is another category, $H, K : \mathscr{B} \to \mathscr{C}$ two functors, and $\beta : H \Rightarrow K$ a natural transformation, we get a natural transformation

$$\beta * \alpha : H \circ F \Rightarrow K \circ G \quad : \quad A \mapsto \beta_{GA} \circ H(\alpha_A) = K(\alpha_A) \circ \beta_{FA} \qquad \text{for every } A \in \mathrm{Ob}(\mathscr{A})$$

called the *Godement product* of α and β ([10, Prop.1.3.4]). Especially, if $H : \mathscr{B} \to \mathscr{C}$ (resp. $H : \mathscr{C} \to \mathscr{A}$) is any functor, we write $H * \alpha$ (resp. $\alpha * H$) instead of $\mathbf{1}_H * \alpha$ (resp. $\alpha * \mathbf{1}_H$).

These two composition laws are related as follows. Suppose that $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ are three categories, $F_1, G_1, H_1 : \mathscr{A} \to \mathscr{B}$ and $F_2, G_2, H_2 : \mathscr{B} \to \mathscr{C}$ are six functors, and we have four natural transformations

$$\alpha_i : F_i \Rightarrow G_i \qquad \beta_i : G_i \Rightarrow H_i \qquad (i = 1, 2).$$

Then we have the identity :

$$(\beta_2 * \beta_1) \circ (\alpha_2 * \alpha_1) = (\beta_2 \circ \alpha_2) * (\beta_1 \circ \alpha_1).$$

The proof is left as an exercise for the reader (see [10, Prop.1.3.5]).

1.1.8. Let $\mathscr{A}$ and $\mathscr{B}$ be two categories, $F : \mathscr{A} \to \mathscr{B}$ a functor. Recall that a functor $G : \mathscr{B} \to \mathscr{A}$ is said to be *left adjoint* to F if there exist bijections

$$\vartheta_{A,B} : \mathrm{Hom}_{\mathscr{A}}(GB, A) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{B}}(B, FA) \qquad \text{for every } A \in \mathrm{Ob}(\mathscr{A}) \text{ and } B \in \mathrm{Ob}(\mathscr{B})$$

and these bijections are natural in both A and B. Then one says that F is *right adjoint* to G, and that (G, F) is an *adjoint pair of functors*.

Especially, to any object B of $\mathscr{B}$ (resp. A of $\mathscr{A}$), the adjoint pair (G, F) assigns a morphism $\vartheta_{GB,B}(\mathbf{1}_{GB}) : B \to FGB$ (resp. $\vartheta^{-1}_{A,FA}(\mathbf{1}_A) : GFA \to A$), whence a natural transformation

$$\eta : \mathbf{1}_{\mathscr{B}} \Rightarrow F \circ G \qquad (\text{resp. } \varepsilon : G \circ F \Rightarrow \mathbf{1}_{\mathscr{A}}) \tag{1.1.9}$$

called the *unit* (resp. *counit*) of the adjunction. These transformations are related by the *triangular identities* expressed by the commutative diagrams :

$$\begin{tikzcd} F \arrow[r, Rightarrow, "\eta * F"] \arrow[dr, equal, "\mathbf{1}_F"'] & FGF \arrow[d, Rightarrow, "F * \varepsilon"] \\ & F \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, Rightarrow, "G * \eta"] \arrow[dr, equal, "\mathbf{1}_G"'] & GFG \arrow[d, Rightarrow, "\varepsilon * G"] \\ & G. \end{tikzcd}$$

Conversely, the existence of natural transformations ε and η as in (1.1.9), which satisfy the above triangular identities, implies that G is left adjoint to F ([10, Th.3.1.5] or [51, Prop.1.5.4]).

Remark 1.1.10. Let (G, F) be an adjoint pair as in (1.1.8), with unit η and counit ε.

(i) Suppose that (H_2, H_1) is another adjoint pair, with $H_1 : \mathscr{B} \to \mathscr{C}$, $H_2 : \mathscr{C} \to \mathscr{B}$ (for some category $\mathscr{C}$), and let η_H (resp. ε_H) be a unit (resp. a counit) for this second adjoint pair. Then clearly $(G \circ H_2, H_1 \circ F)$ is an adjoint pair, and the transformation

$$\widetilde{\varepsilon} := \varepsilon \circ (G * \varepsilon_H * F) \qquad (\text{resp. } \widetilde{\eta} := (H_1 * \eta * H_2) \circ \eta_H)$$

is a counit (resp. a unit) for this adjunction. We say that $\widetilde{\eta}$ and $\widetilde{\varepsilon}$ are the *unit and counit induced by* (η, ε) *and* (η_H, ε_H) .

(ii) Suppose that (G', F') is another adjoint pair, with $F' : \mathscr{A} \to \mathscr{B}$, $G' : \mathscr{B} \to \mathscr{A}$, and let η' (resp. ε') be a unit (resp. a counit) for this second adjoint pair. Suppose moreover that $\tau : F \Rightarrow F'$ is a natural transformation. Then we obtain an adjoint transformation $\tau^\dagger : G' \Rightarrow G$, by the composition :

$$G'B \xrightarrow{G'(\eta_B)} G'FGB \xrightarrow{G'(\tau_{GB})} G'F'GB \xrightarrow{\varepsilon'_{GB}} GB.$$

Conversely, from such $\tau^\dagger$ we can recover τ, hence the rule $\tau \mapsto \tau^\dagger$ establishes a natural bijection from the set of natural transformations $F \Rightarrow F'$, to the set of natural transformations $G' \Rightarrow G$. Notice that this correspondence depends not only on (G, F) and (G', F'), but also on (η, ε) and (η', ε').

(iii) Moreover, using the triangular identities of (1.1.8), it is easily seen that the diagram :

$$\begin{tikzcd} G' \circ F \arrow[r, Rightarrow, "G' * \tau"] \arrow[d, Rightarrow, "\tau^\dagger * F"'] & G' \circ F' \arrow[d, Rightarrow, "\varepsilon'"] \\ G \circ F \arrow[r, Rightarrow, "\varepsilon"] & \mathbf{1}_{\mathscr{A}}. \end{tikzcd}$$

commutes.

(iv) Furthermore, suppose (G'', F'') is another adjoint pair with $F'' : \mathscr{A} \to \mathscr{B}$, $G'' : \mathscr{B} \to \mathscr{A}$ and η'', ε'' are given units and counit for this pair. Let also $\omega : F' \Rightarrow F''$ be a natural transformation; then we may use (η', ε') and (η'', ε'') to define $\omega^\dagger$, and we have :

$$(\omega \circ \tau)^\dagger = \tau^\dagger \circ \omega^\dagger$$

provided (η, ε) and (η'', ε'') are used to define the left-hand side.

(v) Lastly, let (H_1, H_2) and the unit and counit η_H, ε_H be as in (i), and suppose moreover that we have another adjoint pair (H'_1, H'_2) where $H'_1 : \mathscr{B} \to \mathscr{C}$ and $H'_2 : \mathscr{C} \to \mathscr{B}$, with respective unit η'_H and counit ε'_H, and furthermore we are given a natural transformation $\nu : H_1 \Rightarrow H'_1$. Then we get as in (ii) the natural transformation $\nu^\dagger : H'_2 \Rightarrow H_2$, and we have the identity

$$(\nu * \tau)^\dagger = \tau^\dagger * \nu^\dagger$$

provided the left hand-side is defined via $(\widetilde{\eta}, \widetilde{\varepsilon})$ and via the unit and counit for the adjoint pair $(H'_2 \circ G', H'_1 \circ F')$ induced by (η', ε') and $(\eta'_H, \varepsilon'_H)$.

(vi) Especially, if we use (η, ε) and $(\widetilde{\eta}, \widetilde{\varepsilon})$ to define $(H_2 * \tau)^\dagger$, we have :

$$(H_2 * \tau)^\dagger = \tau^\dagger * H_1$$

(all these assertions are exercises for the reader : see also [41, §I.6]).

Proposition 1.1.11. *Let $F : \mathscr{A} \to \mathscr{B}$ be a functor.*

(i) *The following conditions are equivalent :*
  (a) *F is fully faithful and has a fully faithful left adjoint.*
  (b) *F is an equivalence.*

(ii) *Suppose that F admits a left adjoint $G : \mathscr{B} \to \mathscr{A}$, and let $\eta : \mathbf{1}_{\mathscr{B}} \Rightarrow F \circ G$ and $\varepsilon : G \circ F \Rightarrow \mathbf{1}_{\mathscr{A}}$ be a unit and respectively counit for the adjoint pair (G, F). Then :*
  (a) *F (resp. G) is faithful if and only if ε_X (resp. η_Y) is an epimorphism for every $X \in \mathrm{Ob}(\mathscr{A})$ (resp. a monomorphism for every $Y \in \mathrm{Ob}(\mathscr{B})$).*
  (b) *F (resp. G) is fully faithful if and only if ε (resp. η) is an isomorphism of functors.*

(iii) *Suppose that F admits both a left adjoint $G : \mathscr{B} \to \mathscr{A}$ and a right adjoint $H : \mathscr{B} \to \mathscr{A}$. Then G is fully faithful if and only if H is fully faithful.*

Proof. These assertions are [10, Prop.3.4.1, 3.4.2, 3.4.3]; see also [51, Prop.1.5.6]. □

1.1.12. A standard construction associates to any object $X \in \mathrm{Ob}(\mathscr{C})$ a category :

$$\mathscr{C}/X$$

as follows. The objects of $\mathscr{C}/X$ are all the pairs (A, φ) where $A \in \mathrm{Ob}(\mathscr{C})$ and $\varphi : A \to X$ is any morphism of $\mathscr{C}$. For any $\varphi : A \to X$, and $\psi : B \to X$, the morphisms $\mathrm{Hom}_{\mathscr{C}/X}((A, \varphi), (B, \psi))$ are all the commutative diagrams :

$$\begin{tikzcd} A \arrow[rr] \arrow[dr, "\varphi"'] & & B \arrow[dl, "\psi"] \\ & X & \end{tikzcd}$$

of morphisms of $\mathscr{C}$, with composition of morphisms induced by the composition law of $\mathscr{C}$. An object (resp. a morphism) of $\mathscr{C}/X$ is also called an X-*object* (resp. an X-*morphism*) of $\mathscr{C}$. Dually, one defines

$$X/\mathscr{C} := (\mathscr{C}^o/X^o)^o$$

i.e. the objects of $X/\mathscr{C}$ are the pairs (A, φ) with $A \in \mathrm{Ob}(\mathscr{C})$ and $\varphi : X \to A$ any morphism of $\mathscr{C}$. We have an obvious faithful *source* functor :

$$\mathscr{C}/X \to \mathscr{C} \qquad (A, \varphi) \mapsto A \tag{1.1.13}$$

and likewise one obtains a *target* functor $X/\mathscr{C} \to \mathscr{C}$. Moreover, any morphism $f : X \to Y$ in $\mathscr{C}$ induces functors :

$$\begin{array}{lcl} f_* : \mathscr{C}/X \to \mathscr{C}/Y & : & (A, g : A \to X) \mapsto (A, f_* g := f \circ g : A \to Y) \\ f^* : Y/\mathscr{C} \to X/\mathscr{C} & : & (B, h : Y \to B) \mapsto (B, f^* h := h \circ f : X \to B). \end{array} \tag{1.1.14}$$

Furthermore, given a functor $F : \mathscr{C} \to \mathscr{B}$, any $X \in \mathrm{Ob}(\mathscr{C})$ induces functors :

$$\begin{array}{lcl} F_{|X} : \mathscr{C}/X \to \mathscr{B}/FX & : & (A, g) \mapsto (FA, Fg) \\ {}_{X|}F : X/\mathscr{C} \to FX/\mathscr{B} & : & (B, h) \mapsto (FB, Fh). \end{array} \tag{1.1.15}$$

1.1.16. The categories $\mathscr{C}/X$ and $X/\mathscr{C}$ are special cases of the following more general construction. Let $F : \mathscr{A} \to \mathscr{B}$ be any functor. For any $B \in \mathrm{Ob}(\mathscr{B})$, we define $F\mathscr{A}/B$ as the category whose objects are all the pairs (A, f), where $A \in \mathrm{Ob}(\mathscr{A})$ and $f : FA \to B$ is a morphism in $\mathscr{B}$. The morphisms $g : (A, f) \to (A', f')$ are the morphisms $g : A \to A'$ in $\mathscr{A}$ such that $f' \circ Fg = f$. There are well-defined functors :

$$F/B : F\mathscr{A}/B \to \mathscr{B}/B \ : \ (A, f) \mapsto (FA, f) \qquad \text{and} \qquad \iota_B : F\mathscr{A}/B \to \mathscr{A} \ : \ (A, f) \mapsto A.$$

Dually, we define :

$$B/F\mathscr{A} := (F^o\mathscr{A}^o/B^o)^o$$

and likewise one has natural functors :

$$B/F : B/F\mathscr{A} \to B/\mathscr{B} \qquad \text{and} \qquad \iota_B : B/F\mathscr{A} \to \mathscr{A}.$$

Obviously, the category $\mathscr{C}/X$ (resp. $X/\mathscr{C}$) is the same as $\mathbf{1}_\mathscr{C}\mathscr{C}/X$ (resp. $X/\mathbf{1}_\mathscr{C}\mathscr{C}$).

Any morphism $g : B' \to B$ induces functors :

$$\begin{array}{lcl} g/F\mathscr{A} : B/F\mathscr{A} \to B'/F\mathscr{A} & : & (A, f) \mapsto (A, f \circ g) \\ F\mathscr{A}/g : F\mathscr{A}/B' \to F\mathscr{A}/B & : & (A, f) \mapsto (A, g \circ f). \end{array}$$

1.1.17. Let $\mathscr{C}$ be a category; the categories of the form $\mathscr{C}/X$ and $X/\mathscr{C}$ can be faithfully embedded in a single category $\mathsf{Morph}(\mathscr{C})$. The objects of $\mathsf{Morph}(\mathscr{C})$ are all the data (A, B, φ), where $A, B \in \mathrm{Ob}(\mathscr{C})$ and $\varphi : A \to B$ is any morphism of $\mathscr{C}$. If $f : A \to B$ and $f' : A' \to B'$ are two such morphisms, the set $\mathrm{Hom}_{\mathsf{Morph}(\mathscr{C})}((A, B, f), (A', B', f'))$ consists of all the commutative diagrams of morphisms of $\mathscr{C}$:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "g'"] \\ A' \arrow[r, "f'"] & B' \end{tikzcd} \tag{1.1.18}$$

with composition of morphisms induced by the composition law of $\mathscr{C}$, in the obvious way. There are two natural *source* and *target* functors :

$$\mathscr{C} \xleftarrow{\mathsf{s}} \mathsf{Morph}(\mathscr{C}) \xrightarrow{\mathsf{t}} \mathscr{C}$$

such that $\mathsf{s}(A \to B) := A$, $\mathsf{t}(A \to B) := B$ for any object $A \to B$ of $\mathsf{Morph}(\mathscr{C})$, and $\mathsf{s}(1.1.18) = g$, $\mathsf{t}(1.1.18) = g'$. Especially, the functor (1.1.13) is the restriction of s to the subcategory $\mathscr{C}/X$.

1.1.19. Let $\mathscr{C}$ be a category; a very important construction associated to $\mathscr{C}$ is the category

$$\mathscr{C}^\wedge_{\mathsf{U}} := \mathbf{Fun}(\mathscr{C}^o, \mathsf{U}\text{-}\mathbf{Set})$$

whose objects are called the U-*presheaves* on $\mathscr{C}$. The morphisms in $\mathscr{C}^\wedge_{\mathsf{U}}$ are the natural transformations of functors. We usually drop the subscript U, unless we have to deal with more than one universe. Let us just remark that, if U' is another universe, and $\mathsf{U} \subset \mathsf{U}'$, the natural inclusion of categories :

$$\mathscr{C}^\wedge_{\mathsf{U}} \to \mathscr{C}^\wedge_{\mathsf{U}'}$$

is fully faithful. If $\mathscr{C}$ has small Hom-sets (see (1.1.1)), there is a natural functor

$$h : \mathscr{C} \to \mathscr{C}^\wedge$$

(the *Yoneda embedding*) which assigns to any $X \in \mathrm{Ob}(\mathscr{C})$ the functor

$$h_X : \mathscr{C}^o \to \mathbf{Set} \qquad Y \mapsto \mathrm{Hom}_{\mathscr{C}}(Y, X) \quad \text{for every } Y \in \mathrm{Ob}(\mathscr{C})$$

and to any morphism $f : X \to X'$ in $\mathscr{C}$, the natural transformation $h_f : h_X \Rightarrow h_{X'}$ such that

$$h_{f,Y}(g) := f \circ g \qquad \text{for every } Y \in \mathrm{Ob}(\mathscr{C}) \text{ and every } g \in \mathrm{Hom}_{\mathscr{C}}(Y, X).$$

Proposition 1.1.20 (Yoneda's lemma)**.** *With the notation of* (1.1.19)*, we have :*

(i) *The functor h is fully faithful.*

(ii) *Moreover, for every $F \in \mathrm{Ob}(\mathscr{C}^\wedge)$, and every $X \in \mathrm{Ob}(\mathscr{C})$, there is a natural bijection*

$$F(X) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{C}^\wedge}(h_X, F)$$

functorial in both X and F.

Proof. Clearly it suffices to check (ii). However, the sought bijection is obtained explicitly as follows. To a given $a \in F(X)$, we assign the natural transformation $\tau_a : h_X \Rightarrow F$ such that

$$\tau_{a,Y}(f) := Ff(a) \qquad \text{for every } Y \in \mathrm{Ob}(\mathscr{C}) \text{ and every } f \in h_X(Y).$$

Conversely, to a given natural transformation $\tau : h_X \Rightarrow F$ we assign $\tau_X(\mathbf{1}_X) \in F(X)$. As an exercise, the reader can check that these rules establish mutually inverse bijections. The functoriality in F is immediate, and the functoriality in the argument X amounts to the commutativity of the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{C}^\wedge}(h_{X'}, F) \arrow[r, "\sim"] \arrow[d, "\mathrm{Hom}_{\mathscr{C}^\wedge}(h_\varphi, F)"'] & F(X') \arrow[d, "F(\varphi)"] \\
\mathrm{Hom}_{\mathscr{C}^\wedge}(h_X, F) \arrow[r, "\sim"] & F(X)
\end{tikzcd}$$

for every morphism $\varphi : X \to X'$ in $\mathscr{C}$: the verification shall also be left to the reader. $\square$

An object F of $\mathscr{C}^\wedge$ is a *representable presheaf*, if it is isomorphic to h_X, for some $X \in \mathrm{Ob}(\mathscr{C})$. Then, we also say that F is *representable in* $\mathscr{C}$.

1.1.21. We wish to explain some standard constructions of presheaves that are in constant use throughout this work. Namely, let I be a small category, $\mathscr{C}$ a category with small Hom-sets, and X any object of $\mathscr{C}$. We denote by $c_X : I \to \mathscr{C}$ the *constant functor* associated to X :

$$c_X(i) := X \quad \text{for every } i \in \mathrm{Ob}(I) \qquad c_X(\varphi) := \mathbf{1}_X \quad \text{for every } \varphi \in \mathrm{Morph}(I).$$

Any morphism $f : X' \to X$ induces a natural transformation

$$c_f : c_{X'} \Rightarrow c_X \qquad \text{by the rule} \ : \ (c_f)_i := f \text{ for every } i \in I.$$

Definition 1.1.22. With the notation of (1.1.21), let $F : I \to \mathscr{C}$ be any functor.

(i) The *limit* of F is the presheaf on $\mathscr{C}$ denoted

$$\lim_I F : \mathscr{C}^o \to \mathbf{Set}$$

and defined as follows. For any $X \in \mathrm{Ob}(\mathscr{C})$, the set $\lim_I F(X)$ consists of all the natural transformations $c_X \Rightarrow F$; and any morphism $f : X' \to X$ induces the map

$$\lim_I F(f) : \lim_I F(X) \to \lim_I F(X') \qquad \tau \mapsto \tau \circ c_f \quad \text{for every } \tau : c_X \Rightarrow F.$$

(ii) Dually, the *colimit* of F is the presheaf on $\mathscr{C}^o$

$$\operatorname*{colim}_I F := \lim_{I^o} F^o.$$

(iii) We say that $\mathscr{C}$ is *complete* (resp. *cocomplete*) if, for every small category I and every functor $F : I \to \mathscr{C}$, the limit (resp. the colimit) of F is representable in $\mathscr{C}$ (resp. in $\mathscr{C}^o$).

Remark 1.1.23. (i) In the situation of (1.1.21), let $F, F' : I \to \mathscr{C}$ be two functors, and $g : F \Rightarrow F'$ a natural transformation; we deduce a morphism of presheaves on $\mathscr{C}$

$$\lim_I g : \lim_I F \to \lim_I F' \qquad \tau \mapsto g \circ \tau \quad \text{for every } X \in \mathrm{Ob}(\mathscr{C}) \text{ and every } \tau : c_X \Rightarrow F.$$

(ii) Likewise, g induces a morphism of presheaves on $\mathscr{C}^o$

$$\operatorname*{colim}_I g : \operatorname*{colim}_I F' \to \operatorname*{colim}_I F \qquad \tau^o \mapsto \tau^o \circ g^o \quad \text{for every } X \in \mathrm{Ob}(\mathscr{C}) \text{ and } \tau : c_X \Rightarrow F.$$

(iii) With the notation of (ii), suppose that the colimits of F and F' are representable by objects C^o_F, respectively $C^o_{F'}$ of $\mathscr{C}^o$. Then the colimit of g corresponds to a morphism $C^o_{F'} \to C^o_F$ in $\mathscr{C}^o$, *i.e.* a morphism $C_F \to C_{F'}$ in $\mathscr{C}$.

(iv) If $\varphi : I' \to I$ and $H : \mathscr{C} \to \mathscr{C}'$ are any two functors, we obtain a natural transformation

$$\lim_\varphi H : \lim_I F \Rightarrow (\lim_{I'} H \circ F \circ \varphi) \circ H^o$$

by ruling that $\lim_\varphi H(X)(\tau) := H * \tau * \varphi$ for every $X \in \mathrm{Ob}(\mathscr{C})$ and every $\tau : c_X \Rightarrow F$. The reader may spell out the corresponding assertion for colimits.

(v) Let I a small category, $F : \mathscr{C} \to \mathscr{C}'$ and $H : I \to \mathscr{C}$ two functors, and suppose the limit of H is representable by an object L of $\mathscr{C}$, so there exists an isomorphism of presheaves

$$h_L \xrightarrow{\sim} \lim_I H$$

under which, the image of $\mathbf{1}_L \in h_L(L)$ corresponds to a natural transformation $\tau : c_L \Rightarrow H$ from the constant functor $c_L : I \to \mathscr{C}$, to H. There follows a natural transformation

$$F * \tau : c_{FL} \Rightarrow F \circ H$$

of functors $I \to \mathscr{C}'$. In this situation, we say that F *commutes with the limit of* H, if $F * \tau$ induces an isomorphism of functors (defined as in (i))

$$\lim_I F * \tau : h_{FL} \xrightarrow{\sim} \lim_I c_L \xrightarrow{\sim} \lim_I F \circ H$$

in which case FL represents the limit of $F \circ H$. Likewise, we say that F *commutes with the colimit of* H, if the dual condition holds, *i.e.* if F^o commutes with the limit of H^o.

Example 1.1.24. (i) For $i = 1, 2$, let $f_i : A \to B_i$ be two morphisms in a category $\mathscr{C}$ with small Hom-sets; the *push-out* or *coproduct* of f_1 and f_2 is the colimit of the functor $F : I \to \mathscr{C}$, defined as follows. The set $\mathrm{Ob}(I)$ consists of three objects s, t_1, t_2 and $\mathrm{Morph}(I)$ consists of two morphisms $\varphi_i : s \to t_i$ for $i = 1, 2$ (in addition to the identity morphisms of the objects of I); the functor is given by the rule : $Fs := A$, $Ft_i := B_i$ and $F\varphi_i := f_i$ (for $i = 1, 2$).

If $C \in \mathrm{Ob}(\mathscr{C})$ represents the coproduct of f_1 and f_2, we have two morphisms $f'_i : B_i \to C$ ($i = 1, 2$) such that $f'_1 \circ f_1 = f'_2 \circ f_2$; in this case, we say that the commutative diagram :

$$\begin{tikzcd} A \arrow[r, "f_1"] \arrow[d, "f_2"'] & B_1 \arrow[d, "f'_1"] \\ B_2 \arrow[r, "f'_2"] & C \end{tikzcd}$$

is *cocartesian*. Dually one defines the *fibre product* or *pull-back* of two morphisms $g_i : A_i \to B$. If $D \in \mathrm{Ob}(\mathscr{C})$ represents this fibre product, we have morphisms $g'_i : D \to A_i$ such that $g_1 \circ g'_1 = g_2 \circ g'_2$, and we say that the diagram :

$$\begin{tikzcd} D \arrow[r, "g'_1"] \arrow[d, "g'_2"'] & A_1 \arrow[d, "g_1"] \\ A_2 \arrow[r, "g_2"] & B \end{tikzcd}$$

is *cartesian*. The coproduct of f_1 and f_2 is usually called the coproduct of B_1 and B_2 over A, denoted $B_1 \amalg_{(f_1,f_2)} B_2$, or simply $B_1 \amalg_A B_2$, unless the notation gives rise to ambiguities. Likewise one writes $A_1 \times_{(g_1,g_2)} A_2$, or just $A_1 \times_B A_2$ for the fibre product of g_1 and g_2.

(ii) As a special case, if $B := B_1 = B_2$, the coproduct $B \amalg_A B$ is also called the *coequalizer* of f_1 and f_2, and is sometimes denoted $\mathrm{Coequal}(f_1, f_2)$. Likewise, if $A := A_1 = A_2$, the fibre product $A \times_B A$ is also called the *equalizer* of g_1 and g_2, sometimes denoted $\mathrm{Equal}(g_1, g_2)$.

(iii) Furthermore, let $f : A \to B$ be any morphism in $\mathscr{C}$; notice that the identity morphisms of A and B induce natural morphisms of presheaves

$$\pi_f : \mathrm{Coequal}(f, f) \to B \qquad \iota_f : A \to \mathrm{Equal}(f, f).$$

and it is easily seen that f is a monomorphism (resp. an epimorphism) if and only if ι_f (resp. π_f) is an isomorphism in $\mathscr{C}^\wedge$ (here we abuse notation, by writing A and B instead of the corresponding presheaves h_A and h_B).

(iv) Let $I \in \mathsf{U}$ be any small set, and $\underline{B} := (B_i \mid i \in I)$ any family of objects of $\mathscr{C}$. We may regard I as a discrete category (see (1.1.1)), and then the rule $i \mapsto B_i$ yields a functor

$I \to \mathscr{C}$, whose limit (resp. colimit) is called the *product* (resp. *coproduct*) of the family $\underline{B}$, and is denoted $\prod_{i\in I} B_i$ (resp. $\coprod_{i\in I} B_i$).

(v) In the situation of (1.1.21), let $f : F \Rightarrow c_X$ be a natural transformation, and $g : Y \to X$ any morphism in $\mathscr{C}$. Suppose that all fibre products are representable in $\mathscr{C}$, and consider the functor

$$F \times_X Y : I \to \mathscr{C} \qquad i \mapsto F(i) \times_{(f_i,g)} Y$$

(which means that for every $i \in \mathrm{Ob}(I)$ we pick an object of $\mathscr{C}$ representing the above fibre product, and to a morphism $\varphi : i \to j$ in I, we attach the morphism $F(\varphi) \times_X \mathbf{1}_Y$, with obvious notation.) The projections induce a commutative diagram

$$\begin{tikzcd} F \times_X Y \arrow[r, Rightarrow] \arrow[d, Rightarrow] & F \arrow[d, Rightarrow, "f"] \\ c_Y \arrow[r, Rightarrow, "c_g"] & c_X. \end{tikzcd}$$

Suppose moreover that the colimit of F is representable by an object C^o of $\mathscr{C}^o$, and for every $X \in \mathrm{Ob}(\mathscr{C})$, and every f, g as above, the colimit of $F \times_X Y$ is representable by an object C^o_Y of $\mathscr{C}^o$. By remark 1.1.23(iii) we deduce a natural morphism in $\mathscr{C}$:

$$C_Y \to C \times_X Y. \tag{1.1.25}$$

We say that the colimit of F is *universal* if the resulting morphism (1.1.25) is an isomorphism for every $X \in \mathscr{C}$ and every f and g as above. In this case, clearly $C \times_X Y$ represents the colimit of $F \times_X Y$.

(vi) Let $f : X \to Y$ be a morphism in $\mathscr{C}$, and suppose that, for every other morphism $Y' \to Y$, the fibre product $X \times_Y Y'$ is representable in $\mathscr{C}$. In this case, we say that f is a *universal monomorphism* (resp. a *universal epimorphism*) if $f \times_X Y : X \times_Y Y' \to Y'$ is a monomorphism (resp. an epimorphism) for every morphism $Y' \to Y$ in $\mathscr{C}$.

(vii) Suppose that all fibre products are representable in $\mathscr{C}$; in this case, notice that, in view of (iii), the morphism f is a universal epimorphism if and only if f is an epimorphism and the coequalizer of f and f is a universal colimit.

(viii) Suppose that $\mathscr{C}$ is complete and well-powered (see (1.1.3)). Then, for every morphism $f : X \to Y$ we may define the *image* of f, which is a monomorphism :

$$\mathrm{Im}(f) \to Y$$

defined as the smallest of the family $\mathscr{F}$ of subobjects $Y' \to Y$ such that f factors through a (necessarily unique) morphism $X \to Y'$. Indeed choose, for every equivalence class $c \in \mathscr{F}$, a representing monomorphism $Y_c \to Y$, and let I be the full subcategory of $\mathscr{C}/Y$ such that $\mathrm{Ob}(I) = \{Y_c \to Y \mid c \in \mathscr{F}\}$; then I is a small category, and the image of f is more precisely the limit of the inclusion functor $\iota : I \to \mathscr{C}/Y$ (which is representable, since $\mathscr{C}$ is complete).

As an exercise, the reader can show that the resulting morphism $X \to \mathrm{Im}(f)$ is an epimorphism.

Example 1.1.26. Let $\mathscr{C}$ be a category, and $X \in \mathrm{Ob}(\mathscr{C})$.

(i) We say that X is an *initial* (resp. *final*) object of $\mathscr{C}$ if $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ (resp. $\mathrm{Hom}_{\mathscr{C}}(Y, X)$) consists of exactly one element, for every $Y \in \mathrm{Ob}(\mathscr{C})$.

(ii) It is easily seen that an initial object X of $\mathscr{C}$ represents the *empty coproduct* in $\mathscr{C}$, *i.e.* the coproduct of an empty family of objects of $\mathscr{C}$, as in example 1.1.24(iv). Dually, a final object represents the *empty product* in $\mathscr{C}$.

(iii) We say that X is *disconnected*, if there exist $A, B \in \mathrm{Ob}(\mathscr{C})$, neither of which is an initial object of $\mathscr{C}$, and such that X represents the coproduct $A \amalg B$. We say that X is *connected*, if X is not disconnected.

Example 1.1.27. (i) The category $\mathbf{Set}$ is complete and cocomplete, and all colimits in $\mathbf{Set}$ are universal. Hence, all epimorphisms of $\mathbf{Set}$ are universal, in view of example 1.1.24(vii).

(ii) Also the category $\mathbf{Cat}$ is complete and cocomplete. For instance, for any pair of functors

$$\mathscr{A} \xrightarrow{F} \mathscr{C} \xleftarrow{G} \mathscr{B}$$

the fibre product (in the category $\mathbf{Cat}$) of F and G is the category :

$$\mathscr{A} \times_{(F,G)} \mathscr{B}$$

whose set of objects is $\mathrm{Ob}(\mathscr{A}) \times_{\mathrm{Ob}(\mathscr{C})} \mathrm{Ob}(\mathscr{B})$; the morphisms $(A, B) \to (A', B')$ are the pairs (f, g) where $f : A \to A'$ (resp. $g : B \to B'$) is a morphism in $\mathscr{A}$ (resp. in $\mathscr{B}$), such that $Ff = Gg$. If the notation is not ambiguous, we may also denote this category by $\mathscr{A} \times_{\mathscr{C}} \mathscr{B}$. In case $\mathscr{B}$ is a subcategory of $\mathscr{C}$ and G is the natural inclusion functor, we also write $F^{-1}\mathscr{B}$ instead of $\mathscr{A} \times_{\mathscr{C}} \mathscr{B}$. See [10, Prop.5.1.7] for a proof of the cocompleteness of $\mathbf{Cat}$.

1.1.28. Let $I, \mathscr{C}$ be two small categories, $F : I \to \mathscr{C}$ a functor, and G any presheaf on $\mathscr{C}$. We deduce a functor

$$\mathrm{Hom}_{\mathscr{C}^\wedge}(G, h \circ F) : I \to \mathbf{Set} \qquad i \mapsto \mathrm{Hom}_{\mathscr{C}^\wedge}(G, h_{F(i)}) \quad \text{for every } i \in I$$

and by inspecting the definitions, we find a natural isomorphism :

$$\lim_I \mathrm{Hom}_{\mathscr{C}^\wedge}(G, h \circ F) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{C}^\wedge}(G, \lim_I F) \tag{1.1.29}$$

(more precisely, the limit on the left is represented by the set on the right). Likewise, we have a natural isomorphism :

$$\lim_{I^o} \mathrm{Hom}_{\mathscr{C}^\wedge}(h \circ F, G) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{C}^\wedge}(\mathrm{colim}_I F, G). \tag{1.1.30}$$

1.1.31. Suppose that $F : \mathscr{A} \to \mathscr{B}$ is right adjoint to a functor $G : \mathscr{B} \to \mathscr{A}$. Then it is easily seen that F commutes with all representable limits of $\mathscr{A}$, in the sense of remark 1.1.23(v). Dually, G commutes with all representable colimits of $\mathscr{B}$. It is also easy to check that F transforms monomorphisms into monomorphisms, and G transforms epimorphisms into epimorphisms.

Conversely, we have the following :

Theorem 1.1.32. *Let $\mathscr{A}$ be a complete category, $F : \mathscr{A} \to \mathscr{B}$ a functor, and suppose that $\mathscr{A}$ and $\mathscr{B}$ have small* Hom*-sets (see* (1.1.1)*). The following conditions are equivalent :*

(a) *F admits a left adjoint.*
(b) *F commutes with all the limits of $\mathscr{A}$, and every object B of $\mathscr{B}$ admits a* solution set, i.e. *an essentially small subset $S_B \subset \mathrm{Ob}(\mathscr{A})$ such that, for every $A \in \mathrm{Ob}(\mathscr{A})$, every morphism $f : B \to FA$ admits a factorization of the form $f = Fh \circ g$, where $h : A' \to A$ is a morphism in $\mathscr{A}$ with $A' \in S_B$, and $g : B \to FA'$ is a morphism in $\mathscr{B}$.*

Proof. This is [10, Th.3.3.3]. Basically, one would like to construct a left adjoint G explicitly as follows. For any $B \in \mathrm{Ob}(\mathscr{B})$ and any morphism $f : B' \to B$, set :

$$GB := \lim_{B/F\mathscr{A}} \iota_B \qquad Gf := \lim_{f/F\mathscr{A}} \mathbf{1}_{\mathscr{A}}.$$

(notation of (1.1.16)). The problem with this is that the categories $B/F\mathscr{A}$ and $f/F\mathscr{A}$ are not necessarily small, so the above limits do not always exist. The idea is to replace the category $B/F\mathscr{A}$ by its full subcategory $\mathscr{E}_B$, whose objects are all the morphisms $B \to FA$ with $A \in S_B$; then it is easily seen that this is isomorphic to a small category, and the limit over $\mathscr{E}_B$ of the restriction of ι_B yields the sought adjoint : see *loc.cit.* for the details. $\square$

Of course, the "dual" of theorem 1.1.32 yields a criterion for the existence of right adjoints.

1.1.33. The notion of *Kan extension* of a given functor yields another frequently used method to produce (left or right) adjoints, which applies to the following situation. Let $f : \mathscr{A} \to \mathscr{B}$ be a functor, and $\mathscr{C}$ a third category. Then we deduce a natural functor

$$f^* : \mathbf{Fun}(\mathscr{B}, \mathscr{C}) \to \mathbf{Fun}(\mathscr{A}, \mathscr{C}) \quad : \quad G \mapsto G \circ f \qquad (\alpha : G \Rightarrow G') \mapsto \alpha * f.$$

Proposition 1.1.34. *In the situation of* (1.1.33)*, suppose that $\mathscr{A}$ is small, $\mathscr{B}$ and $\mathscr{C}$ have small* Hom*-sets, and $\mathscr{C}$ is cocomplete (resp. complete). Then f^* admits a left (resp. right) adjoint.*

Proof. Indeed, if $\mathscr{C}$ is cocomplete, a left adjoint $f_! : \mathbf{Fun}(\mathscr{A}, \mathscr{C}) \to \mathbf{Fun}(\mathscr{B}, \mathscr{C})$ to f^* is given explicitly as follows. For a given functor $F : \mathscr{A} \to \mathscr{C}$, define $f_! F$ by the rule :

$$B \mapsto \operatorname*{colim}_{f\mathscr{A}/B} F \circ \iota_B \qquad \varphi \mapsto (\operatorname*{colim}_{f\mathscr{A}/\varphi} \mathbf{1}_{\mathscr{C}} : \operatorname*{colim}_{f\mathscr{A}/B} F \circ \iota_B \to \operatorname*{colim}_{f\mathscr{A}/B'} F \circ \iota_{B'})$$

for every $B \in \mathrm{Ob}(\mathscr{B})$ and every morphism $\varphi : B \to B'$ in $\mathscr{B}$ (notation of (1.1.16)). This makes sense, since – under the current assumptions – the categories $f\mathscr{A}/B$ and $f\mathscr{A}/\varphi$ are small.

The construction of a right adjoint, in case $\mathscr{C}$ is complete, is dual to the foregoing, by virtue of the isomorphism of categories :

$$\mathbf{Fun}(\mathscr{D}, \mathscr{C}) \xrightarrow{\sim} \mathbf{Fun}(\mathscr{D}^o, \mathscr{C}^o)^o \qquad \text{for every category } \mathscr{D}.$$

See [10, Th.3.7.2] or [51, Th.2.3.3] for the detailed verifications. $\square$

1.1.35. As an application, suppose that $\mathscr{B}$ is a small category, and $\mathscr{C}$ a category with small Hom-sets; any functor $f : \mathscr{B} \to \mathscr{C}$ induces a functor

$$f^*_{\mathsf{U}} : \mathscr{C}^\wedge_{\mathsf{U}} \to \mathscr{B}^\wedge_{\mathsf{U}} \qquad F \mapsto F \circ f^o$$

and it is easily seen that f^*_{U} commutes with all limits and all colimits. From proposition 1.1.34 we obtain both a left and a right adjoint for f^*_{U}, denoted respectively :

$$f_{\mathsf{U}!} : \mathscr{B}^\wedge_{\mathsf{U}} \to \mathscr{C}^\wedge_{\mathsf{U}} \qquad \text{and} \qquad f_{\mathsf{U}*} : \mathscr{B}^\wedge_{\mathsf{U}} \to \mathscr{C}^\wedge_{\mathsf{U}}.$$

As usual, we drop the subscript U, unless the omission may cause ambiguities.

Notice that the diagram of functors :

(1.1.36)
$$\begin{tikzcd} \mathscr{B} \arrow[r, "h_{\mathscr{B}}"] \arrow[d, "f"'] & \mathscr{B}^\wedge \arrow[d, "f_!"] \\ \mathscr{C} \arrow[r, "h_{\mathscr{C}}"] & \mathscr{C}^\wedge \end{tikzcd}$$

(whose horizontal arrows are the Yoneda imbeddings) is *essentially commutative, i.e.* the two compositions $f_! \circ h_{\mathscr{B}}$ and $h_{\mathscr{C}} \circ f$ are isomorphic functors. Indeed – by of proposition 1.1.20(ii) – for every $B \in \mathrm{Ob}(B)$, the objects $f_! h_B$ and h_{fB} both represent the functor

$$\mathscr{C}^\wedge \to \mathbf{Set} \quad : \quad F \mapsto F(fB).$$

Definition 1.1.37. Let $\mathscr{C}$ be a category.

(i) We say that $\mathscr{C}$ is *finite* if both $\mathrm{Ob}(\mathscr{C})$ and $\mathrm{Morph}(\mathscr{C})$ are finite sets.

(ii) We say that $\mathscr{C}$ is *path-connected*, if $\mathrm{Ob}(\mathscr{C}) \neq \varnothing$ and every two objects X, Y can be connected by a finite sequence of morphisms in $\mathscr{C}$, of arbitrary length :

$$X \to Z_1 \leftarrow Z_2 \to \cdots \leftarrow Z_n \to Y$$

(iii) We say that $\mathscr{C}$ is *directed*, if for every $X, Y \in \mathrm{Ob}(\mathscr{C})$ there exist $Z \in \mathrm{Ob}(\mathscr{C})$ and morphisms $X \to Z$, $Y \to Z$ in $\mathscr{C}$. We say that $\mathscr{C}$ is *codirected*, if $\mathscr{C}^o$ is directed.

(iv) We say that $\mathscr{C}$ is *locally directed* (resp. *locally codirected*) if, for every $X \in \mathrm{Ob}(\mathscr{C})$, the category $X/\mathscr{C}$ is directed (resp. codirected).

(v) We say that $\mathscr{C}$ is *pseudo-filtered* if it is locally directed, and the following holds. For any $X, Y \in \mathrm{Ob}(\mathscr{C})$, and any two morphisms $f, g : X \to Y$, there exists $Z \in \mathrm{Ob}(\mathscr{C})$ and a morphism $h : Y \to Z$ such that $h \circ f = h \circ g$.

(vi) We say that $\mathscr{C}$ is *filtered*, if it is pseudo-filtered and path-connected (in which case, $\mathscr{C}$ is also directed). We say that $\mathscr{C}$ is *cofiltered* if $\mathscr{C}^o$ is filtered (in which case, $\mathscr{C}$ is also codirected).

(vii) A limit $\lim_{\mathscr{C}} F$ is *path-connected* (resp. *codirected*, resp. *locally codirected*, resp. *cofiltered*, resp. *finite*), if $\mathscr{C}$ is path-connected (resp. codirected, resp. locally codirected, resp. cofiltered, resp. finite). Dually, one defines *path-connected*, *directed*, *locally directed*, *filtered* and *finite* colimits.

(viii) We say that a functor $F : \mathscr{A} \to \mathscr{B}$ is *left exact*, if F commutes with all finite limits, in the sense of remark 1.1.23(v). Dually, F is *right exact* if it commutes with all finite colimits. Finally, F is *exact* if it is both left and right exact.

Remark 1.1.38. (i) Notice that, if I is a filtered category, and i is any object of I, the category i/I is again filtered; dually, if I is cofiltered, the category I/i is again cofiltered. Furthermore, let $F : I \to \mathscr{A}$ be any functor; there follow functors $F \circ \mathsf{t} : i/I \to \mathscr{A}$ and $F \circ \mathsf{s} : I/i \to \mathscr{A}$ (notation of (1.1.17)), and we have natural identifications :

$$\operatorname*{colim}_{I} F \xrightarrow{\sim} \operatorname*{colim}_{i/I} F \circ \mathsf{t} \qquad \lim_{I} F \xrightarrow{\sim} \lim_{I/i} F \circ \mathsf{s}.$$

(ii) Let $\mathscr{C}$ be a small category. There is a natural decomposition in $\mathbf{Cat}$:

$$\mathscr{C} \xrightarrow{\sim} \coprod_{i \in I} \mathscr{C}_i$$

where each $\mathscr{C}_i$ is a path-connected category, and I is a small set. This decomposition induces natural isomorphisms in $\mathscr{A}^\wedge$:

$$\operatorname*{colim}_{\mathscr{C}} F \xrightarrow{\sim} \coprod_{i \in I} \operatorname*{colim}_{\mathscr{C}_i} F \circ e_i \qquad \lim_{\mathscr{C}} F \xrightarrow{\sim} \prod_{i \in I} \lim_{\mathscr{C}_i} F \circ e_i$$

for any functor $F : \mathscr{C} \to \mathscr{A}$, where $e_i : \mathscr{C}_i \to \mathscr{C}$ is the natural inclusion functor, for every $i \in I$. The details shall be left to the reader. These simple observations often simplify the calculation of limits and colimits.

(iii) Let $\mathscr{C}$ be a complete category (more generally, a category in which all fibre products are representable), and $F : \mathscr{C} \to \mathscr{B}$ a left exact functor. If follows formally from example 1.1.24(iii) that F transforms monomorphisms into monomorphisms. If F is also conservative, then a morphism $\varphi : X \to Y$ in $\mathscr{C}$ is a monomorphism if and only if the same holds for $F\varphi$.

(iv) Dually, if F is right exact, and all coproducts are representable in $\mathscr{C}$, example 1.1.24(iii) implies that F transforms epimorphism into epimorphisms, and if F is also conservative, then a morphism $\varphi : X \to Y$ in T is an epimorphism if and only if the same holds for $F\varphi$.

(v) In the situation of (1.1.33), suppose that the category $f\mathscr{A}/B$ is finite for every $B \in \mathrm{Ob}(\mathscr{B})$; then, by inspecting the proof of proposition 1.1.34, we see that f^* admits a left adjoint, provided all finite colimits of $\mathscr{C}$ are representable and $\mathscr{C}$ has small Hom-sets. Likewise, we can weaken the condition for the existence of the right adjoint : for the latter, it suffices that all finite limits of $\mathscr{C}$ are representable and $\mathscr{C}$ has small Hom-sets.

1.2. **Tensor categories and abelian categories.** In this section we assemble some basic definitions and results that pertain to abelian categories and other related classes of categories with extra structure.

Definition 1.2.1. A *tensor category* is a datum $\underline{\mathscr{C}} := (\mathscr{C}, \otimes, \Phi, \Psi)$ consisting of a category $\mathscr{C}$, a functor

$$\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C} \quad : \quad (X, Y) \mapsto X \otimes Y$$

and natural isomorphisms :

$$\Phi_{X,Y,Z} : X \otimes (Y \otimes Z) \stackrel{\sim}{\to} (X \otimes Y) \otimes Z \qquad \Psi_{X,Y} : X \otimes Y \stackrel{\sim}{\to} Y \otimes X$$

for every $X, Y, Z \in \mathrm{Ob}(\mathscr{C})$, called respectively the *associativity* and *commutativity constraints* of $\underline{\mathscr{C}}$, that satisfy the following axioms.

(a) *Coherence axiom* : the diagram

$$\begin{tikzcd}
X \otimes (Y \otimes (Z \otimes T)) \arrow[r, "\Phi_{X,Y,Z\otimes T}"] \arrow[d, "X\otimes\Phi_{Y,Z,T}"'] & (X \otimes Y) \otimes (Z \otimes T) \arrow[r, "\Phi_{X\otimes Y,Z,T}"] & ((X \otimes Y) \otimes Z) \otimes T \\
X \otimes ((Y \otimes Z) \otimes T) \arrow[rr, "\Phi_{X,Y\otimes Z,T}"] & & (X \otimes (Y \otimes Z)) \otimes T \arrow[u, "\Phi_{X,Y,Z}\otimes T"']
\end{tikzcd}$$

commutes, for every $X, Y, Z, T \in \mathrm{Ob}(\mathscr{C})$.

(b) *Compatibility axiom* : the diagram

$$\begin{tikzcd}
X \otimes (Y \otimes Z) \arrow[r, "\Phi_{X,Y,Z}"] \arrow[d, "X\otimes\Psi_{Y,Z}"'] & (X \otimes Y) \otimes Z \arrow[r, "\Psi_{X\otimes Y,Z}"] & Z \otimes (X \otimes Y) \arrow[d, "\Phi_{Z,X,Y}"] \\
X \otimes (Z \otimes Y) \arrow[r, "\Phi_{X,Z,Y}"] & (X \otimes Z) \otimes Y \arrow[r, "\Psi_{X,Z\otimes Y}"] & (Z \otimes X) \otimes Y
\end{tikzcd}$$

commutes, for every $X, Y, Z \in \mathrm{Ob}(\mathscr{C})$.

(c) *Commutation axiom* : we have $\Psi_{Y,X} \circ \Psi_{X,Y} = \mathbf{1}_{X\otimes Y}$ for every $X, Y \in \mathrm{Ob}(\mathscr{C})$

(d) *Unit axiom* : there exists an object $U \in \mathrm{Ob}(\mathscr{C})$ and an isomorphism $u : U \stackrel{\sim}{\to} U \otimes U$ such that the functor

$$\mathscr{C} \to \mathscr{C} \quad : \quad X \mapsto U \otimes X \tag{1.2.2}$$

is an equivalence. One says that (U, u) is a *unit object* of $\underline{\mathscr{C}}$.

Lemma 1.2.3. *Let $\underline{\mathscr{C}} := (\mathscr{C}, \otimes, \Phi, \Psi)$ be any tensor category. The diagram*

$$\begin{tikzcd}
X \otimes (Y \otimes Z) \arrow[r, "\Phi_{X,Y,z}"] \arrow[d, "X\otimes\Psi_{Y,Z}"'] & (X \otimes Y) \otimes Z \arrow[r, "\Psi_{X,Y\otimes Z}"] & (Y \otimes X) \otimes Z \\
X \otimes (Z \otimes Y) \arrow[r, "\Phi_{X,Z,Y}"] & (X \otimes Z) \otimes Y \arrow[r, "\Psi_{X\otimes Z,Y}"] & Y \otimes (X \otimes Z) \arrow[u, "\Phi_{Y,X,Z}"']
\end{tikzcd}$$

commutes, for every $X, Y, Z \in \mathrm{Ob}(\mathscr{C})$.

Proof. To ease notation, we shall omit the tensor symbol $\otimes$ between objects, and we shall drop the subscript from Φ and Ψ when we display a diagram. It suffices to consider the diagram

$$\begin{tikzcd}
 & Y(XZ) \arrow[rrr, "\Phi"] \arrow[d, "Y\otimes\Psi"] & & & (YX)Z \arrow[d, "\Psi"'] & \\
 & Y(ZX) \arrow[r, "\Phi"] & (YZ)X \arrow[r, "\Psi\otimes X"] & (ZY)X & Z(YX) \arrow[l, "\Phi"'] & \\
(XZ)Y \arrow[ruu, "\Psi"] \arrow[r, "\Psi\otimes Y"] & (ZX)Y \arrow[u, "\Psi"'] & & & Z(XY) \arrow[lll, "\Phi"'] \arrow[u, "Z\otimes\Psi"] & (XY)Z \arrow[l, "\Psi"'] \arrow[luu, "\Psi\otimes Z"'] \\
X(ZY) \arrow[u, "\Phi"] & & & & & X(YZ) \arrow[lllll, "X\otimes\Psi"'] \arrow[u, "\Phi"']
\end{tikzcd}$$

whose two triangular subdiagrams commute by naturality of Ψ, and whose three rectangular subdiagrams commute by compatibility. $\square$

Definition 1.2.4. Let $\underline{\mathscr{C}} := (\mathscr{C}, \otimes, \Phi, \Psi)$ and $\underline{\mathscr{C}}' := (\mathscr{C}', \otimes', \Phi', \Psi')$ be two tensor categories. A *tensor functor* $\underline{\mathscr{C}} \to \underline{\mathscr{C}}'$ is a pair (F, c) consisting of a functor $F : \mathscr{C} \to \mathscr{C}'$ and a natural isomorphism

$$c_{X,Y} : FX \otimes FY \xrightarrow{\sim} F(X \otimes Y) \qquad \text{for all } X, Y \in \mathrm{Ob}(\mathscr{C})$$

such that the following holds.

(a) For every objects X, Y, Z of $\mathscr{C}$, the diagram

$$\begin{tikzcd} FX \otimes (FY \otimes FZ) \arrow[r, "FX \otimes c_{YZ}"] \arrow[d, "\Phi'_{FX,FY,FZ}"'] & FX \otimes F(Y \otimes Z) \arrow[r, "c_{X,Y\otimes Z}"] & F(X \otimes (Y \otimes Z)) \arrow[d, "F(\Phi_{X,Y,Z})"] \\ (FX \otimes FY) \otimes FZ \arrow[r, "c_{X,Y} \otimes FZ"] & F(X \otimes Y) \otimes FZ \arrow[r, "c_{X\otimes Y,Z}"] & F((X \otimes Y) \otimes Z) \end{tikzcd}$$

commutes.

(b) For all objects X, Y of $\mathscr{C}$, the diagram

$$\begin{tikzcd} FX \otimes FY \arrow[r, "c_{X,Y}"] \arrow[d, "\Psi_{FX,FY}"'] & F(X \otimes Y) \arrow[d, "F(\Psi_{X,Y})"] \\ FY \otimes FX \arrow[r, "c_{Y,X}"] & F(Y \otimes X) \end{tikzcd}$$

commutes.

(c) If (U, u) is a unit object of $\underline{\mathscr{C}}$, then $(FU, c^{-1}_{U,U} \circ Fu)$ is a unit object of $\underline{\mathscr{C}}'$.

Remark 1.2.5. (i) Lemma 1.2.3 illustrates a general principle valid in every tensor categor $\underline{\mathscr{C}}$: namely, say that $X_1, \dots, X_n$ is a sequence of *distinct* objects of $\mathscr{C}$, and X' and X'' are obtained from these two sequences by taking tensor products several times, and in any order, in which case we say that X' and X'' *have no repetitions*. Now, there will be usually various ways to combine the associativity and commutativity constraints, in order to exhibit some isomorphism $X' \xrightarrow{\sim} X''$. However, the resulting isomorphism shall be independent of the way in which it is expressed as such a combination. This follows from a theorem of Mac Lane. To formalize this result, one could observe that, for any set Σ, there exists a universal tensor category T_Σ "generated by Σ", *i.e.* such that – for any other tensor category $\underline{\mathscr{C}}$ – any mapping $\Sigma \to \mathrm{Ob}(\mathscr{C})$ extends uniquely, up to isomorphism, to a tensor functor $T_\Sigma \to \mathscr{C}$. Then Mac Lane's theorem says that, for every set Σ, and every object $X \in \mathrm{Ob}(T_\Sigma)$ that has no repetitions, the group $\mathrm{Aut}_{T_\Sigma}(X)$ is trivial. Instead of relying on such a general result, we shall make *ad hoc* verifications, as in the proof of lemma 1.2.3 and of the forthcoming proposition 1.2.6. However, in view of this principle, in the following we shall often omit a detailed description of the isomorphism that we choose to connect two given objects that are thus related : the reader will be able in any case to produce one isomorphism, and the principle says that these choices cannot be source of ambiguities.

(ii) Let $\mathscr{D}$ be any category, and $\underline{\mathscr{C}}$ a tensor category. Then notice that $\mathbf{Fun}(\mathscr{D}, \mathscr{C})$ inherits from $\underline{\mathscr{C}}$ a natural tensor category structure : we leave to the reader the task of spelling out the details. Moreover notice that, if $(F, c) : \underline{\mathscr{C}}_1 \to \underline{\mathscr{C}}_2$ and $(F', c') : \underline{\mathscr{C}}_2 \to \underline{\mathscr{C}}_3$ are any two tensor functors between tensor categories, then the composition

$$(F' \circ F, (F' * c) \circ (c' * (F \times F))) : \underline{\mathscr{C}}_1 \to \underline{\mathscr{C}}_3$$

is again a tensor functor.

Proposition 1.2.6. *Let (U, u) be a unit object of a tensor category $\underline{\mathscr{C}}$. Then there exists a unique natural isomorphism*

$$u_X : X \xrightarrow{\sim} U \otimes X \qquad \textit{for every } X \in \mathrm{Ob}(\mathscr{C})$$

such that $u_U = u$, and such that the diagrams

$$
\begin{tikzcd}
X \otimes Y \arrow[r, "u_{X\otimes Y}"] \arrow[dr, "u_X \otimes Y"'] & U \otimes (X \otimes Y) \arrow[d, "\Phi_{U,X,Y}"] \\
& (U \otimes X) \otimes Y
\end{tikzcd}
\qquad
\begin{tikzcd}
X \otimes Y \arrow[r, "u_X \otimes Y"] \arrow[d, "X \otimes u_Y"'] & (U \otimes X) \otimes Y \arrow[d, "\Psi_{U,X} \otimes Y"] \\
X \otimes (U \otimes Y) \arrow[r, "\Phi_{X,U,Y}"] & (X \otimes U) \otimes Y
\end{tikzcd}
$$

commute for every $X, Y \in \mathrm{Ob}(\mathscr{C})$.

Proof. Since (1.2.2) is an equivalence, there exists a unique isomorphism u_X fitting into the commutative diagram

$$
\begin{tikzcd}
UX \arrow[r, "u\otimes X"] \arrow[dr, "U \otimes u_X"'] & (UU)X \\
& U(UX). \arrow[u, "\Phi"']
\end{tikzcd}
$$

With this definition, the naturality of the rule : $X \mapsto u_X$ is clear. In order to check the commutativity of the first diagram, it suffices to show that

(1.2.7) $$(U \otimes \Phi_{U,X,Y}) \circ (U \otimes u_{XY}) = U \otimes (u_X \otimes Y).$$

However, set $\Theta := (\Phi_{U,U,X} \otimes Y) \circ \Phi_{U,UX,Y}$; we have :

$$
\begin{aligned}
\Theta \circ (U \otimes (u_X \otimes Y)) &= (\Phi_{U,U,X} \otimes Y) \circ ((U \otimes u_X) \otimes Y) \circ \Phi_{U,X,Y} \\
&= ((u \otimes X) \otimes Y) \circ \Phi_{U,X,Y} \\
&= \Phi_{UU,X,Y} \circ (u \otimes (XY)) \\
&= \Phi_{UU,X,Y} \circ \Phi_{U,U,XY} \circ (U \otimes u_{XY}) \\
&= \Theta \circ (U \otimes \Phi_{U,X,Y}) \circ (U \otimes u_{XY})
\end{aligned}
$$

where the first and third identities hold by naturality of Φ, the second and fourth by the definition of u_X and respectively $u_{X\otimes Y}$, and the fifth by coherence. Since Θ is an isomorphism, we get (1.2.7). Next, in light of the foregoing, the second diagram commutes if and only if the diagram

(1.2.8)
$$
\begin{tikzcd}
XY \arrow[r, "u_{XY}"] \arrow[dr, "X\otimes u_Y"'] & U(XY) \arrow[r, "\Phi"] & (UX)Y \arrow[d, "\Psi\otimes Y"] \\
& X(UY) \arrow[r, "\Phi"] & (XU)Y
\end{tikzcd}
$$

commutes, and it suffices to check that $U \otimes$ (1.2.8) commutes. To this aim, we consider the diagram

$$
\begin{tikzcd}
U(XY) \arrow[rr, "U\otimes(X\otimes u_Y)"] \arrow[dr, "U\otimes u_{XY}"] \arrow[dd, "u\otimes(XY)"'] & & U(X(UY)) \arrow[r, "U\otimes\Phi"] & U((XU)Y) \arrow[d, "U\otimes(\Psi\otimes Y)"] \\
& U(U(XY)) \arrow[rr, "U\otimes\Phi"] \arrow[dl, "\Phi"'] & & U((UX)Y) \arrow[d, "\Phi"] \\
(UU)(XY) \arrow[rr, "\Phi"] & & ((UU)X)Y & (U(UX))Y. \arrow[l, "\Phi\otimes Y"']
\end{tikzcd}
$$

whose lower subdiagram commutes by the coherence axiom for (U, U, X, Y), whose upper subdiagram is equivalent to (1.2.8), since $\Psi^{-1}_{U,X} = \Psi_{X,U}$, and whose triangular subdiagram commutes by definition of u_{XY}. Hence, we are reduced to showing that the outer rectangular subdiagram of the above diagram commutes. However, we have a commutative diagram :

$$\begin{tikzcd}
U(X(UY)) \arrow[d, "\Phi"'] & U(XY) \arrow[l, "U\otimes(X\otimes u_Y)"'] \arrow[r, "u\otimes(XY)"] \arrow[d, "\Phi"] & (UU)(XY) \arrow[d, "\Phi"] \\
(UX)(UY) \arrow[d, "\Psi\otimes(UY)"'] & (UX)Y \arrow[l, "(UX)\otimes u_Y"'] \arrow[r, "(u\otimes X)\otimes Y"] \arrow[d, "\Psi\otimes Y"'] & ((UU)X)Y \arrow[d, "\Psi\otimes Y"] \\
(XU)(UY) & (XU)Y \arrow[l, "(XU)\otimes u_Y"'] \arrow[r, "(X\otimes u)\otimes Y"] & (X(UU))Y \\
& X(UY) \arrow[u, "\Phi"] \arrow[dl, "X\otimes(U\otimes u_Y)"'] \arrow[dr, "X\otimes(u\otimes Y)"] & \\
X(U(UY)) \arrow[uu, "\Phi"] \arrow[rr, "X\otimes\Phi"] & & X((UU)Y) \arrow[uu, "\Phi"']
\end{tikzcd}$$

so we are further reduced to checking the commutativity of the diagram :

$$\begin{tikzcd}
& U(X(UY)) \arrow[r, "U\otimes\Phi"] \arrow[dl, "\Phi"'] & U((XU)Y) \arrow[r, "U\otimes(\Psi\otimes Y)"] & U((UX)Y) \arrow[dr, "\Phi"] & \\
(UX)(UY) \arrow[d, "\Psi\otimes(UY)"'] & U((UY)X) \arrow[u, "U\otimes\Psi"] \arrow[d, "\Phi"'] & U(U(YX)) \arrow[l, "U\otimes\Phi"'] \arrow[r, "U\otimes(U\otimes\Psi)"] \arrow[d, "\Phi"'] & U(U(XY)) \arrow[u, "U\otimes\Phi"'] \arrow[d, "\Phi"'] & (U(UX))Y \arrow[d, "\Phi\otimes Y"] \\
(XU)(UY) & (U(UY))X \arrow[d, "\Psi"'] \arrow[dr, "\Phi\otimes X"] & (UU)(YX) \arrow[r, "(UU)\otimes\Psi"] \arrow[d, "\Phi"'] & (UU)(XY) \arrow[r, "\Phi"] & ((UU)X)Y \arrow[dl, "\Psi\otimes Y"] \\
& X(U(UY)) \arrow[ul, "\Phi"] \arrow[dr, "X\otimes\Phi"'] & ((UU)Y)X \arrow[d, "\Psi"'] & (X(UU))Y & \\
& & X((UU)Y). \arrow[ur, "\Phi"'] & &
\end{tikzcd}$$

However, the leftmost and the lower triangular subdiagrams commute by compatibility, and the upper rightmost subdiagram commutes by coherence. The lower leftmost subdiagram commutes by naturality of Ψ, and the central square sudiagram commutes by naturality of Φ. The remaining central subdiagram commutes by coherence, and the top rectangular subdiagram is of the form $U \otimes D$, where D is a diagram that commutes by virtue of lemma 1.2.3.

The uniqueness of u_X is clear by inspecting the second diagram, with $Y = U$. $\square$

Remark 1.2.9. (i) Keep the notation of proposition 1.2.6. As a consequence of the naturality of u_X, we get a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "u_X"] \arrow[d, "u_X"'] & U \otimes X \arrow[d, "u\otimes u_X"] \\
U \otimes X \arrow[r, "u_{U\otimes X}"] & U \otimes (U \otimes X)
\end{tikzcd}$$

for every object X of $\mathscr{C}$. In other words :

$$u_{U\otimes X} = U \otimes u_X \qquad \text{for every } X \in \mathrm{Ob}(\mathscr{C}).$$

(ii) Let $X = Y = U$ in the second diagram of proposition 1.2.6; we obtain a commutative diagram

$$\begin{tikzcd} U \otimes U \arrow[r, "u\otimes U"] \arrow[d, "U\otimes u"'] & (U \otimes U) \otimes U \arrow[d, "\Psi_{U,U}\otimes U"] \\ U \otimes (U \otimes U) \arrow[r, "\Phi_{U,U,U}"] & (U \otimes U) \otimes U \end{tikzcd}$$

Since $u = u_U$, we may combine with (i), to deduce that $\Psi_{U,U} \otimes U = \mathbf{1}_{(U\otimes U)\otimes U} = \mathbf{1}_{U\otimes U} \otimes U$. By naturality of Ψ, it follows that $U \otimes \Psi_{U,U} = U \otimes \mathbf{1}_{U\otimes U}$, and since (1.2.2) is an equivalence, we conclude that

$$\Psi_{U,U} = \mathbf{1}_{U\otimes U}.$$

Example 1.2.10. Let $\mathscr{C}$ be any category with small Hom-sets, in which finite products are representable. For every pair (X, Y) of objects of $\mathscr{C}$, pick an object $X \otimes Y$ representing their product, and fix also two projections $p_{X,Y} : X \otimes Y \to X$, $q_{X,Y} : X \otimes Y \to Y$ inducing an isomorphism of functors

$$h_{X\otimes Y} \xrightarrow{\sim} h_X \times h_Y \quad : \quad \varphi \mapsto (p_{X,Y}\circ\varphi, q_{X,Y}\circ\varphi) \qquad \text{for every } Z \in \mathrm{Ob}(\mathscr{C}) \text{ and } \varphi \in h_{X\times Y}(Z)$$

(notation of (1.1.19)). If (X', Y') is another such pair, and $(g, h) : (X, Y) \to (X', Y')$ any morphism in $\mathscr{C} \times \mathscr{C}$, then there exists a unique morphism $f : X \otimes Y \to X' \otimes Y'$ such that

$$(p_{X',Y'} \circ f, q_{X',Y'} \circ f) = (g \circ p_{X,Y}, h \circ p_{X,Y})$$

and we set $g \otimes h := f$. These rules define a functor $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$. For every three objects X, Y, Z there is a natural isomorphism $X \otimes (Y \otimes Z) \xrightarrow{\sim} (X \otimes Y) \otimes Z$ that yields an associativity constraint for $\otimes$; namely, we let

$$\Phi_{X,Y,Z} := (p_{X,Y\otimes Z} \otimes (p_{Y,Z} \otimes q_{X,Y\otimes Z})) \otimes (q_{Y,Z} \circ q_{X,Y\otimes Z}).$$

Likewise, we get a commutativity constraint by setting

$$\Psi_{X,Y} := q_{X,Y} \otimes p_{X,Y} \qquad \text{for every } X, Y \in \mathrm{Ob}(\mathscr{C}).$$

The verifications of the axioms of definition 1.2.1 are lengthy but straightforward, and shall be left to the reader. If U is any final object of $\mathscr{C}$ (example 1.1.26(i)), then there exists a unique morphism $u : U \to U \otimes U$ which is easily seen to be an isomorphism, and the pair (U, u) yields a unit for $\otimes$. In this way, any category with finite products and small Hom-sets is naturally endowed with a structure of tensor category. Notice that, for this tensor structure on $\mathscr{C}$ (and indeed, for most of the tensor categories that are found in applications), the existence of functorial isomorphisms u_X fulfilling the conditions of proposition 1.2.6, is self-evident.

Definition 1.2.11. Let $(\mathscr{C}, \otimes, \Phi, \Psi)$ be a tensor category, $X \in \mathrm{Ob}(\mathscr{C})$ any object, and suppose that the functor

$$- \otimes X : \mathscr{C} \to \mathscr{C} \qquad Y \mapsto Y \otimes X$$

admits a right adjoint :

$$\mathscr{H}\!om(X, -) : \mathscr{C} \to \mathscr{C} \qquad Y \mapsto \mathscr{H}\!om(X, Y).$$

Then, we call $\mathscr{H}\!om(X, -)$ the *internal* Hom *functor* for the object X.

Remark 1.2.12. (i) As usual, the internal Hom functor is determined up to unique isomorphism, if it exists. The counit of adjunction is a morphism of $\mathscr{C}$

$$\mathrm{ev}_{X,Y} : \mathscr{H}\!om(X, Y) \otimes X \to Y$$

called the *evaluation morphism*.

(ii) Suppose that every object of $\mathscr{C}$ admits an internal Hom functor; then we say briefly that $\mathscr{C}$ *admits an internal* Hom *functor*, and clearly we get a functor

$$\mathscr{C}^o \times \mathscr{C} \to \mathscr{C} \quad : \quad (X,Y) \mapsto \mathscr{H}om(X,Y) \qquad \text{for every } X,Y \in \mathrm{Ob}(\mathscr{C}).$$

Moreover, for every $X,Y,Z \in \mathrm{Ob}(\mathscr{C})$ the composition

$$\begin{tikzcd}
(\mathscr{H}om(X,Y) \otimes \mathscr{H}om(Y,Z)) \otimes X \arrow[rr, "\sim"] & & (\mathscr{H}om(X,Y) \otimes X) \otimes \mathscr{H}om(Y,Z) \arrow[d, "\mathrm{ev}_{X,Y} \otimes \mathscr{H}om(Y,Z)"] \\
Z & \mathscr{H}om(Y,Z) \otimes Y \arrow[l, "\mathrm{ev}_{Y,Z}"'] & Y \otimes \mathscr{H}om(Y,Z) \arrow[l, "\sim"']
\end{tikzcd}$$

corresponds, by adjunction, to a unique *composition morphism*

$$\mathscr{H}om(X,Y) \otimes \mathscr{H}om(Y,Z) \to \mathscr{H}om(X,Z).$$

(iii) In the situation of (ii), notice that the functor $\mathscr{C} \to \mathscr{C}$ given by the rule : $Z \mapsto \mathscr{H}om(X, \mathscr{H}om(Y,Z))$, for every $Z \in \mathrm{Ob}(\mathscr{C})$, is right adjoint to the functor given by the rule : $Z \mapsto (Z \otimes X) \otimes Y \stackrel{\sim}{\to} T \otimes (X \otimes Y)$. There follows a natural isomorphism

$$\mathscr{H}om(X, \mathscr{H}om(Y,Z)) \stackrel{\sim}{\to} \mathscr{H}om(X \otimes Y, Z) \qquad \text{for every } X,Y,Z \in \mathrm{Ob}(\mathscr{C}).$$

(iv) Moreover, for any unit (U,u) of $\underline{\mathscr{C}}$, we get natural bijections :

$$\mathrm{Hom}_{\mathscr{C}}(U, \mathscr{H}om(X,Y)) \stackrel{\sim}{\to} \mathrm{Hom}_{\mathscr{C}}(U \otimes X, Y) \stackrel{\sim}{\to} \mathrm{Hom}_{\mathscr{C}}(X,Y) \qquad \text{for every } X,Y \in \mathrm{Ob}(\mathscr{C}).$$

Also, for every object Y of $\mathscr{C}$, denote by $u_Y : Y \stackrel{\sim}{\to} U \otimes Y$ the isomorphism given by proposition 1.2.6; for every $X \in \mathrm{Ob}(X)$, it induces natural bijections

$$\mathrm{Hom}_{\mathscr{C}}(Y, \mathscr{H}om(U,X)) \stackrel{\sim}{\to} \mathrm{Hom}_{\mathscr{C}}(Y \otimes U, X) \xrightarrow{\mathrm{Hom}_{\mathscr{C}}(u_Y \circ \Psi_{U,Y}, X)} \mathrm{Hom}_{\mathscr{C}}(Y,X)$$

which correspond, via the Yoneda embedding, to a natural isomorphism

$$\mathscr{H}om(U,X) \stackrel{\sim}{\to} X \qquad \text{for every } X \in \mathrm{Ob}(\mathscr{C}).$$

(v) Let X,Y,Z be any three objects of $\mathscr{C}$; the natural transformation

$$\mathrm{Hom}_{\mathscr{C}}(W \otimes X, Y) \to \mathrm{Hom}_{\mathscr{C}}(W \otimes (X \otimes Z), Y \otimes Z) \qquad \varphi \mapsto (\varphi \otimes Z) \circ \Phi_{W,X,Z}$$

corresponds, via the Yoneda imbedding, to a unique morphism

$$t_{X,Y,Z} : \mathscr{H}om(X,Y) \to \mathscr{H}om(X \otimes Z, Y \otimes Z).$$

The reader can check that $t_{X,Y,Z}$ also corresponds, by adjunction, to the morphism

$$(\mathrm{ev}_{X,Y} \otimes Z) \circ \Phi_{\mathscr{H}om(X,Y),X,Z} : \mathscr{H}om(X,Y) \otimes (X \otimes Z) \to Y \otimes Z.$$

(vi) In the situation of remark 1.2.5(ii), suppose that $\mathscr{C}$ admits an internal Hom functor; then it is easily seen that the resulting tensor category $\mathbf{Fun}(\mathscr{D}, \mathscr{C})$ inherits as well an internal Hom functor, in the obvious way.

The formalism of tensor categories provides the language to deal uniformly with the notions of algebras and their modules that occur in various concrete settings.

Definition 1.2.13. Let $(\mathscr{C}, \otimes, \Phi, \Psi)$ be a tensor category, A and B any two objects of $\mathscr{C}$.

(i) A *left A-module* (resp. a *right B-module*) is a datum (X, μ_X), consisting of an object X of $\mathscr{C}$, and a morphism in $\mathscr{C}$:

$$\mu_X : A \otimes X \to X \qquad (\text{resp. } \mu_X : X \otimes B \to X)$$

called the *scalar multiplication* of X.

(ii) A *morphism of left A-modules* $(X, \mu_X) \to (X', \mu_{X'})$ is a morphism $f : X \to X'$ in $\mathscr{C}$ which makes commute the diagram :

$$\begin{CD} A \otimes X @>{\mu_X}>> X \\ @V{\mathbf{1}_A \otimes f}VV @VV{f}V \\ A \otimes X' @>{\mu_{X'}}>> X'. \end{CD}$$

One defines likewise morphisms of right B-modules.

(iii) An (A, B)-*bimodule* is a datum (X, μ^l_X, μ^r_X) such that (X, μ^l_X) is a left A-module, (X, μ^r_X) is a right B-module, and the scalar multiplications commute, *i.e.* the diagram

$$\begin{tikzcd} A \otimes (X \otimes B) \arrow[rr, "\mathbf{1}_A \otimes \mu^r_X"] \arrow[d, "\Phi_{A,X,B}"'] & & A \otimes X \arrow[d, "\mu^l_X"] \\ (A \otimes X) \otimes B \arrow[r, "\mu^l_X \otimes \mathbf{1}_B"] & X \otimes B \arrow[r, "\mu^r_X"] & X \end{tikzcd}$$

commutes. Of course, a morphism of (A, B)-bimodules must be compatible with both left and right multiplication.

We denote by A-$\mathbf{Mod}_l$ (resp. B-$\mathbf{Mod}_r$, resp. (A, B)-$\mathbf{Mod}$) the category of left A-modules (resp. right B-modules, resp. (A, B)-bimodules). For any two left A-modules (resp. right B-modules, resp. (A, B)-bimodules) X and X', we shall write

$$\mathrm{Hom}_{A_l}(X, X') \qquad (\text{ resp. } \mathrm{Hom}_{B_r}(X, X')\ ) \qquad (\text{ resp. } \mathrm{Hom}_{(A,B)}(X, X')\ )$$

for the set of morphisms of left A-modules (resp. of right B-modules, resp. of (A, B)-bimodules) $X \to X'$.

1.2.14. In the situation of definition 1.2.13, notice that

$$\mathrm{Hom}_{B_r}(X, X') = \mathrm{Equal}(\ \mathrm{Hom}_{\mathscr{C}}(X, X') \overset{\alpha}{\underset{\beta}{\rightrightarrows}} \mathrm{Hom}_{\mathscr{C}}(X \otimes B, X')\ )$$

where α (resp. β) is given by the rule :

$$f \mapsto f \circ \mu_X \qquad (\text{ resp. } f \mapsto \mu_{X'} \circ (f \otimes B)\ ) \qquad \text{for every } f \in \mathrm{Hom}_{\mathscr{C}}(X, X')$$

and similarly for left A-modules. Now, suppose that all equalizers in $\mathscr{C}$ are representable, and that $\mathscr{C}$ admits an internal Hom functor; then we may define

$$\mathscr{H}om_{B_r}(X, X') := \mathrm{Equal}(\ \mathscr{H}om(X, X') \overset{\alpha}{\underset{\beta}{\rightrightarrows}} \mathscr{H}om(X \otimes B, X')\ )$$

where $\alpha := \mathscr{H}om(\mu_X, X')$ and $\beta := \mathscr{H}om(X \otimes B, \mu_{X'}) \circ t_{X,X',B}$ (notation of remark 1.2.12(v)). Then, it is easily seen that the bijections of remark 1.2.12(iv) induce natural identifications

$$\mathrm{Hom}_{\mathscr{C}}(U, \mathscr{H}om_{B_r}(X, X')) \xrightarrow{\sim} \mathrm{Hom}_{B_r}(X, X') \qquad \text{for every } X, X' \in \mathrm{Ob}(B_r\text{-}\mathbf{Mod}).$$

Likewise we may represent in $\mathscr{C}$ the set of morphisms between two left A-modules, and two (A, B)-modules (details left to the reader).

1.2.15. Let $\mathscr{C}$ be a tensor category as in (1.2.14) and A, B, C any three objects of $\mathscr{C}$; suppose that (X, μ^l_X, μ^r_X) is an (A, B)-bimodule, and $(X', \mu^l_{X'}, \mu^r_{X'})$ a (C, B)-bimodule. Then we claim that $\mathscr{H} := \mathscr{H}\!om_{B_r}((X, \mu^r_X), (X', \mu^r_{X'}))$ is naturally a (C, A)-bimodule. For this, we have to exhibit natural morphisms

$$C \otimes \mathscr{H} \xrightarrow{\mu_l} \mathscr{H} \xleftarrow{\mu_r} \mathscr{H} \otimes A$$

fulfilling the condition of definition 1.2.13(iii). However, by adjunction, the datum of μ_l is the same as that of a morphism $C \to \mathscr{H}\!om(\mathscr{H}, \mathscr{H})$, and since the functor $\mathscr{H}\!om(X, -)$ is left exact, the latter is the same as a morphism

$$C \to \mathrm{Equal}(\ \mathscr{H}\!om(\mathscr{H}, \mathscr{H}\!om(X, X')) \underset{\mathscr{H}om(\mathscr{H},\beta)}{\overset{\mathscr{H}om(\mathscr{H},\alpha)}{\rightrightarrows}} \mathscr{H}\!om(\mathscr{H}, \mathscr{H}\!om(X \otimes B, X'))$$

which in turn – by remark 1.2.12(iii) – corresponds to a morphism

$$C \to \mathrm{Equal}(\ \mathscr{H}\!om(\mathscr{H} \otimes X, X') \underset{\mathscr{H}om(\mathscr{H}\otimes\beta,X')}{\overset{\mathscr{H}om(\mathscr{H}\otimes\alpha,X')}{\rightrightarrows}} \mathscr{H}\!om(\mathscr{H} \otimes (X \otimes B), X')$$

and again, the latter is the same as an element of

$$\mathrm{Equal}(\ \mathrm{Hom}_{\mathscr{C}}(C \otimes (\mathscr{H} \otimes X), X') \underset{\mathrm{Hom}_{\mathscr{C}}(C\otimes(\mathscr{H}\otimes\beta),X')}{\overset{\mathrm{Hom}_{\mathscr{C}}(C\otimes(\mathscr{H}\otimes\alpha),X')}{\rightrightarrows}} \mathrm{Hom}_{\mathscr{C}}(C \otimes (\mathscr{H} \otimes (X \otimes B)), X').$$

By unwinding the definition, it is easily seen that the composition

$$\overline{\mu}_l : C \otimes (\mathscr{H} \otimes X) \xrightarrow{\mathrm{ev}_{X,X'}} C \otimes X' \xrightarrow{\mu^l_{X'}} X'$$

lies in the above equalizer, and it provides a left C-module structure for $\mathscr{H}$. Likewise, μ_r shall be the morphism corresponding to the composition

$$\overline{\mu}_r : (\mathscr{H} \otimes X) \otimes A \xrightarrow{\sim} \mathscr{H} \otimes (A \otimes X) \xrightarrow{\mathscr{H}\otimes\mu^l_X} \mathscr{H} \otimes X \xrightarrow{\mathrm{ev}_{X,X'}} X'.$$

Then, the condition that $(\mathscr{H}, \mu_l, \mu_r)$ is a bimodule, comes down to the commutativity of the diagram

$$\begin{tikzcd}
C \otimes ((\mathscr{H} \otimes X) \otimes A) \arrow[rr, "\mathbf{1}_C\otimes\overline{\mu}_r"] \arrow[d, "C\otimes\Phi_{\mathscr{H},X,A}"'] & & C \otimes X' \arrow[d, "\mu^l_{X'}"] \\
C \otimes (\mathscr{H} \otimes (X \otimes A)) \arrow[r, "C\otimes(\mathscr{H}\otimes(\mu^l_X\circ\Psi_{X,A}))"] & C \otimes (\mathscr{H} \otimes X) \arrow[r, "\overline{\mu}_l"] & X'
\end{tikzcd}$$

which is immediate (details left to the reader). We have thus obtained a bifunctor :

(1.2.16) $$\mathscr{H}\!om_{B_r}(-,-) : (A, B)\text{-}\mathbf{Mod}^o \times (C, B)\text{-}\mathbf{Mod} \to (C, A)\text{-}\mathbf{Mod}.$$

Likewise, we may define a bifunctor :

$$\mathscr{H}\!om_{A_l}(-,-) : (A, B)\text{-}\mathbf{Mod}^o \times (A, C)\text{-}\mathbf{Mod} \to (B, C)\text{-}\mathbf{Mod}.$$

1.2.17. Keep the situation of (1.2.15), and suppose moreover that all coequalizers in $\mathscr{C}$ are representable. Fix an (A, B)-bimodule (X, μ^l_X, μ^r_X); the functor

$$(C, B)\text{-}\mathbf{Mod} \to (C, A)\text{-}\mathbf{Mod} \quad : \quad X' \mapsto \mathscr{H}\!om_{B_r}(X, X')$$

admits a left adjoint, the *tensor product*

$$(C, A)\text{-}\mathbf{Mod} \to (C, B)\text{-}\mathbf{Mod} \quad : \quad (X', \mu^l_{X'}, \mu^l_{X'}) \mapsto (X', \mu^l_{X'}, \mu^r_{X'}) \otimes_A (X, \mu^l_X, \mu^r_X)$$

given by the coequalizer (in $\mathscr{C}$) of the morphisms :

$$X' \otimes (A \otimes X) \overset{\mathbf{1}_{X'} \otimes \mu^l_X}{\underset{(\mu^r_{X'} \otimes \mathbf{1}_X) \circ \Phi_{X',A,X}}{\rightrightarrows}} X' \otimes X$$

with scalar multiplications induced by μ^r_X and $\mu^l_{X'}$. Likewise, we have a functor :

$$(A, C)\text{-}\mathbf{Mod} \to (B, C)\text{-}\mathbf{Mod} \quad : \quad (X', \mu^l_{X'}, \mu^l_{X'}) \mapsto (X, \mu^l_X, \mu^r_X) \otimes_A (X', \mu^l_{X'}, \mu^l_{X'})$$

which admits a similar description, and is left adjoint to the functor $X' \mapsto \mathscr{H}om_{A_l}(X, X')$ (verifications left to the reader).

1.2.18. Let (U, u) be unit object for $\underline{\mathscr{C}}$. Notice that, for any object A of $\mathscr{C}$, the rule $(Y, \mu_Y) \mapsto (Y, u_Y^{-1}, \mu_Y)$ (where $u_Y : Y \to U \otimes Y$ is the natural isomorphism supplied by proposition 1.2.6), induces a faithful functor $A\text{-}\mathbf{Mod}_r \to (U, A)\text{-}\mathbf{Mod}$. Letting $C := U$ in (1.2.17), we see that any (A, B)-bimodule X also determines a functor :

$$A\text{-}\mathbf{Mod}_r \to B\text{-}\mathbf{Mod}_r \quad : \quad Y \mapsto Y \otimes_A X$$

and likewise for left A-modules.

Example 1.2.19. If $\mathscr{C} = \mathbf{Set}$ is the category of sets (regarded as a tensor category as in example 1.2.21), then a left A-module is just a set X' with a *left action* of A, *i.e.* a map of sets

$$A \times X' \to X' \quad : \quad (a, x) \mapsto a \cdot x.$$

An (A, A)-bimodule X is a set with both left and right actions of A, such that $(a \cdot x) \cdot a' = a \cdot (x \cdot a')$ for every $a, a' \in A$ and every $x \in X$. With this notation, the tensor product $X' \otimes_A X$ is the quotient $(X' \times X)/\!\sim$, where $\sim$ is the smallest equivalence relation such that $(x'a, x) \sim (x', ax)$ for every $x \in X$, $x' \in X'$ and $a \in A$.

Definition 1.2.20. Let $\underline{\mathscr{C}} := (\mathscr{C}, \otimes, \Phi, \Psi)$ be a tensor category, and (U, u) a unit for $\underline{\mathscr{C}}$.

(i) A $\underline{\mathscr{C}}$*-semigroup* is a datum (M, μ_M) consisting of an object M of $\mathscr{C}$ and a morphism $\mu_M : M \otimes M \to M$, the *multiplication law* of $\underline{M}$, such that (M, μ_M, μ_M) is a (M, M)-bimodule. A morphism of $\underline{\mathscr{C}}$-semigroups is a morphism $\varphi : M \to M'$ in $\mathscr{C}$, such that

$$\mu_{M'} \circ (\varphi \otimes \varphi) = \varphi \circ \mu_M.$$

(ii) A $\underline{\mathscr{C}}$*-monoid* is a datum $\underline{M} := (M, \mu_M, 1_M)$, where (M, μ_M) is a semigroup, and $1_M : U \to M$ is a morphism in $\mathscr{C}$, called the *unit* of $\underline{M}$, such that

$$\mu_M \circ (1_M \otimes \mathbf{1}_M) \circ u_M = \mathbf{1}_M = \mu_M \circ (\mathbf{1}_M \otimes 1_M) \circ u_M$$

where $u_M : M \xrightarrow{\sim} U \otimes M$ is the natural isomorphism provided by proposition 1.2.6. We say that $\underline{M}$ is *commutative*, if

$$\mu_M = \mu_M \circ \Psi_{M,M}.$$

A morphism of monoids $\underline{M} \to \underline{M}'$ is a morphism of semigroups $\varphi : M \to M'$ such that $\varphi \circ 1_M = 1_{M'}$.

Example 1.2.21. (i) Let $\mathscr{C}$ be any category with small Hom-sets, in which finite products are representable, endow $\mathscr{C}$ with the tensor category structure described in example 1.2.10, and pick a final object $1_{\mathscr{C}}$ of $\mathscr{C}$. Then, a $\underline{\mathscr{C}}$-monoid is a datum $\underline{M} := (M, \mu_M, 1_M)$, where $\mu_M : M \times M \to M$ and $1_M : 1_{\mathscr{C}} \to M$ are morphisms of $\mathscr{C}$, and the axioms for μ_M and 1_M can be rephrased as requiring that, for every object X of $\mathscr{C}$, the set $M(X) := \mathrm{Hom}_{\mathscr{C}}(X, M)$, endowed with the composition law :

$$M(X) \times M(X) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{C}}(X, M \times M) \xrightarrow{\mathrm{Hom}_{\mathscr{C}}(X, \mu_M)} M(X) \qquad (m, m') \mapsto m \cdot m'$$

is a (usual) monoid, with unit $\mathrm{Im}\, 1_M(X) \in M(X)$. Of course, $\underline{M}$ is commutative, if and only if $m \cdot m' = m' \cdot m$ for all objects X of $\mathscr{C}$ and every $m, m' \in M(X)$.

(ii) In the situation of (i), a $\underline{\mathscr{C}}$-monoid shall also be called simply a $\mathscr{C}$-monoid. The category of $\mathscr{C}$-monoids admits an initial object which is also a final object, namely $\underline{1}_{\mathscr{C}} := (1_{\mathscr{C}}, \mu_1, \mathbf{1}_1)$, where μ_1 is the (unique) morphism $1_{\mathscr{C}} \times 1_{\mathscr{C}} \to 1_{\mathscr{C}}$. (Most of the above can be repeated with the theory of semigroups replaced by any "algebraic theory" in the sense of [11, Def.3.3.1] : *e.g.* in this way one can define $\mathscr{C}$-groups, $\mathscr{C}$-rings, and so on.)

1.2.22. Let $\underline{\mathscr{C}}$ and U be as in definition 1.2.20, and $\underline{M} := (M, \mu_M, 1_M)$ a $\underline{\mathscr{C}}$-monoid; of course, we are especially interested in the M-modules which are compatible with the unit and multiplication law of M. Hence we define a *left $\underline{M}$-module* as a left M-module (S, μ_S) such that the following diagrams commute :

$$\begin{tikzcd} U \otimes S \arrow[r, equal] \arrow[d, "1_M \otimes \mathbf{1}_S"'] & U \otimes S \\ M \otimes S \arrow[r, "\mu_S"] & S \arrow[u, "u_S"'] \end{tikzcd} \qquad \begin{tikzcd} M \otimes (M \otimes S) \arrow[rr, "\mathbf{1}_M \otimes \mu_S"] \arrow[d, "\Phi_{M,M,S}"'] & & M \otimes S \arrow[d, "\mu_S"] \\ (M \otimes M) \otimes S \arrow[r, "\mu_M \otimes \mathbf{1}_S"] & M \otimes S \arrow[r, "\mu_S"] & S \end{tikzcd}$$

where u_S is the isomorphism given by proposition 1.2.6. Likewise we define right $\underline{M}$-modules, and $(\underline{M}, \underline{N})$-bimodules, if $\underline{N}$ is a second $\underline{\mathscr{C}}$-monoid; especially, $(\underline{M}, \underline{M})$-bimodules shall also be called simply $\underline{M}$-bimodules.

A morphism of left $\underline{M}$-modules $(S, \mu_S) \to (S', \mu_{S'})$ is just a morphism of left M-modules, and likewise for right modules and bimodules. For instance, $\underline{M}$ is a $\underline{M}$-bimodule in a natural way, and an *ideal* of M is a sub-$\underline{M}$-bimodule I of $\underline{M}$. We denote by $\underline{M}$-$\mathbf{Mod}_l$ (resp. $\underline{M}$-$\mathbf{Mod}_r$, resp. $\underline{M}$-$\mathbf{Mod}$) the category of left (resp. right, resp. bi-) $\underline{M}$-modules; more generally, if $\underline{N}$ is a second $\underline{\mathscr{C}}$-monoid, we have the category $(\underline{M}, \underline{N})$-$\mathbf{Mod}$ of the corresponding bimodules.

Example 1.2.23. Take $\mathscr{C} := \mathbf{Set}$, regarded as a tensor category, as in example 1.2.10. Then a $\mathscr{C}$-monoid is just a usual monoid M, and a left M-module is a datum (S, μ_S) consisting of a set S and a *scalar multiplication* $M \times S \to S : (m, s) \mapsto m \cdot s$ such that

$$1 \cdot s = s \quad \text{and} \quad x \cdot (y \cdot s) = (x \cdot y) \cdot s \qquad \text{for every } x, y \in M \text{ and every } s \in S.$$

A morphism $\varphi : (S, \mu_S) \to (T, \mu_T)$ of M-modules is then a map of sets $S \to T$ such that

$$x \cdot \varphi(s) = \varphi(x \cdot s) \qquad \text{for every } x \in M \text{ and every } s \in S$$

Likewise, an ideal of M is a subset $I \subset M$ such that $a \cdot x, x \cdot a \in I$ whenever $a \in I$ and $x \in M$.

Remark 1.2.24. Let $\mathscr{C}$ be a complete and cocomplete category, whose colimits are universal (see example 1.1.24(v)), and $\underline{M}$ a $\mathscr{C}$-monoid (see example 1.2.21(ii)).

(i) The categories $\underline{M}$-$\mathbf{Mod}_l$, $\underline{M}$-$\mathbf{Mod}_r$ and $(\underline{M}, \underline{N})$-$\mathbf{Mod}$ are complete and cocomplete, and the forgetful functor $\underline{M}$-$\mathbf{Mod}_l \to \mathscr{C}$ (resp. the same for right modules and bimodules) commutes with all limits and colimits.

(ii) Notice also that the forgetful functor $\underline{M}$-$\mathbf{Mod}_l \to \mathscr{C}$ is conservative. Together with (i) and remark 1.1.38(iii,iv), this implies that a morphism of left $\underline{M}$-modules is a monomorphism (resp. an epimorphism) if and only if the same holds for the underlying morphism in $\mathscr{C}$ (and likewise for right modules and bimodules).

(iii) For each of these categories, the initial object is just the initial object $\varnothing_{\mathscr{C}}$ of $\mathscr{C}$, endowed with the trivial scalar multiplication. Likewise, the final object is the final object $1_{\mathscr{C}}$ of $\mathscr{C}$, with scalar multiplication given by the unique morphism $M \times 1_{\mathscr{C}} \to 1_{\mathscr{C}}$. Moreover, the forgetful functor $\underline{M}$-$\mathbf{Mod}_l \to \mathscr{C}$ admits a left adjoint, that assigns to any $\Sigma \in \mathrm{Ob}(\mathscr{C})$ the *free $\underline{M}$-module* $\underline{M}^{(\Sigma)}$ generated by Σ; as an object of $\mathscr{C}$, the latter is just $M \times \Sigma$, and the scalar multiplication is derived from the composition law of $\underline{M}$, in the obvious way.

For instance, for any $n \in \mathbb{N}$, and any left (or right or bi-) $\underline{M}$-module S, we denote as usual by $S^{\oplus n}$ the coproduct of n copies of S.

Remark 1.2.25. Let $\mathscr{C}$ be a tensor category, $\underline{M}$, $\underline{N}$, $\underline{P}$, $\underline{Q}$ four $\mathscr{C}$-monoids.

(i) Let S be a $(\underline{M}, \underline{N})$-bimodule, S' a $(\underline{P}, \underline{N})$-bimodule and S'' a $(\underline{P}, \underline{M})$-bimodule. Then it is easily seen that the (P, M)-bimodule (resp. the (P, N)-bimodule) $\mathscr{H}om_{N_r}(S, S')$ (resp. $S'' \otimes_M S$) is actually a $(\underline{P}, \underline{M})$-bimodule (resp. a $(\underline{P}, \underline{N})$-bimodule) and the adjunction of (1.2.17) restricts to an adjunction between the corresponding categories of bimodules : the details shall be left to the reader.

(ii) We have as well the analogue of the usual associativity constraints. Namely, for every $(\underline{M}, \underline{N})$-bimodule S, every $(\underline{N}, \underline{P})$-bimodule S' and every $(\underline{P}, \underline{Q})$-bimodule S'', there is a natural isomorphism

$$(S \otimes_N S') \otimes_P S'' \xrightarrow{\sim} S \otimes_N (S' \otimes_P S'') \qquad \text{in } (\underline{M}, \underline{Q})\text{-}\mathbf{Mod}$$

and natural isomorphisms $M \otimes_M S \xrightarrow{\sim} S \xrightarrow{\sim} S \otimes_N N$ in $(\underline{M}, \underline{N})$-$\mathbf{Mod}$.

(iii) Also, if $\underline{M}$ is commutative, every left (or right) $\underline{M}$-module is naturally a $(\underline{M}, \underline{M})$-bimodule, and we have a commutative constraint

$$S \otimes_M S' \xrightarrow{\sim} S' \otimes_M S \qquad \text{for all left (or right) } \underline{M}\text{-modules}.$$

And taking into account (ii), it is easily seen that $(\underline{M}\text{-}\mathbf{Mod}_l, \otimes_M)$ is a tensor category.

1.2.26. Let $\varphi : \underline{M}_1 \to \underline{M}_2$ be a morphism of $\mathscr{C}$-monoids; we have the $(\underline{M}_1, \underline{M}_2)$-bimodule :

$$M_{1,2} := (M_2, \mu_{M_2} \circ (\varphi \otimes \mathbf{1}_{M_2}), \mu_{M_2}).$$

Letting $X := M_{1,2}$ in (1.2.18), we obtain a *base change functor* for right modules :

$$\underline{M}_1\text{-}\mathbf{Mod}_r \to \underline{M}_2\text{-}\mathbf{Mod}_r \quad : \quad X \mapsto X \otimes_{M_1} M_2 := X \otimes_{M_1} M_{1,2}.$$

The base change is left adjoint to the *restrictions of scalars* associated to φ, *i.e.* the functor :

$$\underline{M}_2\text{-}\mathbf{Mod}_r \to \underline{M}_1\text{-}\mathbf{Mod}_l \quad : \quad (X, \mu_X) \mapsto (X, \mu_X)_{(\varphi)} := (X, \mu_X \circ (\mathbf{1}_X \otimes \varphi))$$

(verifications left to the reader). The same can be repeated, as usual, for left modules; for bimodules, one must take the tensor product on both sides : $X \mapsto M_2 \otimes_{M_1} X \otimes_{M_1} M_2$.

Example 1.2.27. Take $\mathscr{C} = \mathbf{Set}$, and let M be any monoid, Σ any set, and $M^{(\Sigma)}$ the free M-module generated by Σ. From the isomorphism

$$M^{(\Sigma)} \otimes_M \{1\} \xrightarrow{\sim} \{1\}^{(\Sigma)} = \Sigma$$

we see that the cardinality of Σ is an invariant, called the *rank* of the free M-module $M^{(\Sigma)}$, and which we denote $\mathrm{rk}_M M^{(\Sigma)}$.

Definition 1.2.28. (i) A non-empty category $\mathscr{A}$ is *additive*, if the following holds :

(a) For every $A, B \in \mathrm{Ob}(\mathscr{A})$, the set $\mathrm{Hom}_{\mathscr{A}}(A, B)$ is small and carries an abelian group structure (especially, it is not empty).

(b) For every $A, B, C \in \mathrm{Ob}(\mathscr{A})$, the composition law

$$\mathrm{Hom}_{\mathscr{A}}(A, B) \times \mathrm{Hom}_{\mathscr{A}}(B, C) \to \mathrm{Hom}_{\mathscr{A}}(A, C)$$

is a bilinear pairing.

(ii) A functor $F : \mathscr{A} \to \mathscr{B}$ between additive categories is *additive* if it induces group homomorphisms

$$\mathrm{Hom}_{\mathscr{A}}(X, Y) \to \mathrm{Hom}_{\mathscr{B}}(FX, FY) \qquad \varphi \mapsto F\varphi$$

for every $X, Y \in \mathrm{Ob}(\mathscr{A})$.

Remark 1.2.29. Let $\mathscr{A}$ be any additive category.

(i) If $A \in \mathrm{Ob}(\mathscr{A})$ is any object, denote by $\mathbf{0}_A$ the neutral element of the abelian group $\mathrm{End}_{\mathscr{A}}(A)$. Suppose that the equalizer of the pair of morphisms $\mathbf{1}_A, \mathbf{0}_A : A \to A$ is representable by an object 0 of $\mathscr{A}$ (see example 1.1.24(ii)). Then, the datum of a morphism $B \to 0$ is the same as that of a morphism $\varphi : B \to A$ that factors through $\mathbf{0}_A$. By the bilinearity of the Hom-pairing, the latter condition holds if and only if φ is the neutral element of $\mathrm{Hom}_{\mathscr{A}}(B, A)$. We conclude that 0 is a final object in $\mathscr{A}$. Dually, if the coequalizer of the pair $(\mathbf{1}_A, \mathbf{0}_A)$ is representable by some object $0'$ of $\mathscr{A}$, then $0'$ is initial in $\mathscr{A}$. Moreover, if $\mathscr{A}$ admits a final object 0, then it is easily seen that the unique morphism $A \to 0$ is also the coequalizer of the pair $(\mathbf{1}_A, \mathbf{0}_A)$, so 0 is also an initial object. Conversely, if $\mathscr{A}$ admits an initial object, then this object is also final in $\mathscr{A}$, and for any two objects A, B of $\mathscr{A}$, the neutral element $\mathbf{0}_{A,B}$ of $\mathrm{Hom}_{\mathscr{A}}(A, B)$ is the unique morphism that factors through 0. We say that 0 is a *zero object* for $\mathscr{A}$.

(ii) Suppose that $\mathscr{A}$ admits a zero object 0, and moreover that the product $A_1 \times A_2$ is representable in $\mathscr{A}$ for given $A_1, A_2 \in \mathrm{Ob}(\mathscr{A})$. Denote by $p_i : A_1 \times A_2 \to A_i$ ($i = 1, 2$) the projections; then, there are unique morphisms $e_i : A_i \to A_1 \times A_2$ ($i = 1, 2$) such that

$$p_i \circ e_i = \mathbf{1}_{A_i} \qquad \text{for } i = 1, 2 \qquad \text{and} \qquad p_i \circ e_j = \mathbf{0}_{A_j, A_i} \qquad \text{for } i \neq j. \tag{1.2.30}$$

Notice that

$$e_1 \circ p_1 + e_2 \circ p_2 = \mathbf{1}_{A_1 \times A_1}. \tag{1.2.31}$$

Indeed, we have

$$p_i \circ (e_1 \circ p_1 + e_2 \circ p_2) = (p_i \circ e_1 \circ p_1) + (p_i \circ e_2 \circ p_2) = p_i \qquad i = 1, 2$$

by bilinearity of the Hom pairing, and $\mathbf{1}_{A_1 \times A_2}$ is the unique endomorphism φ of $A_1 \times A_2$ such that $p_i \circ \varphi = p_i$ for $i = 1, 2$. It follows that $A_1 \times A_2$ also represents the coproduct $A_1 \amalg A_2$. Indeed, say that $f_i : A_i \to B$, for $i = 1, 2$, are two morphisms to another object B of $\mathscr{A}$, and set $f := f_1 \circ p_1 + f_2 \circ p_2 : A_1 \times A_2 \to B$; it is easily seen that $f \circ e_i = f_i$ for $i = 1, 2$, and by virtue of (1.2.31), the morphism f is the unique one that satisfies these identities. Conversely, if the coproduct of A_1 and A_2 is representable, a similar argument shows that also $A_1 \times A_2$ is representable. We say that $A_1 \times A_2$ is a *biproduct* of A_1 and A_2, and denote it by $A_1 \oplus A_2$.

(iii) Notice that the morphisms $(p_i, e_i \mid i = 1, 2)$ with the identities (1.2.30) and (1.2.31) characterize $A_1 \oplus A_2$ up to unique isomorphism. Namely, say that B is another object of $\mathscr{A}$, for which exist morphisms $p'_i : B \to A_i$ and $e'_i : A_i \to B$ ($i = 1, 2$) such that $p'_i \circ e'_i = \mathbf{1}_{A_i}$ for $i = 1, 2$, and $p'_i \circ e'_j = \mathbf{0}_{A_j A_i}$ for $i \neq j$, and moreover $e'_1 \circ p'_1 + e'_2 \circ p'_2 = \mathbf{1}_B$. Then the pair (e'_1, e'_2) (resp. (p'_1, p'_2)) induces a morphism $e' : A_1 \oplus A_2 \to B$ (resp. $p' : B \to A_1 \oplus A_2$) with

$$p_j \circ p' \circ e' \circ e_i = p'_j \circ e'_i = p_j \circ e_i \qquad \text{for } i, j = 1, 2$$

which – by virtue of the universal properties of the biproduct – implies that $p' \circ e' = \mathbf{1}_{A_1 \oplus A_2}$. Likewise, we may compute

$$\begin{aligned} e' \circ p' &= (e' \circ e_1 \circ p_1 + e' \circ e_2 \circ p_2) \circ (e_1 \circ p_1 \circ p' + e_2 \circ p_2 \circ p') \\ &= e' \circ e_1 \circ p_1 \circ p' + e' \circ e_2 \circ p_2 \circ p' \\ &= e'_1 \circ p'_1 + e'_2 \circ p'_2 = \mathbf{1}_B \end{aligned}$$

whence the contention.

(iv) Suppose that B_1 and B_2 are any other two objects of $\mathscr{A}$ such that $B_1 \oplus B_2$ is also representable; given two morphisms $f_1 : A_1 \to B_1$ and $f_2 : A_2 \to B_2$, we denote by $f_1 \oplus f_2 : A_1 \oplus A_2 \to B_1 \oplus B_2$ the unique morphism such that

$$p_{B,i} \circ (f_1 \oplus f_2) \circ e_{A,i} = f_i \quad \text{for } i = 1, 2, \text{ and} \quad p_{B,i} \circ (f_1 \oplus f_2) \circ e_{A,j} = \mathbf{0}_{A_j, B_i} \quad \text{for } i \neq j.$$

Notice that

$$f_1 \oplus f_2 = (f_1 \oplus \mathbf{0}_{A_2,B_2}) + (\mathbf{0}_{A_1,B_1} \oplus f_2) \tag{1.2.32}$$

(where the sum is taken in the abelian group $\mathrm{Hom}_{\mathscr{A}}(A_1 \oplus A_2, B_1 \oplus B_2)$); indeed, by bilinearity of the Hom pairing, it is easily seen that the right-hand side of (1.2.32) also satisfies the same identities above that define $f_1 \oplus f_2$.

(v) If $f : A \to B$ is any morphism of $\mathscr{A}$, then we define the *kernel* (resp. *cokernel*) of f as the equalizer (resp. coequalizer)

$$\mathrm{Ker}\, f := \mathrm{Equal}(f, \mathbf{0}_{A,B}) \qquad \mathrm{Coker}\, f := \mathrm{Coequal}(f, \mathbf{0}_{A,B}).$$

Suppose that $\mathrm{Ker}\, f$ and $\mathrm{Coker}\, f$ are representable in $\mathscr{A}$ for every such f, and denote by

$$\iota_f : \mathrm{Ker}\, f \to A \qquad \pi_f : B \to \mathrm{Coker}\, f$$

the natural morphisms. Notice that ι_f is a monomorphism, and π_f an epimorphism. Notice also that f factors uniquely as a composition

$$A \xrightarrow{\pi_{\iota_f}} \mathrm{Coker}\, \iota_f \xrightarrow{\beta_f} \mathrm{Ker}\, \pi_f \xrightarrow{\iota_{\pi_f}} B. \tag{1.2.33}$$

(vi) Suppose that $\mathscr{A}$ admits a zero object 0, and let $f : A \to B$ be any morphism; by definition $\mathrm{Ker}\, f$ is the presheaf such that

$$\mathrm{Ker}\, f(C) = \{g : B \to C \mid g \circ f = g \circ \mathbf{0}_{A,B} = \mathbf{0}_{A,C}\}.$$

If f is a monomorphism, the identity $g \circ f = \mathbf{0}_{A,C} = \mathbf{0}_{B,C} \circ f$ implies that $g = \mathbf{0}_{B,C}$, so $\mathrm{Ker}\, f$ is represented by 0. Dually, if f is an epimorphism, then $\mathrm{Coker}\, f$ is represented by 0.

Remark 1.2.34. Let $\mathscr{A}$, $\mathscr{B}$ be any two additive categories that admit a zero object, and $F : \mathscr{A} \to \mathscr{B}$ a functor.

(i) If F is additive, remark 1.2.29(iii) immediately implies that F transforms representable biproducts into representable biproducts. The latter assertion still holds in case F is not necessarily additive, but is either left or right exact. Indeed, suppose that F is left exact, let $A_1 \oplus A_2$ be any biproduct, and let p_i, e_i be the morphisms described in remark 1.2.29(ii); by left exactness, $F(A_1 \oplus FA_2)$ represents the product of FA_1 and FA_2, and any isomorphism $FA_1 \oplus FA_2 \xrightarrow{\sim} F(A_1 \oplus FA_2)$ identifies Fp_1 and Fp_2 with the natural projections. Moreover, F transforms the final object of $\mathscr{A}$ into the final object of $\mathscr{B}$ (see example 1.1.26(ii)); then, by inspecting the argument in remark 1.2.29(ii), it is easily seen that F identifies as well Fe_i with the natural injections $FA_i \to FA_1 \oplus FA_2$, for $i = 1, 2$, so the assertion follows from remark 1.2.29(iii). A similar argument works in case F is right exact.

(ii) Suppose moreover, that all biproducts of $\mathscr{A}$ are representable. Then we claim that the abelian group structure on $\mathrm{Hom}_{\mathscr{A}}(A, B)$ is determined by the category $\mathscr{A}$, *i.e.* if $\mathscr{B}$ is any other additive category, and $F : \mathscr{A} \to \mathscr{B}$ is any equivalence of categories, then F induces group isomorphisms (and not just bijections) on Hom sets. Indeed, let A and B be any two objects of $\mathscr{A}$, and denote by $\Delta_A : A \to A \oplus A$ (resp. $\mu_B : B \oplus B \to B$) the unique morphism such that $p_i \circ \Delta_A = \mathbf{1}_A$ (resp. $\mu_B \circ e_i = \mathbf{1}_B$) for $i = 1, 2$. Then we have

$$f_1 + f_2 = \mu_B \circ (f_1 \oplus f_2) \circ \Delta_A \qquad \text{for every } f_1, f_2 : A \to B$$

where $f_1 + f_2$ denotes the sum in the abelian group $\mathrm{Hom}_{\mathscr{A}}(A, B)$. Indeed, since clearly $\mathbf{0}_{A_1,B_1} \oplus \mathbf{0}_{A_2,B_2} = \mathbf{0}_{A_1 \oplus A_2, B_1 \oplus B_2}$, identity (1.2.32) reduces to checking that $f_1 = \mu_B \circ (f_1 \oplus \mathbf{0}_{A_2,B_2}) \circ \Delta_A$ (and likewise for f_2), which follows easily from (1.2.31) : details left to the reader.

(iii) Combining (i) and (ii) we see that, if all biproducts of $\mathscr{A}$ are representable, and F is either left or right exact, then F is additive. More generally, we see that for F to be additive, it suffices that F sends the zero object of $\mathscr{A}$ to the zero object of $\mathscr{B}$, and F commutes with the biproducts of the form $A \oplus A$, for every $A \in \mathrm{Ob}(\mathscr{A})$.

Definition 1.2.35. An *abelian category* is an additive category $\mathscr{A}$ such that the following holds:

(a) All the kernels and cokernels of $\mathscr{A}$ are representable (especially, $\mathscr{A}$ admits an initial and final object 0).
(b) For every morphism f of $\mathscr{A}$, the morphism β_f of (1.2.33) is an isomorphism.
(c) The product of any finite family of objects of $\mathscr{A}$ is representable in $\mathscr{A}$ (see example 1.1.24(iv)).

Remark 1.2.36. Let $\mathscr{A}$ be any additive category.

(i) For any other additive category $\mathscr{B}$, let us denote by $\mathbf{Add}(\mathscr{B},\mathscr{A})$ the full subcategory of $\mathbf{Fun}(\mathscr{B},\mathscr{A})$ whose objects are the additive functors. Notice that, if $\mathscr{C}$ is a small category, then $\mathbf{Fun}(\mathscr{C},\mathscr{A})$ is an additive category; indeed, if $\tau,\sigma : F \Rightarrow G$ are two natural transformations between functors $F, G : \mathscr{C} \to \mathscr{A}$, then we obtain a natural transformation $\tau+\sigma$ from F to G, by the rule : $(\tau+\sigma)_X := \tau_X + \sigma_X$ for every $X \in \mathrm{Ob}(\mathscr{C})$ (where the sum denotes the addition law of $\mathrm{Hom}_{\mathscr{A}}(FX, GX)$). Clearly, this rule yields an abelian group structure, and the composition of natural transformation defines a bilinear pairing $(\tau,\tau') \mapsto \tau\circ\tau'$ on the resulting groups of natural transformations (verification left to the reader).

(ii) Especially, if $\mathscr{B}$ is a small additive category, then also $\mathbf{Add}(\mathscr{B},\mathscr{A})$ is an additive category. Moreover, if $\mathscr{A}$ is an abelian category, then $\mathbf{Fun}(\mathscr{C},\mathscr{A})$ is an abelian category, for every small category $\mathscr{C}$: details left to the reader.

(iii) By definition, for every $A, B \in \mathrm{Ob}(\mathscr{A})$, the set $h_A(B)$ carries an abelian group structure, such that the presheaf h_A factors through an additive functor $h^\dagger_A : \mathscr{A}^o \to \mathbb{Z}\text{-}\mathbf{Mod}$ from $\mathscr{A}^o$ to the category of abelian groups, and the forgetful functor $\mathbb{Z}\text{-}\mathbf{Mod} \to \mathbf{Set}$. Hence, the Yoneda imbedding factors through a fully faithful *group-valued Yoneda imbedding*

$$h^\dagger : \mathscr{A} \to \mathbf{Add}(\mathscr{A}^o, \mathbb{Z}\text{-}\mathbf{Mod}).$$

In view of (ii), we conclude that every small additive category is a full subcategory of an abelian category. Moreover, Yoneda's lemma extends *verbatim* to the group-valued case : namely, by inspecting the proof of proposition 1.1.20, we see that, for every $A \in \mathrm{Ob}(\mathscr{A})$ and every *additive* functor $F : \mathscr{A}^o \to \mathbb{Z}\text{-}\mathbf{Mod}$ there are natural isomorphisms of abelian groups

$$F(A) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{Add}(\mathscr{A}^o,\mathbb{Z}\text{-}\mathbf{Mod})}(h_A, F). \tag{1.2.37}$$

(iv) Suppose that $f : \mathscr{A} \to \mathscr{B}$ is any functor between small additive categories; then the arguments of (1.1.35) extend *verbatim* to the present situation : namely, the induced functor

$$f^* : \mathbf{Fun}(\mathscr{B}^o, \mathbb{Z}\text{-}\mathbf{Mod}) \to \mathbf{Fun}(\mathscr{A}^o, \mathbb{Z}\text{-}\mathbf{Mod})$$

admits both left and right adjoints, denoted respectively $f_!$ and f_*, and we have :

Proposition 1.2.38. *In the situation of remark* 1.2.36(iv)*, suppose that f is additive. Then :*

(i) *Both f^*, $f_!$ and f_* are additive functors, and restrict to functors*

$$\mathbf{Add}(\mathscr{B}^o, \mathbb{Z}\text{-}\mathbf{Mod}) \underset{f_!\ \ f_*}{\overset{f^*}{\rightleftarrows}} \mathbf{Add}(\mathscr{A}^o, \mathbb{Z}\text{-}\mathbf{Mod}).$$

(ii) *The resulting diagram of functors*

$$\begin{tikzcd} \mathscr{A} \arrow[r, "h^\dagger"] \arrow[d, "f"'] & \mathbf{Add}(\mathscr{A}^o, \mathbb{Z}\text{-}\mathbf{Mod}) \arrow[d, "f_!"] \\ \mathscr{B} \arrow[r, "h^\dagger"] & \mathbf{Add}(\mathscr{B}^o, \mathbb{Z}\text{-}\mathbf{Mod}) \end{tikzcd}$$

is essentially commutative.

Proof. (i): Since every left (resp. right) adjoint functor is right (resp. left) exact, remark 1.2.34(iii) says that f^*, f_* and $f_!$ are additive. Next, a simple inspection shows that f^* transforms additive functors into additive functors. Let now $F : \mathscr{A}^o \to \mathbb{Z}\text{-}\mathbf{Mod}$ be an additive functor, $B \in \mathrm{Ob}(\mathscr{B})$ any object, and set $G := f_! F$; from the proof of proposition 1.1.34, we see that

$$GB = \operatorname*{colim}_{\psi: B \to fA} FA$$

where the colimit ranges over the small category $f\mathscr{A}^o/B$ of all pairs (A, ψ) consisting of an object A of $\mathscr{A}$, and a morphism $\psi : B \to fA$ in $\mathscr{B}$. Denote by $0_{\mathscr{A}}$ and $0_{\mathscr{B}}$ the zero objects of $\mathscr{A}$ and $\mathscr{B}$; we wish to show that G is additive, and according to remark 1.2.34(iii), it suffices to check that $G(0_{\mathscr{B}}) = 0$, and that the natural morphism $G(B \oplus B) \to GB \oplus GB$ (deduced from the projections $p_i : B \oplus B \to B$) is an isomorphism, for every $B \in \mathrm{Ob}(\mathscr{B})$.

However, notice that the functor $\iota_{0_{\mathscr{B}}} : f\mathscr{A}^o/0_{\mathscr{B}} \to \mathscr{A}^o$ is an isomorphism of categories (notation of (1.1.16)); whence a natural isomorphism

$$G(0_{\mathscr{B}}) \xrightarrow{\sim} \operatorname*{colim}_{\mathscr{A}^o} F \xrightarrow{\sim} F(0_{\mathscr{A}}) = 0$$

where the last identity holds, since F is additive. Next, for any $B_1, B_2 \in \mathrm{Ob}(\mathscr{B})$ consider the functor

$$\Phi : (f\mathscr{A}^o/B_1) \times (f\mathscr{A}^o/B_2) \to f\mathscr{A}^o/B_1 \oplus B_2 \qquad ((A_1, \psi_1), (A_2, \psi_2)) \mapsto (A_1 \oplus A_2, \psi_1 \oplus \psi_2).$$

Claim 1.2.39. For any category $\mathscr{C}$, and any functor $H : f\mathscr{A}^o/B_1 \oplus B_2 \to \mathscr{C}$, the functor Φ induces an isomorphism

$$\operatorname*{colim}_{(f\mathscr{A}^o/B_1)\times(f\mathscr{A}^o/B_2)} H \circ \Phi \xrightarrow{\sim} \operatorname*{colim}_{f\mathscr{A}^o/(B_1 \oplus B_2)} H.$$

Proof of the claim. Let X be any object of $\mathscr{C}$, and $\tau : H \circ \Phi \Rightarrow H \circ c_X$ a natural transformation, where $c_X : f\mathscr{A}^o/B_1 \oplus B_2 \to \mathscr{C}$ is the constant functor with value X; since Φ is faithful, it suffices to check that τ extends uniquely to a natural transformation $\tau' : H \Rightarrow c_X$.

However, let $(A, \psi : B_1 \oplus B_2 \to fA)$ be any object of $f\mathscr{A}^o/B_1 \oplus B_2$, and denote by $\psi_i : B_i \to fA$ (for $i = 1, 2$) the composition of ψ with the natural monomorphism $e_i : B_i \to B_1 \oplus B_2$. Since f is additive, the morphism $\mu_A : A \oplus A \to A$ defines a morphism

$$\mu_A^o : (A, \psi) \to (A \oplus A, \psi_1 \oplus \psi_2) \qquad \text{in } f\mathscr{A}^o/B_1 \oplus B_2$$

and we may set

$$\tau'_{(A,\psi)} := \tau_{(A \oplus A, \psi_1 \oplus \psi_2)} \circ H(\mu_A^o).$$

Suppose that $\beta^o : (A, \psi) \to (A', \psi')$ is any morphism in $f\mathscr{A}^o/B_1 \oplus B_2$; there follows a commutative diagram in $\mathscr{B}$

$$\begin{tikzcd}
B \arrow[r, "\psi'_1 \oplus \psi'_2"] \arrow[dr, "\psi_1 \oplus \psi_2"'] & fA' \oplus fA' \arrow[r, "\mu_{fA'}"] \arrow[d, "f\beta \oplus f\beta"'] & fA' \arrow[d, "f\beta"] \\
& fA \oplus fA \arrow[r, "\mu_{fA}"] & fA
\end{tikzcd}$$

which allows to compute

$$\begin{aligned}
\tau'_{(A',\psi')} \circ H(\beta^o) &= \tau_{(A' \oplus A', \psi'_1 \oplus \psi'_2)} \circ H(\mu^o_{A'}) \circ H(\beta^o) \\
&= \tau_{(A' \oplus A', \psi'_1 \oplus \psi'_2)} \circ H(\beta^o \oplus \beta^o) \circ H(\mu^o_A) \\
&= \tau_{(A \oplus A, \psi_1 \oplus \psi_2)} \circ H(\mu^o_A) = \tau'_{(A,\psi)}
\end{aligned}$$

so τ' is indeed a natural transformation, and clearly it is the unique one extending τ. $\Diamond$

In light of claim 1.2.39, we are reduced to checking that the natural morphism

$$\operatorname*{colim}_{(f\mathscr{A}^o/B_1)\times(f\mathscr{A}^o/B_2)} F\circ\iota_{B_1\oplus B_2}\circ\Phi\to GB_1\oplus GB_2$$

is an isomorphism, for any $B_1, B_2 \in \mathrm{Ob}(\mathscr{B})$. The latter assertion follows easily by inspecting the definitions, since F is additive. Lastly, a similar argument shows that f_*F is additive, whenever the same holds for F : the reader can spell out the proof as an exercise.

(ii): One may argue as in (1.1.35) : in view of (1.2.37), we see that, for every object A of $\mathscr{A}$, both $h^\dagger_{fA}$ and $f_!h^\dagger_A$ represent the same functor : details left to the reader. $\square$

Definition 1.2.40. An *abelian tensor category* is a tensor category $(\mathscr{C}, \otimes, \Phi, \Psi)$ such that $\mathscr{C}$ is an abelian category, and the functor $\otimes$ induces bilinear pairings

$$\mathrm{Hom}_{\mathscr{C}}(A, B)\times\mathrm{Hom}_{\mathscr{C}}(A', B')\to\mathrm{Hom}_{\mathscr{C}}(A\otimes A', B\otimes B') \quad : \quad (f, g)\mapsto f\otimes g$$

for every $A, A', B, B' \in \mathrm{Ob}(\mathscr{C})$.

Remark 1.2.41. Let $(\mathscr{A}, \otimes, \Phi, \Psi)$ be a tensor category, such that $\mathscr{A}$ is abelian. If $\mathscr{A}$ admits an internal Hom functor, then the functor $-\otimes A$ is right exact, and the functor $\mathscr{H}om(A, -)$ is left exact for every $A \in \mathrm{Ob}(\mathscr{A})$, so both are additive, by virtue of remark 1.2.34(iii). Especially, $\mathscr{A}$ is an abelian tensor category, in this case.

Lemma 1.2.42. *Let $\mathscr{A}$ be any abelian category, an $\Sigma \subset \mathrm{Ob}(\mathscr{A})$ a small subset. We have :*

(i) *There exists a small full abelian subcategory $\mathscr{B}$ of $\mathscr{A}$ such that $\Sigma\subset\mathrm{Ob}(\mathscr{B})$.*

(ii) *If $\mathscr{A}$ is small, there exists a complete and cocomplete abelian tensor category $(\mathscr{C}, \otimes)$ with internal* Hom *functor, and a fully faithful additive functor $\mathscr{A}\to\mathscr{C}$.*

Proof. (i): Let $\mathscr{B}_0$ be the full subcategory of $\mathscr{A}$ such that $\mathrm{Ob}(\mathscr{B}_0) = \Sigma$; clearly $\mathscr{B}_0$ is small. Next, for any subcategory $\mathscr{D}$ of $\mathscr{A}$, denote by $\mathscr{D}'$ a subcategory of $\mathscr{A}$ obtained as follows. For every morphism φ of $\mathscr{D}$, we pick objects in $\mathscr{A}$ representing the kernel and cokernel of φ, and for any two objects of $\mathscr{D}$, we pick an object in $\mathscr{A}$ representing their product; let $\Sigma'\subset\mathrm{Ob}(\mathscr{A})$ be the resulting subset. Then $\mathscr{D}'$ is the full subcategory of $\mathscr{A}$ such that $\mathrm{Ob}(\mathscr{D}') = \mathrm{Ob}(\mathscr{D})\cup\Sigma'$. It is easily seen that $\mathscr{D}'$ is small, whenever the same holds for $\mathscr{D}$. Then we set inductively $\mathscr{B}_{i+1} := \mathscr{B}'_i$ for every $i\in\mathbb{N}$. The full subcategory $\mathscr{B}$ of $\mathscr{A}$ with $\mathrm{Ob}(\mathscr{B}) = \bigcup_{i\in\mathbb{N}}\mathrm{Ob}(\mathscr{B}_i)$ is still small, and it is abelian, by construction.

(ii): We let $\mathscr{C} := \mathbf{Fun}(\mathscr{A}, \mathbb{Z}\text{-}\mathbf{Mod})$. Then $\mathscr{C}$ is an abelian category, by virtue of remark 1.2.36(ii), and since $\mathbb{Z}$-$\mathbf{Mod}$ is complete and cocomplete, the same holds for $\mathscr{C}$; moreover, the standard tensor product of abelian groups defines a tensor category structure with internal Hom functor on $\mathbb{Z}$-$\mathbf{Mod}$, and the latter is inherited by $\mathscr{C}$ (remarks 1.2.5(ii) and 1.2.12(vi)). It is clear that these two structures amount to an abelian tensor category, and the group-valued Yoneda imbedding is the sought fully faithful functor. $\square$

1.2.43. Let $\mathscr{A}$ be a small abelian category, and $h^\dagger : \mathscr{A}\to\mathscr{A}^\dagger := \mathbf{Fun}(\mathscr{A}^o, \mathbb{Z}\text{-}\mathbf{Mod})$ the fully faithful group-valued Yoneda imbedding. For every abelian group G, denote by $G_{\mathscr{A}} : \mathscr{A}^o\to\mathbb{Z}\text{-}\mathbf{Mod}$ the constant functor with value G : so, $G_{\mathscr{A}}(A) := A$ for every $A\in\mathrm{Ob}(\mathscr{A})$, and $G_{\mathscr{A}}(\varphi) := \mathbf{1}_A$ for every morphism φ in $\mathscr{A}$. Since $\mathscr{A}^\dagger$ is an abelian tensor category (see the proof of lemma 1.2.42(ii)), we may form the tensor product

$$G\otimes_{\mathbb{Z}} A := G_{\mathscr{A}}\otimes h^\dagger_A \qquad \text{for every } A\in\mathrm{Ob}(\mathscr{A}).$$

We claim that, if G is finitely generated, $G\otimes_{\mathbb{Z}} A$ lies in the essential image of $h^\dagger$. Indeed, this is clear if G is free of finite rank, since in that case $G\otimes_{\mathbb{Z}} A$ is a finite direct sum of copies of A; in the general case, we may write G as a cokernel of a map $L_1\to L_2$ of free abelian groups of finite rank, and since the functor $-\otimes h^\dagger_A$ is right exact, we see that $G\otimes_{\mathbb{Z}} A$ is the cokernel of a morphism of $\mathscr{A}$, so it is represented by an object of $\mathscr{A}$. Moreover, if $\varphi : G\to H$ is any morphism of abelian groups, we have an obvious induced morphism $\varphi_{\mathscr{A}} : G_{\mathscr{A}} \to H_{\mathscr{A}}$, whence a morphism $\varphi \otimes_{\mathbb{Z}} A := \varphi_{\mathscr{A}} \otimes h^{\dagger}_A$.

Thus, after replacing $G \otimes_{\mathbb{Z}} A$ by an isomorphic object, we obtain a well defined functor

$$\mathbb{Z}\text{-}\mathbf{Mod}_{\mathrm{fg}} \times \mathscr{A} \to \mathscr{A} \qquad (G, A) \mapsto G \otimes_{\mathbb{Z}} A \tag{1.2.44}$$

where $\mathbb{Z}\text{-}\mathbf{Mod}_{\mathrm{fg}}$ is the full subcategory of $\mathbb{Z}\text{-}\mathbf{Mod}$ whose objects are the finitely generated abelian groups. This functor is not unique, but any two such functors are naturally isomorphic.

Remark 1.2.45. Keep the notation of (1.2.43); we have :

(i) From the construction, it is clear that (1.2.44) is a *biadditive* functor, *i.e.*, for every abelian group G, and every $A \in \mathrm{Ob}(\mathscr{A})$, the restrictions $G \otimes -$ and $- \otimes A$ of (1.2.44) are additive.

(ii) Suppose that $\mathscr{A}$ is cocomplete; since the tensor product is right exact, it follows easily that (1.2.44) extends to the whole of $\mathbb{Z}\text{-}\mathbf{Mod}$: details left to the reader.

(iii) On the other hand, using Zorn's lemma, (1.2.44) can be defined even in case $\mathscr{A}$ is not small : again, we leave the details to the reader.

1.3. 2**-categories.** In dealing with categories, the notion of equivalence is much more central than the notion of isomorphism. On the other hand, equivalence of categories is usually not preserved by the standard categorical operations discussed thus far. For instance, consider the following :

Example 1.3.1. Let $\mathscr{C}$ be the category with $\mathrm{Ob}(\mathscr{C}) = \{a, b\}$, and whose only morphisms are $\mathbf{1}_a$, $\mathbf{1}_b$ and $u : a \to b$, $v : b \to a$. Then necessarily $u \circ v = \mathbf{1}_b$ and $v \circ u = \mathbf{1}_a$. Let $\mathscr{C}_a$ (resp. $\mathscr{C}_b$) be the unique subcategory of $\mathscr{C}$ with $\mathrm{Ob}(\mathscr{C}_a) = \{a\}$ (resp. $\mathrm{Ob}(\mathscr{C}_b) = \{b\}$). Clearly both inclusion functors $\mathscr{C}_a \to \mathscr{C} \leftarrow \mathscr{C}_b$ are equivalences. However, $\mathscr{C}_a \times_{\mathscr{C}} \mathscr{C}_b$ is the empty category; especially, this fibre product is not equivalent to $\mathscr{C} = \mathscr{C} \times_{\mathscr{C}} \mathscr{C}$.

It is therefore natural to seek a new framework for the manipulation of categories and functors "up to equivalences", and thus more consonant with the very spirit of category theory. Precisely such a framework is provided by the theory of 2-categories, which we proceed to present.

1.3.2. The category $\mathbf{Cat}$, together with the category structure on the sets $\mathrm{Fun}(-, -)$ (as in (1.1.7)), provides the first example of a 2-*category*. The latter is the datum of :

- A set $\mathrm{Ob}(\mathscr{A})$, whose elements are called the *objects of* $\mathscr{A}$.
- For every $A, B \in \mathrm{Ob}(\mathscr{A})$, a category $\mathscr{A}(A, B)$. The objects of $\mathscr{A}(A, B)$ are called 1-*cells* or *arrows*, and are designated by the usual arrow notation $f : A \to B$. Given $f, g \in \mathrm{Ob}(\mathscr{A}(A, B))$, we shall write $f \Rightarrow g$ to denote a morphism from f to g in $\mathscr{A}(A, B)$. Such morphisms are called 2-*cells*. The composition of 2-cells $\alpha : f \Rightarrow g$ and $\beta : g \Rightarrow h$ shall be denoted by $\beta \odot \alpha : f \Rightarrow h$.
- For every $A, B, C \in \mathrm{Ob}(\mathscr{A})$, a *composition bifunctor* :
$$c_{ABC} : \mathscr{A}(A, B) \times \mathscr{A}(B, C) \to \mathscr{A}(A, C).$$
Given 1-cells $A \xrightarrow{f} B \xrightarrow{g} C$, we write $g \circ f := c_{ABC}(f, g)$.
Given two 2-cells $\alpha : f \Rightarrow g$ and $\beta : h \Rightarrow k$, respectively in $\mathscr{A}(A, B)$ and $\mathscr{A}(B, C)$, we use the notation
$$\beta * \alpha := c_{ABC}(\alpha, \beta) : h \circ f \Rightarrow k \circ g.$$
Also, if h is any 1-cell of $\mathscr{A}(B, C)$, we usually write $h * \alpha$ instead of $\mathbf{1}_h * \alpha$. Likewise, we set $\beta * f := \beta * \mathbf{1}_f$, for every 1-cell f in $\mathscr{A}(A, B)$.
- For every element $A \in \mathrm{Ob}(\mathscr{A})$, a *unit functor* :
$$u_A : \mathbf{1} \to \mathscr{A}(A, A)$$

where $\mathbf{1} := (*, \mathbf{1}_*)$ is the terminal element of $\mathbf{Cat}$. Hence u_A is the datum of an object:

$$\mathbf{1}_A \in \mathrm{Ob}(\mathscr{A}(A,A))$$

and its identity endomorphism, which we shall denote by $i_A : \mathbf{1}_A \to \mathbf{1}_A$.

The bifunctors c_{ABC} are required to satisfy an *associativity axiom*, which says that the diagram:

$$\begin{tikzcd}
\mathscr{A}(A,B)\times\mathscr{A}(B,C)\times\mathscr{A}(C,D) \arrow[r, "\mathbf{1}\times c_{BCD}"] \arrow[d, "c_{ABC}\times\mathbf{1}"'] & \mathscr{A}(A,B)\times\mathscr{A}(B,D) \arrow[d, "c_{ABD}"] \\
\mathscr{A}(A,C)\times\mathscr{A}(C,D) \arrow[r, "c_{ACD}"] & \mathscr{A}(A,D)
\end{tikzcd}$$

commutes for every $A, B, C, D \in \mathrm{Ob}(\mathscr{A})$. Likewise, the functor u_A is required to satisfy a *unit axiom*; namely, the diagram :

$$\begin{tikzcd}
\mathbf{1}\times\mathscr{A}(A,B) \arrow[d, "u_A\times\mathbf{1}_{\mathscr{A}(A,B)}"'] & \mathscr{A}(A,B) \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d, equal] & \mathscr{A}(A,B)\times\mathbf{1} \arrow[d, "\mathbf{1}_{\mathscr{A}(A,B)}\times u_B"] \\
\mathscr{A}(A,A)\times\mathscr{A}(A,B) \arrow[r, "c_{AAB}"] & \mathscr{A}(A,B) & \mathscr{A}(A,B)\times\mathscr{A}(B,B) \arrow[l, "c_{ABB}"']
\end{tikzcd}$$

commutes for every $A, B \in \mathrm{Ob}(\mathscr{A})$.

1.3.3. In a 2-category $\mathscr{A}$, it makes sense to speak of *adjoint pair of* 1*-cells*, or of the *Kan extension of a* 1*-cell* : see [10, Def.7.1.2, 7.1.3] for the definitions. In the same vein, we say that a 1-cell $f : A \to B$ of $\mathscr{A}$ is an *equivalence* if there exists a 1-cell $g : B \to A$ and invertible 2-cells $g \circ f \Rightarrow \mathbf{1}_A$ and $f \circ g \Rightarrow \mathbf{1}_B$.

Definition 1.3.4. ([10, Def.7.5.1]) Let $\mathscr{A}$ and $\mathscr{B}$ be two 2-categories. A *pseudo-functor* $F : \mathscr{A} \to \mathscr{B}$ is the datum of :

- For every $A \in \mathrm{Ob}(\mathscr{A})$, an object $FA \in \mathrm{Ob}(\mathscr{B})$.
- For every $A, B \in \mathrm{Ob}(\mathscr{A})$, a functor :

  $$F_{AB} : \mathscr{A}(A,B) \to \mathscr{B}(FA, FB).$$

  We shall often omit the subscript, and write only Ff instead of $F_{AB}f : FA \to FB$, for a 1-cell $f : A \to B$.
- For every $A, B, C \in \mathrm{Ob}(\mathscr{A})$, a natural isomorphism γ_{ABC} between two functors $\mathscr{A}(A,B) \times \mathscr{A}(B,C) \to \mathscr{B}(FA, FC)$ as indicated by the (not necessarily commutative) diagram :

  $$\begin{tikzcd}
  \mathscr{A}(A,B)\times\mathscr{A}(B,C) \arrow[r, "c_{ABC}"] \arrow[d, "F_{AB}\times F_{BC}"'] \arrow[dr, phantom, "\gamma_{ABC}\nearrow"] & \mathscr{A}(A,C) \arrow[d, "F_{AC}"] \\
  \mathscr{B}(FA,FB)\times\mathscr{B}(FB,FC) \arrow[r, "c_{FA,FB,FC}"'] & \mathscr{B}(FA,FC).
  \end{tikzcd}$$

  To ease notation, for every $(f, g) \in \mathscr{A}(A,B) \times \mathscr{A}(B,C)$, we shall write $\gamma_{f,g}$ instead of $(\gamma_{ABC})_{(f,g)} : c_{FA,FB,FC}(F_{AB}f, F_{BC}g) \to F_{AC}(c_{ABC}(f, g))$.
- For every $A \in \mathrm{Ob}(\mathscr{A})$, a natural isomorphism δ_A between functors $\mathbf{1} \to \mathscr{B}(FA, FA)$, as indicated by the (not necessarily commutative) diagram :

  $$\begin{tikzcd}
  \mathbf{1} \arrow[r, "u_A"] \arrow[d, equal] \arrow[dr, phantom, "\delta_A\nearrow"] & \mathscr{A}(A,A) \arrow[d, "F_{AA}"] \\
  \mathbf{1} \arrow[r, "u_{FA}"'] & \mathscr{B}(FA,FA).
  \end{tikzcd}$$

The system $(\delta_\bullet, \gamma_{\bullet\bullet\bullet})$ is called the *coherence constraint* for F. This datum is required to satisfy:

- A *composition axiom*, which says that the diagram :

$$\begin{tikzcd} Fh \circ Fg \circ Ff \arrow[r, Rightarrow, "\mathbf{1}_{Fh} * \gamma_{f,g}"] \arrow[d, Rightarrow, "\gamma_{g,h} * \mathbf{1}_{Ff}"'] & Fh \circ (F(g \circ f)) \arrow[d, Rightarrow, "\gamma_{g \circ f, h}"] \\ F(h \circ g) \circ Ff \arrow[r, Rightarrow, "\gamma_{f, h \circ g}"] & F(h \circ g \circ f) \end{tikzcd}$$

  commutes for every sequence of arrows : $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ in $\mathscr{A}$.

- A *unit axiom*, which says that the diagrams :

$$\begin{tikzcd} Ff \circ \mathbf{1}_{FA} \arrow[r, Rightarrow, "\mathbf{1}_{Ff} * \delta_A"] \arrow[d, Rightarrow, "\mathbf{1}_{Ff}"'] & Ff \circ F\mathbf{1}_A \arrow[d, Rightarrow, "\gamma_{\mathbf{1}_A, f}"] \\ Ff \arrow[r, Rightarrow, "\mathbf{1}_{Ff}"] & F(f \circ \mathbf{1}_A) \end{tikzcd} \qquad \begin{tikzcd} \mathbf{1}_{FB} \circ Ff \arrow[r, Rightarrow, "\delta_B \circ \mathbf{1}_{Ff}"] \arrow[d, Rightarrow, "\mathbf{1}_{Ff}"'] & F(\mathbf{1}_B) \circ Ff \arrow[d, Rightarrow, "\gamma_{f, \mathbf{1}_B}"] \\ Ff \arrow[r, Rightarrow, "\mathbf{1}_{Ff}"] & F(\mathbf{1}_B \circ f) \end{tikzcd}$$

  commute for every arrow $f : A \to B$ (where, to ease notation, we have written δ_A instead of $(\delta_A)_* : \mathbf{1}_{FA} \to F_{AA}\mathbf{1}_A$, and likewise for δ_B).

Definition 1.3.5. Consider two pseudo-functors $F, G : \mathscr{A} \to \mathscr{B}$ between 2-categories $\mathscr{A}$, $\mathscr{B}$. A *pseudo-natural transformation* $\alpha : F \Rightarrow G$ is the datum of :

- For every object A of $\mathscr{A}$, a 1-cell $\alpha_A : FA \to GA$.
- For every pair of objects A, B of $\mathscr{A}$, a natural isomorphism τ_{AB} between two functors $\mathscr{A}(A, B) \to \mathscr{B}(FA, GB)$, as indicated by the (not necessarily commutative) diagram:

$$\begin{tikzcd} \mathscr{A}(A,B) \arrow[r, "F_{AB}"] \arrow[d, "G_{AB}"'] \arrow[dr, phantom, "\tau_{AB} \nearrow"] & \mathscr{B}(FA, FB) \arrow[d, "\mathscr{B}(\mathbf{1}_{\mathscr{B}}, \alpha_B)"] \\ \mathscr{B}(GA, GB) \arrow[r, "\mathscr{B}(\alpha_A, \mathbf{1}_{\mathscr{B}})"'] & \mathscr{B}(FA, GB) \end{tikzcd}$$

  where $\mathscr{B}(\alpha_A, \mathbf{1}_{\mathscr{B}})$ is the functor obtained by fixing α_A in the first argument of the composition bifunctor $c_{FA,GA,GB} : \mathscr{B}(FA, GA) \times \mathscr{B}(GA, GB) \to \mathscr{B}(FA, GB)$, and likewise for $\mathscr{B}(\mathbf{1}_{\mathscr{B}}, \alpha_B)$. The datum $\tau_{\bullet\bullet}$ is called the *coherence constraint* for α.

This datum is required to satisfy the following *coherence axioms* (in which we denote by (δ^F, γ^F) and (δ^G, γ^G) the coherence constraints for F and respectively G) :

- For every $A \in \mathrm{Ob}(\mathscr{A})$, the diagram :

$$\begin{tikzcd} \alpha_A \arrow[r, Rightarrow, "\mathbf{1}_{\alpha_A}"] \arrow[d, Rightarrow, "\mathbf{1}_{\alpha_A}"'] & \mathbf{1}_{GA} \circ \alpha_A \arrow[r, Rightarrow, "\delta^G_A * \mathbf{1}_{\alpha_A}"] & G(\mathbf{1}_A) \circ \alpha_A \arrow[d, Rightarrow, "\tau_{\mathbf{1}_A}"] \\ \alpha_A \circ \mathbf{1}_{FA} \arrow[rr, Rightarrow, "\mathbf{1}_{\alpha_A} * \delta^F_A"] & & \alpha_A \circ F(\mathbf{1}_A) \end{tikzcd}$$

  commutes.

- For each pair of arrows $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathscr{A}$, the diagram :

$$\begin{tikzcd} Gg \circ Gf \circ \alpha_A \arrow[r, Rightarrow, "\mathbf{1}_{Gg} * \tau_f"] \arrow[d, Rightarrow, "\gamma^G_{f,g} * \mathbf{1}_{\alpha_A}"'] & Gg \circ \alpha_B \circ Ff \arrow[r, Rightarrow, "\tau_g * \mathbf{1}_f"] & \alpha_C \circ Fg \circ Ff \arrow[d, Rightarrow, "\mathbf{1}_{\alpha_C} * \gamma^F_{f,g}"] \\ G(g \circ f) \circ \alpha_A \arrow[rr, Rightarrow, "\tau_{g \circ f}"] & & \alpha_C \circ F(g \circ f) \end{tikzcd}$$

  commutes.

If $\alpha : F \Rightarrow G$ and $\beta : G \Rightarrow H$ are two pseudo-natural transformations, we may define the composition $\beta \odot \alpha : F \Rightarrow H$ which is the pseudo-natural transformation given by the rule $A \mapsto \beta_A \circ \alpha_A$ for every $A \in \mathrm{Ob}(\mathscr{A})$. The coherence constraint of $\beta \odot \alpha$ is given by the rule :

$$(A, B) \mapsto (\beta_B * \tau^\alpha_{AB}) \odot (\tau^\beta_{AB} * \alpha_A) \qquad \text{for every } A, B \in \mathrm{Ob}(\mathscr{A})$$

where $\tau^\alpha_{\bullet\bullet}$ (resp. $\tau^\beta_{\bullet\bullet}$) denotes the coherence constraint of α (resp. of β).

Example 1.3.6. (i) Any category $\mathscr{A}$ can be regarded as a 2-category in a natural way : namely, for any two objects A and B of $\mathscr{A}$ one lets $\mathscr{A}(A, B)$ be the discrete category $\mathrm{Hom}_\mathscr{A}(A, B)$; hence the only 2-cells of $\mathscr{A}$ are the identities $\mathbf{1}_f : f \Rightarrow f$, for every morphism $f : A \to B$. The composition bifunctor c_{ABC} is of course given (on 1-cells) by the composition law for morphisms of $\mathscr{A}$. Likewise, the functor u_A assigns to every object A its identity endomorphism.

(ii) In the same vein, every functor between usual categories, is a pseudo-functor between the corresponding 2-categories as in (i); of course, the coherence constraint consists of identities. Finally, every natural transformation of usual functors can be regarded naturally as a pseudo-natural transformation between the corresponding pseudo-functors.

(iii) As it has already been mentioned, the category $\mathbf{Cat}$ is naturally a 2-category. Namely, for any three small categories $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$, the 1-cells in $\mathbf{Cat}(\mathscr{A}, \mathscr{B})$ are the functors from $\mathscr{A}$ to $\mathscr{B}$, and the 2-cells are the natural transformations between these functors. The composition law $c_{\mathscr{A}\mathscr{B}\mathscr{C}}$ is defined on 1-cells by the usual composition of functors, and on 2-cells as in (1.1.7).

(iv) The standard constructions on categories admit analogues for 2-categories. However, if $\mathscr{A}$ is a 2-category, there are several inequivalent candidates for the *opposite* 2-*category* $\mathscr{A}^o$: one can reverse the 1-cells, one can reverse the 2-cells, *i.e.* replace the categories $\mathscr{A}(A, B)$ by their opposites, or do both. We leave to the reader the task of spelling out the definition(s).

(v) Likewise, if X is any object of $\mathscr{A}$, one may define a 2-category $\mathscr{A}/X$. Its objects are the arrows in $\mathscr{A}$ of the form $f : A \to X$. If $g : B \to X$ is another such arrow, $\mathscr{A}/X(f, g)$ is the subcategory of $\mathscr{A}(f, g)$ whose objects are all the arrows $h : A \to B$ such that $g \circ h = f$, and whose morphisms are all the 2-cells α of $\mathscr{A}(f, g)$ such that $\mathbf{1}_g * \alpha = \mathbf{1}_f$. The composition and unit functors for $\mathscr{A}/X$ are the restrictions of the corresponding ones for $\mathscr{A}$.

(vi) Consider 2-categories $\mathscr{A}$, $\mathscr{B}$. For every object B of $\mathscr{B}$, one may define the *constant pseudo-functor* with value B : this is the pseudo-functor

$$\mathsf{F}_B : \mathscr{A} \to \mathscr{B} \qquad \text{such that} \qquad \mathsf{F}_B(A) := B \quad \mathsf{F}_B(f) := \mathbf{1}_B \quad \mathsf{F}_B(\alpha) := i_B$$

for every $A \in \mathrm{Ob}(\mathscr{A})$, every 1-cell f, and every 2-cell α of $\mathscr{A}$. The coherence constraint for F_B consists of identities. Given a pseudo-functor $F : \mathscr{A} \to \mathscr{B}$, a *pseudo-cone on F with vertex B* is a pseudo-natural transformation $\mathsf{F}_B \Rightarrow F$. Especially, every 1-cell $f : B \to B'$ induces a pseudo-cone :

$$\mathsf{F}_f : \mathsf{F}_B \Rightarrow \mathsf{F}_{B'} \quad : \quad (\mathsf{F}_f)_A := f \quad \text{for every } A \in \mathrm{Ob}(\mathscr{A})$$

whose coherence constraint consists of identities.

Dually, a *pseudo-cocone on F with vertex B* is a pseudo-natural transformation $F \Rightarrow \mathsf{F}_B$, and F_f can thus be viewed as a pseudo-cocone on F with vertex B'.

Definition 1.3.7. Consider two pseudo-functors $F, G : \mathscr{A} \to \mathscr{B}$ between 2-categories $\mathscr{A}$, $\mathscr{B}$, and two pseudo-natural transformations $\alpha, \beta : F \Rightarrow G$. A *modification* $\Xi : \alpha \rightsquigarrow \beta$ is a family :

$$\Xi_A : \alpha_A \Rightarrow \beta_A$$

of 2-cells of $\mathscr{B}$, for every object A of $\mathscr{A}$. Such a family is required to satisfy the following condition. For every pair of 1-cells $f, g : A \to A'$ of $\mathscr{A}$, and every 2-cell $\gamma : f \Rightarrow g$, the equality

$$(\Xi_{A'} * F\gamma) \odot \tau^\alpha_{AA',f} = \tau^\beta_{AA',g} \odot (G\gamma * \Xi_A)$$

holds in $\mathscr{B}$, where $\tau^\alpha_{\bullet\bullet}$ (resp. $\tau^\beta_{\bullet\bullet}$) denotes the coherence constraint for α (resp. for β).

1.3.8. If $\Xi, \Theta : \alpha \rightsquigarrow \beta$ are two modifications between pseudo-natural transformations $\alpha, \beta : F \Rightarrow G$ of pseudo-functors $F, G : \mathscr{A} \to \mathscr{B}$, we may define the composition

$$\Xi \circ \Theta : \alpha \rightsquigarrow \beta \quad : \quad A \mapsto \Xi_A \odot \Theta_A \quad \text{for every } A \in \mathrm{Ob}(A).$$

We may then consider the category :

$$\mathsf{PsNat}(F, G)$$

whose objects are the pseudo-natural transformations $F \Rightarrow G$, and whose morphisms are the modifications $\alpha \rightsquigarrow \beta$ between them. For instance, $\mathsf{PsNat}(\mathsf{F}_B, F)$ (resp. $\mathsf{PsNat}(F, \mathsf{F}_B)$) is the category of pseudo-cones (resp. pseudo-cocones) on F with vertex B (example 1.3.6(vi)).

1.3.9. Let $\mathscr{A}, \mathscr{B}$ be two 2-categories; using the modification, we can endow the category $\mathsf{PsFun}(\mathscr{A}, \mathscr{B})$ of pseudo-functors $\mathscr{A} \to \mathscr{B}$, with a natural structure of 2-category. Namely, for any two pseudo-functors $F, G : \mathscr{A} \to \mathscr{B}$, the 1-cells $F \to G$ of $\mathsf{PsFun}(\mathscr{A}, \mathscr{B})$ are the pseudo-natural transformations $F \Rightarrow G$ of two such functors; of course, for fixed F and G, the category structure on the set of 1-cells $F \to G$ is precisely the one of $\mathsf{PsNat}(F, G)$, *i.e.* the 2-cells of $\mathsf{PsFun}(\mathscr{A}, \mathscr{B})$ are the modifications. The composition functor

$$\mathsf{PsNat}(F, G) \times \mathsf{PsNat}(G, H) \to \mathsf{PsNat}(F, H)$$

assigns, to any two modifications $\Xi : \alpha \rightsquigarrow \beta$ and $\Xi' : \alpha' \rightsquigarrow \beta'$, the modification

$$\Xi' * \Xi : \alpha' \circ \alpha \rightsquigarrow \beta' \circ \beta \qquad A \mapsto \Xi'_A * \Xi_A \qquad \text{for every } A \in \mathrm{Ob}(\mathscr{A}).$$

With this notation, we say that a pseudo-natural transformation of pseudo-functors $\mathscr{A} \to \mathscr{B}$ is a *pseudo-natural equivalence* if it is an equivalence in the 2-category $\mathsf{PsFun}(\mathscr{A}, \mathscr{B})$ (in the sense of (1.3.3)).

Definition 1.3.10. Let $\mathscr{A}$ and $\mathscr{B}$ be two 2-categories, and $F : \mathscr{A} \to \mathscr{B}$ a pseudo-functor.

(i) We say that F is a 2-*equivalence* from $\mathscr{A}$ to $\mathscr{B}$ if the following holds :
- For every $A, B \in \mathrm{Ob}(\mathscr{A})$, the functor F_{AB} is an equivalence (notation of definition 1.3.4).
- For every $A' \in \mathrm{Ob}(\mathscr{B})$ there exists $A \in \mathrm{Ob}(\mathscr{A})$ and an equivalence $FA \to A'$.

(ii) Let $G : \mathscr{B} \to \mathscr{A}$ be a pseudo-functor. We say that G is *right* 2-*adjoint* to F if the following holds.
- For every $A \in \mathrm{Ob}(\mathscr{A})$ and $B \in \mathrm{Ob}(\mathscr{B})$ there exists an equivalence of categories
$$\vartheta_{AB} : \mathrm{Hom}_{\mathscr{B}}(FA, B) \to \mathrm{Hom}_{\mathscr{A}}(A, GB).$$
- For every pair of 1-cells $f : A' \to A$ in $\mathscr{A}$, $g : B \to B'$ in $\mathscr{B}$, there exists a natural isomorphism of functors
$$\begin{tikzcd} \mathrm{Hom}_{\mathscr{B}}(FA, B) \arrow[r, "\mathrm{Hom}_{\mathscr{B}}(Ff,g)"] \arrow[d, "\vartheta_{AB}"'] \arrow[dr, phantom, "\tau_{f,g} \nearrow"] & \mathrm{Hom}_{\mathscr{B}}(FA', B') \arrow[d, "\vartheta_{A'B'}"] \\ \mathrm{Hom}_{\mathscr{A}}(A, GB) \arrow[r, "\mathrm{Hom}_{\mathscr{A}}(f,Gg)"] & \mathrm{Hom}_{\mathscr{A}}(A', GB'). \end{tikzcd}$$
- For every pair of compositions of 1-cells $A'' \xrightarrow{f'} A' \xrightarrow{f} A$ in $\mathscr{A}$, $B \xrightarrow{g} B' \xrightarrow{g'} B''$, we have the identity
$$(\tau_{f',g'} * \mathrm{Hom}_{\mathscr{B}}(Ff, g)) \circ (\mathrm{Hom}_{\mathscr{A}}(f', Gg') * \tau_{f,g}) = \tau_{f \circ f', g' \circ g}.$$

In this case, we also say that (F, G) is a 2-adjoint pair of pseudo-functors.

Remark 1.3.11. Let $F : \mathscr{A} \to \mathscr{B}$ be a pseudo-functor.

(i) The reader may show that F is a 2-equivalence if and only if there exists a pseudo-functor $G : \mathscr{B} \to \mathscr{A}$ and two pseudo-natural equivalences $\mathbf{1}_\mathscr{A} \Rightarrow G \circ F$ and $F \circ G \Rightarrow \mathbf{1}_\mathscr{B}$.

(ii) Suppose that F admits a right 2-adjoint $G : \mathscr{A} \to \mathscr{B}$, and let $\vartheta_{\bullet\bullet}$ and $\tau_{\bullet\bullet}$ be the corresponding data as in definition 1.3.10(ii). Using these data, one can define a unit $\eta : \mathbf{1}_\mathscr{A} \Rightarrow F \circ G$ and counit $\varepsilon : G \circ F \Rightarrow \mathbf{1}_\mathscr{B}$, that are pseudo-natural transformations fulfilling triangular identities as in (1.1.8).

(iii) Conversely, the existence of pseudo-natural transformation ε, η as in (ii), fulfilling the above mentioned triangular identities, implies that G is right 2-adjoint to F (details left to the reader).

Definition 1.3.12. Let $\mathscr{A}$ be a 2-category, $F : \mathscr{A} \to \mathscr{B}$ a pseudo-functor.

(i) A 2-*limit* of F is a pair :
$$2\text{-}\lim_{\mathscr{A}} F := (L, \pi)$$
consisting of an object L of $\mathscr{B}$ and a pseudo-cone $\pi : \mathsf{F}_L \Rightarrow F$, such that the functor :
$$\mathscr{B}(B, L) \to \mathsf{PsNat}(\mathsf{F}_B, F) \quad : \quad f \mapsto \pi \odot \mathsf{F}_f$$
is an equivalence of categories, for every $B \in \mathrm{Ob}(\mathscr{B})$.

(ii) Dually, a 2-*colimit* of F is a pair :
$$2\text{-}\mathop{\mathrm{colim}}_{\mathscr{A}} F := (L, \pi)$$
consisting of an object L of $\mathscr{B}$ and a pseudo-cocone $\pi : F \Rightarrow \mathsf{F}_L$, such that the functor:
$$\mathscr{B}(L, B) \to \mathsf{PsNat}(F, \mathsf{F}_B) \quad : \quad f \mapsto \mathsf{F}_f \odot \pi$$
is an equivalence of categories, for every $B \in \mathrm{Ob}(\mathscr{B})$.

As usual, if the 2-limit exists, it is unique up to (non-unique) equivalence : if (L', π') is another 2-limit, there exists an equivalence $h : L \to L'$ and an isomorphism $\beta : \pi' \odot \mathsf{F}_h \xrightarrow{\sim} \pi$; moreover, the pair (h, β) is unique up to unique isomorphism, in a suitable sense, that the reader may spell out, as an exercise. A similar remark holds for 2-colimits.

(iii) We say that $\mathscr{B}$ is 2-*complete* (resp. 2-*cocomplete*) if, for every small 2-category $\mathscr{A}$, every pseudo-functor $\mathscr{A} \to \mathscr{B}$ admits a 2-limit (resp. a 2-colimit).

Remark 1.3.13. To be in keeping with the terminology of [10, Ch.VII], we should write pseudo-bilimit instead of 2-limit (and likewise for 2-colimit). The term “2-limit” denotes in *loc.cit.* a related notion, which makes it unique up to isomorphism, not just up to equivalence. However, the notion introduced in definition 1.3.12 is the one that occurs most frequently in applications.

The following lemma 1.3.14 indicates that the framework of 2-categories does indeed provide an adequate answer to the issues raised in (1.3).

Lemma 1.3.14. *Let $\mathscr{A}$ be a 2-category, $F, G : \mathscr{A} \to \mathscr{B}$ two pseudo-functors, $\omega : F \Rightarrow G$ a pseudo-natural equivalence, and suppose that the 2-limit of F exists. Then the same holds for the 2-limit of G, and there is a natural equivalence in $\mathscr{B}$:*
$$2\text{-}\lim_{\mathscr{A}} F \xrightarrow{\sim} 2\text{-}\lim_{\mathscr{A}} G.$$
More precisely, if (L, π) is a pair with $L \in \mathrm{Ob}(\mathscr{B})$ and a pseudo-cone $\pi : \mathsf{F}_L \Rightarrow F$ representing the 2-limit of F, then the pair $(L, \omega \circ \pi)$ represents the 2-limit of G.

A dual assertion holds for 2-colimits.

Proof. It is easily seen that the rule $\alpha \mapsto \omega \odot \alpha$ induces an equivalence of categories :
$$\mathsf{PsNat}(\mathsf{F}_B, F) \to \mathsf{PsNat}(\mathsf{F}_B, G).$$
The claim is an immediate consequence. □

Proposition 1.3.15. *For every small category $\mathscr{B}$, the 2-category $\mathbf{Cat}/\mathscr{B}$ is 2-complete and 2-cocomplete.*

Proof. We only show 2-completeness, and we leave the proof of 2-cocompleteness as an exercise for the reader. Notice that, in the special case where $\mathscr{B}$ is the terminal object of $\mathbf{Cat}$, the assertion means that $\mathbf{Cat}$ is 2-complete and 2-cocomplete. Let :

$$F : \mathscr{A} \to \mathbf{Cat} \quad : \quad a \mapsto F_a \qquad \text{for every } a \in \mathrm{Ob}(\mathscr{A})$$

be a pseudo-functor from a small 2-category $\mathscr{A}$; we define a category $\mathscr{L}_F$ as follows :

- The objects of $\mathscr{L}_F$ are all the systems $(X_\bullet, \xi^X_\bullet)$, where $X_a \in \mathrm{Ob}(F_a)$ for every $a \in \mathrm{Ob}(\mathscr{A})$ and $\xi^X_f : F_{ab}(f)(X_a) \xrightarrow{\sim} X_b$ is an isomorphism in F_b, for every morphism $f : a \to b$ in $\mathscr{A}$. The data $(X_\bullet, \xi^X_\bullet)$ are required to fulfill the following conditions.
  (a) If $f : a \to b$ and $g : b \to c$ are any two morphisms in $\mathscr{A}$, the diagram :
$$\begin{tikzcd} F_{bc}(g) \circ F_{ab}(f)(X_a) \arrow[r, "F_{bc}(g)(\xi^X_f)"] \arrow[d, "\gamma_{(f,g)}(X_a)"'] & F_{bc}(g)(X_b) \arrow[d, "\xi^X_g"] \\ F_{ac}(g \circ f)(X_a) \arrow[r, "\xi^X_{g\circ f}(X_a)"] & X_c \end{tikzcd}$$
  commutes, where γ denotes the coherence constraint of F.
  (b) Moreover, $\xi^X_{\mathbf{1}_a} = \mathbf{1}_{X_a}$ for every $a \in \mathrm{Ob}(\mathscr{A})$, where $\mathbf{1}_a$ is the image of the unit functor $u_a : \mathbf{1} \to \mathscr{A}$.
- The morphisms $(X_\bullet, \xi^X_\bullet) \to (Y_\bullet, \xi^Y_\bullet)$ in $\mathscr{L}_F$ are the systems of morphisms $t_\bullet := (t_a : X_a \to Y_a \mid a \in \mathrm{Ob}(\mathscr{A}))$ such that the diagram :
$$\begin{tikzcd} F_{ab}(f)(X_a) \arrow[r, "F_{ab}(f)(t_a)"] \arrow[d, "\xi^X_f"'] & F_{ab}(f)(Y_a) \arrow[d, "\xi^Y_f"] \\ X_b \arrow[r, "t_b"] & Y_b \end{tikzcd}$$
  commutes for every morphism $f : a \to b$ in $\mathscr{A}$.

Next, we define a pseudo-cone $\pi : \mathsf{F}_{\mathscr{L}_F} \Rightarrow F$ as follows.

- For every $a \in \mathrm{Ob}(\mathscr{A})$, we let $\pi_a : \mathscr{L}_F \to F_a$ be the functor given by the rule :
$$(X_\bullet, \xi^X_\bullet) \mapsto X_a \qquad \text{and} \qquad t_\bullet \mapsto t_a.$$
- For every morphism $f : a \to b$ in $\mathscr{A}$, we let $\tau_f : F_{ab}(f) \circ \pi_a \Rightarrow \pi_b$ be the natural transformation given by the rule :
$$(\tau_f)_{(X_\bullet, \xi^X_\bullet)} := \xi^\bullet_f.$$

The verification that $(\pi_\bullet, \tau_\bullet)$ satisfies the coherence axioms for a pseudo-natural transformation, is straightforward, using the foregoing conditions (a) and (b). We claim that $(\mathscr{L}_F, \pi)$ is a 2-limit of F. Indeed, suppose that $\mathscr{C}$ is another small category, and $\alpha : \mathsf{F}_\mathscr{C} \Rightarrow F$ is a pseudo-cone with vertex $\mathscr{C}$. By definition, α is the datum of functors $\alpha_a : \mathscr{C} \to F_a$ for every $a \in \mathrm{Ob}(\mathscr{A})$, and natural isomorphisms $\sigma_f : F_{ab}(f) \circ \alpha_a \Rightarrow \alpha_b$ for every morphism $f : a \to b$ in $\mathscr{A}$ fulfilling the usual coherence axioms. To such datum we attach a functor $\mathscr{L}_\alpha : \mathscr{C} \to \mathscr{L}_F$ by the following rule. For every $X \in \mathrm{Ob}(\mathscr{C})$, we set :

$$\mathscr{L}_\alpha(X) := (\alpha_\bullet X, (\sigma_\bullet)_X)$$

and to every morphism $g : X \to Y$ we assign the compatible system $(\alpha_a g \mid a \in \mathrm{Ob}(\mathscr{A}))$. It is then immediate to see that $\pi \odot \mathsf{F}_{\mathscr{L}_\alpha} = \alpha$, and $\mathscr{L}_{\pi \odot \mathsf{F}_G} = G$ for every functor $G : \mathscr{C} \to \mathscr{L}_F$. $\square$

1.3.16. Let $\mathscr{A}$ be any 2-category. We shall say that a diagram of objects and arrows in $\mathscr{A}$:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\ C \arrow[r, "k"] & D \end{tikzcd} \tag{1.3.17}$$

is *essentially commutative*, if there exists an invertible 2-cell $\alpha : h \circ f \Rightarrow k \circ g$. Let L be the small category with $\mathrm{Ob}(L) := \{0, 1, 2\}$, and whose set of arrows consists of the identity morphisms, and two more arrows $1 \to 0$ and $2 \to 0$. An essentially commutative diagram (1.3.17) can be regarded as a pseudo-cone π with vertex A, on the functor $F : L \to \mathscr{A}$ such that $F(0) := D$, $F(1) := B$, $F(2) := C$, $F(1 \to 0) := h$ and $F(2 \to 0) := k$. We say that (1.3.17) is 2*-cartesian* if (A, π) is a 2-limit of the functor F.

For instance, let $\mathscr{A} := \mathbf{Cat}$; by inspecting the proof of proposition 1.3.15, we see that (1.3.17) is a 2-cartesian diagram, if and only if the functors f and g and the 2-cell α induce an equivalence from the small category A, to the category whose objects are all data of the form $X := (b, c, \xi)$, where $b \in \mathrm{Ob}(B)$, $c \in \mathrm{Ob}(C)$, and $\xi : h(b) \xrightarrow{\sim} k(c)$ is an isomorphism. If $X' := (b', c', \xi')$ is another such datum, the morphisms $X \to X'$ are the pairs (φ, ψ), where $\varphi : b \to b'$ (resp. $\psi : c \to c'$) is a morphism in B (resp. in C), and $\xi' \circ h(\varphi) = k(\varphi') \circ \xi$.

This category shall be called the 2*-fibre product* of the functors k and h, and shall be denoted

$$B \overset{2}{\underset{(h,k)}{\times}} C$$

or sometimes, just $B \overset{2}{\times}_D C$, if there is no danger of ambiguity.

1.4. **Fibrations.** We keep the assumptions and notation of section 1.1; especially, $\mathbf{Cat}$ is synonimous with U-$\mathbf{Cat}$. Let $\varphi : \mathscr{A} \to \mathscr{B}$ be a functor, $f : A' \to A$ a morphism in $\mathscr{A}$, and set

$$g := \varphi f : \varphi A' \to \varphi A.$$

We say that f is *φ-cartesian*, or – slightly abusively – that f is *$\mathscr{B}$-cartesian*, if the induced commutative diagram of sets (notation of (1.1.14)) :

$$\begin{tikzcd} \mathrm{Hom}_{\mathscr{A}}(X, A') \arrow[r, "f_*"] \arrow[d, "\varphi"'] & \mathrm{Hom}_{\mathscr{A}}(X, A) \arrow[d, "\varphi"] \\ \mathrm{Hom}_{\mathscr{B}}(\varphi X, \varphi A') \arrow[r, "g_*"] & \mathrm{Hom}_{\mathscr{B}}(\varphi X, \varphi A) \end{tikzcd}$$

is cartesian for every $X \in \mathrm{Ob}(\mathscr{A})$. In this case, one also says that f *is an inverse image of* A over g, or – slightly abusively – that A' *is an inverse image of* A over g. One verifies easily that the composition of two $\mathscr{B}$-cartesian morphisms is again $\mathscr{B}$-cartesian.

Definition 1.4.1. Let $\varphi : \mathscr{A} \to \mathscr{B}$ be a functor, and B any object of $\mathscr{B}$.

(i) The *fibre of* φ *over* B is the category $\varphi^{-1}B$ whose objects are all the $A \in \mathrm{Ob}(\mathscr{A})$ such that $\varphi A = B$, and whose morphisms $f : A' \to A$ are the elements of $\mathrm{Hom}_{\mathscr{A}}(A', A)$ such that $\varphi f = \mathbf{1}_B$. We denote by :

$$\iota_B : \varphi^{-1}B \to \mathscr{A}$$

the natural faithful imbedding of $\varphi^{-1}B$ into $\mathscr{A}$.

(ii) We say that φ is a *fibration* if, for every morphism $g : B' \to B$ in $\mathscr{B}$, and every $A \in \mathrm{Ob}(\varphi^{-1}B)$, there exists an inverse image $f : A' \to A$ of A over g. In this case, we also say that $\mathscr{A}$ is a *fibred $\mathscr{B}$-category*.

Example 1.4.2. Let $\mathscr{C}$ be any category, X any object of $\mathscr{C}$.

(i) The functor (1.1.13) is a fibration, and all the morphisms in $\mathscr{C}/X$ are $\mathscr{C}$-cartesian. The easy verification shall be left to the reader.
(ii) The source functor $\mathsf{s} : \mathsf{Morph}(\mathscr{C}) \to \mathscr{C}$ is a fibration (notation of (1.1.17)); more precisely, the s-cartesian morphisms are the square diagrams (1.1.18) in which g' is an isomorphism. For any $B \in \mathrm{Ob}(\mathscr{B})$, the fibre $\mathsf{s}^{-1}B$ is the category $B/\mathscr{C}$.
(iii) Suppose that all fibre products are representable in $\mathscr{C}$. Then also the target functor $\mathsf{t} : \mathsf{Morph}(\mathscr{C}) \to \mathscr{C}$ is a fibration; more precisely, the t-cartesian morphisms are the square diagrams (1.1.18) which are cartesian (*i.e.* fibred). For any $B \in \mathrm{Ob}(\mathscr{B})$, the fibre $\mathsf{t}^{-1}B$ is the category $\mathscr{C}/B$.

Definition 1.4.3. Let $\mathscr{A}$, $\mathscr{A}'$ and $\mathscr{B}$ be three small categories, $\varphi : \mathscr{A} \to \mathscr{B}$ and $\varphi' : \mathscr{A}' \to \mathscr{B}$ two functors, which we regard as objects of $\mathbf{Cat}/\mathscr{B}$; let also $F : \mathscr{A} \to \mathscr{A}'$ be a $\mathscr{B}$-functor, *i.e.* a morphism $\varphi \to \varphi'$ in $\mathbf{Cat}/\mathscr{B}$. We say that F is *cartesian* (resp. *strongly cartesian*), if it sends $\mathscr{B}$-cartesian morphisms of $\mathscr{A}$ (resp. all morphisms of $\mathscr{A}$), to $\mathscr{B}$-cartesian morphisms in $\mathscr{A}'$. We denote by :

$$\mathsf{Cart}_{\mathscr{B}}(\mathscr{A}, \mathscr{A}')$$

the category whose objects are the strongly cartesian $\mathscr{B}$-functors $F : \mathscr{A} \to \mathscr{A}'$, and whose morphisms are the natural transformations :

$$\alpha : F \Rightarrow G \qquad \text{such that} \qquad \varphi' * \alpha = \mathbf{1}_{\varphi}. \tag{1.4.4}$$

Remark 1.4.5. The reader might prefer to reserve the notation $\mathsf{Cart}_{\mathscr{B}}(\mathscr{A}, \mathscr{A}')$ for the larger category of all cartesian functors. However, in our applications only the categories of strongly cartesian functors arise, hence we may economize a few adverbs.

Besides, in many cases we shall consider $\mathscr{B}$-categories $\mathscr{A}$ whose every morphism is $\mathscr{B}$-cartesian (this happens, *e.g.* when $\mathscr{A}$ is a subcategory of $\mathscr{B}$); in such situations, obviously every cartesian functor $\mathscr{A} \to \mathscr{A}'$ is strongly cartesian.

1.4.6. Let $\varphi : \mathscr{A} \to \mathscr{B}$ be a fibration, and $g : B' \to B$ any morphism in $\mathscr{B}$. Suppose we have chosen, for every object $A \in \mathrm{Ob}(\varphi^{-1}B)$, an inverse image :

$$g_A : g^*A \to A$$

of A over g (so $g^*A \in \mathrm{Ob}(\varphi^{-1}B')$ and $\varphi(g_A) = g$). Then the rule $A \mapsto g^*A$ extends naturally to a functor :

$$g^* : \varphi^{-1}B \to \varphi^{-1}B'.$$

Namely, by definition, for every morphism $h : A' \to A$ in $\varphi^{-1}B$ there exists a unique morphism $g^*h : g^*A' \to g^*A$, such that :

$$h \circ g_{A'} = g_A \circ g^*h \tag{1.4.7}$$

and if $k : A'' \to A'$ is another morphism in $\varphi^{-1}B$, the uniqueness of g^*k implies that :

$$g^*k \circ g^*h = g^*(k \circ h).$$

Moreover, (1.4.7) also means that the rule $A \mapsto g_A$ defines a natural transformation :

$$g_\bullet : \iota_{B'} \circ g^* \Rightarrow \iota_B$$

(notation of definition 1.4.1(i)).

1.4.8. Let $\varphi : \mathscr{A} \to \mathscr{B}$ be a fibration between small categories, and $B'' \xrightarrow{h} B' \xrightarrow{g} B$ two morphisms in $\mathscr{B}$. Proceeding as in (1.4.6), we may attach to g and h three functors :

$$g^* : \varphi^{-1}B \to \varphi^{-1}B' \qquad h^* : \varphi^{-1}B' \to \varphi^{-1}B'' \qquad (g \circ h)^* : \varphi^{-1}B \to \varphi^{-1}B''$$

as well as natural transformations :

$$g_\bullet : \iota_{B'} \circ g^* \Rightarrow \iota_B \qquad h_\bullet : \iota_{B''} \circ h^* \Rightarrow \iota_{B'} \qquad (g \circ h)_\bullet : \iota_{B''} \circ (g \circ h)^* \Rightarrow \iota_B$$

and by inspecting the constructions, one easily finds a unique natural isomorphism :

$$\gamma_{h,g} : h^* \circ g^* \Rightarrow (g \circ h)^*$$

which fits into a commutative diagram :

$$\begin{tikzcd}
\iota_{B''} \circ h^* \circ g^* \arrow[r, Rightarrow, "h_\bullet * g^*"] \arrow[d, Rightarrow, "\iota_{B''} * \gamma_{h,g}"'] & \iota_{B'} \circ g^* \arrow[d, Rightarrow, "g_\bullet"] \\
\iota_{B''} \circ (g \circ h)^* \arrow[r, Rightarrow, "(g\circ h)_\bullet"] & \iota_B .
\end{tikzcd}$$

It follows that the rule which assigns to each $B \in \mathrm{Ob}(\mathscr{B})$ the small category $\varphi^{-1}B$, to each morphism g in $\mathscr{B}$ a chosen functor g^*, and to each pair of morphisms (g, h) as above, the natural isomorphism $\gamma_{h,g}$, defines a pseudo-functor

$$\mathsf{c} : \mathscr{B}^o \to \mathbf{Cat}. \tag{1.4.9}$$

Notice as well that, for every $B \in \mathrm{Ob}(\mathscr{B})$, we may choose $(\mathbf{1}_B)^*$ to be the identity functor of $\varphi^{-1}B$; then the isomorphism

$$(\delta_B)_* : \varphi^{-1}B \to \varphi^{-1}B$$

required by definition 1.3.4, shall also be the identity. (Here we view $\mathscr{B}^o$ as a 2-category, as explained in example 1.3.6(i); also, the natural 2-category structure on $\mathbf{Cat}$ is the one of example 1.3.6(iii)). The datum of a pseudo-functor c as in (1.4.9) is called a *cleavage* (in french: "clivage") for the fibration φ.

Example 1.4.10. Let $\mathscr{B}$ be any category.

(i) Let $\varphi : \mathscr{A} \to \mathscr{B}$ be a fibration, and $\psi : \mathscr{C} \to \mathscr{B}$ any functor. Then the induced functor $\mathscr{A} \times_{(\varphi,\psi)} \mathscr{C} \to \mathscr{C}$ is also a fibration. The cartesian morphisms of $\mathscr{A} \times_{(\varphi,\psi)} \mathscr{C}$ are the pairs (f, g) where f is a cartesian morphism of $\mathscr{A}$, g is a morphism of $\mathscr{C}$, and $\varphi(f) = \psi(g)$.

(ii) A composition of fibrations is again a fibration. Also, for $i = 1, 2$, let $\mathscr{A}_i \to \mathscr{B}$ be two fibrations; combining with (i) we see that $\mathscr{A}_1 \times_{\mathscr{B}} \mathscr{A}_2 \to \mathscr{B}$ is also a fibration.

(iii) Let F be a presheaf on $\mathscr{B}$. To F we may attach a fibration $\varphi_F : \mathscr{A}_F \to \mathscr{B}$ as follows. The objects of $\mathscr{A}_F$ are all the pairs (X, s), where $X \in \mathrm{Ob}(\mathscr{B})$ and $s \in F(X)$. A morphism $(X, s) \to (X', s')$ is a morphism $f : X \to X'$ in $\mathscr{B}$, such that $F(f)(s') = s$. The functor φ_F is defined by the rule : $\varphi_F(X, s) = X$ for every $(X, s) \in \mathrm{Ob}(\mathscr{A}_F)$. For any $X \in \mathrm{Ob}(\mathscr{B})$, the fibre $\varphi_F^{-1}(X)$ is (naturally isomorphic to) the discrete category $F(X)$ (see (1.1.1)). Notice also that every morphism in $\mathscr{A}_F$ is cartesian. This construction is a special case of (1.4.15).

(iv) Let F and G be presheaves on $\mathscr{B}$. With the notation of (iii), there is a natural bijection:

$$\mathrm{Hom}_{\mathscr{C}^\wedge}(F, G) \xrightarrow{\sim} \mathsf{Cart}_{\mathscr{B}}(\mathscr{A}_F, \mathscr{A}_G).$$

Namely, to a morphism $\psi : F \to G$ one assigns the functor $\mathscr{A}_\psi : \mathscr{A}_F \to \mathscr{A}_G$ such that $\mathscr{A}_\psi(X, s) = (X, \psi(X)(s))$ for every $(X, s) \in \mathrm{Ob}(\mathscr{A}_F)$.

(v) For instance, if $\mathscr{B}$ has small Hom-sets, and F is representable by an object X of $\mathscr{B}$, then one checks easily that $\mathscr{A}_F$ is naturally isomorphic to $\mathscr{C}/X$ and the fibration $\mathscr{A}_F$ is equivalent to that of example 1.4.2(i).

(vi) Furthermore, for $i = 1, 2$, let $G_i \to F$ be two morphisms of presheaves on $\mathscr{B}$; then we have a natural isomorphism of fibrations over $\mathscr{B}$:

$$\mathscr{A}_{G_1} \times_{\mathscr{A}_F} \mathscr{A}_{G_2} \stackrel{\sim}{\to} \mathscr{A}_{G_1 \times_F G_2}$$

(details left to the reader).

1.4.11. Let $\mathscr{A}'' \to \mathscr{B}$ be another small $\mathscr{B}$-category, and $F : \mathscr{A} \to \mathscr{A}'$ any $\mathscr{B}$-functor; we obtain a natural functor :

$$\mathsf{Cart}_{\mathscr{B}}(F, \mathscr{A}'') : \mathsf{Cart}_{\mathscr{B}}(\mathscr{A}', \mathscr{A}'') \to \mathsf{Cart}_{\mathscr{B}}(\mathscr{A}, \mathscr{A}'') \qquad G \mapsto G \circ F.$$

To any morphism $\alpha : G \Rightarrow G'$ in $\mathsf{Cart}_{\mathscr{B}}(\mathscr{A}', \mathscr{A}'')$, the functor $\mathsf{Cart}_{\mathscr{B}}(F, \mathscr{A}'')$ assigns the natural transformation $\alpha * F : G \circ F \Rightarrow G' \circ F$.

Likewise, any cartesian $\mathscr{B}$-functor $G : \mathscr{A}' \to \mathscr{A}''$ induces a functor :

$$\mathsf{Cart}_{\mathscr{B}}(\mathscr{A}, G) : \mathsf{Cart}_{\mathscr{B}}(\mathscr{A}, \mathscr{A}') \to \mathsf{Cart}_{\mathscr{B}}(\mathscr{A}, \mathscr{A}'') \qquad F \mapsto G \circ F$$

which assigns to any morphism $\beta : F \Rightarrow F'$ in $\mathsf{Cart}_{\mathscr{B}}(\mathscr{A}, \mathscr{A}')$, the natural transformation $G * \beta : G \circ F \Rightarrow G \circ F'$.

Furthermore, in case $\varphi' : \mathscr{A}' \to \mathscr{B}$ is a fibration, we have a natural equivalence of categories

$$\mathsf{Cart}_{\mathscr{A}}(\mathscr{A}, p') : \mathsf{Cart}_{\mathscr{A}}(\mathscr{A}, \mathscr{A} \times_{(\varphi, \varphi')} \mathscr{A}') \stackrel{\sim}{\to} \mathsf{Cart}_{\mathscr{B}}(\mathscr{A}, \mathscr{A}') \tag{1.4.12}$$

(where $p' : \mathscr{A} \times_{(\varphi, \varphi')} \mathscr{A}' \to \mathscr{A}'$ is the natural projection functor).

Proposition 1.4.13. *Let $\varphi : \mathscr{A} \to \mathscr{B}$ and $\varphi' : \mathscr{A}' \to \mathscr{B}$ be two fibrations, and $F : \mathscr{A} \to \mathscr{A}'$ a cartesian $\mathscr{B}$-functor. We have :*

(i) *The following conditions are equivalent :*
   (a) *F is a* fibrewise equivalence, i.e. *for any $B \in \mathrm{Ob}(\mathscr{B})$ the restriction $\varphi^{-1}B \to \varphi'^{-1}B$ of the functor F, is an equivalence.*
   (b) *F is a $\mathscr{B}$-equivalence,* i.e. *an equivalence in the 2-category U'-$\mathbf{Cat}/\mathscr{B}$ (in the sense of* (1.3.3)*) where U' is any universe relative to which the categories $\mathscr{A}$, $\mathscr{A}'$ and $\mathscr{B}$ are small.*

(ii) *If the equivalent conditions of* (i) *hold, the functor $\mathsf{Cart}_{\mathscr{B}}(\mathscr{C}, F)$ is an equivalence, for every $\mathscr{B}$-category $\mathscr{C} \to \mathscr{B}$.*

Proof. Condition (b) means that F admits a quasi-inverse G which is a $\mathscr{B}$-functor. The proof is similar to that of [11, Prop.8.4.2], and shall be left as an exercise for the reader. ☐

Definition 1.4.14. Let $\mathscr{B}$ be any small category.

(i) A *split fibration* over $\mathscr{B}$ is a pair (φ, c) consisting of a fibration $\varphi : \mathscr{A} \to \mathscr{B}$ and a cleavage c for φ, such that c is a functor.

(ii) For $i = 1, 2$, let $(\varphi_i : \mathscr{A}_i \to \mathscr{B}, \mathsf{c}_i)$ be two split fibrations, and $F : \mathscr{A}_1 \to \mathscr{A}_2$ a cartesian functor. For every $B \in \mathrm{Ob}(\mathscr{B})$, let $F_B : \varphi_1^{-1}B \to \varphi_2^{-1}B$ denote the restriction of F. We say that F is *a split cartesian functor* $(\varphi_1, \mathsf{c}_1) \to (\varphi_2, \mathsf{c}_2)$ if the rule : $B \mapsto F_B$ defines a natural transformation $\mathsf{c}_1 \Rightarrow \mathsf{c}_2$.

1.4.15. Let $\mathscr{B}$ be a small category. Clearly, the collection of all fibrations $\mathscr{A} \to \mathscr{B}$ (resp. of split fibrations $(\varphi : \mathscr{A} \to \mathscr{B}, \mathsf{c})$) with $\mathscr{A}$ small, forms a category $\mathsf{Fib}(\mathscr{B})$ (resp. $\mathsf{sFib}(\mathscr{B})$) whose morphisms are the cartesian functors (resp. the split cartesian functors). Furthermore, we have an obvious forgetful functor :

$$\mathsf{sFib}(\mathscr{B}) \to \mathsf{Fib}(\mathscr{B}) \qquad (\varphi, \mathsf{c}) \mapsto \varphi. \tag{1.4.16}$$

On the other hand, we have as well a natural functor

$$\mathsf{C} : \mathsf{Fib}(\mathscr{B}) \to \mathsf{sFib}(\mathscr{B})$$

defined as follows. For a fibration $\varphi : \mathscr{A} \to \mathscr{B}$, we let $\mathsf{C}(\mathscr{A}/\mathscr{B})$ be the category whose objects are all the pairs (B, G), where $B \in \mathrm{Ob}(\mathscr{B})$ and $G : \mathscr{B}/B \to \mathscr{A}$ is a cartesian functor (recall that $\mathscr{B}/B$ is fibred over $\mathscr{B}$, by example 1.4.2(i)). The morphisms $(B, G) \to (B', G')$ are the pairs (f, α), where $f : B \to B'$ is a morphism of $\mathscr{B}$, and $\alpha : G \Rightarrow G' \circ f_*$ is a morphism of $\mathsf{Cart}_{\mathscr{B}}(\mathscr{B}/B, \mathscr{A})$. The rule $(B, G) \mapsto B$ defines a functor $\mathsf{C}(\varphi) : \mathsf{C}(\mathscr{A}/\mathscr{B}) \to \mathscr{B}$, and it is easily seen that $\mathsf{C}(\varphi)$ is a fibration; indeed, for any morphism $f : B' \to B$ and any object (B, G) of $\mathsf{C}(\mathscr{A}/\mathscr{B})$, the pair $(B', G \circ f_*)$ is an inverse image of (B, G) over f (details left to the reader). Notice that, with this choice of inverse images, the resulting cleavage for $\mathsf{C}(\varphi)$ is actually a functor. Notice as well that the rule : $(\mathscr{A} \to \mathscr{B}) \mapsto \mathsf{C}(\mathscr{A}/\mathscr{B})$ is functorial in the fibration $\mathscr{A}$. Namely, every cartesian functor $F : \mathscr{A}_1 \to \mathscr{A}_2$ of fibred $\mathscr{B}$-categories induces a cartesian functor $\mathsf{C}(F/\mathscr{B}) : \mathsf{C}(\mathscr{A}_1/\mathscr{B}) \to \mathsf{C}(\mathscr{A}_2/\mathscr{B})$ via the rule : $(B, G) \mapsto (B, F \circ G)$ for every object (B, G) of $\mathsf{C}(\mathscr{A}_1/\mathscr{B})$, and if $F' : \mathscr{A}_2 \to \mathscr{A}_3$ is another cartesian functor of fibered $\mathscr{B}$-categories, we have the identity

$$\mathsf{C}(F' \circ F/\mathscr{B}) = \mathsf{C}(F'/\mathscr{B}) \circ \mathsf{C}(F/\mathscr{B}).$$

Let us spell out the cleavage of $\mathsf{C}(\varphi)$: this is the functor

$$\mathsf{Cart}_{\mathscr{B}}(\mathscr{B}/-, \mathscr{A}) : \mathscr{B}^o \to \mathsf{U}'\text{-}\mathbf{Cat}$$

(for a suitable universe U') that assigns to any $B \in \mathrm{Ob}(\mathscr{B}^o)$ the category $\mathsf{Cart}_{\mathscr{B}}(\mathscr{B}/B, \mathscr{A})$, and to any morphism $B' \xrightarrow{g} B$ in $\mathscr{B}$ the functor :

$$\mathsf{Cart}_{\mathscr{B}}(g_*, \mathscr{A}) : \mathsf{Cart}_{\mathscr{B}}(\mathscr{B}/B, \mathscr{A}) \to \mathsf{Cart}_{\mathscr{B}}(\mathscr{B}/B', \mathscr{A}).$$

Moreover, let $F : \mathscr{A}_1 \to \mathscr{A}_2$ be a cartesian functor of fibred $\mathscr{B}$-categories. Then we obtain a natural transformation of cleavages :

$$\mathsf{Cart}_{\mathscr{B}}(\mathscr{B}/-, \mathscr{A}_1) \Rightarrow \mathsf{Cart}_{\mathscr{B}}(\mathscr{B}/-, \mathscr{A}_2) \qquad B \mapsto \mathsf{Cart}_{\mathscr{B}}(\mathscr{B}/B, F).$$

Summing up, this shows that the rule $(\varphi : \mathscr{A} \to \mathscr{B}) \mapsto (\mathsf{C}(\varphi), \mathsf{Cart}_{\mathscr{B}}(\mathscr{B}/-, \mathscr{A}))$ defines the sought functor C.

Theorem 1.4.17. *With the notation of* (1.4.15)*, we have :*

(i) *The functor* C *is right* 2*-adjoint to* (1.4.16) *(see definition* 1.3.10(ii)*).*
(ii) *The counit of this pair of* 2*-adjoint functors is a pseudo-natural equivalence.*

Proof. According to remark 1.3.11(iii), it suffices to exhibit a unit and a counit fulfilling triangular identities as in (1.1.8). To this aim, let $\varphi : \mathscr{A} \to \mathscr{B}$ be a fibration, and consider the natural cartesian functor of $\mathscr{B}$-categories

$$\mathsf{ev}_\bullet : \mathsf{C}(\mathscr{A}/\mathscr{B}) \to \mathscr{A}$$

that assigns to every object (B, G) of $\mathsf{C}(\mathscr{A}/\mathscr{B})$ its evaluation $G\mathbf{1}_B \in \mathrm{Ob}(\varphi^{-1}B)$, and to any morphism $(f, \alpha) : (B, G) \to (B', G')$, the morphism $\alpha_B : G\mathbf{1}_B \to G'f$.

Claim 1.4.18. The functor $\mathsf{ev}_\bullet$ is a $\mathscr{B}$-equivalence.

Proof of the claim. It suffices to show that $\mathsf{ev}_\bullet$ is a fibrewise equivalence (proposition 1.4.13(i)). However, for every $B \in \mathrm{Ob}(\mathscr{B})$, an inverse equivalence $\beta_B : \varphi^{-1}B \to \mathsf{Cart}_{\mathscr{B}}(\mathscr{B}/B, \mathscr{A})$ can be constructed explicitly by choosing, for a fixed $A \in \mathrm{Ob}(\varphi^{-1}B)$, and every object $g : B' \to B$ of $\mathscr{B}/B$, an inverse image $g_A : g^*A \to A$ of A over g. These choices determine β_B on objects, via the rule :

$$\beta_B(A) : \mathrm{Ob}(\mathscr{B}/B) \to \mathrm{Ob}(\mathscr{A}) \quad : \quad g \mapsto g^*A.$$

Then the image of $\beta_B(A)$ is uniquely determined on every morphism of $\mathscr{B}/B$, by the universal property of g_A (cp. (1.4.6)). By the same token, one sees that $\beta_B(A)(f)$ is $\mathscr{B}$-cartesian, for every morphism f of $\mathscr{B}/B$, *i.e.* $\beta_B(A)$ is a cartesian functor. Finally, the rule $A \mapsto \beta_B(A)$ is functorial on $\varphi^{-1}B$: for the details, see [11, Prop.8.2.7]. $\diamondsuit$

Clearly every cartesian functor $F : \mathscr{A}_1 \to \mathscr{A}_2$ of fibred $\mathscr{B}$-categories yields a commutative diagram of $\mathscr{B}$-categories :

$$\begin{tikzcd} \mathsf{C}(\mathscr{A}_1/\mathscr{B}) \arrow[r, "\mathsf{C}(F/\mathscr{B})"] \arrow[d, "\mathsf{ev}_\bullet"'] & \mathsf{C}(\mathscr{A}_2/\mathscr{B}) \arrow[d, "\mathsf{ev}_\bullet"] \\ \mathscr{A}_1 \arrow[r, "F"] & \mathscr{A}_2 \end{tikzcd}$$

hence $\mathsf{ev}_\bullet$ is our candidate pseudo-natural counit, and claim 1.4.18 implies that (ii) holds for this choice of counit. Next, let $(\varphi : \mathscr{A} \to \mathscr{B}, \mathsf{c})$ be any split fibration. To define a unit, we need to exhibit a natural split cartesian functor :

$$(\varphi, \mathsf{c}) \to (\mathsf{C}(\mathscr{A}/\mathscr{B}), \mathsf{Cart}_{\mathscr{B}}(\mathscr{B}/-, \mathscr{A})). \tag{1.4.19}$$

This is obtained as follows. Let A be any object of $\mathscr{A}$, and set $B := \varphi(A)$; to A we assign the unique cartesian functor F_A determined on objects by the rule : $F_A(f) := f^*A$, for every $f \in \mathrm{Ob}(\mathscr{B}/B)$; of course, f^* denotes the pull-back functor provided by the split cleavage c. It is easily seen that the rule $A \mapsto F_A$ extends to a unique split cartesian functor (1.4.19). Lastly, we entrust to the reader the verification that the unit and counit thus defined do fulfill the triangular identities. □

Remark 1.4.20. Let $\varphi : \mathscr{A} \to \mathscr{B}$ be a fibration. Since the equivalence $\mathsf{ev}_\bullet$ of claim 1.4.18 is independent of choices, one might hope that the rule : $B \mapsto (\mathsf{ev}_\bullet : \mathsf{C}(\varphi)^{-1}B \to \varphi^{-1}B)$ extends to a natural isomorphism between functors $\mathscr{B}^o \to \mathsf{U}'\text{-}\mathbf{Cat}$. However, since (1.4.9) is only a pseudo-functor, the best one can achieve is a pseudo-natural equivalence :

$$\mathsf{ev} : \mathsf{Cart}_{\mathscr{B}}(\mathscr{B}/-, \mathscr{A}) \Rightarrow \mathsf{c}$$

which will be uniquely determined, for every given choice of a cleavage c for the fibration φ.

Proposition 1.4.21. *Let $F : \mathscr{A} \to \mathscr{B}$ be a functor, $\mathscr{C} \to \mathscr{B}$ a fibration, and suppose that, for every $B \in \mathrm{Ob}(\mathscr{B})$, the induced functor $\mathsf{Cart}_{\mathscr{B}}(F/B, \mathscr{C})$ is an equivalence (notation of* (1.1.16)*). Then the functor $\mathsf{Cart}_{\mathscr{B}}(F, \mathscr{C})$ is an equivalence.*

Proof. Let $\mathscr{D}$ denote the category whose objects are all the pairs (B, G), where $B \in \mathrm{Ob}(\mathscr{B})$ and $G : F\mathscr{A}/B \to \mathscr{C}$ is a strictly cartesian functor. The morphisms in $\mathscr{D}$ are defined as for the category $\mathsf{C}(\mathscr{C})$ introduced in the proof of theorem 1.4.17. It is easily seen that the rule : $(B, G) \mapsto G$ determines a fibration $\mathscr{D} \to \mathscr{B}$. Moreover, the natural transformation

$$\mathsf{Cart}_{\mathscr{B}}(F/-, \mathscr{C}) : \mathsf{Cart}_{\mathscr{B}}(\mathscr{B}/-, \mathscr{C}) \Rightarrow \mathsf{Cart}_{\mathscr{B}}(F\mathscr{A}/-, \mathscr{C})$$

induces a cartesian functor of $\mathscr{B}$-categories $G : \mathsf{C}(\mathscr{C}) \to \mathscr{D}$, whence a commutative diagram :

$$\begin{tikzcd} \mathsf{Cart}_{\mathscr{B}}(\mathscr{B}, \mathscr{C}) \arrow[r, "\mathsf{Cart}_{\mathscr{B}}(F,\mathscr{C})"] \arrow[d, "\alpha"'] & \mathsf{Cart}_{\mathscr{B}}(\mathscr{A}, \mathscr{C}) \arrow[d, "\beta"] \\ \mathsf{Cart}_{\mathscr{B}}(\mathscr{B}, \mathsf{C}(\mathscr{C})) \arrow[r, "\mathsf{Cart}_{\mathscr{B}}(\mathscr{B},G)"] & \mathsf{Cart}_{\mathscr{B}}(\mathscr{B}, \mathscr{D}). \end{tikzcd} \tag{1.4.22}$$

Namely, α assigns to any strongly cartesian $\mathscr{B}$-functor $F : \mathscr{B} \to \mathscr{C}$ the strongly cartesian functor $\alpha(F) : \mathscr{B} \to \mathsf{C}(\mathscr{C})$, given by the rule :

$$B \mapsto (F \circ \mathsf{s}_B : \mathscr{B}/B \to \mathscr{C}) \qquad \text{for every } B \in \mathrm{Ob}(\mathscr{B})$$

where $\mathsf{s}_B : \mathscr{B}/B \to \mathscr{B}$ is the functor (1.1.13). Likewise, β sends a strongly cartesian functor $G : \mathscr{A} \to \mathscr{C}$ to the strongly cartesian functor $\beta(G) : \mathscr{B} \to \mathscr{D}$ given by the rule :

$$B \mapsto (F \circ \iota_B : F\mathscr{A}/B \to \mathscr{C}) \qquad \text{for every } B \in \mathrm{Ob}(\mathscr{B})$$

where $\iota_B : F\mathscr{A}/B \to \mathscr{A}$ is defined as in (1.1.16).

Our assumption then means that G is a fibrewise equivalence, in which case, proposition 1.4.13(ii) ensures that the bottom arrow of (1.4.22) is an equivalence. We leave to the reader the verification that both vertical arrows are equivalences as well, after which the assertion follows. $\square$

1.5. **Sieves and descent theory.** This section develops the basics of descent theory, in the general framework of fibred categories. For later use, it is convenient to introduce the notion of n-*faithful functor*, for all integers $n \leq 2$. Namely : if $n < 0$, every functor is n-faithful; a functor $F : \mathscr{A} \to \mathscr{B}$ (between any two categories $\mathscr{A}$ and $\mathscr{B}$) is 0-faithful, if it is faithful (definition 1.1.4); F is 1-faithful, if it is fully faithful; finally, we say that F is 2-faithful, if it is an equivalence.

Definition 1.5.1. Let $\mathscr{B}$ and $\mathscr{C}$ be two categories, $F : \mathscr{B} \to \mathscr{C}$ a functor.

(i) A *sieve* of $\mathscr{C}$ is a full subcategory $\mathscr{S}$ of $\mathscr{C}$ such that the following holds. If $A \in \mathrm{Ob}(\mathscr{S})$, and $B \to A$ is any morphism in $\mathscr{C}$, then $B \in \mathrm{Ob}(\mathscr{S})$.

(ii) If $S \subset \mathrm{Ob}(\mathscr{C})$ is any subset, there is a smallest sieve $\mathscr{S}_S$ of $\mathscr{C}$ such that $S \subset \mathrm{Ob}(\mathscr{S}_S)$; we call $\mathscr{S}_S$ the *sieve generated by* S.

(iii) If $\mathscr{S}$ is a sieve of $\mathscr{C}$, the *inverse image of* $\mathscr{S}$ *under* F is the full subcategory $F^{-1}\mathscr{S}$ of $\mathscr{B}$ with $\mathrm{Ob}(F^{-1}\mathscr{S}) = \{B \in \mathrm{Ob}(\mathscr{B}) \mid FB \in \mathrm{Ob}(\mathscr{S})\}$ (notice that $F^{-1}\mathscr{S}$ is a sieve).

(iv) If $f : X \to Y$ is any morphism in $\mathscr{C}$, and $\mathscr{S}$ is any sieve of $\mathscr{C}/Y$, we shall write $\mathscr{S} \times_Y f$ for the inverse image of $\mathscr{S}$ under the functor f_* (notation of (1.1.14)).

Example 1.5.2. For instance, suppose that $\mathscr{S}$ is the sieve of $\mathscr{C}/Y$ generated by a family :

$$\{Y_i \to Y \mid i \in I\} \subset \mathrm{Ob}(\mathscr{C}/Y).$$

If the fibre product $X_i := X \times_Y Y_i$ is representable in $\mathscr{C}$ for every $i \in I$, then $\mathscr{S} \times_Y f$ is the sieve generated by the family of induced projections $\{X_i \to X \mid i \in I\} \subset \mathrm{Ob}(\mathscr{C}/X)$.

1.5.3. Let $\mathscr{C}$ be a category with small Hom-sets, X an object of $\mathscr{C}$, and $\mathscr{S}$ a sieve of the category $\mathscr{C}/X$; we define the presheaf $h_{\mathscr{S}}$ on $\mathscr{C}$ by ruling that

$$h_{\mathscr{S}}(Y) := \{f \in \mathrm{Hom}_{\mathscr{C}}(Y, X) \mid (Y, f) \in \mathrm{Ob}(\mathscr{S})\} \qquad \text{for every } Y \in \mathrm{Ob}(\mathscr{C}).$$

For a given morphism $f : Y' \to Y$ in $\mathscr{C}$, the map $h_{\mathscr{S}}(f)$ is just the restriction of $\mathrm{Hom}_{\mathscr{C}}(f, X)$. Hence $h_{\mathscr{S}}$ is a subobject of h_X (notation of (1.1.19)), and indeed the rule $\mathscr{S} \mapsto h_{\mathscr{S}}$ sets up a natural bijection between the subobjects of h_X in $\mathscr{C}^\wedge$ and the sieves of $\mathscr{C}/X$. The inverse mapping sends a subobject F of h_X to the full subcategory $\mathscr{S}_F$ of $\mathscr{C}/X$ such that :

$$\mathrm{Ob}(\mathscr{S}_F) = \bigcup_{Y \in \mathrm{Ob}(\mathscr{C})} \{(Y, f) \mid f \in F(Y)\}.$$

In the notation of (1.1.16), this is naturally isomorphic to the category $h\mathscr{C}/F$ and it is easy to check that it is indeed a sieve of $\mathscr{C}/X$.

Example 1.5.4. In the situation of (1.5.3), let $S := \{X_i \to X \mid i \in I\}$ be any family of morphisms in $\mathscr{C}$. Then S generates the sieve $\mathscr{S}$ if and only if :

$$h_{\mathscr{S}} = \bigcup_{i \in I} \mathrm{Im}(h_{X_i} \to h_X).$$

(Notice that the above union is well defined even in case I is not small.)

In the same way, one sees that the sieves of $\mathscr{C}$ are in natural bijection with the subobjects of the final object $1_{\mathscr{C}}$ of $\mathscr{C}^\wedge$. Moreover, it is easy to check that, for every sieve $\mathscr{S}$ of $\mathscr{C}/X$, and every morphism $f : Y \to X$ in $\mathscr{C}$, the above correspondence induces a natural identification of subobjects of h_Y :

$$h_{\mathscr{S} \times_X f} = h_{\mathscr{S}} \times_{h_X} h_Y.$$

Since the Yoneda imbedding is fully faithful, we shall often abuse notation, to identify an object X of $\mathscr{C}$ with the corresponding representable presheaf h_X; with this notation, we may write $h_{\mathscr{S}\times_X f} = h_{\mathscr{S}} \times_X Y$.

1.5.5. Let $\mathscr{C}$ be a small category, and $\mathscr{S}$ the sieve of $\mathscr{C}$ generated by a subset $S \subset \mathrm{Ob}(\mathscr{C})$. Say that $S = \{S_i \mid i \in I\}$ for a set I of the universe U; for every $i \in I$, there is a faithful imbedding $\varepsilon_i : \mathscr{C}/S_i \to \mathscr{S}$, and for every pair $(i, j) \in I \times I$, we define

$$\mathscr{C}/S_{ij} := \mathscr{C}/S_i \times_{(\varepsilon_i,\varepsilon_j)} \mathscr{C}/S_j$$

(notation of (1.1.27)). Hence, the objects of $\mathscr{C}/S_{ij}$ are all the triples (X, g_i, g_j), where $X \in \mathrm{Ob}(\mathscr{C})$ and $g_l \in \mathrm{Hom}_{\mathscr{C}}(X, S_l)$ for $l = i, j$. The natural projections :

$$\pi^1_{ij*} : \mathscr{C}/S_{ij} \to \mathscr{C}/S_i \qquad \pi^0_{ij*} : \mathscr{C}/S_{ij} \to \mathscr{C}/S_j$$

are faithful imbeddings. We deduce a natural diagram of categories :

$$\coprod_{(i,j)\in I\times I} \mathscr{C}/S_{ij} \overset{\partial_0}{\underset{\partial_1}{\rightrightarrows}} \coprod_{i\in I} \mathscr{C}/S_i \xrightarrow{\varepsilon} \mathscr{S}$$

where :

$$\partial_0 := \coprod_{(i,j)\in I\times I} \pi^0_{ij*} \qquad \partial_1 := \coprod_{(i,j)\in I\times I} \pi^1_{ij*} \qquad \varepsilon := \coprod_{i\in I} \varepsilon_i.$$

Remark 1.5.6. Notice that, under the current assumptions, the product $S_{ij} := S_i \times S_j$ is not necessarily representable in $\mathscr{C}$. In case it is, we may consider another category, also denoted $\mathscr{C}/S_{ij}$, namely the category of S_{ij}-objects of $\mathscr{C}$ (as in (1.1.12)). The latter is naturally isomorphic to the category with the same name introduced in (1.5.5). Moreover, under this natural isomorphism, the projections π^0_{ij*} and π^1_{ij*} are identified with the functors induced by the natural morphisms $\pi^0_{ij} : S_{ij} \to S_j$ and respectively $\pi^1_{ij} : S_{ij} \to S_i$. Hence, in this case, the notation of (1.5.5) is compatible with (1.1.14).

Lemma 1.5.7. *With the notation of* (1.5.5)*, the functor ε induces an isomorphism between $\mathscr{S}$ and the coequalizer (in the category* $\mathbf{Cat}$*) of the pair of functors* (∂_0, ∂_1).

Proof. Let $\mathscr{A}$ be any other object of $\mathbf{Cat}$, and $F : \coprod_{i\in I} \mathscr{C}/S_i \to \mathscr{A}$ a functor such that $F \circ \partial_0 = F \circ \partial_1$. We have to show that F factors uniquely through ε. To this aim, we construct explicitly a functor $G : \mathscr{S} \to \mathscr{A}$ such that $G \circ \varepsilon = F$. First of all, by the universal property of the coproduct, F is the same as a family of functors $(F_i : \mathscr{C}/S_i \to \mathscr{A} \mid i \in I)$, and the assumption on F amounts to the system of identities :

$$F_i \circ \pi^1_{ij*} = F_j \circ \pi^0_{ij*} \qquad \text{for every } i, j \in I. \tag{1.5.8}$$

Hence, let $X \in \mathrm{Ob}(\mathscr{S})$; by assumption there exists $i \in I$ and a morphism $f : X \to S_i$ in $\mathscr{C}$, so we may set $GX := F_i f$. In case $g : X \to S_j$ is another morphism in $\mathscr{C}$, we deduce an object $h := (X, f, g) \in \mathrm{Ob}(\mathscr{C}/S_{ij})$, so $f = \pi^1_{ij*}h$ and $g = \pi^0_{ij*}h$; then (1.5.8) shows that $F_i f = F_j g$, *i.e.* GX is well-defined.

Next, let $\varphi : X \to Y$ be any morphism in $\mathscr{S}$; choose $i \in I$ and a morphism $f_Y : Y \to S_i$, and set $f_X := f_Y \circ \varphi$. We let $G\varphi := F_i(\varphi : f_X \to f_Y)$. Arguing as in the foregoing, one verifies easily that $G\varphi$ is independent of all the choices, and then it follows easily that $G(\psi \circ \varphi) = G\psi \circ G\varphi$ for every other morphism $\psi : Y \to Z$ in $\mathscr{S}$. It is also clear that $G\mathbf{1}_X = \mathbf{1}_{GX}$, whence the contention. ☐

Example 1.5.9. Let $\mathscr{B}$ be any category and X any object of $\mathscr{B}$.

(i) Let $\varphi : \mathscr{A} \to \mathscr{B}$ be a fibration, and $\mathscr{S}$ any sieve of $\mathscr{A}$. By restriction, φ induces a functor $\varphi_{|\mathscr{S}} : \mathscr{S} \to \mathscr{B}$, and it is straightforward to see that $\varphi_{|\mathscr{S}}$ is again a fibration.

(ii) In the situation of example 1.4.10(iii), suppose that $F = h_{\mathscr{S}}$ for some sieve $\mathscr{S}$ of $\mathscr{B}/X$ (notation of (1.5.3)). Then $\mathscr{A}_F$ is naturally isomorphic to $\mathscr{S}$, and φ_F is naturally identified to the restriction $\mathscr{S} \to \mathscr{B}$ of the functor (1.1.13).

1.5.10. In the situation of (1.5.5), suppose that the set of generators S is the whole of $\mathrm{Ob}(\mathscr{S})$; in this case, the augmentation ε can also be used to produce the following 2-categorical presentation of $\mathscr{S}$. Consider the functor

$$G_{\mathscr{S}} : \mathscr{S} \to \mathbf{Cat}/\mathscr{C} \quad : \quad Y \mapsto \mathscr{C}/Y \qquad (Z \xrightarrow{f} Y) \mapsto (\mathscr{C}/Z \xrightarrow{f_*} \mathscr{C}/Y).$$

We have a natural cocone

$$\widehat{\varepsilon} : G_{\mathscr{S}} \Rightarrow \mathsf{F}_{\mathscr{S}}$$

where $\mathsf{F}_{\mathscr{S}}$ is the constant functor $\mathscr{S} \to \mathbf{Cat}/\mathscr{C}$ with value $\mathscr{S}$, and $\widehat{\varepsilon}_X : \mathscr{C}/X \to \mathscr{S}$ is the faithful imbedding as in (1.5.5), for every $X \in \mathrm{Ob}(\mathscr{S})$. We may then state :

Lemma 1.5.11. *The functor $\widehat{\varepsilon}$ induces an equivalence of categories :*

$$2\text{-}\operatorname*{colim}_{\mathscr{S}} G_{\mathscr{S}} \xrightarrow{\sim} \mathscr{S}.$$

Proof. For a given small $\mathscr{C}$-category $\mathscr{A}$, a pseudo-cocone $\varphi_\bullet : G_{\mathscr{S}} \Rightarrow \mathsf{F}_{\mathscr{A}}$ is the datum of a system of functors

$$\varphi_X : \mathscr{C}/X \to \mathscr{A} \qquad \text{for every } X \in \mathrm{Ob}(\mathscr{S})$$

and natural isomorphisms

$$\tau_f : \varphi_Z \Rightarrow \varphi_Y \circ f_* \qquad \text{for every } (f : Z \to Y) \in \mathrm{Morph}(\mathscr{S})$$

such that

$$\tau_{g\circ f} = (\tau_g * f_*) \circ \tau_f \qquad \text{for every composition } Z \xrightarrow{f} Y \xrightarrow{g} X \text{ in } \mathscr{S}. \tag{1.5.9}$$

To such $\varphi_\bullet$ we associate a functor $\varphi^\dagger : \mathscr{S} \to \mathscr{A}$, by the rule :

$$\varphi^\dagger(X) := \varphi_X(\mathbf{1}_X) \qquad \text{for every } X \in \mathrm{Ob}(\mathscr{S}).$$

Let $g : Y \to X$ be any morphism in $\mathscr{S}$, and denote by $(g_{/X} : g \to \mathbf{1}_X) \in \mathrm{Hom}_{\mathscr{C}/X}(g, \mathbf{1}_X)$ the element corresponding to g; we have a morphism $\tau_g(\mathbf{1}_Y) : \varphi_Y(\mathbf{1}_Y) \to \varphi_X(g_*\mathbf{1}_Y) = \varphi_X(g)$, and we set

$$\varphi^\dagger(g) := \varphi_X(g_{/X}) \circ \tau_g(\mathbf{1}_Y) : \varphi^\dagger(Y) \to \varphi^\dagger(X).$$

Let us check that $\varphi^\dagger$ is indeed a functor on $\mathscr{S}$: for any two morphisms $f : Z \to Y$ and $g : Y \to X$ we may compute

$$\begin{aligned}
\varphi^\dagger(g) \circ \varphi^\dagger(f) &= \varphi_X(g_{/X}) \circ \tau_g(\mathbf{1}_Y) \circ \varphi_Y(f_{/Y}) \circ \tau_f(\mathbf{1}_Z) \\
&= \varphi_X(g_{/X}) \circ \varphi_X(f_{/X}) \circ \tau_g(f) \circ \tau_f(\mathbf{1}_Z) \\
&= \varphi_X(g \circ f_{/X}) \circ \tau_{g\circ f}(\mathbf{1}_Z) \\
&= \varphi^\dagger(g \circ f)
\end{aligned}$$

where the second identity follows from the naturality of τ_g, and the third follows from (1.5.9). It is easily seen that the rule : $\varphi_\bullet \mapsto \varphi^\dagger$ defines a functor

$$\mathsf{PsNat}(G_{\mathscr{S}}, \mathsf{F}_{\mathscr{A}}) \to \mathbf{Fun}(\mathscr{S}, \mathscr{A})$$

such that $(\mathsf{F}_\psi \circ \widehat{\varepsilon})^\dagger = \psi$ for every functor $\psi : \mathscr{S} \to \mathscr{A}$. Lastly, for every $\varphi_\bullet$ as above, the pseudo-cocone $\mathsf{F}_{\varphi^\dagger} \circ \widehat{\varepsilon}$ is naturally isomorphic to $\varphi_\bullet$, so the claim follows (details left to the reader). $\square$

Definition 1.5.12. Let $\varphi : \mathscr{A} \to \mathscr{B}$ be a fibration between small categories, B an object of $\mathscr{B}$, $\mathscr{S}$ a sieve of $\mathscr{B}/B$, and denote by $\iota_{\mathscr{S}} : \mathscr{S} \to \mathscr{B}/B$ the fully faithful imbedding.

(i) For $i \in \{0,1,2\}$, we say that $\mathscr{S}$ is a *sieve of φ-i-descent*, if the restriction functor :

$$\mathsf{Cart}_{\mathscr{B}}(\iota_{\mathscr{S}}, \mathscr{A}) : \mathsf{Cart}_{\mathscr{B}}(\mathscr{B}/B, \mathscr{A}) \to \mathsf{Cart}_{\mathscr{B}}(\mathscr{S}, \mathscr{A})$$

is i-faithful. (Here $\mathscr{B}/B$ is fibred over $\mathscr{B}$ as in example 1.4.2(i).)

(ii) For $i \in \{0,1,2\}$, we say that $\mathscr{S}$ is a *sieve of universal φ-i-descent* if, for every morphism $f : B' \to B$ of $\mathscr{B}$, the sieve $\mathscr{S} \times_B f$ is of φ-i-descent (notation of definition 1.5.1(iv)).

(iii) Let $f : B' \to B$ be a morphism in $\mathscr{B}$. We say that f is a *morphism of φ-i-descent* (resp. a *morphism of universal φ-i-descent*), if the sieve generated by $\{f\}$ is of φ-i-descent (resp. of universal φ-i-descent).

Example 1.5.13. (i) Consider the fibration $\mathrm{s} : \mathsf{Morph}(\mathscr{C}) \to \mathscr{C}$ of example 1.4.2(ii). A sieve of s-1-descent (resp. of universal s-1-descent) is also called an *epimorphic sieve* (resp. a *universal epimorphic sieve*), and a sieve of s-2-descent (resp. of universal s-2-descent) is also called a *strict epimorphic sieve* (resp. a *universal strict epimorphic sieve*).

(ii) Notice that, for every $X \in \mathrm{Ob}(\mathscr{C})$ there is a natural equivalence of categories :

$$X/\mathscr{C} \xrightarrow{\sim} \mathsf{Cart}_{\mathscr{C}}(\mathscr{C}/X, \mathsf{Morph}(\mathscr{C})).$$

Namely, to a morphism $f : X \to Y$ in $\mathscr{C}$ one assigns the cartesian functor $f_* : \mathscr{C}/X \to \mathscr{C}/Y \subset \mathsf{Morph}(\mathscr{C})$ (notation of (1.1.14)). An essential inverse for this equivalence is given by the rule : $F \mapsto F(\mathbf{1}_X)$ for every cartesian functor $F : \mathscr{C}/X \to \mathsf{Morph}(\mathscr{C})$.

(iii) More generally, for every sieve $\mathscr{S} \subset \mathscr{C}/X$ there is a natural faithful functor :

$$h_{\mathscr{S}}/h\mathscr{C} \to \mathsf{Cart}_{\mathscr{C}}(\mathscr{S}, \mathsf{Morph}(\mathscr{C})). \tag{1.5.14}$$

Indeed, denote by $h_{|\mathscr{S}} : \mathscr{S} \xrightarrow{\sim} h\mathscr{C}/h_{\mathscr{S}}$ the isomorphism of categories provided by (1.5.3). Then (1.5.14) assigns to any object $\varphi : h_{\mathscr{S}} \to h_Y$ of $h_{\mathscr{S}}/h\mathscr{C}$ the functor

$$(h\mathscr{C}/\varphi) \circ h_{|\mathscr{S}} : \mathscr{S} \to h\mathscr{C}/h_Y \qquad (g : Z \to X) \mapsto \varphi \circ h_{|\mathscr{S}}(g)$$

(notation of (1.1.16), and the Yoneda imbedding identifies the category $h\mathscr{C}/h_Y$ with $\mathscr{C}/Y \subset \mathsf{Morph}(\mathscr{C})$). Under the faithful imbedding (1.5.14) and the equivalence of (ii), the restriction functor $\mathsf{Cart}_{\mathscr{C}}(\iota_{\mathscr{S}}, \mathsf{Morph}(\mathscr{C}))$ corresponds to the composition :

$$X/\mathscr{C} \xrightarrow{h} h_X/h\mathscr{C} \xrightarrow{i^*} h_{\mathscr{S}}/h\mathscr{C}.$$

where $i : h_{\mathscr{S}} \to h_X$ is the natural inclusion of presheaves.

1.5.15. In the situation of definition 1.5.12, suppose that $\mathscr{S}$ is the sieve generated by a set of objects $\{S_i \to B \mid i \in I\} \subset \mathrm{Ob}(\mathscr{B}/B)$. There follows a natural diagram of categories (notation of (1.5.5)) :

$$\mathsf{Cart}_{\mathscr{B}}(\mathscr{S}, \mathscr{A}) \xrightarrow{\varepsilon^*} \prod_{i\in I} \mathsf{Cart}_{\mathscr{B}}(\mathscr{B}/S_i, \mathscr{A}) \underset{\partial_1^*}{\overset{\partial_0^*}{\rightrightarrows}} \prod_{(i,j)\in I\times I} \mathsf{Cart}_{\mathscr{B}}(\mathscr{B}/S_{ij}, \mathscr{A})$$

where $\varepsilon^* := \mathsf{Cart}_{\mathscr{B}}(\varepsilon, \mathscr{A})$ and $\partial_i^* := \mathsf{Cart}_{\mathscr{B}}(\partial_i, \mathscr{A})$, for $i = 0, 1$. With this notation, lemma 1.5.7 easily implies that ε^* induces an isomorphism between $\mathsf{Cart}_{\mathscr{B}}(\mathscr{S}, \mathscr{A})$ and the equalizer (in the category $\mathbf{Cat}$) of the pair of functors $(\partial_0^*, \partial_1^*)$.

Example 1.5.16. A family of morphisms $(f_i : X_i \to X \mid i \in I)$ in a category $\mathscr{C}$ is called an *epimorphic* (resp. *strict epimorphic*, resp. *universal strict epimorphic*) family if it generates a sieve of $\mathscr{C}/X$ with the same property. In view of (1.5.15) and example 1.5.13(ii), we see that such a family is epimorphic (resp. strict epimorphic) if and only if the following holds. For every $Y \in \mathrm{Ob}(\mathscr{C})$, the natural map

$$\prod_{i\in I} f_{i*} : \mathrm{Hom}_{\mathscr{C}}(X, Y) \to \prod_{i\in I} \mathrm{Hom}_{\mathscr{C}}(X_i, Y)$$

is injective (resp. and its image consists of all the systems of morphisms $(g_i : X_i \to Y \mid i \in I)$ such that, for every $Z \in \mathrm{Ob}(\mathscr{C})$, every $i, j \in I$, and every pair of morphisms $h_i : Z \to X_i$, $h_j : Z \to X_j$ with $f_i \circ h_i = f_j \circ h_j$, we have $g_i \circ h_i = g_j \circ h_j$).

We say that the family $(f_i \mid i \in I)$ is *effective epimorphic*, if it is strict epimorphic, and moreover the fibre products $X_i \times_X X_j$ are representable in $\mathscr{C}$ for every $i, j \in I$. This is the same as saying that the map $\prod_{i\in I} f_{i*}$ identifies $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ with the equalizer of the two natural maps

$$\prod_{i\in I} \mathrm{Hom}_{\mathscr{C}}(X_i, Y) \rightrightarrows \prod_{(i,j)\in I\times I} \mathrm{Hom}_{\mathscr{C}}(X_i \times_X X_j, Y).$$

A family $(f_i \mid i \in I)$ as above is called *universal epimorphic* (resp. *universal effective epimorphic*) if (a) the fibre products $Y_i := X_i \times_X Y$ are representable in $\mathscr{C}$, for every $i \in I$ and every morphism $Y \to X$ in $\mathscr{C}$, and (b) all the resulting families $(Y_i \to Y \mid i \in I)$ are still epimorphic (resp. effective epimorphic).

1.5.17. We would like to exploit the presentation (1.5.15) of $\mathsf{Cart}_{\mathscr{B}}(\mathscr{S}, \mathscr{A})$, in order to translate definition 1.5.12 in terms of the fibre categories $\varphi^{-1}S_i$ and $\varphi^{-1}S_{ij}$. The problem is that such a translation must be carried out via a pseudo-natural equivalence (namely ev), and such equivalences do not respect a presentation as above in terms of equalizers in the category $\mathbf{Cat}$. What we need is to upgrade our presentation of $\mathscr{S}$ to a new one, which is preserved by pseudo-natural transformations. This is achieved as follows. Resume the general situation of (1.5.5). For every $i, j, k \in I$, set $\mathscr{C}/S_{ijk} := \mathscr{C}/S_{ij} \times_{\mathscr{C}} \mathscr{C}/S_k$. We have a natural diagram of categories :

$$\coprod_{(i,j,k)\in I^3} \mathscr{C}/S_{ijk} \overset{\partial_0}{\underset{\partial_2}{\overset{-\partial_1\to}{\rightrightarrows}}} \coprod_{(i,j)\in I^2} \mathscr{C}/S_{ij} \overset{\partial_0}{\underset{\partial_1}{\rightrightarrows}} \coprod_{i\in I} \mathscr{C}/S_i \xrightarrow{\varepsilon} \mathscr{S} \tag{1.5.18}$$

where ∂_0 is the coproduct of the natural projections $\pi^0_{ijk*} : \mathscr{C}/S_{ijk} \to \mathscr{C}/S_{jk}$ for every $i, j, k \in I$, and likewise ∂_1 (resp. ∂_2) is the coproduct of the projections $\pi^1_{ijk*} : \mathscr{C}/S_{ijk} \to \mathscr{C}/S_{ik}$ (resp. $\pi^2_{ijk*} : \mathscr{C}/S_{ijk} \to \mathscr{C}/S_{ij}$). We can view (1.5.18) as an *augmented* 2*-truncated semi-simplicial object* in $\mathbf{Cat}/\mathscr{C}$, *i.e.* a functor :

$$F_{\mathscr{S}} : (\Sigma_2^\wedge)^o \to \mathbf{Cat}/\mathscr{C}.$$

from the opposite of the category $\Sigma_2^\wedge$ whose objects are the ordered sets $\varnothing$, $\{0\}$, $\{0,1\}$ and $\{0,1,2\}$, and whose morphisms are the non-decreasing injective maps (this is a subcategory of the category $\Delta_2^\wedge$ of (4.2)).

Remark 1.5.19. Suppose that finite products are representable in $\mathscr{B}$, and for every $i, j, k \in I$, set $S_{ij} := S_i \times S_j$, and $S_{ijk} := S_{ij} \times S_k$. Just as in remark 1.5.6, the category $\mathscr{C}/S_{ijk}$ of S_{ijk}-objects of $\mathscr{C}$ is naturally isomorphic to the category with the same name introduced in (1.5.17), and under this isomorphism, the functors π^0_{ijk*} are identified with the functors arising from the natural projections $\pi^0_{ijk} : S_{ijk} \to S_{jk}$ (and likewise for π^1_{ijk*} and π^2_{ijk*}). For this reason, even in case these products are not representable in $\mathscr{C}$, we shall abuse notation, and write $X \to S_{ij}$ (resp. $X \to S_{ijk}$) to signify an object of $\mathscr{C}/S_{ij}$ (resp. of $\mathscr{C}/S_{ijk}$).

With this notation, denote by Σ_2 the full subcategory of $\Sigma_2^\wedge$ whose objects are the non-empty sets; we have the following 2-category analogue of lemma 1.5.7 :

Proposition 1.5.20. *The augmentation ε induces an equivalence of categories :*

$$2\text{-}\operatorname*{colim}_{\Sigma_2^o} F_{\mathscr{S}} \xrightarrow{\sim} \mathscr{S}.$$

Proof. Here we regard Σ_2^o (resp. $\mathbf{Cat}/\mathscr{C}$) as a 2-category, as in example 1.3.6(i) (resp. example 1.3.6(iii),(v)). Now, for a given small $\mathscr{C}$-category $\mathscr{A}$, a pseudo-cocone $\varphi_\bullet : F_\mathscr{S} \Rightarrow \mathsf{F}_\mathscr{A}$ is a datum $(\varphi_\bullet, \alpha_\bullet, \beta_\bullet)$ consisting of a system of $\mathscr{C}$-functors :

$$\varphi_0 : \coprod_{i\in I} \mathscr{C}/S_i \to \mathscr{A} \qquad \varphi_1 : \coprod_{(i,j)\in I^2} \mathscr{C}/S_{ij} \to \mathscr{A} \qquad \varphi_2 : \coprod_{(i,j,k)\in I^3} \mathscr{C}/S_{ijk} \to \mathscr{A}$$

together with natural $\mathscr{C}$-isomorphisms (*i.e.* invertible 2-cells in $\mathbf{Cat}/\mathscr{C}$) :

$$\alpha_i : \varphi_0 \circ \partial_i \Rightarrow \varphi_1 \qquad \beta_j : \varphi_1 \circ \partial_j \Rightarrow \varphi_2 \qquad (i = 0,1 \quad j = 0,1,2)$$

related by the simplicial identities :

$$\begin{aligned}
\beta_1 \circ (\alpha_0 * \partial_1) &= \beta_0 \circ (\alpha_0 * \partial_0) \\
\beta_2 \circ (\alpha_0 * \partial_2) &= \beta_0 \circ (\alpha_1 * \partial_0) \\
\beta_2 \circ (\alpha_1 * \partial_2) &= \beta_1 \circ (\alpha_1 * \partial_1).
\end{aligned}$$

We consider the natural $\mathscr{C}$-isomorphism :

$$\omega_\varphi := \alpha_1^{-1} \circ \alpha_0 : \varphi_0 \circ \partial_0 \Rightarrow \varphi_0 \circ \partial_1.$$

The above simplicial identities yield :

$$\begin{aligned}
\omega_\varphi * \partial_0 &= (\alpha_0 * \partial_2)^{-1} \circ \beta_2^{-1} \circ \beta_1 \circ (\alpha_0 * \partial_1) \\
\omega_\varphi * \partial_1 &= (\alpha_1 * \partial_2)^{-1} \circ \beta_2^{-1} \circ \beta_1 \circ (\alpha_0 * \partial_1)
\end{aligned}$$

which in turn imply the (cocycle) identity :

$$(\omega_\varphi * \partial_2) \circ (\omega_\varphi * \partial_0) = \omega_\varphi * \partial_1.$$

Hence, let us denote by $d(\mathscr{A})$ the category whose objects are all pairs (φ, ω) consisting of a $\mathscr{C}$-functor $\varphi : \coprod_{i\in I} \mathscr{C}/S_i \to \mathscr{A}$ and a natural $\mathscr{C}$-isomorphism $\omega : \varphi \circ \partial_0 \Rightarrow \varphi \circ \partial_1$ fulfilling the cocycle identity : $(\omega * \partial_2) \circ (\omega * \partial_0) = \omega * \partial_1$; the morphisms $\alpha : (\varphi, \omega) \to (\varphi', \omega')$ in $d(\mathscr{A})$ are the natural $\mathscr{C}$-transformations (*i.e.* 2-cells in $\mathbf{Cat}/\mathscr{C}$) $\alpha : \varphi \Rightarrow \varphi'$, such that the diagram :

$$\begin{tikzcd}
\varphi \circ \partial_0 \arrow[r, Rightarrow, "\omega"] \arrow[d, Rightarrow, "\alpha * \partial_0"'] & \varphi \circ \partial_1 \arrow[d, Rightarrow, "\alpha * \partial_1"] \\
\varphi' \circ \partial_0 \arrow[r, Rightarrow, "\omega'"] & \varphi' \circ \partial_1
\end{tikzcd}$$

commutes. It is easily seen that the rule $(\varphi_\bullet, \alpha_\bullet, \beta_\bullet) \mapsto (\varphi := \alpha_1^{-1} \circ \alpha_0, \omega_\varphi)$ extends to a functor:

$$d_\mathscr{A} : \mathsf{PsNat}(F_\mathscr{S}, \mathsf{F}_\mathscr{A}) \to d(\mathscr{A}).$$

The functor $d_\mathscr{A}$ is an equivalence of $\mathscr{C}$-categories; indeed, a quasi-inverse $\mathscr{C}$-functor :

$$e_\mathscr{A} : d(\mathscr{A}) \to \mathsf{PsNat}(F_\mathscr{S}, \mathsf{F}_\mathscr{A})$$

can be constructed as follows. Given any object (φ, ω) of $d(\mathscr{A})$, set :

$$\begin{gathered}
\varphi_0 := \varphi \qquad \varphi_1 := \varphi \circ \partial_0 \qquad \varphi_2 := \varphi_1 \circ \partial_0 \\
\alpha_0 := \mathbf{1}_{\varphi_1} \qquad \alpha_1 := \omega^{-1} \qquad \beta_0 := \mathbf{1}_{\varphi_2} =: \beta_1 \qquad \beta_2 := \alpha_1 * \partial_0.
\end{gathered}$$

Using the cocycle identity for ω, one verifies easily that the datum $e_\mathscr{A}(\varphi, \omega) := (\varphi_\bullet, \alpha_\bullet, \beta_\bullet)$ is a pseudo-cocone; moreover, the construction is obviously functorial in (φ, ω), and there are natural isomorphisms $d_\mathscr{A} \circ e_\mathscr{A} \Rightarrow \mathbf{1}$ and $e_\mathscr{A} \circ d_\mathscr{A} \Rightarrow \mathbf{1}$.

Next, consider the $\mathscr{C}$-category $\mathscr{R}$ whose objects are the same as the objects of $\coprod_{i\in I} \mathscr{C}/S_i$, and with morphisms given by the rule :

$$\mathrm{Hom}_\mathscr{R}(X \to S_i, Y \to S_j) := \mathrm{Hom}_\mathscr{C}(X, Y)$$

for any pair of objects $(X \to S_i, Y \to S_j)$. We have obvious $\mathscr{C}$-functors

$$\varepsilon' : \coprod_{i\in I} \mathscr{C}/S_i \to \mathscr{R} \quad \text{and} \quad \delta : \mathscr{R} \to \mathscr{S} \quad : \quad (X \to S_i) \mapsto X$$

where ε' is the identity on objects, and it is clear that δ is an equivalence of $\mathscr{C}$-categories (*i.e.* it is an equivalence, when regarded as a 1-cell in $\mathbf{Cat}/\mathscr{C}$). Moreover, there is a natural $\mathscr{C}$-isomorphism :

$$i : \varepsilon' \circ \partial_0 \Rightarrow \varepsilon' \circ \partial_1$$

which assigns to every object $X \to S_{ij}$ of $\mathscr{C}/S_{ij}$ the identity map $i_{(X\to S_{ij})} := \mathbf{1}_X$. It is easily seen that the pair (ε', i) is an object of $d(\mathscr{R})$, whence a pseudo-cocone

$$\varepsilon'_\bullet := e_{\mathscr{R}}(\varepsilon', i) : F_{\mathscr{S}} \Rightarrow \mathsf{F}_{\mathscr{R}}.$$

Clearly $\varepsilon = \delta \circ \varepsilon'$, hence it suffices to show that the composed functor :

$$\mathbf{Cat}/\mathscr{C}(\mathscr{R}, \mathscr{A}) \to \mathsf{PsNat}(F_{\mathscr{S}}, \mathsf{F}_{\mathscr{A}}) \to d(\mathscr{A}) \quad : \quad G \mapsto d_{\mathscr{A}}(\mathsf{F}_G \odot \varepsilon'_\bullet) \tag{1.5.21}$$

is an isomorphism for every small $\mathscr{C}$-category $\mathscr{A}$. By inspecting the construction, we see that the latter assigns to every $\mathscr{C}$-functor $G : \mathscr{R} \to \mathscr{A}$ the pair $(G \circ \varepsilon', G * i)$. To conclude, we shall exhibit an inverse for (1.5.21).

Namely, let (φ, ω) be any object of $d(\mathscr{A})$; we set :

$$Gf := \varphi f \qquad \text{for every } f \in \mathrm{Ob}(\mathscr{R}).$$

Now, suppose that $f_X : X \to S_i$ and $f_Y : Y \to S_j$ are two objects of $\mathscr{R}$, and $h : X \to Y$ is an element of $\mathrm{Hom}_{\mathscr{R}}(f_X, f_Y)$; the pair $(f_X, f'_X := f_Y \circ h)$ determines a morphism $f'_X \times f_X : X \to S_{ji}$ with $\partial_0(f'_X \times f_X) = f_X$ and $\partial_1(f'_X \times f_X) = f'_X$ (notation of remark 1.5.19); hence we may define Gh as the composition :

$$Gh : Gf_X \xrightarrow{\omega_{f'_X \times f_X}} \varphi f'_X \xrightarrow{\varphi(h/S_j)} \varphi f_Y = Gf_Y$$

where h/S_j denotes h, regarded as an element of $\mathrm{Hom}_{\mathscr{C}/S_j}(f'_X, f_Y)$.

Next, let $f_Z : Z \to S_k$ be another object of $\mathscr{R}$, and $g : Y \to Z$ any element of $\mathrm{Hom}_{\mathscr{R}}(f_Y, f_Z)$; we have to verify that $G(g \circ h) = Gg \circ Gh$. As in the foregoing, we deduce morphisms $X \to S_j$, $X \to S_k$ and $Y \to S_k$, as well as their products $X \to S_{ji}$, $X \to S_{ki}$, and $Y \to S_{kj}$, whence a diagram :

$$\begin{tikzcd}
Gf_X := \varphi(X \to S_i) \arrow[r, "\omega_{(X\to S_{ji})}"] \arrow[dr, "\omega_{(X\to S_{ki})}"'] & \varphi(X \to S_j) \arrow[r, "\varphi(h/S_j)"] \arrow[d, "\omega_{(X\to S_{kj})}"] & Gf_Y := \varphi(Y \to S_j) \arrow[d, "\omega_{(Y\to S_{kj})}"] \\
& \varphi(X \to S_k) \arrow[r, "\varphi(h/S_k)"] \arrow[dr, "\varphi(g\circ h/S_k)"'] & \varphi(Y \to S_k) \arrow[d, "\varphi(g/S_k)"] \\
& & Gf_Z := \varphi(Z \to S_k).
\end{tikzcd} \tag{1.5.22}$$

The sought identity amounts to asserting that (1.5.22) commutes, which can be easily verified, using the cocycle condition for ω, and the obvious identities :

$$\partial_0(h/S_{kj}) = h/S_j \quad \text{and} \quad \partial_1(h/S_{kj}) = h/S_k.$$

Finally, notice that $G\mathbf{1}_f = G(\mathbf{1}_f \circ \mathbf{1}_f) = G\mathbf{1}_f \circ G\mathbf{1}_f$, for every $f \in \mathrm{Ob}(\mathscr{R})$; since $G\mathbf{1}_f$ is an isomorphism, it follows that $G\mathbf{1}_f = \mathbf{1}_{Gf}$.

Hence G is a functor $\mathscr{R} \to \mathscr{A}$; furthermore, the rule $(\varphi, \omega) \mapsto G$ is clearly functorial, and a simple inspection shows that $G \circ \varepsilon' = \varphi$ and $G * i = \omega$. Conversely, if we apply the foregoing procedure to a pair of the form $(G \circ \varepsilon', G * i)$, we obtain back the functor G. $\square$

1.5.23. Resume the situation of (1.5.15), and notice that all the categories appearing in (1.5.18) are fibred over $\mathscr{B}$: indeed, every morphism in each of these categories is cartesian, hence all the functors appearing in (1.5.18) are cartesian. Let us consider now the functor :

$$\mathsf{Cart}_{\mathscr{B}}(-,\mathscr{A}) : (\mathbf{Cat}/\mathscr{B})^o \to \mathbf{Cat}$$

that assigns to every $\mathscr{B}$-category $\mathscr{C}$ the category $\mathsf{Cart}_{\mathscr{B}}(\mathscr{C},\mathscr{A})$. With the notation of (1.5.15), we deduce a functor :

$$\mathsf{Cart}_{\mathscr{B}}(F_{\mathscr{S}},\mathscr{A}) : \Sigma_2^\wedge \to \mathbf{Cat}$$

and in light of the foregoing observations, proposition 1.5.20 easily implies that $\mathsf{Cart}_{\mathscr{B}}(\varepsilon,\mathscr{A})$ induces an equivalence of categories :

$$\text{(1.5.24)} \qquad 2\text{-}\lim_{\overleftarrow{\Sigma_2}} \mathsf{Cart}_{\mathscr{B}}(F_{\mathscr{S}},\mathscr{A}) \xrightarrow{\sim} \mathsf{Cart}_{\mathscr{B}}(\mathscr{S},\mathscr{A}).$$

Next, suppose that the fibre products $S_{ij} := S_i \times S_j$ and $S_{ijk} := S_{ij} \times S_k$ are representable in $\mathscr{B}$ (see remark 1.5.19); in this case, we may compose with the pseudo-equivalence ev of remark 1.4.20 : combining with lemma 1.3.14 we finally obtain an equivalence between the category $\mathsf{Cart}_{\mathscr{B}}(\mathscr{S},\mathscr{A})$, and the 2-limit of the pseudo-functor $\mathsf{d} := \mathsf{ev} \circ \mathsf{Cart}_{\mathscr{B}}(F_{\mathscr{S}},\mathscr{A}) : \Sigma_2 \to \mathbf{Cat}$:

$$\prod_{i\in I} \varphi^{-1}S_i \underset{\partial^1}{\overset{\partial^0}{\rightrightarrows}} \prod_{(i,j)\in I^2} \varphi^{-1}S_{ij} \underset{\partial^2}{\overset{\partial^0}{\underset{\partial^1}{\rightrightarrows}}} \prod_{(i,j,k)\in I^3} \varphi^{-1}S_{ijk}.$$

Of course, the coface operators ∂^s on $\prod_{i\in I} \varphi^{-1}S_i$ decompose as products of pull-back functors:

$$\pi_{ij}^{0*} : \varphi^{-1}S_j \to \varphi^{-1}S_{ij} \qquad \pi_{ij}^{1*} : \varphi^{-1}S_i \to \varphi^{-1}S_{ij}$$

attached – via the chosen cleavage c of φ – to the projections $\pi_{ij}^0 : S_{ij} \to S_j$ and $\pi_{ij}^1 : S_{ij} \to S_i$ (and likewise for the components π_{ijk}^{t*} of the other coface operators).

1.5.25. By inspecting the proof of proposition 1.3.15, we may give the following explicit description of this 2-limit. Namely, it is the category whose objects are the data

$$\underline{X} := (X_i, X_{ij}, X_{ijk}, \xi_i^u, \xi_{ij}^s, \xi_{ijk}^t \mid i,j,k \in I; s \in \{0,1\}; u,t \in \{0,1,2\})$$

where :

$$X_i \in \mathrm{Ob}(\varphi^{-1}S_i) \quad X_{ij} \in \mathrm{Ob}(\varphi^{-1}S_{ij}) \quad X_{ijk} \in \mathrm{Ob}(\varphi^{-1}S_{ijk}) \qquad \text{for every } i,j,k \in I$$

and for every $i,j,k \in I$:

$$\begin{array}{lll}
\xi_i^0 : (\pi_{ik}^1\pi_{ijk}^1)^* X_i \xrightarrow{\sim} X_{ijk} & \xi_{ij}^0 : \pi_{ij}^{0*} X_j \xrightarrow{\sim} X_{ij} & \xi_{ijk}^0 : \pi_{ijk}^{0*} X_{jk} \xrightarrow{\sim} X_{ijk} \\
\xi_j^1 : (\pi_{jk}^1\pi_{ijk}^0)^* X_j \xrightarrow{\sim} X_{ijk} & \xi_{ij}^1 : \pi_{ij}^{1*} X_i \xrightarrow{\sim} X_{ij} & \xi_{ijk}^1 : \pi_{ijk}^{1*} X_{ik} \xrightarrow{\sim} X_{ijk} \\
\xi_k^2 : (\pi_{jk}^0\pi_{ijk}^0)^* X_k \xrightarrow{\sim} X_{ijk} & & \xi_{ijk}^2 : \pi_{ijk}^{2*} X_{ij} \xrightarrow{\sim} X_{ijk}
\end{array}$$

are isomorphisms related by the cosimplicial identities :

$$\begin{array}{ll}
\xi_{ijk}^0 \circ \pi_{ijk}^{0*}\xi_{jk}^0 = \xi_k^2 \circ \gamma_X^{00} & \xi_{ijk}^1 \circ \pi_{ijk}^{1*}\xi_{ik}^0 = \xi_k^2 \circ \gamma_X^{01} \\
\xi_{ijk}^2 \circ \pi_{ijk}^{2*}\xi_{ij}^0 = \xi_j^1 \circ \gamma_X^{02} & \xi_{ijk}^0 \circ \pi_{ijk}^{0*}\xi_{jk}^1 = \xi_j^1 \circ \gamma_X^{10} \\
\xi_{ijk}^2 \circ \pi_{ijk}^{2*}\xi_{ij}^1 = \xi_i^0 \circ \gamma_X^{12} & \xi_{ijk}^1 \circ \pi_{ijk}^{1*}\xi_{ik}^1 = \xi_i^0 \circ \gamma_X^{11}
\end{array}$$

where $\gamma^{st} := \gamma_{\mathsf{d}(\partial^s),\mathsf{d}(\partial^t)} : \mathsf{d}(\partial^s) \circ \mathsf{d}(\partial^t) \Rightarrow \mathsf{d}(\partial^s \circ \partial^t)$ denotes the coherence constraint of the cleavage c, for any pair of arrows (∂^s, ∂^t) in the category Σ_2. The morphisms $\underline{X} \to \underline{Y}$ in this category are the systems of morphisms :

$$(X_i \to Y_i, X_{ij} \to Y_{ij}, X_{ijk} \to Y_{ijk} \mid i,j,k \in I)$$

that are compatible in the obvious way with the various isomorphisms. However, one may argue as in the proof of proposition 1.5.20, to replace this category by an equivalent one which admits a handier description : given a datum $\underline{X}$, one can make up an isomorphic datum $\underline{X}^* := (X_i, X^*_{ij}, X^*_{ijk}, \eta_i, \eta_{ij}, \eta_{ijk})$, by the rule :

$$X^*_{ij} := \pi^{1*}_{ij} X_i \qquad X^*_{ijk} := (\pi^1_{ik} \circ \pi^1_{ijk})^* X_i$$

$$\begin{array}{lll} \eta^0_i := \mathbf{1} & \eta^1_{ij} := \mathbf{1} & \eta^0_{ijk} := \eta^1_j \circ \gamma^{10}_X \\ \eta^1_j := (\xi^0_i)^{-1} \circ \xi^1_j & \eta^0_{ij} := (\xi^1_{ij})^{-1} \circ \xi^0_{ij} & \eta^1_{ijk} := \gamma^{11}_X \\ \eta^2_k := (\xi^0_i)^{-1} \circ \xi^2_k & & \eta^2_{ijk} := \gamma^{12}_X. \end{array}$$

The cosimplicial identities for this new object are subsumed into a single cocycle identity for $\omega_{ij} := \eta^0_{ij}$. Summing up, we arrive at the following description of our 2-limit :

- The objects are all the systems $\underline{X} := (X_i, \omega^X_{ij} \mid i,j \in I)$ where $X_i \in \mathrm{Ob}(\varphi^{-1}S_i)$ for every $i \in I$, and
$$\omega^X_{ij} : \pi^{0*}_{ij} X_j \xrightarrow{\sim} \pi^{1*}_{ij} X_i$$
is an isomorphism in $\varphi^{-1}S_{ij}$, for every $i,j \in I$, fulfilling the cocycle identity :
$$p^{2*}_{ijk}\omega^X_{ij} \circ p^{0*}_{ijk}\omega^X_{jk} = p^{1*}_{ijk}\omega^X_{ik} \qquad \text{for every } i,j,k \in I$$
where, for every $i,j,k \in I$, and $t = 0,1,2$ we have set :
$$p^{t*}_{ijk}\omega^X_{\bullet\bullet} := \gamma^{1t}_X \circ \pi^{t*}_{ijk}\omega^X_{\bullet\bullet} \circ (\gamma^{0t}_X)^{-1}.$$
- The morphisms $\underline{X} \to \underline{Y}$ are the systems of morphisms $(f_i : X_i \to Y_i \mid i \in I)$ with :

(1.5.26) $$\omega^Y_{ij} \circ \pi^{0*}_{ij} f_j = \pi^{1*}_{ij} f_i \circ \omega^X_{ij} \qquad \text{for every } i,j \in I.$$

1.5.27. We shall call any pair $(X_\bullet, \omega_\bullet)$ of the above form, a *descent datum for the fibration* φ, *relative to the family* $\underline{S} := (\pi_i : S_i \to B \mid i \in I)$ *and the cleavage* c. The category of such descent data shall be denoted:

$$\mathsf{Desc}(\varphi, \underline{S}, \mathsf{c}).$$

Sometimes we may also denote it by $\mathsf{Desc}(\mathscr{A}, \underline{S}, \mathsf{c})$, if the notation is not ambiguous. Of course, two different choices of cleavage lead to equivalent categories of descent data, so usually we omit mentioning explicitly c, and write simply $\mathsf{Desc}(\varphi, \underline{S})$ or $\mathsf{Desc}(\mathscr{A}, \underline{S})$. The foregoing discussion can be summarized, by saying that there is a commutative diagram of categories :

(1.5.28) $$\begin{tikzcd} \mathsf{Cart}_{\mathscr{B}}(\mathscr{B}/B, \mathscr{A}) \arrow[r, "\mathsf{Cart}_{\mathscr{B}}(\iota_{\mathscr{S}}, \mathscr{A})"] \arrow[d, "\mathsf{ev}_B"'] & \mathsf{Cart}_{\mathscr{B}}(\mathscr{S}, \mathscr{A}) \arrow[d, "\delta_{\underline{S}}"] \\ \varphi^{-1}B \arrow[r, "\rho_{\underline{S}}"] & \mathsf{Desc}(\varphi, \underline{S}, \mathsf{c}) \end{tikzcd}$$

whose vertical arrows are equivalences, and where $\rho_{\underline{S}}$ is determined by c. Explicitly, $\rho_{\underline{S}}$ assigns to every object C of $\varphi^{-1}B$ the pair $(C_\bullet, \omega^C_\bullet)$ where $C_i := \pi^*_i C$, and ω^C_{ij} is defined as the composition :

$$\pi^{0*}_{ij} \circ \pi^*_j C \xrightarrow[\sim]{\gamma_{(\pi^0_{ij}, \pi_j)}} (\pi_j \circ \pi^0_{ij})^* C = (\pi_i \circ \pi^1_{ij})^* C \xrightarrow[\sim]{\gamma^{-1}_{(\pi^1_{ij}, \pi_i)}} \pi^{1*}_{ij} \circ \pi^*_i C$$

where $\gamma_{(\pi^1_{ij}, \pi_i)}$ and $\gamma_{(\pi^0_{ij}, \pi_j)}$ are the coherence constraints for the cleavage c (see (1.4.8)). The descent datum $(X_\bullet, \omega_\bullet)$ is said to be *effective*, if it lies in the essential image of $\rho_{\underline{S}}$.

We also have an obvious functor :

$$\mathsf{p}_{\underline{S}} : \mathsf{Desc}(\varphi, \underline{S}) \to \prod_{i \in I} \varphi^{-1} S_i \qquad (X_i, \omega_{ij} \mid i,j \in I) \mapsto (X_i \mid i \in I)$$

such that :

$$\mathsf{p}_{\underline{S}} \circ \rho_{\underline{S}} = \prod_{i \in I} \pi^*_i : \varphi^{-1}B \to \prod_{i \in I} \varphi^{-1} S_i.$$

1.5.29. Furthermore, for every morphism $f : B' \to B$ in $\mathscr{B}$, set

$$\underline{S} \times_B f := (\pi_i \times_B B' : S_i \times_B B' \to B' \mid i \in I)$$

which is a generating family for the sieve $\mathscr{S} \times_B f$ (notation of (1.1.12)); then we deduce a pseudo-natural transformation of pseudo-functors $(\mathscr{B}/B)^o \to \mathbf{Cat}$:

$$\rho : \mathsf{c} \circ i_B^o \Rightarrow \mathsf{Desc}(\varphi, \underline{S} \times_B -, \mathsf{c}) \qquad (f : B' \to B) \mapsto \rho_{\underline{S} \times_B f}$$

(where $i_B : \mathscr{B}/B \to \mathscr{B}$ is the functor (1.1.13)) fitting into a commutative diagram :

$$\begin{tikzcd}
\mathsf{Cart}_{\mathscr{B}}(\mathscr{B}/-, \mathscr{A}) \circ i_B^o \arrow[r, Rightarrow, "\mathsf{Cart}_{\mathscr{B}}(\iota_{\mathscr{S} \times_B -}, \mathscr{A})"] \arrow[d, Rightarrow, "\mathsf{ev} * i_B^o"'] & \mathsf{Cart}_{\mathscr{B}}(\mathscr{S} \times_B -, \mathscr{A}) \arrow[d, Rightarrow, "\delta_{\underline{S} \times_B -}"] \\
\mathsf{c} \circ i_B^o \arrow[r, Rightarrow, "\rho"] & \mathsf{Desc}(\varphi, \underline{S} \times_B -, \mathsf{c})
\end{tikzcd}$$

using which, one can figure out the pseudo-functoriality of the rule : $f \mapsto \mathsf{Desc}(\varphi, \underline{S} \times_B f, \mathsf{c})$. Namely, every pair of objects $f : C \to B$ and $f' : C' \to B$, and any morphism $h : C' \to C$ in $\mathscr{B}/B$, yield a commutative diagram :

$$\begin{tikzcd}
S_j \times_B C' \arrow[d, "h_j := S_j \times_B h"'] & S_{ij} \times_B C' \arrow[l, "\widetilde{\pi}^0_{ij}"'] \arrow[r, "\widetilde{\pi}^1_{ij}"] \arrow[d, "h_{ij} := S_{ij} \times_B h"'] & S_i \times_B C' \arrow[r, "\widetilde{\pi}_i"] \arrow[d, "h_i := S_i \times_B h"] & C' \arrow[d, "h"] \\
S_j \times_B C & S_{ij} \times_B C \arrow[l, "\pi^0_{ij}"] \arrow[r, "\pi^1_{ij}"'] & S_i \times_B C \arrow[r, "\pi_i"'] & C.
\end{tikzcd}$$

Hence one obtains a functor :

$$\mathsf{Desc}(\varphi, h, \mathsf{c}) : \mathsf{Desc}(\varphi, \underline{S} \times_B f, \mathsf{c}) \to \mathsf{Desc}(\varphi, \underline{S} \times_B f', \mathsf{c})$$

by the rule :

$$(X_i, \omega^X_{ij} \mid i, j \in I) \mapsto (h_i^* X_i, \widetilde{\omega}^X_{ij} \mid i, j \in I)$$

where $\widetilde{\omega}^X_{ij}$ is the isomorphism that makes commute the diagram :

$$\begin{tikzcd}
h^*_{ij} \pi^{0*}_{ij} X_j \arrow[r, "\gamma_{(h_{ij}, \pi^0_{ij})}"] \arrow[d, "h^*_{ij} \omega_{ij}"'] & (\pi^0_{ij} \circ h_{ij})^* X_j & \widetilde{\pi}^{0*}_{ij} \circ h_j^* X_j \arrow[l, "\gamma_{(\widetilde{\pi}^0_{ij}, h_j)}"'] \arrow[d, "\widetilde{\omega}^X_{ij}"] \\
h^*_{ij} \pi^{1*}_{ij} X_i \arrow[r, "\gamma_{(h_{ij}, \pi^1_{ij})}"] & (\pi^1_{ij} \circ h_{ij})^* X_i & \widetilde{\pi}^{1*}_{ij} \circ h_i^* X_i \arrow[l, "\gamma_{(\widetilde{\pi}^1_{ij}, h_i)}"']
\end{tikzcd}$$

and if $f'' : C'' \to B$ is a third object, with a morphism $g : C'' \to C'$, we have a natural isomorphism of functors :

$$\mathsf{Desc}(\varphi, g, \mathsf{c}) \circ \mathsf{Desc}(\varphi, h, \mathsf{c}) \Rightarrow \mathsf{Desc}(\varphi, h \circ g, \mathsf{c})$$

which is induced by the cleavage c, in the obvious fashion.

Theorem 1.5.30. *For $i = 1, 2$, let $\varphi_i : \mathscr{A}_i \to \mathscr{B}$ be two fibrations, $F : \mathscr{A}_1 \to \mathscr{A}_2$ a cartesian functor of $\mathscr{B}$-categories, B an object of $\mathscr{B}$ and $\mathscr{S}$ a sieve of $\mathscr{B}/B$ generated by the family $(S_i \to B \mid i \in I)$. We assume that S_{ij} and S_{ijk} are representable in $\mathscr{B}$, for every $i, j, k \in I$ (see remark* 1.5.19*); then we have :*

- (i) *For $n \in \{0, 1, 2\}$ and every $i, j, k \in I$, suppose that*
  - (a) *$\mathscr{S}$ is a sieve both of φ_1-n-descent and of φ_2-$(n-1)$-descent.*
  - (b) *The restriction $F_i : \varphi_1^{-1} S_i \to \varphi_2^{-1} S_i$ of F is n-faithful.*
  - (c) *The restriction $F_{ij} : \varphi_1^{-1} S_{ij} \to \varphi_2^{-1} S_{ij}$ of F is $(n-1)$-faithful.*
  - (d) *The restriction $F_{ijk} : \varphi_1^{-1} S_{ijk} \to \varphi_2^{-1} S_{ijk}$ of F is $(n-2)$-faithful.*

  Then the restriction $F_B : \varphi_1^{-1} B \to \varphi_2^{-1} B$ of F is n-faithful.

(ii) *Suppose that the functors F_{ij} are fully faithful, and the functors F_{ijk} are faithful, for every $i, j, k \in I$. Then the natural commutative diagram*

$$\begin{array}{ccc} \mathsf{Cart}_{\mathscr{B}}(\mathscr{S}, \mathscr{A}_1) & \xrightarrow{\mathsf{Cart}_{\mathscr{B}}(\mathscr{S},F)} & \mathsf{Cart}_{\mathscr{B}}(\mathscr{S}, \mathscr{A}_2) \\ \downarrow & & \downarrow \\ \prod_{i\in I} \varphi_1^{-1} S_i & \xrightarrow{\prod_{i\in I} F_i} & \prod_{i\in I} \varphi_2^{-1} S_i \end{array}$$

is 2*-cartesian.*

Proof. (i): In view of theorem 1.4.17, we may assume that both $\mathscr{A}_1$ and $\mathscr{A}_2$ are split fibrations (with a suitable choice of cleavages), and F is a split cartesian functor. Recall that the latter condition means the following. For every morphism $g : X \to Y$ in $\mathscr{B}$, the induced diagram

$$\begin{array}{ccc} \varphi_1^{-1} Y & \xrightarrow{g^*} & \varphi_1^{-1} X \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F} \\ \varphi_2^{-1} Y & \xrightarrow{g^*} & \varphi_2^{-1} X \end{array}$$

commutes (where the horizontal arrows are the pull-back functor given by the chosen cleavages). In this situation, we have a commutative diagram

$$\begin{array}{ccc} \varphi_1^{-1} B & \xrightarrow{\rho_{\underline{S}}} & \mathsf{Desc}(\varphi_1, \underline{S}) \\ {\scriptstyle F_B}\downarrow & & \downarrow{\scriptstyle \underline{F}} \\ \varphi_2^{-1} B & \xrightarrow{\rho_{\underline{S}}} & \mathsf{Desc}(\varphi_2, \underline{S}) \end{array} \tag{1.5.31}$$

whose right vertical arrow is the functor given by the rule :

$$\underline{X} := (X_i, \omega_{ij} \mid i, j \in I) \mapsto \underline{F}(\underline{X}) := (F_i X_i, F_{ij}\omega_{ij} \mid i, j \in I).$$

for every object $\underline{X}$ of $\mathsf{Desc}(\varphi_1, \underline{S})$. By assumption, the top horizontal arrow of (1.5.31) is n-faithful, and the bottom horizontal arrow is $(n-1)$-faithful. We need to prove that the left vertical arrow is n-faithful, and it is easily seen that this will follow, once we have shown that the same holds for the right vertical arrow.

Suppose first that $n = 0$; we have to show that $\underline{F}$ is faithful. However, let $\underline{X}$ and $\underline{Y}$ be two objects of $\mathsf{Desc}(\varphi_1, \underline{S})$, and $\underline{h}_1, \underline{h}_2 : \underline{X} \to \underline{Y}$ two morphisms. By definition, $\underline{h}_t$ (for $t = 1, 2$) is a compatible system $(h_{t,i} : X_i \to Y_i \mid i \in I)$, where each $h_{t,i}$ is a morphism in $\varphi_1^{-1} S_i$. Then, $\underline{F}(\underline{h}_t)$ is the compatible system $(F_i h_{t,i} \mid i \in I)$. Thus, the condition $\underline{F}(\underline{h}_1) = \underline{F}(\underline{h}_2)$ translates the system of identities $F_i h_{1,i} = F_i h_{2,i}$ for every $i \in I$. By assumption, each F_i is faithful, therefore $\underline{h}_1 = \underline{h}_2$, as stated.

For $n = 1$, assumption (d) is empty, (b) means that F_i is fully faithful, and (c) means that F_{ij} is faithful for every $i, j \in I$. In light of the previous case, we have only to show that $\underline{F}$ is full. Hence, let $\underline{X}, \underline{Y}$ be as in the foregoing, and $(h_i : F_i X_i \to F_i Y_i \mid i \in I)$ a morphism $\underline{F}(\underline{X}) \to \underline{F}(\underline{Y})$ in $\mathsf{Desc}(\varphi_2, \underline{S})$. By assumption, for every $i \in I$ we may find a unique morphism $f_i : X_i \to Y_i$ such that $F_i f_i = h_i$. It remains only to check that the system $(f_i \mid i \in I)$ fulfills condition (1.5.26), and since the functors F_{ij} are faithful, it suffices to verify that F_{ij}(1.5.26) holds. However, since F is split cartesian, we have :

$$F_{ij} \circ \pi_{ij}^{0*} f_j = \pi_{ij}^{0*} \circ F_j f_j \qquad F_{ij} \circ \pi_{ij}^{1*} f_i = \pi_{ij}^{1*} \circ F_i f_i$$

hence we reduce to showing that $(F_{ij}\omega_{ij}^Y) \circ \pi_{ij}^{0*} h_j = \pi_{ij}^{1*} h_i \circ (F_{ij}\omega_{ij}^X)$, which holds by assumption.

Next, we consider assertion (ii) : the contention is that the functors $\underline{F}$ and :

$$\mathsf{p}_{1,\underline{S}} : \mathsf{Desc}(\varphi_1, \underline{S}) \to \prod_{i\in I} \varphi_1^{-1} S_i$$

as in (1.5.27), induce an equivalence $(\mathsf{p}_{1,\underline{S}}, \underline{F})$ between $\mathsf{Desc}(\varphi_1, \underline{S})$ and the category $\mathscr{C}$ consisting of all data of the form $\underline{G} := (G_i, H_i, \alpha_i, \omega_{ij}^H \mid i, j \in I)$, where $G_i \in \mathrm{Ob}(\varphi_1^{-1}S_i)$, $H_i \in \mathrm{Ob}(\varphi_2^{-1}S_i)$, $\alpha_i : F_iG_i \stackrel{\sim}{\to} H_i$ are isomorphisms in $\varphi_2^{-1}S_i$, and $\underline{H} := (H_i, \omega_{ij}^H \mid i, j \in I)$ is an object of $\mathsf{Desc}(\varphi_2, \underline{S})$. However, given an object as above, set :

$$\omega_{ij}^{H'} := (\pi_{ij}^{1*}\alpha_i^{-1}) \circ \omega_{ij}^H \circ (\pi_{ij}^{0*}\alpha_j).$$

Since φ_2 is a split fibration, one verifies easily that the datum $\underline{H}' := (H'_i := F_iG_i, \omega_{ij}^{H'} \mid i, j \in I)$ is an object of $\mathsf{Desc}(\varphi_2, \underline{S})$ isomorphic to $\underline{H}$, and the new datum $(F_i, H'_i, \mathbf{1}_{H'_i}, \omega_{ij}^{H'} \mid i, j \in I)$ is isomorphic to $\underline{G}$; hence $\mathscr{C}$ is equivalent to the category $\mathscr{C}'$ whose objects are all data of the form $(G_i, \omega_{ij} \mid i, j \in I)$ where $G_i \in \mathrm{Ob}(\varphi_1^{-1}S_i)$, and $(F_iG_i, \omega_{ij} \mid i, j \in I)$ is an object of $\mathsf{Desc}(\varphi_2, \underline{S})$. By assumption F_{ij} is fully faithful, and F is a split cartesian functor; hence we may find unique isomorphisms $\widetilde{\omega}_{ij}^G : \pi_{ij}^{0*}G_j \stackrel{\sim}{\to} \pi_{ij}^{1*}G_i$ such that $\widetilde{\omega}_{ij} = F_{ij}\widetilde{\omega}_{ij}^G$. We claim that the datum $(G_i, \omega_{ij}^G \mid i, j \in I)$ is an object of $\mathsf{Desc}(\varphi_1, \underline{S})$, *i.e.* the isomorphisms ω_{ij}^G satisfy the cocycle condition

$$\pi_{ijk}^{2*}\omega_{ij}^G \circ \pi_{ijk}^{0*}\omega_{jk}^G = \pi_{ijk}^{1*}\omega_{ik}^G \qquad \text{for every } i, j, k \in I. \tag{1.5.32}$$

To check this identity, since by assumption the functors F_{ijk} are faithful, it suffices to see that F_{ijk}(1.5.32) holds, which is clear, since the cocycle condition holds for the isomorphisms ω_{ij} (and since F is split cartesian). This shows that $(\mathsf{p}_{1,\underline{S}}, \underline{F})$ is essentially surjective. Next, since the functor $\mathsf{p}_{1,\underline{S}}$ is faithful, the same holds for $(\mathsf{p}_{1,\underline{S}}, \underline{F})$. Finally, let

$$\underline{G} := (G_i, \omega_{ij} \mid i, j \in I) \qquad \underline{G}' := (G'_i, \omega'_{ij} \mid i, j \in I)$$

be two objects of $\mathscr{C}'$. A morphism $\underline{G} \to \underline{G}'$ consists of a system $(\alpha_i : G_i \to G'_i \mid i \in I)$ of morphisms such that $(F_i\alpha_i \mid i \in I)$ is a morphism

$$(F_iG_i, \omega_{ij} \mid i, j \in I) \to (F_iG'_i, \omega'_{ij} \mid i, j \in I)$$

in $\mathsf{Desc}(\varphi_2, \underline{S})$. To show that $(\mathsf{p}_{1\underline{S}}, \underline{F})$ is full, and since we know already that this functor is essentially surjective, we may assume that there exist $(G_i, \omega_{ij}^G \mid i, j \in I)$, $(G'_i, \omega_{ij}^{G'} \mid i, j \in I)$ in $\mathsf{Desc}(\varphi_1, \underline{S})$ such that $\omega_{ij} = F_{ij}\omega_{ij}^G$ and $\omega'_{ij} = F_{ij}\omega_{ij}^{G'}$ for every $i, j \in I$; in this case, it suffices to verify the identity

$$\omega_{ij}^{G'} \circ \pi_{ij}^{0*}\alpha_j = \pi_{ij}^{1*}\alpha_i \circ \omega_{ij}^G \qquad \text{for every } i, j \in I. \tag{1.5.33}$$

Again, the faithfulness of F_{ij} reduces to checking that F_{ij}(1.5.33) holds, which is clear, since F is split cartesian.

Lastly, notice that the case $n = 2$ of assertion (i) is a formal consequence of (ii). □

1.5.34. In the situation of (1.5.27), let $\mathscr{S}$ be the sieve generated by the family $\underline{S}$, and $g : B' \to B$ any morphism in $\mathscr{B}$. We let :

$$B'_i := B' \times_B S_i \quad B'_{ij} := B' \times_B S_{ij} \quad B'_{ijk} := B' \times_B S_{ijk} \qquad \text{for every } i, j, k \in I$$

and denote $g_i : B'_i \to S_i$, $g_{ij} : B'_{ij} \to S_{ij}$ and $g_{ijk} : B'_{ijk} \to S_{ijk}$ the induced projections.

Corollary 1.5.35. *With the notation of* (1.5.34)*, let* $n \in \{0, 1, 2\}$*. The following holds :*

(i) $\mathscr{S}$ *is a sieve of* φ*-*n*-descent, if and only if* $\rho_{\underline{S}}$ *is* n*-faithful (see* (1.5.28)*).*

(ii) *Suppose that :*

  (a) $\mathscr{S}$ *is a sieve of universal* φ*-*n*-descent.*

  (b) *The pull-back functors* $g_i^* : \varphi^{-1}S_i \to \varphi^{-1}B'_i$ *are* n*-faithful.*

(c) *The pull-back functors* $g_{ij}^* : \varphi^{-1}S_{ij} \to \varphi^{-1}B'_{ij}$ *are* $(n-1)$*-faithful.*
(d) *The pull-back functors* $g_{ijk}^* : \varphi^{-1}S_{ijk} \to \varphi^{-1}B'_{ijk}$ *are* $(n-2)$*-faithful.*
Then the pull-back functor g^* *is* n*-faithful.*

(iii) *Suppose that the functors* g_{ij}^* *are fully faithful, and the functors* g_{ijk}^* *are faithful, for every* $i, j, k \in I$. *Then the natural essentially commutative diagram :*

$$\begin{tikzcd}
\mathsf{Desc}(\varphi, \underline{S}) \arrow[r, "\mathsf{Desc}(\varphi, g)"] \arrow[d, "\mathsf{p}_{\underline{S}}"'] & \mathsf{Desc}(\varphi, \underline{S} \times_B B') \arrow[d, "\mathsf{p}_{\underline{S} \times_B B'}"] \\
\prod_{i\in I} \varphi^{-1}S_i \arrow[r, "\prod_{i\in I} g_i^*"] & \prod_{i\in I} \varphi^{-1}B'_i
\end{tikzcd}$$

is 2*-cartesian (see* (1.3.16)*).*

Proof. (i) follows by inspecting (1.5.28).

(ii): Thanks to theorem 1.4.17, we may assume that φ is a split fibration. Now, set

$$\mathscr{C} := \mathsf{Morph}(\mathscr{B}) \qquad \mathscr{A}_1 := \mathscr{C} \times_{(\mathsf{t},\varphi)} \mathscr{A} \qquad \mathscr{A}_2 := \mathscr{C} \times_{(\mathsf{s},\varphi)} \mathscr{A}$$

where $\mathsf{s}, \mathsf{t} : \mathscr{C} \to \mathscr{B}$ are the source and target functors (see (1.1.17)). The natural projections $\varphi_i : \mathscr{A}_i \to \mathscr{C}$ (for $i = 1, 2$) are two fibrations (see example 1.4.10(i)). Moreover, t induces a functor $\mathsf{t}_{|g} : \mathscr{C}/g \to \mathscr{B}/B$ (notation of (1.1.15)) and we let $\mathscr{S}/g := \mathsf{t}_{|g}^{-1}\mathscr{S}$, which is the sieve of $\mathscr{C}/g$ generated by the cartesian diagrams

$$D_i \quad : \quad \begin{tikzcd}
B'_i \arrow[r, "g_i"] \arrow[d] & S_i \arrow[d] \\
B' \arrow[r, "g"] & B
\end{tikzcd} \qquad \text{for every } i \in I.$$

Notice that the products $D_{ij} := D_i \times D_j$ are represented in $\mathscr{C}/g$ by the diagrams

$$\begin{tikzcd}
B'_{ij} \arrow[r, "g_{ij}"] \arrow[d] & S_{ij} \arrow[d] \\
B' \arrow[r, "g"] & B
\end{tikzcd} \qquad \text{for every } i, j \in I$$

and likewise one may represent the triple products $D_{ijk} := D_{ij} \times D_k$.

By definition, the objects of $\mathscr{A}_1$ (resp. $\mathscr{A}_2$) are the pairs $(h : X \to Y, a)$, where h is a morphism in $\mathscr{B}$ and $a \in \mathrm{Ob}(\varphi^{-1}Y)$ (resp. $a \in \mathrm{Ob}(\varphi^{-1}X)$). A morphism of $\mathscr{A}_1$ (resp. of $\mathscr{A}_2$)

$$(h : X \to Y, a) \to (h' : X' \to Y', a')$$

is a datum (f_1, f_2, t), where $f_1 : X \to X'$ and $f_2 : Y \to Y'$ are morphisms in $\mathscr{B}$ with $f_2 \circ h = h' \circ f_1$, and $t : a \to f_2^* a'$ (resp. $t : a \to f_1^* a'$) is a morphism in $\varphi^{-1}Y$ (resp. in $\varphi^{-1}X$). Now, we define a functor $F : \mathscr{A}_1 \to \mathscr{A}_2$ of $\mathscr{C}$-categories, by the rule :

- $(h, a) \mapsto (h, h^*a)$ for every $(h, a) \in \mathrm{Ob}(\mathscr{A}_1)$.
- $(f_1, f_2, t) \mapsto (f_1, f_2, h^*t)$ for every morphism (f_1, f_2, t) of $\mathscr{A}_1$ as above. Notice that, if $t : a \to f_2^* a'$ is a morphism in $\varphi^{-1}Y$, then $h^*t : h^*a \to h^*f_2^*a' = f_1^*h'^*a'$ is a morphism of $\varphi^{-1}Y'$, since φ is a split fibration.

Notice that a morphism (f_1, f_2, t) of either $\mathscr{A}_1$ or $\mathscr{A}_2$ is cartesian if and only if t is an isomorphism; especially, it is clear that F is a cartesian functor. Moreover, for every object $h : X \to Y$ of $\mathscr{C}$, the restriction $\varphi_1^{-1}h \to \varphi_2^{-1}h$ of F is isomorphic to the pull-back functor $h^* : \varphi^{-1}Y \to \varphi^{-1}h$. Especially, conditions (b)–(d) say that the restriction $F_i : \varphi_1^{-1}g_i \to \varphi_2^{-1}g_i$ (resp. $F_{ij} : \varphi_1^{-1}g_{ij} \to \varphi_2^{-1}g_{ij}$, resp. $F_{ijk} : \varphi_1^{-1}g_{ijk} \to \varphi_2^{-1}g_{ijk}$) are n-faithful (resp. $(n-1)$-faithful, resp. $(n-2)$-faithful). In light of theorem 1.5.30(i), we are then reduced to showing

Claim 1.5.36. $\mathscr{S}/g$ is a sieve both of φ_1-n-descent and of φ_2-n-descent.

Proof of the claim. Let $\mathscr{D}$ be any (small) category; we remark first that a functor $\mathscr{D} \to \mathscr{A}_1$ is the same as a pair of functors $(H : \mathscr{D} \to \mathscr{A}, K : \mathscr{D} \to \mathscr{C})$ such that $\varphi \circ H = \mathsf{t} \circ K$, and likewise one can describe the functors $\mathscr{D} \to \mathscr{A}_2$. Then, it is easily seen that the functors

$$\mathsf{Cart}_{\mathscr{B}}(\mathscr{B}/B, \mathscr{A}) \to \mathsf{Cart}_{\mathscr{C}}(\mathscr{C}/g, \mathscr{A}_1) \qquad G \mapsto (G \circ \mathsf{t}, \mathsf{t})$$
$$\mathsf{Cart}_{\mathscr{B}}(\mathscr{B}/B', \mathscr{A}) \to \mathsf{Cart}_{\mathscr{C}}(\mathscr{C}/g, \mathscr{A}_2) \qquad G \mapsto (G \circ \mathsf{s}, \mathsf{s})$$

are equivalences, and induce equivalences

$$\mathsf{Cart}_{\mathscr{B}}(\mathscr{S}, \mathscr{A}) \to \mathsf{Cart}_{\mathscr{C}}(\mathscr{S}/g, \mathscr{A}_1)$$
$$\mathsf{Cart}_{\mathscr{B}}(\mathscr{S} \times_B g, \mathscr{A}) \to \mathsf{Cart}_{\mathscr{C}}(\mathscr{S}/g, \mathscr{A}_2)$$

(details left to the reader). The claim follows immediately. □

1.5.37. Quite generally, if $\varphi : \mathscr{A} \to \mathscr{B}$ is a fibration over a category $\mathscr{B}$ that admits fibre products, the descent data for φ (relative to a fixed cleavage c) also form a fibration :

$$\mathsf{D}\varphi : \varphi\text{-}\mathsf{Desc} \to \mathsf{Morph}(\mathscr{B}).$$

Namely, for every morphism $f : T' \to T$ of $\mathscr{B}$, the fibre over f is the category $\mathsf{Desc}(\varphi, f)$ of all descent data (f, A, ξ) relative to the family $\{f\}$, and the cleavage c, so A is an object of $\varphi^{-1}T'$ and $\xi : p_1^*A \xrightarrow{\sim} p_2^*A$ is an isomorphism in the category $\varphi^{-1}(T' \times_T T')$ satisfying the usual cocycle condition (here $p_1, p_2 : T' \times_T T' \to T'$ denote the two natural morphisms). Given two objects $\underline{A} := (f : T' \to T, A, \xi)$, $\underline{A}' := (g : W' \to W, A', \zeta)$ of φ-Desc, the morphisms $\underline{A} \to \underline{A}'$ are the data (h, α) consisting of a commutative diagram :

$$\begin{tikzcd} W' \arrow[r, "h"] \arrow[d, "g"'] & T' \arrow[d, "f"] \\ W \arrow[r] & T \end{tikzcd}$$

and a morphism $\alpha : A \to A'$ such that $\varphi(\alpha) = h$ and $p_1^*(\alpha) \circ \xi = \zeta \circ p_2^*(\alpha)$.

We have a natural cartesian functor of fibrations :

$$\begin{tikzcd} \mathscr{A} \times_{(\varphi,\mathsf{t})} \mathsf{Morph}(\mathscr{B}) \arrow[rr, "\mathsf{d}"] \arrow[dr, "\mathsf{p}"'] & & \varphi\text{-}\mathsf{Desc} \arrow[dl, "\mathsf{D}\varphi"] \\ & \mathsf{Morph}(\mathscr{B}) & \end{tikzcd}$$

where $\mathsf{t} : \mathsf{Morph}(\mathscr{B}) \to \mathscr{B}$ is the target functor (see (1.1.17) and example 1.1.27(ii)). Namely, to any pair $(T, f : S \to \varphi T)$ with $T \in \mathrm{Ob}(\mathscr{A})$ and $f \in \mathrm{Ob}(\mathsf{Morph}(\mathscr{B}))$, one assigns the canonical descent datum $\mathsf{d}(T, f) := \rho_{\{f\}}(T)$ in $\mathsf{Desc}(\varphi, f)$ associated to the pair as in (1.5.28).

Corollary 1.5.38. *In the situation of* (1.5.37)*, let $f : B' \to B$ be a morphism of $\mathscr{B}$, and $\mathscr{S}$ a sieve of $\mathscr{B}/B$, generated by a family $(S_i \to B \mid i \in I)$. Let $n \in \{0, 1, 2\}$, and suppose that :*

(a) *$\mathscr{S}$ is a sieve of universal φ-n-descent.*
(b) *For every $i \in I$, the morphism $S_i \times_B f$ is of φ-n-descent.*
(c) *For every $i, j \in I$, the morphism $S_{ij} \times_B f$ is of φ-$(n-1)$-descent.*
(d) *For every $i, j, k \in I$, the morphism $S_{ijk} \times_B f$ is of φ-$(n-2)$-descent.*

Then f is a morphism of φ-n-descent.

Proof. In view of corollary 1.5.35(i), it is easily seen that a morphism $g : T' \to T$ in $\mathscr{B}$ is of φ-n-descent if and only if the restriction $\varphi^{-1}T \to \mathsf{D}\varphi^{-1}g$ of d is n-faithful. Set $\mathscr{C} := \mathsf{Morph}(\mathscr{B})$; as in the proof of corollary 1.5.35(ii), the functor t induces a functor $\mathsf{t}_{|f} : \mathscr{C}/f \to \mathscr{B}/B$, and we let $\mathscr{S}/f := \mathsf{t}_{|f}^{-1}\mathscr{S}$. With this notation, theorem 1.5.30(i) reduces to showing :

Claim 1.5.39. The sieve $\mathscr{S}/f$ is both of p-n-descent and of $\mathsf{D}\varphi$-n-descent.

Proof of the claim. By claim 1.5.36, it is already known that $\mathscr{S}/f$ is of p-n-descent. To show that $\mathscr{S}/f$ is of $\mathsf{D}\varphi$-n-descent, we consider the commutative diagram

$$\begin{tikzcd} \mathsf{Cart}_{\mathscr{B}}(\mathscr{B}/B', \mathscr{A}) \arrow[r] \arrow[d] & \mathsf{Cart}_{\mathscr{C}}(\mathscr{C}/f, \varphi\text{-}\mathsf{Desc}) \arrow[d] \\ \mathsf{Cart}_{\mathscr{B}}(\mathscr{S} \times_B f, \mathscr{A}) \arrow[r] & \mathsf{Cart}_{\mathscr{C}}(\mathscr{S}/f, \varphi\text{-}\mathsf{Desc}) \end{tikzcd}$$

whose left (resp. right) vertical arrow is induced by the inclusion $\mathscr{S} \times_B f \to \mathscr{B}/B'$ (resp. $\mathscr{S}/f \to \mathscr{S} \times_B f$) and whose top horizontal arrow is defined as follows. Given a cartesian functor $G : \mathscr{B}/B' \to \mathscr{A}$, we let $\mathsf{D}G : \mathscr{C}/f \to \varphi$-$\mathsf{Desc}$ be the unique cartesian functor determined on the objects of $\mathscr{C}/f$ by the rule :

$$\left(\begin{tikzcd} T' \arrow[r, "g"] \arrow[d, "h"'] & T \arrow[d] \\ B' \arrow[r, "f"] & B \end{tikzcd}\right) \mapsto \mathsf{d}(G(h), g).$$

We leave to the reader the verification the rule $G \mapsto \mathsf{D}G$ extends to a well defined functor, and then there exists a unique (similarly defined) bottom horizontal arrow that makes commute the foregoing diagram. Moreover, both horizontal arrow thus obtained are equivalences of categories. The claim follows. □

1.6. **Profinite groups and Galois categories.** Quite generally, for any profinite group P, let P-$\mathbf{Set}$ denote the category of discrete finite sets, endowed with a continuous left action of P (the morphisms in P-$\mathbf{Set}$ are the P-equivariant maps). Any continuous group homomorphism $\omega : P \to Q$ of profinite groups induces a *restriction functor*

$$\mathrm{Res}(\omega) : Q\text{-}\mathbf{Set} \to P\text{-}\mathbf{Set}$$

in the obvious way. In case the notation is not ambiguous, one writes also Res^P_Q for this functor.

For any two profinite groups P and Q, we denote by

$$\mathrm{Hom}_{\mathrm{cont}}(P, Q)$$

the set of all continuous group homomorphisms $P \to Q$. If φ_1, φ_2 are two such group homomorphisms, we say that φ_1 is *conjugate* to φ_2, and we write $\varphi_1 \sim \varphi_2$, if there exists an inner automorphism ω of G, such that $\varphi_2 = \omega \circ \varphi_1$. Clearly the trivial map $\pi \to G$ (whose image is the neutral element of G), is the unique element of a distinguished conjugacy class.

1.6.1. Let P be any profinite group; for any (discrete) finite group G, consider the pointed set $\mathrm{Hom}_{\mathrm{cont}}(P, G)/\!\sim$ of conjugacy classes of continuous group homomorphisms $P \to G$. This is also denoted

$$H^1_{\mathrm{cont}}(P, G)$$

and called the first *non-abelian continuous cohomology group* of P with coefficients in G (so G is regarded as a P-module with trivial P-action). Clearly the formation of $H^1(P, G)$ is covariant on the argument G, and controvariant for continuous homomorphisms of profinite groups.

Lemma 1.6.2. *Let $\varphi : P \to P'$ be a continuous homomorphism of profinite groups, and suppose that the induced map of pointed sets :*

$$H^1_{\mathrm{cont}}(P', G) \to H^1_{\mathrm{cont}}(P, G) \quad : \quad f \mapsto f \circ \varphi$$

is bijective, for every finite group G. Then φ is an isomorphism of topological groups.

Proof. First we show that φ is injective. Indeed, let $x \in P$ be any element; we may find an open normal subgroup $H \subset P$ such that $x \notin H$; taking $G := P/H$, we deduce that the projection $P \to P/H$ factors through φ and a group homomorphism $f : P' \to P/H$, hence $x \notin \mathrm{Ker}\,\varphi$, as claimed. Moreover, let $H' := \mathrm{Ker}\,f$; clearly $H' \cap P = H$, so the topology of P is induced from that of P'. It remains only to show that φ is surjective; to this aim, we consider any continuous surjection $f' : P' \to G'$ onto a finite group, and it suffices to show that the restriction of f' to φP is still surjective. Indeed, let G be the image of φP in G', denote by $i : G \to G'$ the inclusion map, and let $f : P \to G$ be the unique continuous map such that $i \circ f = f' \circ \varphi$; by assumption, there exists a continuous group homomorphism $g : P' \to G$ such that $f = g \circ \varphi$. On the other hand, $(i \circ g) \circ \varphi = f' \circ \varphi$, hence the conjugacy class of $i \circ g$ equals the conjugacy class of f', especially $i \circ g$ is surjective, hence the same holds for i, as required. $\square$

1.6.3. Let G be a profinite group, and $H \subset G$ an open subgroup. It is easily seen that H is also a profinite group, with the topology induced from G. Morever, the restriction functor

$$\mathrm{Res}^H_G : G\text{-}\mathbf{Set} \to H\text{-}\mathbf{Set}$$

admits a left adjoint

$$\mathrm{Ind}^G_H : H\text{-}\mathbf{Set} \to G\text{-}\mathbf{Set}.$$

Namely, to any finite set Σ with a continuous left action of H, one assigns the set $\mathrm{Ind}^G_H\Sigma := G \times \Sigma/\!\sim$, where $\sim$ is the equivalence relation such that

$$(gh, \sigma) \sim (g, h\sigma) \qquad \text{for every } g \in G,\ h \in H \text{ and } \sigma \in \Sigma.$$

The left G-action on $\mathrm{Ind}^G_H\Sigma$ is given by the rule : $(g', (g, \sigma)) \mapsto (g'g, \sigma)$ for every $g, g' \in G$ and $\sigma \in \Sigma$. It is easily seen that this action is continuous, and the reader may check that the functor Ind^G_H is indeed left adjoint to Res^H_G.

1.6.4. Moreover, let 1 denote the final object of H-$\mathbf{Set}$; notice that $\mathrm{Ind}^G_H 1 = G/H$, the set of orbits of G under its right translation action by H. Hence, for any finite set Σ with continuous H-action, the unique map $t_\Sigma : \Sigma \to 1$ yields a G-equivariant map

$$\mathrm{Ind}^G_H t_\Sigma : \mathrm{Ind}^G_H \to G/H$$

and therefore Ind^G_H factors through a functor

$$H\text{-}\mathbf{Set} \to G\text{-}\mathbf{Set}/(G/H). \tag{1.6.5}$$

It is easily seen that (1.6.5) is an equivalence. Indeed, one obtains a natural quasi-inverse, by the rule : $(f : \Sigma \to G/H) \mapsto f^{-1}(H)$. The detailed verification shall be left to the reader.

Definition 1.6.6. ([42, Exp.V, Def.5.1]) Let $\mathscr{C}$ be a category, and $F : \mathscr{C} \to \mathbf{Set}$ a functor.

(i) We say that $\mathscr{C}$ is a *Galois category*, if $\mathscr{C}$ is equivalent to P-$\mathbf{Set}$, for some profinite group P. We denote $\mathbf{Galois}$ the category whose objects are all the Galois categories, and whose morphisms are the exact functors between Galois categories.

(ii) We say that F is a *fibre functor*, if F is exact and conservative, and $F(X)$ is a finite set for every $X \in \mathrm{Ob}(\mathscr{C})$.

(iii) We denote $\mathbf{fibre.Fun}$ the 2-*category of fibre functors*, defined as follows :
  - (a) The objects are all the pairs $(\mathscr{C}, F)$ consisting of a Galois category $\mathscr{C}$ and a fibre functor F for $\mathscr{C}$.
  - (b) The 1-cells $(\mathscr{C}_1, F_1) \to (\mathscr{C}_2, F_2)$ are all the pairs (G, β) consisting of an exact functor $G : \mathscr{C}_1 \to \mathscr{C}_2$ and an isomorphism of functors $\beta : F_1 \xrightarrow{\sim} F_2 \circ G$.
  - (c) And for every pair of 1-cells $(G', \beta'), (G, \beta) : (\mathscr{C}_1, F_1) \to (\mathscr{C}_2, F_2)$, the 2-cells $(G', \beta') \to (G, \beta)$ are the isomorphisms $\gamma : G' \xrightarrow{\sim} G$ such that $(F_2 * \gamma) \circ \beta' = \beta$.

Composition of 1-cells and 2-cells is defined in the obvious way. We shall also denote simply by G a 1-cell (G, β) as in (b), such that $F_1 = F_2 \circ G$ and β is the identity automorphism of F_1.

1.6.7. Notice that any Galois category $\mathscr{C}$ admits a fibre functor : indeed, if P is any profinite group, the forgetful functor

$$\mathsf{f}_P : P\text{-}\mathbf{Set} \to \mathbf{Set}$$

fulfills the conditions of definition 1.6.6(ii), therefore the same holds for the functor $\mathsf{f}_P \circ \beta$, if $\beta : \mathscr{C} \to P\text{-}\mathbf{Set}$ is any equivalence. For any Galois category $\mathscr{C}$ and any fibre functor $F : \mathscr{C} \to \mathbf{Set}$, we denote

$$\pi_1(\mathscr{C}, F)$$

the group of automorphisms of F, and we call it the *fundamental group of $\mathscr{C}$ pointed at F*. By definition, for every $X \in \mathrm{Ob}(\mathscr{C})$, the finite set $F(X)$ is endowed with a natural left action of $\pi_1(\mathscr{C}, F)$. For every $X \in \mathrm{Ob}(\mathscr{C})$ and every $\xi \in F(X)$, the stabilizer $H_{X,\xi} \subset \pi_1(\mathscr{C}, F)$ is a subgroup of finite index, and we endow $\pi_1(\mathscr{C}, F)$ with the coarsest group topology for which all such $H_{X,\xi}$ are open subgroups. The resulting topological group $\pi_1(\mathscr{C}, F)$ is profinite, and its natural left action on every $F(X)$ is continuous. Thus, F upgrades to a functor denoted

$$F^\dagger : \mathscr{C} \xrightarrow{\sim} \pi_1(\mathscr{C}, F)\text{-}\mathbf{Set}.$$

A basic result states that $F^\dagger$ is an equivalence ([42, Exp.V, Th.4.1]).

Example 1.6.8. Let P be any profinite group. Then there is an obvious injective map

$$P \to \pi_1(P\text{-}\mathbf{Set}, \mathsf{f}_P)$$

and [42, Exp.V, Th.4.1] implies that this map is an isomorphism of profinite groups. In other words, the group P can be recovered, up to unique isomorphism, from the category P-**Set** together with its forgetful functor f_P.

Remark 1.6.9. Let $\mathscr{C}, \mathscr{C}'$ be two Galois categories, and $F : \mathscr{C} \to \mathbf{Set}$ a fibre functor.

(i) Any exact functor $G : \mathscr{C}' \to \mathscr{C}$ induces a continuous group homomorphism :

$$\pi_1(G) : \pi_1(\mathscr{C}, F) \to \pi_1(\mathscr{C}', F \circ G) \qquad \omega \mapsto \omega * G.$$

(ii) Furthermore, any isomorphism $\beta : F' \xrightarrow{\sim} F$ of fibre functors of $\mathscr{C}$ induces an isomorphism of profinite groups :

$$\pi_1(\beta) : \pi_1(\mathscr{C}', F) \xrightarrow{\sim} \pi_1(\mathscr{C}', F) \qquad \omega \mapsto \beta^{-1} \circ \omega \circ \beta$$

(see [42, Exp.V, §4] for all these generalities).

(iii) Let now $(G, \beta) : (\mathscr{C}_1, F_1) \to (\mathscr{C}_2, F_2)$ be a 1-cell of $\mathbf{fibre}.\mathbf{Fun}$. Combining (i) and (ii), we deduce a natural continuous group homomorphism

$$\pi_1(G, \beta) : \pi_1(\mathscr{C}_2, F_2) \xrightarrow{\pi_1(G)} \pi_1(\mathscr{C}_1, F_2 \circ G) \xrightarrow{\pi_1(\beta)} \pi_1(\mathscr{C}_1, F_1).$$

Proposition 1.6.10. *With the notation of remark* 1.6.9*, the rule that assigns :*

- *To any object $(\mathscr{C}, F)$ of $\mathbf{fibre}.\mathbf{Fun}$, the profinite group $\pi_1(\mathscr{C}, F)$*
- *To any 1-cell (G, β) of $\mathbf{fibre}.\mathbf{Fun}$, the continuous map $\pi_1(G, \beta)$*

defines a pseudo-functor

$$\pi_1 : \mathbf{fibre}.\mathbf{Fun} \to \mathbf{pf}.\mathbf{Grp}^o$$

from the 2-category of fibre functors, to the opposite of the category of profinite groups (and continuous group homomorphisms).

Proof. (Here we regard $\mathbf{pf.Grp}$ as a 2-category with trivial 2-cells : see example 1.3.6(i)). Let

$$(\mathscr{C}_1, F_1) \xrightarrow{(G,\beta_G)} (\mathscr{C}_2, F_2) \xrightarrow{(H,\beta_H)} (\mathscr{C}_3, F_3)$$

be any pair of (composable) 1-cells; functoriality on 1-cells amounts to the identity :

$$\pi_1(G, \beta_G) \circ \pi_1(H, \beta_H) = \pi_1(H \circ G, (\beta_H * G) \circ \beta_G)$$

whose detailed verification we leave to the reader. Next, let $\gamma : (G', \beta') \to (G, \beta)$ be a 2-cell between 1-cells $(G', \beta'), (G, \beta) : (\mathscr{C}_1, F_1) \to (\mathscr{C}_2, F_2)$; we have to check that $\pi_1(G, \beta) = \pi_1(G', \beta')$. This identity boils down to the commutativity of the diagram :

$$\begin{tikzcd} \pi_1(\mathscr{C}_2, F_2) \arrow[r, "\pi_1(G')"] \arrow[d, "\pi_1(G)"'] & \pi_1(\mathscr{C}_1, F_2 \circ G') \arrow[d, "\pi_1(\beta')"] \\ \pi_1(\mathscr{C}_2, F_2 \circ G) \arrow[ur, "\pi_1(F_2*\gamma)"] \arrow[r, "\pi_1(\beta)"] & \pi_1(\mathscr{C}_1, F_1). \end{tikzcd}$$

However, the commutativity of the lower triangular subdiagram is clear, hence we are reduced to checking the commutativity of the upper triangular subdiagram; the latter is a special case of the following more general :

Claim 1.6.11. Let $\mathscr{C}$ and $\mathscr{C}'$ be two Galois categories, $G, G' : \mathscr{C}' \to \mathscr{C}$ two exact functors, $\beta : G' \xrightarrow{\sim} G$ an isomorphism, and $F : \mathscr{C} \to \mathbf{Set}$ a fibre functor. Then the induced diagram of profinite groups

$$\begin{tikzcd} & \pi_1(\mathscr{C}, F) \arrow[dl, "\pi_1(G)"'] \arrow[dr, "\pi_1(G')"] & \\ \pi_1(\mathscr{C}', F \circ G) \arrow[rr, "\pi_1(F*\beta)"] & & \pi_1(\mathscr{C}', F \circ G') \end{tikzcd}$$

commutes.

Proof of the claim. Left to the reader. ☐

Example 1.6.12. Let $\omega : P \to Q$ be a continuous group homomorphism between profinite groups. Clearly $\mathsf{f}_P \circ \mathrm{Res}(\omega) = \mathsf{f}_Q$, and it is easily seen that the resulting diagram

$$\begin{tikzcd} P \arrow[r, "\omega"] \arrow[d] & Q \arrow[d] \\ \pi_1(P\text{-}\mathbf{Set}, \mathsf{f}_P) \arrow[r, "\pi_1(\mathrm{Res}(\omega))"] & \pi_1(Q\text{-}\mathbf{Set}, \mathsf{f}_Q) \end{tikzcd}$$

commutes, where the vertical arrows are the natural identifications given by example 1.6.8 : the verification is left as an exercise to the reader.

1.6.13. Let $\underline{P} := (P_i \mid i \in I)$ be a cofiltered system of profinite groups, with continuous transition maps, and denote by P the limit of this system, in the category of groups. Then P is naturally a closed subgroup of $Q := \prod_{i \in I} P_i$, and the topology $\mathscr{T}$ induced by the inclusion map $P \to Q$ makes it into a compact and complete topological group. Moreover, since the topology of Q is profinite, the same holds for the topology $\mathscr{T}$ (details left to the reader). It is then easily seen that the resulting topological group $(P, \mathscr{T})$ is the limit of the system $\underline{P}$ in the category of profinite groups.

Proposition 1.6.14. *In the situation of* (1.6.13), *the natural functor*

$$2\text{-}\operatorname*{colim}_{i \in I} P_i\text{-}\mathbf{Set} \to P\text{-}\mathbf{Set}$$

is an equivalence.

Proof. The functor is obviously faithful; let us show that it is also full. Indeed, let $j \in I$ be any index, Σ, Σ' two objects of P_j-$\mathbf{Set}$, and $\varphi : \Sigma \to \Sigma'$ a P-equivariant map; we need to show that φ is already P_i-equivariant, for some index $i \in I$. To this aim, we may as usual replace I by I/j, and assume that j is the final element of I. We may also find an open normal subgroup $H_j \subset P_j$ that acts trivially on both Σ and Σ'. Then, for every $i \in I$, we let $H_i \subset P_i$ be the preimage of P_j, and we set $\overline{P}_i := P_i/H_i$. Let also $\overline{P} := P/H$, where $H \subset P$ is the preimage of H_j; by construction, φ is $\overline{P}$-equivariant. Clearly, we may find $i \in I$ such that the image of $\overline{P}$ in the finite group $\overline{P}_j$ equals the image of $\overline{P}_i$, and for such index i, the induced map $\overline{P} \to \overline{P}_i$ is an isomorphism. Especially, φ is P_i-equivariant, as sought.

Lastly, we show essential surjectivity. Indeed, let Σ be any object of P-$\mathbf{Set}$; we have to show that the P-action on Σ is the restriction of a continuous P_i-action, for a suitable $i \in I$. However, we may find a normal open subgroup $H \subset P$ that acts trivially on Σ. Then there exists a normal open subgroup $L \subset Q$ (notation of (1.6.13)) such that $P \cap L \subset H$. We may also assume that there exists a finite subset $J \subset I$ and for every $i \in J$ an open normal subgroup $L_i \subset P_i$ such that $L = \prod_{i \in J} L_i \times \prod_{i \in I \setminus J} P_i$. Pick an index $j \in I$ that admits morphisms $f_i : j \to i$ in I, for every $i \in J$, and let $L'_i \subset P_j$ denote the preimage of L_i under the corresponding map $P_j \to P_i$. Finally, set $H_j := \bigcap_{i \in J} L'_i$. By construction, H contains the preimage of H_j in P, and we may therefore assume that H is this preimage. We may replace as usual I by I/j, and assume that j is the final element of I. Then, for every $i \in I$, we let H_i denote the preimage of H_j in P_i, and we set $\overline{P}_i := P_i/H_i$. Set as well $\overline{P} := P/H$. Clearly, there exists $i \in I$ such that the image of the induced map $\overline{P} \to \overline{P}_j$ equals the image of $\overline{P}_i$; for such index i, the induced map $\overline{P} \to \overline{P}_i$ is an isomorphism. Thus, Σ the restriction of an object of P_i-$\mathbf{Set}$, as wished. $\square$

1.6.15. We consider now a situation that generalizes slightly that of (1.6.13). Namely, let I be a small filtered category, and

$$(\mathscr{C}_\bullet, F_\bullet) : I \to \mathbf{fibre.Fun} \qquad i \mapsto (\mathscr{C}_i, F_i)$$

a pseudo-functor. By proposition 1.6.10, the composition of π_1 and $(\mathscr{C}_\bullet, F_\bullet)$ is a functor

$$\pi_1(\mathscr{C}_\bullet, F_\bullet) : I^o \to \mathbf{pf.Grp} \qquad i \mapsto P_i := \pi_1(\mathscr{C}_i, F_i).$$

Let P denote the limit (in $\mathbf{pf.Grp}$) of the cofiltered system $P_\bullet$, and set

$$\mathscr{C} := 2\text{-}\operatorname*{colim}_I \mathscr{C}_\bullet$$

where the 2-colimit is formed in the 2-category of small categories. We may then state :

Corollary 1.6.16. *In the situation of* (1.6.15), *there exists a natural equivalence :*

$$\mathscr{C} \xrightarrow{\sim} P\text{-}\mathbf{Set}.$$

Proof. Recall that $(\mathscr{C}_\bullet, F_\bullet)$ is the datum of isomorphisms

$$\beta_\varphi : F_j \xrightarrow{\sim} F_i \circ \mathscr{C}_\varphi \qquad \text{for every morphism } \varphi : j \to i \text{ in } I$$

and 2-cells :

$$\tau_{\psi,\varphi} : (\mathscr{C}_{\psi\circ\varphi}, \beta_{\psi\circ\varphi}) \xrightarrow{\sim} (\mathscr{C}_\psi, \beta_\psi) \circ (\mathscr{C}_\varphi, \beta_\varphi) \qquad \text{for every composition } j \xrightarrow{\varphi} i \xrightarrow{\psi} k$$

that – by definition – satisfy the identities :

$$(\beta_\psi * \mathscr{C}_\varphi) \circ \beta_\varphi = (F_k * \tau_{\psi,\varphi}) \circ \beta_{\psi\circ\varphi} \qquad \text{for every composition } j \xrightarrow{\varphi} i \xrightarrow{\psi} k \tag{1.6.17}$$

as well as the composition identities :

$$((\mathscr{C}_\mu, \beta_\mu) * \tau_{\psi,\varphi}) \circ \tau_{\mu,\psi\circ\varphi} = (\tau_{\mu,\psi} * (\mathscr{C}_\varphi, \beta_\varphi)) \circ \tau_{\mu\circ\psi,\varphi} \qquad \text{for compositions } j \xrightarrow{\varphi} i \xrightarrow{\psi} k \xrightarrow{\mu} l.$$

Let $P_\bullet$-$\mathbf{Set} : I \to \mathbf{Cat}$ denote the functor given by the rule : $i \mapsto P_i$-$\mathbf{Set}$ for every $i \in \mathrm{Ob}(I)$, and $\varphi \mapsto \mathrm{Res}(P_\varphi)$, where $P_\varphi := \pi_1(\mathscr{C}_\varphi, \beta_\varphi)$ for every morphism φ of I. In view of proposition

1.6.14 and lemma 1.3.14, it suffices to show that the rule : $i \mapsto F_i^\dagger$ for every $i \in \mathrm{Ob}(I)$ (notation of (1.6.7)), extends to a pseudo-natural isomorphism

$$F_\bullet^\dagger : \mathscr{C}_\bullet \xrightarrow{\sim} P_\bullet\text{-}\mathbf{Set}.$$

Indeed, let $\varphi : j \to i$ be any morphism of I, and X any object of $\mathscr{C}_j$; we remark that the bijection

$$\beta_\varphi(X) : \mathrm{Res}(P_\varphi)(F_j^\dagger X) \xrightarrow{\sim} F_i^\dagger \circ \mathscr{C}_\varphi(X)$$

is P_i-equivariant; the proof amounts to unwinding the definitions, and shall be left to the reader. Hence we get an isomorphism of functors $\beta_\varphi^\dagger : \mathrm{Res}(P_\varphi) \circ F_j^\dagger \xrightarrow{\sim} F_i^\dagger \circ \mathscr{C}_\varphi$, and from (1.6.17) we deduce the identities :

$$(\beta_\psi^\dagger * \mathscr{C}_\varphi) \circ (\mathrm{Res}(P_\varphi) * \beta_\varphi^\dagger) = (F_k^\dagger * \tau_{\psi,\varphi}) \circ \beta_{\psi\circ\varphi}^\dagger \qquad \text{for every composition } j \xrightarrow{\varphi} i \xrightarrow{\psi} k.$$

The latter show that the system $\beta_\bullet^\dagger$ fulfills the coherence axiom for a pseudo-natural transformation, as required. □

Remark 1.6.18. (i) Keep the situation of (1.6.15), and let $a_\bullet : \mathscr{C}_\bullet \Rightarrow \mathscr{C}$ be the universal pseudo-cocone induced by the pseudo-functor $\mathscr{C}_\bullet$. We may regard the pseudo-functor $(\mathscr{C}_\bullet, F_\bullet)$ as a pseudo-cocone $F_\bullet : \mathscr{C}_\bullet \Rightarrow \mathbf{Set}$ whose vertex is the category $\mathbf{Set}$. Then, by the universal property of colimits, we get a functor $F : \mathscr{C} \to \mathbf{Set}$ and an isomorphism

$$\sigma_\bullet : F * a_\bullet \xrightarrow{\sim} F_\bullet.$$

On the other hand, let $r_\bullet : P_\bullet\text{-}\mathbf{Set} \Rightarrow P\text{-}\mathbf{Set}$ be the natural cocone (so r_i is the restriction functor corresponding to the natural map $P \to P_i$, for every $i \in \mathrm{Ob}(I)$); the equivalence G of corollary 1.6.16 is deduced from the pseudo-cocone $r_\bullet \circ F_\bullet^\dagger : \mathscr{C}_\bullet \to P\text{-}\mathbf{Set}$ (where $F_\bullet^\dagger$ is as in the proof of corollary 1.6.16), so we have an isomorphism of pseudo-functors

$$t_\bullet : G * a_\bullet \xrightarrow{\sim} r_\bullet \circ F_\bullet^\dagger$$

whence an isomorphism

$$\mathsf{f}_P * t_\bullet : (\mathsf{f}_P \circ G) * a_\bullet \xrightarrow{\sim} \mathsf{f}_P * (r_\bullet \circ F_\bullet^\dagger) = F_\bullet$$

(notation of (1.6.7)). There follows an isomorphism $F * a_\bullet \xrightarrow{\sim} (\mathsf{f}_P \circ G) * a_\bullet$; by the universal property of the 2-colimit, the latter must come from a unique isomorphism $\vartheta : F \xrightarrow{\sim} \mathsf{f}_P \circ G$. Especially, we see that F is also a fibre functor, and we get a pseudo-cocone

$$(a_\bullet, \sigma_\bullet) : (\mathscr{C}_\bullet, F_\bullet) \Rightarrow (\mathscr{C}, F).$$

It is now immediate that $(\mathscr{C}, F)$ is the 2-colimit (in the 2-category $\mathbf{fibre}.\mathbf{Fun}$) of the pseudo-functor $(\mathscr{C}_\bullet, F_\bullet)$, and $(a_\bullet, \sigma_\bullet)$ is the corresponding universal pseudo-cocone.

(ii) Likewise, $r_\bullet$ may be regarded as a universal pseudo-cocone

$$r_\bullet : (P_\bullet\text{-}\mathbf{Set}, \mathsf{f}_{P_\bullet}) \Rightarrow (P\text{-}\mathbf{Set}, \mathsf{f}_P)$$

(with trivial coherence constraint), and the coherence constraint $\beta_\bullet^\dagger$ as in the proof of corollary 1.6.16 yields a pseudo-natural equivalence

$$F_\bullet^\dagger : (\mathscr{C}_\bullet, F_\bullet) \xrightarrow{\sim} (P_\bullet\text{-}\mathbf{Set}, \mathsf{f}_{P_\bullet})$$

as well as an isomorphism

$$(G, \vartheta) * (a_\bullet, \sigma_\bullet) \xrightarrow{\sim} r_\bullet \circ F_\bullet^\dagger.$$

Thus, for every $i \in \mathrm{Ob}(I)$ we get a 2-cell of $\mathbf{fibre}.\mathbf{Fun}$:

$$(G, \vartheta) \circ (a_i, \sigma_i) \to r_i \circ F_i^\dagger$$

whence – by proposition 1.6.10 – a commutative diagram of profinite groups :

$$\begin{tikzcd} P \arrow[rr] \arrow[d, "\pi_1(G,\vartheta)"'] & & P_i \arrow[d, "\pi_1(F_i^\dagger)"] \\ \pi_1(\mathscr{C},F) \arrow[rr, "\pi_1(a_i,\sigma_i)"] & & \pi_1(\mathscr{C}_i,F_i). \end{tikzcd}$$

1.6.19. Let $(\mathscr{C},F)$ be a fibre functor, and X a connected object of $\mathscr{C}$ (example 1.1.26(iii)); pick any $\xi \in F(X)$, and let $H_\xi \subset \pi_1(\mathscr{C},F)$ be the stabilizer of ξ for the natural left action of $\pi_1(\mathscr{C},F)$ on $F(X)$. For every object $f : Y \to X$ of $\mathscr{C}/X$, we set

$$F_\xi(f) := F(f)^{-1}(\xi) \subset F(Y).$$

It is clear that the rule $f \mapsto F_\xi(f)$ yields a functor $F_\xi^\dagger : \mathscr{C}/X \to H_\xi\text{-}\mathbf{Set}$, which we call the *subfunctor of* $F_{|X}$ *selected by* ξ.

Proposition 1.6.20. *In the situation of* (1.6.19), *we have :*

(i) $\mathscr{C}/X$ *is also a Galois category, and* $F_\xi := \mathsf{f}_{H_\xi} \circ F_\xi^\dagger$ *is a fibre functor for* $\mathscr{C}/X$.

(ii) *The functor* $F_\xi^\dagger$ *induces a natural isomorphism of profinite groups :*

$$\pi_1(F_\xi^\dagger) : H_\xi \stackrel{\sim}{\to} \pi_1(\mathscr{C}/X, F_\xi).$$

Proof. The fibre functor F induces an equivalence of categories

$$\mathscr{C}/X \stackrel{\sim}{\to} \pi_1(\mathscr{C},F)\text{-}\mathbf{Set}/F(X).$$

On the other hand, since X is connected, there exists a unique isomorphism $\omega : F(X) \stackrel{\sim}{\to} G/H_\xi$ of G-sets such that $\omega(\xi) = H_\xi$, and then the discussion of (1.6.4) yields an equivalence

$$\pi_1(\mathscr{C},F)\text{-}\mathbf{Set}/F(X) \stackrel{\sim}{\to} H_\xi\text{-}\mathbf{Set}.$$

A simple inspection shows that the resulting equivalence $\mathscr{C}/X \stackrel{\sim}{\to} H_\xi\text{-}\mathbf{Set}$ is none else than the functor $F_\xi^\dagger$, so the assertion follows from remark 1.6.9(i) and example 1.6.8. ☐

1.6.21. Let $(\mathscr{C},F)$ be a fibre functor, and let us now fix a cleavage $\mathsf{c} : \mathscr{C}^o \to \mathbf{Cat}$ for the fibred category $\mathsf{t} : \mathsf{Morph}(\mathscr{C}) \to \mathscr{C}$ (see example 1.4.2(iii)). Also, let I be a small cofiltered category, and $X_\bullet : I \to \mathscr{C}$ a functor such that X_i is a connected object of $\mathscr{C}$, for every $i \in \mathrm{Ob}(I)$; we pick an element

$$\xi_\bullet \in \lim_I F \circ X_\bullet.$$

In other words, $\xi_\bullet := (\xi_i \in F(X_i) \mid i \in I)$ is a compatible system of elements such that

$$F(\varphi)(\xi_j) = \xi_i \qquad \text{for every morphism } \varphi : j \to i \text{ in } I.$$

For every $i \in I$, we denote by $H_i \subset \pi_1(\mathscr{C},F)$ the stabilizer of ξ_i for the left action of $\pi_1(\mathscr{C},F)$ on $F(X_i)$. Clearly, any morphism $j \to i$ induces an inclusion $H_j \subset H_i$. Furthermore, let

$$F_i^\dagger : \mathscr{C}/X_i \to H_i\text{-}\mathbf{Set} \qquad \text{for every } i \in I$$

be the subfunctor selected by ξ_i, and set $F_i := \mathsf{f}_{H_i} \circ F_i^\dagger$. Let $\varphi : j \to i$ be a morphism of I; to the corresponding morphism $X_\varphi : X_j \to X_i$, the cleavage c associates a pull-back functor

$$X_\varphi^* : \mathscr{C}/X_i \to \mathscr{C}/X_j.$$

Especially, for any object $Y \in \mathrm{Ob}(\mathscr{C}/X_i)$ we have the cartesian diagram in $\mathscr{C}$:

$$\begin{tikzcd} X_\varphi^*(Y) \arrow[r] \arrow[d] & Y \arrow[d] \\ X_j \arrow[r, "X_\varphi"] & X_i \end{tikzcd}$$

whence, since F is exact, a natural bijection :

$$F(X_\varphi^*(Y)) \xrightarrow{\sim} F(Y) \times_{F(X_i)} F(X_j)$$

which in turns yields a bijection :

$$F_j^\dagger(X_\varphi^*(Y)) \xrightarrow{\sim} F(Y) \times_{F(X_i)} \{\xi_j\} = F_i^\dagger(Y) \times \{\xi_j\}.$$

That is, we have a natural isomorphism of functors :

$$\alpha_\varphi^\dagger : F_j^\dagger \circ X_\varphi^* \xrightarrow{\sim} \mathrm{Res}_{H_i}^{H_j} \circ F_i^\dagger \qquad (1.6.22)$$

and since the X_φ^* are exact functors, it is easily seen that the isomorphisms $\alpha_\varphi := \mathsf{f}_j * \alpha_\varphi^\dagger$ yield a pseudo-functor

$$I \to \mathbf{fibre}.\mathbf{Fun} \qquad i \mapsto (\mathscr{C}/X_i, F_i) \qquad \varphi \mapsto (X_\varphi^*, \alpha_\varphi). \qquad (1.6.23)$$

Moreover, $\alpha_\varphi^\dagger$ can be seen as a 2-cell of $\mathbf{fibre}.\mathbf{Fun}$: $F_j^\dagger \circ (X_\varphi^*, \alpha_\varphi) \xrightarrow{\sim} \mathrm{Res}_{H_j}^{H_i} \circ F_i^\dagger$, whence a commutative diagram of profinite groups :

$$\begin{tikzcd} H_j \arrow[r] \arrow[d, "\pi_1(F_j^\dagger)"'] & H_i \arrow[d, "\pi_1(F_i^\dagger)"] \\ \pi_1(\mathscr{C}/X_j, F_j) \arrow[r, "\pi_1(X_\varphi^*,\alpha_\varphi)"] & \pi_1(\mathscr{C}/X_i, F_i) \end{tikzcd}$$

whose top horizontal arrow is the inclusion map. Especially, notice that the map $\pi_1(X_\varphi^*, \alpha_\varphi)$ does not depend on the chosen cleavage; this can also be seen by remarking that any two cleavages c, c' are related by a pseudo-natural isomorphism $\mathsf{c} \xrightarrow{\sim} \mathsf{c}'$ (details left to the reader).

1.6.24. Let $(\mathscr{C}/X, F_\xi)$ be the 2-colimit of the pseudo-functor (1.6.23), as in remark 1.6.18(i), and fix a corresponding universal pseudo-cocone $(a_\bullet, \sigma_\bullet) : (\mathscr{C}/X_\bullet, F_\bullet) \Rightarrow (\mathscr{C}/X, F_\xi)$. We may then state :

Corollary 1.6.25. *In the situation of* (1.6.24)*, there exists a natural isomorphism of profinite groups :*

$$H := \bigcap_{i \in \mathrm{Ob}(I)} H_i \xrightarrow{\sim} \pi_1(\mathscr{C}/X, F_\xi)$$

which fits into a commutative diagram :

$$\begin{tikzcd} H \arrow[r] \arrow[d] & H_i \arrow[d, "\pi_1(F_i^\dagger)"] \\ \pi_1(\mathscr{C}/X, F_\xi) \arrow[r, "\pi_1(a_i,\sigma_i)"] & \pi_1(\mathscr{C}/X_i, F_i) \end{tikzcd} \qquad \textit{for every } i \in \mathrm{Ob}(I) \qquad (1.6.26)$$

whose top horizontal arrow is the inclusion map.

Proof. By corollary 1.6.16, we have an isomorphism of $\pi_1(\mathscr{C}/X, F_\xi)$ with the limit of the cofiltered system $(\pi_1(\mathscr{C}/X, F_i) \mid i \in \mathrm{Ob}(I))$; on the other hand, the discussion of (1.6.21) shows that the latter system is naturally isomorphic to the system $(H_i \mid i \in \mathrm{Ob}(I))$. Lastly, the commutativity of (1.6.26) follows from remark 1.6.18(ii). $\square$

2. Sites and topoi

In this chapter, we assemble some generalities concerning sites and topoi. The main reference for this material is [3]. As in the previous sections, we fix a universe U, and small means U-small throughout. Especially, a presheaf on any category takes its values in U, unless explicitly stated otherwise.

2.1. **Topologies and sites.** Let $\mathscr{C}$ be a small category; we wish to begin with a closer investigation of the category $\mathscr{C}^\wedge$ of presheaves on $\mathscr{C}$, introduced in (1.1.19). First, we remark that $\mathscr{C}^\wedge$ is complete and cocomplete, and for every $X \in \mathrm{Ob}(\mathscr{C})$, the functor

$$\mathscr{C}^\wedge \to \mathbf{Set} \quad : \quad F \mapsto F(X)$$

commutes with all limits and all colimits (in other words, the limits and colimits in $\mathscr{C}^\wedge$ are computed argumentwise) : see [10, Th.2.5.14 and Cor.2.15.4]. As a corollary, a morphism in $\mathscr{C}^\wedge$ is an isomorphism if and only if it is both a monomorphism and an epimorphism, since the same holds in the category $\mathbf{Set}$. Likewise, a morphism of presheaves $F \to G$ is a monomorphism (resp. an epimorphism) if and only if the induced map of sets $F(X) \to G(X)$ is injective (resp. surjective) for every $X \in \mathrm{Ob}(\mathscr{C})$: see [10, Cor.2.15.3]. Furthermore, the filtered colimits in $\mathscr{C}^\wedge$ commute with all finite limits, again because the same holds in $\mathbf{Set}$ ([10, Th.2.13.4]). For the same reason, all colimits and all epimorphisms are universal in $\mathscr{C}^\wedge$ (see example 1.1.24(v,vii)).

It is also easily seen that $\mathscr{C}^\wedge$ is well-powered : indeed, for every presheaf F on $\mathscr{C}$, and every $X \in \mathrm{Ob}(\mathscr{C})$, the set of subsets of $F(X)$ is small, and a subobject of F is just a compatible system of subsets $F'(X) \subset F(X)$, for X ranging over the small set of objects of $\mathscr{C}$. Likewise one sees that $\mathscr{C}^\wedge$ is co-well-powered.

Hence, for every morphism $f : F \to G$ in $\mathscr{C}^\wedge$, the image of f is well defined (see example 1.1.24(viii)); more concretely, $\mathrm{Im}(f) \subset G$ is the subobject defined by the rule :

$$X \mapsto \mathrm{Im}(F(X) \to G(X)) \qquad \text{for every } X \in \mathrm{Ob}(\mathscr{C}).$$

Denote by $\{*\}$ a final object of $\mathbf{Set}$ (*i.e.* any choice of a set with a single element); the initial (resp. final) object of $\mathscr{C}^\wedge$ is the presheaf $\varnothing_{\mathscr{C}}$ (resp. $1_{\mathscr{C}}$) such that $\varnothing_{\mathscr{C}}(X) = \varnothing$ (resp. $1_{\mathscr{C}}(X) = \{*\}$) for every $X \in \mathrm{Ob}(\mathscr{C})$.

Lemma 2.1.1. *Let $\mathscr{C}$ be a small category, F a presheaf on $\mathscr{C}$. We have a natural isomorphism:*

$$\operatorname*{colim}_{h\mathscr{C}/F} h/F \xrightarrow{\sim} \mathbf{1}_F \qquad \textit{in the category } \mathscr{C}^\wedge/F$$

where $h : \mathscr{C} \to \mathscr{C}^\wedge_{\mathsf{U}}$ is the Yoneda embedding (notation of (1.1.16) *and* (1.1.19)*).*

Proof. To begin with, notice that, under the current assumptions, $h\mathscr{C}/F$ is a small category. Let $\mathsf{s} : \mathscr{C}^\wedge/F \to \mathscr{C}^\wedge$ be the source functor as in (1.1.13); let G be any presheaf on $\mathscr{C}$; according to (1.1.30) we have a natural bijection :

$$\mathrm{Hom}_{\mathscr{C}^\wedge}(\operatorname*{colim}_{h\mathscr{C}/F} \mathsf{s} \circ h/F, G) \xrightarrow{\sim} S := \lim_{(h\mathscr{C}/F)^o} \mathrm{Hom}_{\mathscr{C}^\wedge}(\mathsf{s} \circ h/F, G).$$

By inspecting the definitions, we see that the elements of S are in natural bijection with the compatible systems of the form $(f_\sigma \in G(X) \mid X \in \mathrm{Ob}(\mathscr{C}), \sigma \in F(X))$, such that $(G\psi)(f_\sigma) = f_{(F\psi)(\sigma)}$ for every morphism $\psi : X' \to X$ in $\mathscr{C}$. Such a system defines a unique natural transformation $F \Rightarrow G$, hence F and the above colimit represent the same presheaf on the category $(\mathscr{C}^\wedge_{\mathsf{U}'})^o$, and the assertion follows. $\square$

As a special case of lemma 2.1.1, consider any object X of a small category $\mathscr{C}$, any sieve $\mathscr{S}$ of $\mathscr{C}/X$, and take $F := h_{\mathscr{S}}$ (notation of (1.5.3)); recalling the isomorphism of categories $h\mathscr{C}/F \xrightarrow{\sim} \mathscr{S}$, we deduce a natural isomorphism in $\mathscr{C}^\wedge$:

$$\operatorname*{colim}_{\mathscr{S}} h \circ \mathsf{s} \xrightarrow{\sim} h_{\mathscr{S}} \tag{2.1.2}$$

where $\mathsf{s} : \mathscr{S} \to \mathscr{C}$ is the restriction of the functor (1.1.13).

Proposition 2.1.3. *In the situation of* (1.1.35), *let F be any presheaf on $\mathscr{B}$. We have :*

(i) *$f_! F$ is naturally isomorphic to the presheaf on $\mathscr{C}$ given by the rule :*

$$(2.1.4) \qquad Y \mapsto \operatorname*{colim}_{(Y/f\mathscr{B})^o} F \circ \iota^o_Y \qquad \varphi \mapsto \big(\operatorname*{colim}_{(\varphi/f\mathscr{B})^o} \mathbf{1}_{\mathscr{B}} : \operatorname*{colim}_{(Y/f\mathscr{B})^o} F \circ \iota^o_Y \to \operatorname*{colim}_{(Y'/f\mathscr{B})^o} F \circ \iota^o_{Y'}\big)$$

for every $Y \in \mathrm{Ob}(\mathscr{C})$ and every morphism $\varphi : Y' \to Y$ in $\mathscr{C}$ (notation of (1.1.16)*).*

(ii) *$f_* F$ is naturally isomorphic to the presheaf on $\mathscr{C}$ given by the rule :*

$$Y \mapsto \lim_{(f\mathscr{B}/Y)^o} F \circ \iota^o_Y \qquad \varphi \mapsto \big(\lim_{(f\mathscr{B}/\varphi)^o} \mathbf{1}_{\mathscr{B}} : \lim_{(f\mathscr{B}/Y)^o} F \circ \iota^o_Y \to \lim_{(f\mathscr{B}/Y')^o} F \circ \iota^o_{Y'}\big)$$

for every $Y \in \mathrm{Ob}(\mathscr{C})$ and every morphism $\varphi : Y' \to Y$ in $\mathscr{C}$.

(iii) *If all the finite limits of $\mathscr{B}$ are representable, and f is left exact, then the functor $f_!$ is left exact.*

Proof. (i), (ii): The above expressions are derived directly from the proof of proposition 1.1.34.

(iii): Under these assumptions, the category $Y/f\mathscr{B}$ is cofiltered for every $Y \in \mathrm{Ob}(\mathscr{C})$, hence $(Y/f\mathscr{B})^o$ is filtered. However, the filtered colimits in the category $\mathbf{Set}$ commute with all finite limits, so the assertion follows from (i). $\square$

2.1.5. In the situation of (1.1.35), let V be a universe such that U $\subset$ V. As a first corollary of proposition 2.1.3(i,ii) we get an essentially commutative diagram of categories

$$\begin{tikzcd}
\mathscr{C}^\wedge_{\mathsf{U}} \arrow[d] & \mathscr{B}^\wedge_{\mathsf{U}} \arrow[l, "f_{\mathsf{U}*}"'] \arrow[r, "f_{\mathsf{U}!}"] \arrow[d] & \mathscr{C}^\wedge_{\mathsf{U}} \arrow[d] \\
\mathscr{C}^\wedge_{\mathsf{U}} & \mathscr{B}^\wedge_{\mathsf{V}} \arrow[l, "f_{\mathsf{V}*}"'] \arrow[r, "f_{\mathsf{V}!}"] & \mathscr{C}^\wedge_{\mathsf{V}}
\end{tikzcd}$$

whose vertical arrows are the inclusion functors. Let us remark :

Lemma 2.1.6. *Let $f : \mathscr{B} \to \mathscr{C}$ be a functor between small categories. We have :*

(i) *f is fully faithful if and only if the same holds for $f_!$, if and only if the same holds for f_*.*

(ii) *Suppose that f admits a right adjoint $g : \mathscr{C} \to \mathscr{B}$. Then there are natural isomorphisms of functors :*

$$f^* \stackrel{\sim}{\to} g_! \qquad f_* \stackrel{\sim}{\to} g^*.$$

Proof. (i): By proposition 1.1.11(iii), $f_!$ is fully faithful if and only if the same holds for f_*.

Now, suppose that f is fully faithful. We have to show that the unit of adjunction $F \to f^* f_! F$ is an isomorphism, for every $F \in \mathrm{Ob}(\mathscr{B}^\wedge)$ (proposition 1.1.11(ii)). Since both f^* and $f_!$ commute with arbitrary colimits, lemma 2.1.1 reduces to the case where $F = h_Y$ for some $Y \in \mathrm{Ob}(\mathscr{B})$ In this case, taking into account (1.1.36), we see that the unit of adjunction is the map given by the composition :

$$h_Y(Z) = \mathrm{Hom}_{\mathscr{B}}(Z, Y) \xrightarrow{\omega} \mathrm{Hom}_{\mathscr{C}}(fZ, fY) = (f^* h_{fY})(Z) \qquad \text{for every } Z \in \mathrm{Ob}(\mathscr{B})$$

where ω is the map given by f, which is a bijective by assumption, whence the claim.

Conversely, if $f_!$ is fully faithful, then (1.1.36) and the full faithfulness of the Yoneda imbeddings, imply that f is fully faithful.

(ii): It suffices to show the first stated isomorphism, since the second one will follow by adjunction. However, let $X \in \mathrm{Ob}(\mathscr{B})$, $F \in \mathrm{Ob}(\mathscr{C}^\wedge)$, and let $s : h\mathscr{C}/F \to \mathscr{C}$ be the functor given by the rule : $(h_Y \to F) \mapsto Y$. Taking into account lemma 2.1.1 and (1.1.36), we may compute :

$$\begin{aligned} f^*F(X) = F(fX) &\xrightarrow{\sim} \operatorname*{colim}_{h\mathscr{C}/F} \mathrm{Hom}_{\mathscr{C}}(f(X), s) \\ &\xrightarrow{\sim} \operatorname*{colim}_{h\mathscr{C}/F} \mathrm{Hom}_{\mathscr{B}}(X, g \circ s) \\ &\xrightarrow{\sim} \operatorname*{colim}_{h\mathscr{C}/F} (g_! \circ h \circ s)(X) \\ &\xrightarrow{\sim} g_! F(X) \end{aligned}$$

whence the contention. □

Example 2.1.7. (i) Let $\mathscr{C}$ be a category, X any object of $\mathscr{C}$, denote by $\iota_X : \mathscr{C}/X \to \mathscr{C}$ the functor of (1.1.13), and suppose that $\mathscr{C}$ is V-small, for some universe V containing U. By inspecting (2.1.4) we obtain a natural isomorphism :

$$(2.1.8) \qquad (\iota_X)_{\mathsf{V}!}F(Y) \xrightarrow{\sim} \{(a, \varphi) \mid \varphi \in \mathrm{Hom}_{\mathscr{C}}(Y, X),\ a \in F(\varphi)\}$$

for every V-presheaf F on $\mathscr{C}/X$ and every $Y \in \mathrm{Ob}(\mathscr{C})$. If $\psi : Z \to Y$ is any morphism of $\mathscr{C}$, the corresponding map $(\iota_X)_{\mathsf{V}!}F(Y) \to (\iota_X)_{\mathsf{V}!}F(Z)$ is given by the rule :

$$(2.1.9) \qquad (a, \varphi) \mapsto (F(\psi)(a), \varphi \circ \psi) \qquad \text{for every } (a, \varphi) \in (\iota_X)_{\mathsf{V}!}F(Y).$$

(ii) Especially, if $\mathscr{C}$ has small Hom-sets and F is a U-presheaf on $\mathscr{C}/X$, then we see that $(\iota_X)_{\mathsf{V}!}F$ is a U-presheaf. Since the inclusion $(\mathscr{C}/X)^\wedge_{\mathsf{U}} \to (\mathscr{C}/X)^\wedge_{\mathsf{V}}$ is fully faithful, we deduce that the target of (2.1.8) can be used to define a left adjoint $(\iota_X)_{\mathsf{U}!} : (\mathscr{C}/X)^\wedge_{\mathsf{U}} \to \mathscr{C}^\wedge_{\mathsf{U}}$ to $(\iota_X)^*_{\mathsf{U}}$. (As usual the latter shall often be denoted just $\iota_{X!}$). We also deduce from (2.1.8) that $\iota_{X!}$ transforms monomorphisms to monomorphisms, and more generally, that it commutes with fibre products. On the other hand, it does not generally preserve final objects, hence it is not generally exact.

(iii) More precisely, (2.1.8) yields a natural isomorphism :

$$\iota_{X!}(1_{\mathscr{C}/X}) \xrightarrow{\sim} h_X.$$

It follows that $\iota_{X!}$ is the composition of a functor

$$e_X : (\mathscr{C}/X)^\wedge \to \mathscr{C}^\wedge/h_X$$

and the functor $\iota_{h_X} : \mathscr{C}^\wedge/h_X \to \mathscr{C}^\wedge$. Let F be any presheaf on $\mathscr{C}/X$; in view of (2.1.9), we see that the corresponding morphism $e_X(F) : \iota_{X!}F \to h_X$ is given by the rule : $(a, \varphi) \mapsto \varphi$ for every $Y \in \mathrm{Ob}(\mathscr{C})$ and every $(a, \varphi) \in \iota_{X!}F(Y)$. We claim that e_X is an equivalence. Indeed, let G be another presheaf on $\mathscr{C}/X$, and $f : e_X(F) \to e_X(G)$ a morphism in $\mathscr{C}^\wedge/h_X$; the foregoing description of $e_X(F)$ shows that f is the datum of a system of maps $f_\varphi : F(\varphi) \to G(\varphi)$, for $\varphi : Y \to X$ ranging over the objects of $\mathscr{C}/X$, subject to the condition that $f_{\varphi\circ\psi} \circ F(\psi) = G(\psi) \circ f_\varphi$ for every morphism $\psi : Z \to Y$ of X-objects. Such a datum is obviously nothing else than a morphism $F \to G$ in $(\mathscr{C}/X)^\wedge$, so this shows already that e_X is fully faithful.

Next, the datum of a morphism $g : F \to h_X$ in $\mathscr{C}^\wedge$ amounts to a compatible system of partitions $F(Y) = \bigcup_\varphi F(Y)_\varphi$, for every $Y \in \mathrm{Ob}(\mathscr{C})$, with φ ranging over $\mathrm{Hom}_{\mathscr{C}}(Y, X)$; namely, $F(Y)_\varphi := g(Y)^{-1}(\varphi)$ for every such φ; the rule $\varphi \mapsto F(\iota_X(Y))_\varphi$ then defines a presheaf G on $\mathscr{C}/X$ with a natural isomorphism $\iota_X(G) \xrightarrow{\sim} F$, all of which shows that e_X is essentially surjective, as claimed. The quasi-inverse just constructed, can be described more compactly as the functor that assigns to $g : F \to h_X$ the presheaf given by the rule :

$$(Y \xrightarrow{\varphi} X) \mapsto \mathrm{Hom}_{\mathscr{C}^\wedge/h_X}((h_Y \xrightarrow{h_\varphi} h_X), g).$$

Definition 2.1.10. Let $\mathscr{C}$ be a category.

(i) A *topology* on $\mathscr{C}$ is the datum, for every $X \in \mathrm{Ob}(\mathscr{C})$, of a set $J(X)$ of sieves of $\mathscr{C}/X$, fulfilling the following conditions :

(a) (Stability under base change) For every morphism $f : Y \to X$ of $\mathscr{C}$, and every $\mathscr{S} \in J(X)$, the sieve $\mathscr{S} \times_X f$ lies in $J(Y)$.
(b) (Local character) Let X be any object of $\mathscr{C}$, and $\mathscr{S}$, $\mathscr{S}'$ two sieves of $\mathscr{C}/X$, with $\mathscr{S} \in J(X)$. Suppose that, for every object $f : Y \to X$ of $\mathscr{S}$, the sieve $\mathscr{S}' \times_X f$ lies in $J(Y)$. Then $\mathscr{S}' \in J(X)$.
(c) For every $X \in \mathrm{Ob}(\mathscr{C})$, we have $\mathscr{C}/X \in J(X)$.

(ii) In the situation of (i), the elements of $J(X)$ shall be called the *sieves covering* X. Moreover, say that $\mathscr{S}$ is the sieve of $\mathscr{C}/X$ generated by a family $(f_i : X_i \to X \mid i \in I)$ of morphisms. If $\mathscr{S}$ is a sieve covering X, we say that the family $(f_i \mid i \in I)$ *covers* X, or that it is a *covering family of* X.
(iii) The datum $(\mathscr{C}, J)$ of a category $\mathscr{C}$ and a topology $J := (J(X) \mid X \in \mathrm{Ob}(\mathscr{C}))$ on $\mathscr{C}$ is called a *site*, and then $\mathscr{C}$ is also called the *category underlying* the site $(\mathscr{C}, J)$. We say that $(\mathscr{C}, J)$ is a *small site*, if $\mathscr{C}$ is a small category.
(iv) The set of all topologies on $\mathscr{C}$ is partially ordered by inclusion: given two topologies J_1 and J_2 on $\mathscr{C}$, we say that J_1 is *finer* than J_2, if $J_2(X) \subset J_1(X)$ for every $X \in \mathrm{Ob}(\mathscr{C})$.

Remark 2.1.11. Let $(\mathscr{C}, J)$ be any site.

(i) It is easily seen that any finite intersection of sieves covering an object X, again covers X. Indeed, say that $\mathscr{S}_1$ and $\mathscr{S}_2$ are two sieves covering X; set $\mathscr{S} := \mathscr{S}_1 \cap \mathscr{S}_2$ and let $f : Y \to X$ be any object of $\mathscr{S}_1$. Then $\mathscr{S} \times_X f = \mathscr{S}_2 \times_X f$.

(ii) Also, any sieve of $\mathscr{C}/X$ containing a covering sieve is again a covering sieve. Indeed, if $\mathscr{S} \subset \mathscr{S}'$, then $\mathscr{S}' \times_X f = \mathscr{C}/Y$ for every object $f : Y \to X$ of $\mathscr{S}$.

(iii) Let $(f_i : Y_i \to X \mid i \in I)$ be a family of objects of $\mathscr{C}/X$ that generates a sieve $\mathscr{S}$ covering X, and for every $i \in I$, let $(g_{ij} : Z_{ij} \to Y_i \mid i \in J_i)$ a family of objects of $\mathscr{C}/Y_i$ that generates a sieve $\mathscr{S}_i$ covering Y_i. Then the family $(f_i \circ g_{ij} : Z_{ij} \to X \mid i \in I, \ j \in J_i)$ generates a sieve $\mathscr{S}'$ covering X. Indeed, say that $f : Y \to X$ lies in $\mathscr{S}$, and pick $i \in I$ such that f factors through f_i and a morphism $g : Y \to Y_i$; then it is easily seen that $\mathscr{S}_i \times_{Y_i} g \subset \mathscr{S}' \times_X f$.

Example 2.1.12. Let $F : \mathscr{A} \to \mathscr{B}$ be a fibration between two categories. For every $X \in \mathrm{Ob}(\mathscr{B})$, let $J_F(X)$ denote the set of all sieves $\mathscr{S} \subset \mathscr{B}/X$ of universal F-2-descent. (To make this definition, we have to choose a universe V such that $\mathscr{A}$ and $\mathscr{B}$ are V-small, but clearly the resulting J_F does not depend on V.) Then we claim that J_F is a topology on $\mathscr{B}$. Indeed, it is clear that J_F fulfills conditions (a) and (c) of definition 2.1.10(i). In order to conclude, it suffices therefore to show the following :

Lemma 2.1.13. *In the situation of example* 2.1.12*, let $\mathscr{S}' \subset \mathscr{S}$ be two sieves of $\mathscr{B}/X$, and suppose that, for every object $f : Y \to X$ of $\mathscr{S}$, the sieve $\mathscr{S}' \times_X f$ lies in $J_F(Y)$. Then $\mathscr{S}' \in J_F(X)$ if and only if $\mathscr{S} \in J_F(X)$.*

Proof. Up to replacing U by a larger universe, we may assume that $\mathscr{B}$ and $\mathscr{C}$ are small. Notice then that our assumptions are preserved under any base change $X' \to X$ in $\mathscr{B}$, hence it suffices to show that $\mathscr{S}'$ is a sieve of F-2-descent if and only if the same holds for $\mathscr{S}$. To this aim, for every small category $\mathscr{A}$ we construct a functor

$$\vartheta : \mathsf{Cart}_{\mathscr{B}}(\mathscr{S}', \mathscr{A}) \to 2\text{-}\lim_{\overleftarrow{\mathscr{S}}} \mathsf{Cart}_{\mathscr{B}}(G_{\mathscr{S}}, \mathscr{A})$$

where $G_{\mathscr{S}} : \mathscr{S} \to \mathbf{Cat}/\mathscr{B}$ is the functor introduced in 1.5.10. Indeed, let $\varphi : \mathscr{S}' \to \mathscr{A}$ be a cartesian functor; for every object $f : Y \to X$ of $\mathscr{S}$, denote

$$\mathscr{B}/Y \xleftarrow{j_f} \mathscr{S}' \times_X f \xrightarrow{i_f} \mathscr{S}'$$

the natural functors. By assumption, there exist cartesian functors $\varphi_f : \mathscr{B}/Y \to \mathscr{A}$, and natural isomorphisms

$$\alpha_f : \varphi_f \circ j_f \Rightarrow \varphi \circ i_f$$

for every such f. Moreover, for every morphism $g : Z \to Y$ in $\mathscr{S}$, there exists a unique isomorphism of functors

$$\omega_{f,g} : \varphi_f \circ g_* \Rightarrow \varphi_{f\circ g}$$

fitting into the commutative diagram :

$$\begin{tikzcd}
\varphi_f \circ g_* \circ j_{f\circ g} \arrow[rr, equal] \arrow[d, Rightarrow, "\omega_{f,g} * j_{f\circ g}"'] & & \varphi_f \circ j_f \circ i_g \arrow[d, Rightarrow, "\alpha_f * i_g"] \\
\varphi_{f\circ g} \circ j_{f\circ g} \arrow[r, Rightarrow, "\alpha_{f\circ g}"] & \varphi \circ i_{f\circ g} \arrow[r, equal] & \varphi \circ i_f \circ i_g .
\end{tikzcd}$$

The uniqueness of $\omega_{f,g}$ implies that the datum $(\varphi_f, \omega_{f,g} \mid f \in \mathrm{Ob}(\mathscr{S}), g \in \mathrm{Morph}(\mathscr{S}))$ defines a pseudo-natural transformation $\varphi_\bullet : G_\mathscr{S} \Rightarrow \mathsf{F}_\mathscr{A}$. Moreover, if $\tau : \varphi \Rightarrow \varphi'$ is any natural transformation of cartesian functors $\mathscr{S}' \to \mathscr{A}$, there exists, for every $f \in \mathrm{Ob}(\mathscr{S})$, a unique natural transformation $\tau_f : \varphi_f \Rightarrow \varphi'_f$ fitting into the commutative diagram :

$$\begin{tikzcd}
\varphi_f \circ j_f \arrow[r, Rightarrow, "\tau_f * j_f"] \arrow[d, Rightarrow, "\alpha_f"'] & \varphi'_f \circ j_f \arrow[d, Rightarrow, "\alpha'_f"] \\
\varphi \circ i_f \arrow[r, Rightarrow, "\tau * j_f"] & \varphi' \circ i_f .
\end{tikzcd}$$

In other words, the rule $\varphi \mapsto \varphi_\bullet$ defines a functor ϑ as sought.

Notice next that, if $f : Y \to X$ lies in $\mathrm{Ob}(\mathscr{S}')$, then $\mathscr{S}' \times_X f = \mathscr{B}/Y$; it follows that the restriction to $\mathscr{S}'$ of the functor $G_\mathscr{S}$ agrees with the functor $G_{\mathscr{S}'}$, and this restriction operation induces a functor

$$\rho : 2\text{-}\varprojlim_{\mathscr{S}} \mathsf{Cart}_\mathscr{B}(G_\mathscr{S}, \mathscr{A}) \to 2\text{-}\varprojlim_{\mathscr{S}'} \mathsf{Cart}_\mathscr{B}(G_{\mathscr{S}'}, \mathscr{A}).$$

Summing up, we arrive at the essentially commutative diagram :

$$\begin{tikzcd}
\mathsf{Cart}_\mathscr{B}(\mathscr{S}, \mathscr{A}) \arrow[r, "\mathsf{Cart}_\mathscr{B}(\iota, \mathscr{A})"] \arrow[d, "\omega"'] & \mathsf{Cart}_\mathscr{B}(\mathscr{S}', \mathscr{A}) \arrow[d, "\omega'"] \arrow[dl, "\vartheta"'] \\
2\text{-}\varprojlim\limits_{\mathscr{S}} \mathsf{Cart}_\mathscr{B}(G_\mathscr{S}, \mathscr{A}) \arrow[r, "\rho"] & 2\text{-}\varprojlim\limits_{\mathscr{S}'} \mathsf{Cart}_\mathscr{B}(G_{\mathscr{S}'}, \mathscr{A})
\end{tikzcd}$$

in which $\iota : \mathscr{S}' \to \mathscr{S}$ is the inclusion functor, and where ω and ω' are equivalences deduced from lemma 1.5.11. A little diagram chase shows that $\mathsf{Cart}_\mathscr{B}(\iota, \mathscr{A})$ must then also be an equivalence, whence the contention. $\square$

2.1.14. Suppose now that $\mathscr{C}$ has small Hom-sets. Then, in view of the discussion in (1.5.3), a topology can also be defined by assigning, to any object X of $\mathscr{C}$, a family $J'(X)$ of subobjects of h_X, such that :

- (a) For every $X \in \mathrm{Ob}(\mathscr{C})$, every $R \in J'(X)$, and every morphism $Y \to X$ in $\mathscr{C}$, the fibre product $R \times_X Y$ lies in $J'(Y)$.
- (b) Say that $X \in \mathrm{Ob}(\mathscr{C})$, and let R, R' be two subobjects of h_X, such that $R \in J'(X)$. Suppose that, for every $Y \in \mathrm{Ob}(\mathscr{C})$, and every morphism $f : h_Y \to R$, we have $R' \times_X Y \in J'(Y)$. Then $R' \in J'(X)$.
- (c) $h_X \in J'(X)$ for every $X \in \mathrm{Ob}(\mathscr{C})$.

In this case, the elements of $J'(X)$ are naturally called the *subobjects covering* X. This viewpoint is adopted in the following :

Definition 2.1.15. Let V be a universe, $C := (\mathscr{C}, J)$ be a site, and suppose that the category $\mathscr{C}$ has V-small Hom-sets. Let also $F \in \mathrm{Ob}(\mathscr{C}^\wedge_\mathsf{V})$.

(i) We say that F is a *separated* V-*presheaf* (resp. a V-*sheaf*) on C, if for every $X \in \mathrm{Ob}(\mathscr{C})$ and every subobject R covering X, the induced morphism :

$$F(X) = \mathrm{Hom}_{\mathscr{C}^\wedge_\mathsf{V}}(h_X, F) \to \mathrm{Hom}_{\mathscr{C}^\wedge_\mathsf{V}}(R, F)$$

is injective (resp. is bijective). For $\mathsf{V} = \mathsf{U}$, we shall just say separated presheaf instead of separated U-presheaf, and likewise for sheaves.

(ii) The full subcategory of $\mathscr{C}^\wedge_\mathsf{V}$ consisting of all V-sheaves (resp. all separated presheaves) on C is denoted $C^\sim_\mathsf{V}$ (resp. $C^{\mathrm{sep}}_\mathsf{V}$). For $\mathsf{V} = \mathsf{U}$, this category will be usually denoted simply $C^\sim$ (resp. C^{sep}).

2.1.16. In the situation of definition 2.1.15(i), say that $R = h_\mathscr{S}$ for some sieve $\mathscr{S}$ covering X, and let $(S_i \to X \mid i \in I)$ be a generating family for $\mathscr{S}$. Combining lemma 1.5.7 and examples 1.4.10(iv), 1.5.9(ii), we get a natural isomorphism :

$$\mathrm{Hom}_{\mathscr{C}^\wedge}(R, F) \xrightarrow{\sim} \mathrm{Equal}\left(\prod_{i\in I} \mathsf{Cart}_\mathscr{C}(\mathscr{C}/S_i, \mathscr{A}_F) \rightrightarrows \prod_{(i,j)\in I\times I} \mathsf{Cart}_\mathscr{C}(\mathscr{C}/S_{ij}, \mathscr{A}_F)\right)$$

which – again by example 1.4.10(iv,v,vi) – we may rewrite more simply as :

$$\mathrm{Hom}_{\mathscr{C}^\wedge}(R, F) \xrightarrow{\sim} \mathrm{Equal}\left(\prod_{i\in I} F(S_i) \rightrightarrows \prod_{(i,j)\in I\times I} \mathrm{Hom}_{\mathscr{C}^\wedge}(h_{S_i} \times_{h_X} h_{S_j}, F)\right).$$

The above equalizer can be described explicitly as follows. It consists of all the systems

$$(a_i \mid i \in I) \qquad \text{with } a_i \in F(S_i) \text{ for every } i \in I$$

such that, for every $i, j \in I$, every object $Y \to X$ of $\mathscr{C}/X$, and every pair $(g_i : Y \to S_i, g_j : Y \to S_j)$ of morphisms in $\mathscr{C}/X$, we have :

$$F(g_i)(a_i) = F(g_j)(a_j). \tag{2.1.17}$$

If the fibred product $S_{ij} := S_i \times_X S_j$ is representable in $\mathscr{C}$, the latter expression takes the more familiar form :

$$\mathrm{Hom}_{\mathscr{C}^\wedge}(R, F) \xrightarrow{\sim} \mathrm{Equal}\left(\prod_{i\in I} F(S_i) \rightrightarrows \prod_{(i,j)\in I\times I} F(S_{ij})\right).$$

Remark 2.1.18. Let $\mathscr{C}$ be a category with small Hom-sets.

(i) The arguments in (2.1.16) yield also the following. A presheaf F on $\mathscr{C}$ is separated (resp. is a sheaf) on C, if and only if every covering sieve of C is a sieve of 1-descent (resp. of 2-descent) for the fibration $\varphi_F : \mathscr{A}_F \to \mathscr{C}$ of example 1.4.10(iii).

(ii) Let F be a presheaf on $\mathscr{C}$. We deduce from (i) and example 2.1.12 that the topology $J^F := J_{\varphi_F}$ is the finest on $\mathscr{C}$ for which F is a sheaf. A subobject $R \subset h_X$ (for any $X \in \mathrm{Ob}(\mathscr{C})$) lies in $J^F(X)$ if and only if the natural map $F(X') \to \mathrm{Hom}_{\mathscr{C}^\wedge}(R \times_X X', F)$ is bijective for every morphism $X' \to X$ in $\mathscr{C}$.

(iii) More generally, if $(F_i \mid i \in I)$ is any family of presheaves on $\mathscr{C}$, we see that there exists a finest topology for which each F_i is a sheaf : namely, the intersection of the topologies J^{F_i} as in (ii).

(iv) As an important special case, we deduce the existence of a finest topology J on $\mathscr{C}$ such that all representable presheaves are sheaves on $(\mathscr{C}, J)$. This topology is called the *canonical topology* on $\mathscr{C}$. The foregoing shows that a sieve of $\mathscr{C}/X$ is universal strict epimorphic (see example 1.5.13) if and only if it covers X in the canonical topology of $\mathscr{C}$.

2.1.19. Suppose furthermore, that $\mathscr{C}$ is small; then, directly from definition 2.1.15 (and from (1.1.29)), we see that the category $C^\sim$ is complete, and the fully faithful inclusion $C^\sim \to \mathscr{C}^\wedge$ commutes with all limits; moreover, given a presheaf F on $\mathscr{C}$, it is possible to construct a solution set for F relative to this functor, and therefore one may apply theorem 1.1.32 to produce a left adjoint. However, a more direct and explicit construction of the left adjoint can be given; the latter also provides some additional information which is hard to extract from the former method. Namely, we have :

Theorem 2.1.20. *In the situation of* (2.1.19), *the following holds :*

(i) *The inclusion functor* $i : C^\sim \to \mathscr{C}^\wedge$ *admits a left adjoint*

$$\mathscr{C}^\wedge \to C^\sim \qquad F \mapsto F^a. \tag{2.1.21}$$

For every presheaf F, *we call* F^a *the* sheaf associated to F.

(ii) *Morever,* (2.1.21) *is an exact functor.*

Proof. We begin with the following :

Claim 2.1.22. Let F be a separated presheaf on $\mathscr{C}$, and $\mathscr{S}_1 \subset \mathscr{S}_2$ two sieves covering some $X \in \mathrm{Ob}(\mathscr{C})$. Then the natural map $\mathrm{Hom}_{\mathscr{C}^\wedge}(h_{\mathscr{S}_2}, F) \to \mathrm{Hom}_{\mathscr{C}^\wedge}(h_{\mathscr{S}_1}, F)$ is injective.

Proof of the claim. We may find a generating family $(f_i : S_i \to X \mid i \in I_2)$ for $\mathscr{S}_2$, and a subset $I_1 \subset I_2$, such that $(f_i \mid i \in I_1)$ generates $\mathscr{S}_1$. Let $s, s' : h_{\mathscr{S}_2} \to F$, whose images agree in $\mathrm{Hom}_{\mathscr{C}^\wedge}(h_{\mathscr{S}_1}, F)$. By (2.1.16), s and s' correspond to families $(s_i \mid i \in I_2)$, $(s'_i \mid i \in I_2)$ with $s_i, s'_i \in F(S_i)$ for every $i \in I_2$, fulfilling the system of identities (2.1.17), and the foregoing condition means that $s_i = s'_i$ for every $i \in I_1$. We need to show that $s_i = s'_i$ for every $i \in I_2$. Hence, let $i \in I_2$ be any element; by assumption, the natural map

$$F(S_i) \to \mathrm{Hom}_{\mathscr{C}^\wedge}(h_{\mathscr{S}_1} \times_X S_i, F)$$

is injective. However, the objects of $\mathscr{S}_1 \times_X f_i$ are all the morphisms $g_i : Y \to S_i$ in $\mathscr{C}$ such that $f_i \circ g_i = f_j \circ g_j$ for some $j \in I_1$ and some $g_j : Y \to S_j$ in $\mathscr{C}$. If we apply the identities (2.1.17) to these maps g_i, g_j, we deduce that :

$$F(g_i)(s_i) = F(g_j)(s_j) = F(g_j)(s'_j) = F(g_i)(s'_i).$$

In other words, s_i and s'_i have the same image in $\mathrm{Hom}_{\mathscr{C}^\wedge}(h_{\mathscr{S}_1} \times_X S_i, F)$, hence they agree, as claimed. ◊

Next, for every $X \in \mathrm{Ob}(\mathscr{C})$, denote by $\mathbf{J}(X)$ the full subcategory of $\mathbf{Cat}/(\mathscr{C}/X)$ such that $\mathrm{Ob}(\mathbf{J}(X)) = J(X)$. Notice that $\mathbf{J}(X)$ is small and cofiltered, for every such X. Define a functor $\mathsf{h} : \mathbf{J}(X) \to \mathscr{C}^\wedge$ by the rule : $\mathscr{S} \mapsto h_{\mathscr{S}}$ for every $\mathscr{S} \in J(X)$; to an inclusion of sieves $\mathscr{S}' \subset \mathscr{S}$ there corresponds the natural monomorphism $h_{\mathscr{S}'} \to h_{\mathscr{S}}$ of subobjects of h_X.

For a given presheaf F on $\mathscr{C}$, set

$$F^+(X) := \operatorname*{colim}_{\mathbf{J}(X)^o} \mathrm{Hom}_{\mathscr{C}^\wedge}(\mathsf{h}^o, F).$$

For a morphism $f : Y \to X$ in $\mathscr{C}$ and a sieve $\mathscr{S} \in J(X)$, the natural projection $h_{\mathscr{S}} \times_X Y \to h_{\mathscr{S}}$ induces a map :

$$\mathrm{Hom}_{\mathscr{C}^\wedge}(h_{\mathscr{S}}, F) \to \mathrm{Hom}_{\mathscr{C}^\wedge}(h_{\mathscr{S}} \times_X Y, F)$$

whence a map $F^+(X) \to F^+(Y)$, after taking colimits. In other words, we have a functor :

$$\mathscr{C}^\wedge \to \mathscr{C}^\wedge \qquad F \mapsto F^+. \tag{2.1.23}$$

with a natural transformation $F(X) \to F^+(X)$, since $\mathscr{C}/X \in J(X)$ for every $X \in \mathrm{Ob}(\mathscr{C})$.

Claim 2.1.24. (i) The functor (2.1.23) is left exact.

(ii) For every $F \in \mathrm{Ob}(\mathscr{C}^\wedge)$, the presheaf F^+ is separated.

(iii) If F is a separated presheaf on $\mathscr{C}$, then F^+ is a sheaf on $C^\sim$.

Proof of the claim. (i) is clear, since $\mathbf{J}(X)^o$ is filtered for every $X \in \mathrm{Ob}(\mathscr{C})$.

(ii): Let $s, s' \in F^+(X)$, and suppose that the images of s and s' agree in $\mathrm{Hom}_{\mathscr{C}^\wedge}(h_{\mathscr{S}}, F^+)$, for some $\mathscr{S} \in J(X)$. We may find a sieve $\mathscr{T} \in J(X)$ such that s and s' come from elements $\overline{s}, \overline{s}' \in \mathrm{Hom}_{\mathscr{C}^\wedge}(h_{\mathscr{T}}, F)$. Let $(g_i : S_i \to X \mid i \in I)$ be a family of generators for $\mathscr{S}$; in view of (2.1.16), the images of s and s' agree in $F^+(S_i)$ for every $i \in I$. The latter means that, for every $i \in I$, there exists $\mathscr{S}_i \in J(S_i)$, refining $\mathscr{T} \times_X g_i$, such that the images of $\overline{s}$ and $\overline{s}'$ agree in $\mathrm{Hom}_{\mathscr{C}^\wedge}(h_{\mathscr{S}_i}, F)$. For every $i \in I$, let $(g_{i\lambda} : T_{i\lambda} \to S_i \mid \lambda \in \Lambda_i)$ be a family of generators for $\mathscr{S}_i$, and consider the sieve $\mathscr{T}'$ of $\mathscr{C}/X$ generated by $(g_i \circ g_{i\lambda} : T_{i\lambda} \to X \mid i \in I, \lambda \in \Lambda_i)$. Then $\mathscr{T}'$ covers X (remark 2.1.11(iii)) and refines $\mathscr{T}$, and the images of $\overline{s}$ and $\overline{s}'$ agree in $\mathrm{Hom}_{\mathscr{C}^\wedge}(h_{\mathscr{T}'}, F)$ (as one sees easily, again by virtue of (2.1.16)). This shows that $s = s'$, whence the contention.

(iii): In view of (ii), it suffices to show that the natural map $F^+(X) \to \mathrm{Hom}_{\mathscr{C}^\wedge}(h_{\mathscr{S}}, F^+)$ is surjective for every $\mathscr{S} \in J(X)$. Hence, say that $s \in \mathrm{Hom}_{\mathscr{C}^\wedge}(h_{\mathscr{S}}, F^+)$, and let $(S_i \mid i \in I)$ be a generating family for $\mathscr{S}$. By (2.1.16), s corresponds to a system $(s_i \in F^+(S_i) \mid i \in I)$ such that the following holds. For every $i, j \in I$, and every pair of morphisms $u_i : Y \to S_i$ and $u_j : Y \to S_j$ in $\mathscr{C}/X$, we have

$$F^+(u_i)(s_i) = F^+(u_j)(s_j). \tag{2.1.25}$$

For every $i \in I$, let $\mathscr{S}_i \in J(S_i)$ such that s_i is the image of some $\overline{s}_i \in \mathrm{Hom}_{\mathscr{C}^\wedge}(h_{\mathscr{S}_i}, F)$. For every u_i, u_j as above, set $\mathscr{S}_{ij} := (\mathscr{S}_i \times_{S_i} u_i) \cap (\mathscr{S}_j \times_{S_j} u_j)$; since F is separated, (2.1.25) and claim 2.1.22 imply that the images of $\overline{s}_i$ and $\overline{s}_j$ agree in $\mathrm{Hom}_{\mathscr{C}^\wedge}(h_{\mathscr{S}_{ij}}, F)$, for every $i, j \in I$.

However, for every $i \in I$, let $(g_{i\lambda} : T_{i\lambda} \to S_i \mid \lambda \in \Lambda_i)$ be a generating family for $\mathscr{S}_i$; then $\overline{s}_i$ corresponds to a compatible system of sections $\overline{s}_{i\lambda} \in F(T_{i\lambda})$, and $\mathscr{S}_{ij}$ is the sieve of all morphisms $f : Z \to Y$ such that

$$u_i \circ f = g_{i\lambda} \circ f'_i \qquad \text{and} \qquad u_j \circ f = g_{j\mu} \circ f'_j$$

for some $\lambda \in \Lambda_i$, $\mu \in \Lambda_j$ and some $f'_i : Z \to T_{i\lambda}$, $f'_j : Z \to T_{j\mu}$, so by construction we have

$$F(f'_i)(\overline{s}_{i\lambda}) = F(f'_j)(\overline{s}_{j\mu}) \qquad \text{for every } i, j \in I \text{ and } \lambda \in \Lambda_i,\ \mu \in \Lambda_\mu. \tag{2.1.26}$$

Finally, let $\mathscr{T}$ be the sieve of $\mathscr{C}/X$ generated by $(g_i \circ g_{i\lambda} : T_{i\lambda} \to X \mid i \in I, \lambda \in \Lambda_i)$; then $\mathscr{T}$ covers X (remark 2.1.11(iii)), and (2.1.26) shows that the system $(F(g_{i\lambda})(\overline{s}_{i\lambda}) \mid i \in I, \lambda \in \Lambda_i)$ defines an element of $\mathrm{Hom}_{\mathscr{C}^\wedge}(h_{\mathscr{T}}, F)$, whose image in $F^+(X)$ agrees with s. ◊

From claim 2.1.24 we see that the rule : $F \mapsto F^a := (F^+)^+$ defines a left exact functor $\mathscr{C}^\wedge \to C^\sim$, with natural transformations $\eta_F : F \Rightarrow i(F^a)$ for every $F \in \mathrm{Ob}(\mathscr{C}^\wedge)$ and $\varepsilon_G : (iG)^a \Rightarrow G$ for every $G \in \mathrm{Ob}(C^\sim)$ fulfilling the triangular identities of (1.1.8). The theorem follows. □

Remark 2.1.27. Let $C := (\mathscr{C}, J)$ be a small site.

(i) It has already been remarked that $C^\sim$ is complete, and from theorem 2.1.20 we also deduce that $C^\sim$ is cocomplete, and the functor (2.1.21) (resp. the inclusion functor $i : C^\sim \to \mathscr{C}^\wedge$) commutes with all colimits (resp. with all limits); more precisely, if $F : I \to C^\sim$ is any functor from a small category I, we have a natural isomorphism in $C^\sim$ (resp. in $\mathscr{C}^\wedge$) :

$$\operatorname*{colim}_I F \xrightarrow{\sim} (\operatorname*{colim}_I i \circ F)^a \qquad (\text{resp. } \ i(\lim_I F) \xrightarrow{\sim} \lim_I i \circ F).$$

Especially, limits in $C^\sim$ are computed argumentwise (see (1.1.31)). Moreover, it follows that all colimits and all epimorphisms are universal in $C^\sim$ (see example 1.1.24(v,vii)), and filtered colimits in $C^\sim$ commute with finite limits, since the same holds in $\mathscr{C}^\wedge$.

(ii) Furthermore, $C^\sim$ is well-powered and co-well-powered, since the same holds for $\mathscr{C}^\wedge$. Especially, every morphism $f : F \to G$ in $C^\sim$ admits a well defined image (example 1.1.24(viii)). Such an image can be constructed explicitly as the subobject $(\mathrm{Im}\, i(f))^a$ (details left to the reader).

(iii) By composing with the Yoneda embedding, we obtain a functor
$$h^a : \mathscr{C} \to C^\sim \quad : \quad X \mapsto (h_X)^a \qquad \text{for every } X \in \mathrm{Ob}(\mathscr{C})$$
and lemma 2.1.1 yields a natural isomorphism :
$$\operatorname*{colim}_{h\mathscr{C}/F} h^a_X \xrightarrow{\sim} F \qquad \text{for every sheaf } F \text{ on } C.$$
(iv) From the proof of claim 2.1.24 it is clear that the functor
$$\mathscr{C}^\wedge \to C^{\mathrm{sep}} \qquad F \mapsto F^{\mathrm{sep}} := \mathrm{Im}(F \to F^+)$$
is left adjoint to the inclusion $C^{\mathrm{sep}} \to \mathscr{C}^\wedge$. Moreover, we have a natural identification :
$$F^a \xrightarrow{\sim} (F^{\mathrm{sep}})^+ \qquad \text{for every } F \in \mathrm{Ob}(\mathscr{C}^\wedge).$$
(v) Let V be a universe such that $\mathsf{U} \subset \mathsf{V}$; from the definitions, it is clear that the fully faithful inclusion $\mathscr{C}^\wedge_{\mathsf{U}} \subset \mathscr{C}^\wedge_{\mathsf{V}}$ restricts to a fully faithful inclusion
$$C^\sim_{\mathsf{U}} \subset C^\sim_{\mathsf{V}}.$$
Moreover, by inspecting the proof of theorem 2.1.20, we deduce an essentially commutative diagram of categories :
$$\begin{tikzcd} \mathscr{C}^\wedge_{\mathsf{U}} \arrow[r] \arrow[d] & C^\sim_{\mathsf{U}} \arrow[d] \\ \mathscr{C}^\wedge_{\mathsf{V}} \arrow[r] & C^\sim_{\mathsf{V}} \end{tikzcd}$$
whose vertical arows are the inclusion functors, and whose horizontal arrows are the functors $F \mapsto F^a$.

In practice, one often encounters sites that are not small, but which share many of the properties of small sites. These more general situations are encompassed by the following :

Definition 2.1.28. Let $C := (\mathscr{C}, J)$ be a site.

(i) A *topologically generating family* for C is a subset $G \subset \mathrm{Ob}(\mathscr{C})$, such that, for every $X \in \mathrm{Ob}(\mathscr{C})$, the family
$$G/X := \bigcup_{Y \in G} \mathrm{Hom}_{\mathscr{C}}(Y, X) \subset \mathrm{Ob}(\mathscr{C}/X)$$
generates a sieve covering X.

(ii) We say that C is a U-*site*, if $\mathscr{C}$ has small Hom-sets, and C admits a small topologically generating family. In this case, we also say that J is a U-*topology* on $\mathscr{C}$.

2.1.29. Let $C =: (\mathscr{C}, J)$ be a U-site, and G a small topologically generating family for C. For every $X \in \mathrm{Ob}(\mathscr{C})$, denote by $J_G(X) \subset J(X)$ the set of all sieves covering X which are generated by a subset of G/X (notation of definition 2.1.28(i)).

Lemma 2.1.30. *With the notation of* (2.1.29), *for every $X \in \mathrm{Ob}(\mathscr{C})$ the following holds :*

(i) $J_G(X)$ *is a small set.*

(ii) $J_G(X)$ *is a cofinal subset of the set* $J(X)$ *(where the latter is partially ordered by inclusion of sieves).*

Proof. (i) is left to the reader.

(ii): Let $\mathscr{S}$ be any sieve covering X, and say that $\mathscr{S}$ is generated by a family $(f_i : S_i \to X \mid i \in I)$ of objects of $\mathscr{C}/X$ (indexed by some not necessarily small set I). Let $\mathscr{S}'$ be the sieve generated by
$$\bigcup_{i \in I} \{f_i \circ g \mid g \in G/S_i\}.$$

It is easily seen that $\mathscr{S}' \subset \mathscr{S}$ and $\mathscr{S}' \in J_G(X)$. □

Remark 2.1.31. (i) In the situation of (2.1.29), let V be a universe with $\mathsf{U} \subset \mathsf{V}$, and such that C is a V-small site. For every $X \in \mathrm{Ob}(\mathscr{C})$, denote by $\mathbf{J}_G(X)$ the full subcategory of $\mathbf{J}(X)$ such that $\mathrm{Ob}(\mathbf{J}_G(X)) = J_G(X)$, and let $\mathsf{h}_G : \mathbf{J}_G(X) \to \mathscr{C}^\wedge$ be the restriction of h (notation of the proof of theorem 2.1.20). Lemma 2.1.30 implies that the natural map :

$$\operatorname*{colim}_{\mathbf{J}_G(X)^o} \mathrm{Hom}_{\mathscr{C}^\wedge}(\mathsf{h}^o_G, F) \to F^+(X)$$

is bijective. Therefore, if F is a U-presheaf, $F^+(X)$ is essentially U-small, and then the same holds for $F^a(X)$. In other words, the restriction to $\mathscr{C}^\wedge_\mathsf{U}$ of the functor $\mathscr{C}^\wedge_\mathsf{V} \to C^\sim_\mathsf{V} : F \mapsto F^a$, factors through $C^\sim$.

(ii) We deduce that theorem 2.1.20 holds, more generally, when C is an arbitrary U-site. Likewise, a simple inspection shows that remark 2.1.27(i,iv,v) holds when C is only assumed to be a U-site.

Proposition 2.1.32. *Let $C := (\mathscr{C}, J)$ be a U-site. The following holds :*

(i) *A morphism in $C^\sim$ is an isomorphism if and only if it is both a monomorphism and an epimorphism.*
(ii) *All epimorphisms in $C^\sim$ are universal effective.*

Proof. (i): Let $\varphi : F \to G$ be a monomorphism in $C^\sim$; then the morphism of presheaves $i(\varphi) : iF \to iG$ is also a monomorphism, and it is easily seen that the cocartesian diagram

$$\mathscr{D} \quad : \quad \begin{tikzcd} iF \arrow[r, "i(\varphi)"] \arrow[d, "i(\varphi)"'] & iG \arrow[d, "\alpha"] \\ iG \arrow[r] & iG \amalg_{iF} iG \end{tikzcd}$$

is also cartesian, hence the same holds for the induced diagram of sheaves $\mathscr{D}^a$. If moreover, φ is an epimorphism, then α^a is an isomorphism, hence the same holds for $\varphi = (i(\varphi))^a$.

(ii): Let $f : F \to G$ be an epimorphism in $C^\sim$; in view of remarks 2.1.27(i) and 2.1.31(ii), it suffices to show that f is effective. However, set $G' := \mathrm{Im}(i(f))$, and let $p_i : F \times_G F \to F$ (for $i = 1, 2$) be the two projections. Suppose $\varphi : F \to X$ is a morphism in $C^\sim$ such that $\varphi \circ p_1 = \varphi \circ p_2$; since $i(F \times_G F) = iF \times_{iG} iF = iF \times_{G'} iF$, the morphism $i(\varphi)$ factors through a (unique) morphism $\psi : G' \to X$. On the other hand, it is easily seen that $(G')^a = G$, hence φ factors through the morphism $\psi^a : G \to X$. □

Remark 2.1.33. Proposition 2.1.32(i) implies that every morphism $\varphi : X \to Y$ in $C^\sim$ factors uniquely (up to unique isomorphism) as the composition of an epimorphism followed by a monomorphism. Indeed; such a factorization is provided by the natural morphisms $X \to \mathrm{Im}(\varphi)$ and $\mathrm{Im}(\varphi) \to Y$ (see example 1.1.24(viii)). If $X \xrightarrow{\varphi'} Z \to Y$ is another such factorization, then by definition φ' factors through a unique monomorphism $\psi : \mathrm{Im}(\varphi) \to Z$. However, ψ is an epimorphism, since the same holds for φ'. Hence ψ is an isomorphism.

Proposition 2.1.34. *Let $(\mathscr{C}, J)$ be a U-site, $X \in \mathrm{Ob}(\mathscr{C})$, and R any subobject of h_X. The following conditions are equivalent :*

(a) *The inclusion map $i : R \to h_X$ induces an isomorphism on associated sheaves*

$$i^a : R^a \xrightarrow{\sim} h^a_X.$$

(b) *R covers X.*

Proof. (b)⇒(a) : By definition, the natural map $R^a(X) \to \mathrm{Hom}_{\mathscr{C}^\wedge}(R, R^a)$ is bijective, hence there exists a morphism $f : h_X \to R^a$ in $\mathscr{C}^\wedge$ whose composition with i is the unit of adjunction $R \to R^a$. Therefore, $f^a : h_X^a \to R^a$ is a left inverse for i^a. On the other hand, we have a commutative diagram :

$$\begin{tikzcd} \mathrm{Hom}_{\mathscr{C}^\wedge}(h_X^a, h_X^a) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathscr{C}^\wedge}(R^a, h_X^a) \arrow[d] \\ h_X^a(X) \arrow[r] & \mathrm{Hom}_{\mathscr{C}^\wedge}(R, h_X^a) \end{tikzcd}$$

whose bottom and vertical arrows are bijective, so that the same holds also for the top arrow. Set $g := i^a \circ f^a$, and notice that $g \circ i^a = i^a$, therefore g must be the identity of h_X^a, whence the contention.

(a)⇒(b): Let $\eta_X : h_X \to h_X^a$ be the unit of adjunction, and set $j := (i^a)^{-1} \circ \eta_X : h_X \to R^a$. By remarks 2.1.27(iv) and 2.1.31(ii), we may find a covering subobject $i_1 : R_1 \to h_X$, and a morphism $j_1 : R_1 \to R^{\mathrm{sep}}$ whose image in $\mathrm{Hom}_{\mathscr{C}^\wedge}(R_1, R^a)$ equals $j \circ i_1$. Denote by $\eta'_X : h_X \to h_X^{\mathrm{sep}}$ the unit of adjunction; by construction, the two morphisms

$$i^{\mathrm{sep}} \circ j_1, \eta'_X \circ i_1 : R_1 \to h_X^{\mathrm{sep}}$$

have the same image in $\mathrm{Hom}_{\mathscr{C}^\wedge}(R_1, h_X^a)$. This means that there exists a subobject $i_2 : R_2 \to R_1$ covering X, such that $i^{\mathrm{sep}} \circ j_1 \circ i_2 = \eta'_X \circ i_1 \circ i_2$.

Next, let $Y_\bullet := (Y_\lambda \to X \mid \lambda \in \Lambda)$ be a generating family for the sieve of $\mathscr{C}/X$ corresponding to R_2. There follows, for every $\lambda \in \Lambda$, a commutative diagram :

$$\begin{tikzcd} h_{Y_\lambda} \arrow[r, "j_\lambda"] \arrow[d, "i_\lambda"'] & R^{\mathrm{sep}} \arrow[d, "i^{\mathrm{sep}}"] \\ h_X \arrow[r, "\eta'_X"] & h_X^{\mathrm{sep}}. \end{tikzcd} \tag{2.1.35}$$

Then, for every $\lambda \in \Lambda$ there exists a covering subobject $s_\lambda : R_\lambda \to h_{Y_\lambda}$ such that j_λ lifts to some $t_\lambda : R_\lambda \to R$, and we pick a generating family $(Z_{\lambda\mu} \to Y_\lambda \mid \mu \in \Lambda_\lambda)$ for the sieve of $\mathscr{C}/Y_\lambda$ corresponding to R_λ; after replacing $Y_\bullet$ by the resulting family $(Z_{\lambda\mu} \to X \mid \lambda \in \Lambda,\ \mu \in \Lambda_\lambda)$ (which still covers X, by virtue of remark 2.1.11(iii)), we may assume that (2.1.35) lifts to a commutative diagram

$$\begin{tikzcd} h_{Y_\lambda} \arrow[r, "t_\lambda"] \arrow[d, "i_\lambda"'] & R \arrow[d] \\ h_X \arrow[r, "\eta'_X"] & h_X^{\mathrm{sep}}. \end{tikzcd}$$

for every $\lambda \in \Lambda$. Then there exists a covering subobject $s'_\lambda : R'_\lambda \to h_{Y_\lambda}$ such that $i \circ t_\lambda \circ s'_\lambda = i_\lambda \circ s'_\lambda$ in $\mathrm{Hom}_{\mathscr{C}^\wedge}(R'_\lambda, h_X)$. Finally, set

$$R' := \bigcup_{\lambda \in \Lambda} \mathrm{Im}(i_\lambda \circ s'_\lambda : R'_\lambda \to h_X)$$

(notice that $R' \in \mathrm{Ob}(\mathscr{C}_{\mathsf{U}}^\wedge)$ even in case Λ is not a small set). It is easily seen that R' is a covering subobject of X, and the inclusion map $R' \to h_X$ factors through R, so R covers X as well. □

Definition 2.1.36. Let $(\mathscr{C}, J)$ be a site, such that $\mathscr{C}$ has small Hom-sets, and let $\varphi : F \to G$ be a morphism in $\mathscr{C}^\wedge$.

(i) We say that φ is a *covering morphism* if, for every $X \in \mathrm{Ob}(\mathscr{C})$ and every morphism $h_X \to G$ in $\mathscr{C}^\wedge$, the image of the induced morphism $F \times_G h_X \to h_X$ is a covering subobject of X.

(ii) We say that φ a *bicovering morphism* if both φ and the morphism $G \to G \times_F G$ induced by φ, are covering morphisms.

Example 2.1.37. In the situation of definition 2.1.36, let $S := \{X_i \mid i \in I\}$ be a family of morphisms in $\mathscr{C}$, and pick a universe V containing U, such that I is V-small. Using example 1.5.4, it is easily seen that S covers X if and only if the induced morphism in $\mathscr{C}^\wedge_\mathsf{V}$

$$\coprod_{i \in I} h_{X_i} \to h_X$$

is a covering morphism.

Corollary 2.1.38. *Let $C := (\mathscr{C}, J)$ be a U-site, and $\varphi : F \to G$ a morphism in $\mathscr{C}^\wedge$. Then φ is a covering (resp. bicovering) morphism if and only if $\varphi^a : F^a \to G^a$ is an epimorphism (resp. is an isomorphism) in $C^\sim$.*

Proof. Let V be a universe with $\mathsf{U} \subset \mathsf{V}$, and such that $\mathscr{C}$ is V-small. Clearly φ is a covering (resp. bicovering) morphism in $\mathscr{C}^\wedge_\mathsf{U}$ if and only if the same holds for the image of φ under the fully faithful inclusion $\mathscr{C}^\wedge_\mathsf{U} \subset \mathscr{C}^\wedge_\mathsf{V}$. Hence, we may replace U by V, and assume that C is a small site.

Now, suppose that φ^a is an epimorphism; let X be any object of $\mathscr{C}$, and $h_X \to G$ a morphism. Since the epimorphisms of $C^\sim$ are universal (remark 2.1.27(i)), the induced morphism

$$(\varphi \times_G X)^a : (F \times_G h_X)^a \to h^a_X$$

is an epimorphism. Let $R \subset h_X$ be the image of $\varphi \times_G h_X$; then the induced morphism $R^a \to h^a_X$ is both a monomorphism and an epimorphism, so it is an isomorphism, by proposition 2.1.32(i). Hence R is a covering subobject, according to proposition 2.1.34.

Conversely, suppose that φ is a covering morphism. By remark 2.1.27(iii), G is the colimit of a family $(h^a_{X_i} \mid i \in I)$ for certain $X_i \in \mathrm{Ob}(\mathscr{C})$. By definition, the image R_i of the induced morphism $\varphi \times_G X_i : F \times_G h_{X_i} \to h_{X_i}$ covers X_i, for every $i \in I$. Now, the induced morphism $F \times_G h_{X_i} \to R_i$ is an epimorphism, and the morphism $R^a_i \to h^a_{X_i}$ is an isomorphism (proposition 2.1.34), hence $(\varphi \times_G X_i)^a$ is an epimorphism, and then the same holds for

$$\operatorname*{colim}_{i \in I} (\varphi \times_G X_i)^a : \operatorname*{colim}_{i \in I} F^a \times_{G^a} h^a_{X_i} \to \operatorname*{colim}_{i \in I} h^a_{X_i} = G^a$$

which is isomorphic to φ^a, since the colimits of $C^\sim$ are universal (remark 2.1.27(i)); so φ^a is an epimorphism.

Next, if φ is a bicovering morphism, the foregoing shows that φ^a is an epimorphism, and the induced morphism $G^a \to G^a \times_{F^a} G^a$ is both an epimorphism and a monomorphism, hence it is an isomorphism (proposition 2.1.32(i)), therefore φ^a is monomorphism (remark 1.1.38(iii)). So finally, φ^a is an isomorphism, again by proposition 2.1.32(i). Conversely, if φ^a is an isomorphism, the reader may show in the same way, that both φ and the induced morphism $G \to G \times_F G$ are covering morphisms. $\square$

Definition 2.1.39. Let $C = (\mathscr{C}, J)$ and $C' = (\mathscr{C}', J')$ be two sites, and $g : \mathscr{C} \to \mathscr{C}'$ a functor on the underlying categories.

(i) We say that g is *continuous* for the topologies J and J', if the following holds. For every universe V such that $\mathscr{C}$ and $\mathscr{C}'$ have V-small Hom-sets, and every V-sheaf F on C', the V-presheaf $g^*_\mathsf{V} F$ is a V-sheaf on C (notation of (1.1.35)). In this case, g clearly induces a functor

$$\widetilde{g}_{\mathsf{V}*} : C'^\sim_\mathsf{V} \to C^\sim_\mathsf{V}$$

such that the diagram of functors :

$$\begin{tikzcd} C'^{\sim}_{\mathsf{V}} \arrow[r, "\widetilde{g}_{\mathsf{V}*}"] \arrow[d, "i_{C'}"'] & C^{\sim}_{\mathsf{V}} \arrow[d, "i_C"] \\ \mathscr{C}'^{\wedge}_{\mathsf{V}} \arrow[r, "g^*_{\mathsf{V}}"] & \mathscr{C}^{\wedge}_{\mathsf{V}} \end{tikzcd}$$

commutes (where the vertical arrows are the natural fully faithful embeddings).

(ii) We say that g is *cocontinuous* for the topologies J and J' if the following holds. For every $X \in \mathrm{Ob}(\mathscr{C})$, and every covering sieve $\mathscr{S}' \in J'(gX)$, the sieve $g^{-1}_{|X}\mathscr{S}'$ covers X (notation of definition 1.5.1(iii) and (1.1.15)).

Lemma 2.1.40. *In the situation of definition* 2.1.39*, the following conditions are equivalent :*

(a) *g is a continuous functor.*

(b) *There exists a universe V, such that $\mathscr{C}$ is V-small, $\mathscr{C}'$ is a V-site, and for every V-sheaf F on C', the V-presheaf $g^*_{\mathsf{V}}F$ is a V-sheaf.*

Proof. Obviously, (a)⇒(b). For the converse, we notice :

Claim 2.1.41. Let V be a universe such that $\mathscr{C}$ and $\mathscr{C}'$ are V-small. The following conditions are equivalent :

(c) For every V-sheaf F on C', the V-presheaf $g^*_{\mathsf{V}}F$ is a V-sheaf.

(d) For every $X \in \mathrm{Ob}(\mathscr{C})$, and every covering subobject $R \subset h_X$, the induced morphism $g_{\mathsf{V}!}R \to h_{g(X)}$ is a bicovering morphism in $\mathscr{C}'^{\wedge}_{\mathsf{V}}$ (notation of (1.1.35)).

Proof of the claim. (c)⇒(d): By assumption, for every sheaf F on C', the natural map

$$g^*_{\mathsf{V}}F(X) \to \mathrm{Hom}_{\mathscr{C}^\wedge}(R, g^*_{\mathsf{V}}F)$$

is bijective. By adjunction, it follows that the natural map $F(g(X)) \to \mathrm{Hom}_{\mathscr{C}'^\wedge}(g_{\mathsf{V}!}R, F)$ is also bijective, therefore the morphism $(g_{\mathsf{V}!}R)^a \to h^a_{g(X)}$ is an isomorphism, whence (d), in light of corollary 2.1.38. The proof that (d)⇒(c) is analogous, and shall be left as an exercise for the reader. ◊

Hence, suppose that (b) holds, and let V' be another universe such that $\mathsf{V} \subset \mathsf{V}'$; it follows easily from claim 2.1.41 and remark 2.1.27(v) that condition (b) holds also for V' instead of V.

Let V' be any universe such that $\mathscr{C}$ and $\mathscr{C}'$ have V'-small Hom-sets, and pick any universe V'' such that $\mathsf{V} \cup \mathsf{V}' \subset \mathsf{V}''$. By the foregoing, for every V''-sheaf F, the V''-presheaf $g^*_{\mathsf{V}''}F$ is a V''-sheaf; if F is a V-sheaf, then obviously we conclude that $g^*_{\mathsf{V}}F$ is a V-sheaf. □

Lemma 2.1.42. *In the situation of definition* 2.1.39*, consider the following conditions :*

(a) *g is continuous.*

(b) *For every covering family $(X_i \to X \mid i \in I)$ in C, the family $(gX_i \to gX \mid i \in I)$ covers gX in C'.*

(c) *For every small covering family $(X_i \to X \mid i \in I)$ in C, the family $(gX_i \to gX \mid i \in I)$ covers gX in C'.*

(d) *For every universe V such that $\mathscr{C}$ and $\mathscr{C}'$ have V-small sets, g induces a functor*

$$g^{\mathrm{sep}}_{\mathsf{V}*} : C'^{\mathrm{sep}}_{\mathsf{V}} \to C^{\mathrm{sep}}_{\mathsf{V}}$$

such that the diagram of functors :

$$\begin{tikzcd} C'^{\mathrm{sep}}_{\mathsf{V}} \arrow[r, "\widetilde{g}_{\mathsf{V}*}"] \arrow[d, "i_{C'}"'] & C^{\mathrm{sep}}_{\mathsf{V}} \arrow[d, "i_C"] \\ \mathscr{C}'^{\wedge}_{\mathsf{V}} \arrow[r, "g^*_{\mathsf{V}}"] & \mathscr{C}^{\wedge}_{\mathsf{V}} \end{tikzcd}$$

commutes (where the vertical arrows are the natural fully faithful embeddings).

Then (a)⇒(b)⇔(d)⇒(c). *Moreover, if all fibre products are representable in* $\mathscr{C}$*, and* g *commutes with fibre products, then* (b)⇒(a). *Furthermore, if* C *is a* U*-site, then* (c)⇒(b).

Proof. Obviously (b)⇒(c).

(a)⇒(b): After replacing U by a larger universe, we may assume that I is a small set and both $\mathscr{C}$ and $\mathscr{C}'$ are small. Let F be any sheaf on C'; by assumption, g^*F is a sheaf on C, hence the natural map :

$$F(gX) = g^*F(X) \to \prod_{i\in I} g^*F(X_i) = \prod_{i\in I} F(gX_i)$$

is injective (by (2.1.16)). This means that the induced morphism

$$\coprod_{i\in I} h^a_{gX_i} \to h^a_{gX} \tag{2.1.43}$$

is an epimorphism in $C^\sim$. Then the assertion follows from corollary 2.1.38 and example 2.1.37.

(d)⇒(b) is similar : we may assume that I, $\mathscr{C}$ and $\mathscr{C}'$ are small. If F is any separated presheaf on C', then by assumption g^*F is separated on C, and arguing as in the foregoing, we deduce that the induced morphism $\amalg_{i\in I} h^{\mathrm{sep}}_{gX_i} \to h^{\mathrm{sep}}_{gX}$ is an epimorphism in C^{sep}, and then (2.1.43) is an epimorphism in $C^\sim$, so we conclude, again by corollary 2.1.38 and example 2.1.37.

(b)⇒(d): let F be a separated V-presheaf on C', and $(X_i \to X \mid i \in I)$ any covering family in C; in view of (2.1.16), it suffices to show that the induced map

$$F(gX) = g^*_{\mathsf{V}}F(X) \to \prod_{i\in I} g^*_{\mathsf{V}}F(X_i) = \prod_{i\in I} F(gX_i)$$

is injective. But this is clear, since $(gX_i \to gX \mid i \in I)$ is a covering family in C'.

Next, suppose that (b) holds, the fibre products in $\mathscr{C}$ are representable, and g commutes with all fibre products. For every $i, j \in I$, set $X_{ij} := X_i \times_X X_j$. To show that (a) holds, it suffices – in view of (2.1.16) – to prove :

Claim 2.1.44. The natural map

$$g^*F(X) \to \mathrm{Hom}_{\mathscr{C}^\wedge}\left(\mathrm{Coequal}\left(\coprod_{i,j\in I} h_{X_{ij}} \rightrightarrows \coprod_{i\in I} h_{X_i}\right), g^*F\right)$$

is bijective.

Proof of the claim. Since $g_!$ is right exact, and due to (1.1.36), this is the same as the natural map

$$F(gX) \to \mathrm{Hom}_{\mathscr{C}^\wedge}\left(\mathrm{Coequal}\left(\coprod_{i,j\in I} h_{gX_{ij}} \rightrightarrows \coprod_{i\in I} h_{gX_i}\right), F\right).$$

However, by assumption $gX_{ij} = gX_i \times_{gX} gX_j$, and then the claim follows by applying (2.1.16) to the covering family $(gX_i \to gX \mid i \in I)$. ◊

Lastly, suppose that C is a U-site; in order to show that (c)⇒(b), we remark more precisely :

Claim 2.1.45. Let C be a U-site, $\mathscr{F} := (\varphi_i : X_i \to X \mid i \in I)$ any covering family. Then there exists a small set $J \subset I$ such that the subfamily $(\varphi_i \mid i \in J)$ covers X.

Proof of the claim. Let $\mathscr{S} \subset \mathscr{C}/X$ be the sieve generated by $\mathscr{S}$. By lemma 2.1.30, we may find a small covering family $\mathscr{F}' := (\psi_i : X'_i \to X \mid i \in I')$ (*i.e.* such that I' is small), that generates a sieve $\mathscr{S}' \subset \mathscr{S}$. Then, for every $i \in I'$ we may find $\gamma(i) \in I$ such that ψ_i factors through $\varphi_{\gamma(i)}$. The subset $J := \gamma I'$ will do. ◊

Let $\mathscr{F}$ and J be as in claim 2.1.45; then the sieve $g\mathscr{S}$ generated by $(g(\varphi_i) \mid i \in I)$ contains the sieve $g\mathscr{S}'$ generated by $(g(\varphi_i) \mid i \in J)$. Especially, if $g\mathscr{S}'$ is a covering sieve, the same holds for $g\mathscr{S}$, whence the contention. $\square$

2.1.46. In the situation of definition 2.1.39, let V be a universe with $\mathsf{U} \subset \mathsf{V}$, such that C is a V-site, and $\mathscr{C}'$ has V-small Hom-sets. Then we may define a functor

$$\breve{g}^*_{\mathsf{V}} : C'^{\sim}_{\mathsf{V}} \to C^{\sim}_{\mathsf{V}} \qquad F \mapsto (g^* \circ i_{C'} F)^a$$

(where $i_{C'} : C'^{\sim}_{\mathsf{V}} \to \mathscr{C}'^{\wedge}_{\mathsf{V}}$ is the forgetful functor). As usual, when $\mathsf{V} = \mathsf{U}$ we often omit the subscript U. With this notation, we have :

Lemma 2.1.47. *In the situation of definition* 2.1.39*, let* V *be a universe with* $\mathsf{U} \subset \mathsf{V}$*, such that* $\mathscr{C}$ *is* V*-small, and* $\mathscr{C}'$ *has* V*-small* Hom*-sets. Then the following conditions are equivalent :*

(a) g *is a cocontinuous functor.*

(b) *For every* V*-sheaf* F *on* C*, the* V*-presheaf* g_*F *is a* V*-sheaf on* C'*.*

(Notation of (1.1.35)*.) When these conditions hold, the restriction of* $g_{\mathsf{V}*}$ *is a functor*

$$\breve{g}_{\mathsf{V}*} : C^{\sim}_{\mathsf{V}} \to C'^{\sim}_{\mathsf{V}}$$

which is right adjoint to $\breve{g}^*_{\mathsf{V}}$*.*

Proof. After replacing V by U, we may assume that $\mathscr{C}$ is small, and $\mathscr{C}'$ has small Hom-sets. To begin with, we remark :

Claim 2.1.48. Let X be any object of $\mathscr{C}$, and $\mathscr{T}' \subset \mathscr{C}'/gX$ a covering sieve; to ease notation, set $\mathscr{T} := g^{-1}_{|X}\mathscr{T}'$. Then :

$$h_{\mathscr{T}} = g^*h_{\mathscr{T}'} \times_{g^*h_{gX}} h_X.$$

Proof of the claim. Left to the reader. $\diamondsuit$

(b)⇒(a): The assumption implies that, for every sheaf F on C, every $X \in \mathrm{Ob}(\mathscr{C})$, and every covering sieve $\mathscr{S}' \subset \mathscr{C}'/gX$, the natural map

$$g_*F(gX) \to \mathrm{Hom}_{\mathscr{C}'^{\wedge}}(h_{\mathscr{S}'}, g_*F)$$

is bijective. By adjunction, the same then holds for the natural map

$$\mathrm{Hom}_{\mathscr{C}^{\wedge}}(g^*h_{gX}, F) \to \mathrm{Hom}_{\mathscr{C}^{\wedge}}(g^*h_{\mathscr{S}'}, F)$$

so the induced morphism $g^*h_{\mathscr{S}'} \to g^*h_{gX}$ is bicovering (proposition 2.1.34). Also, this morphism is a monomorphism (since g^* commutes with all limits); therefore, after base change along the unit of adjunction $h_X \to g^*h_{gX}$, we deduce a covering monomorphism

$$g^*h_{\mathscr{S}'} \times_{g^*h_{gX}} h_X \to h_X.$$

Then the contention follows from claim 2.1.48.

(a)⇒(b): Let F be any sheaf on C; we have to show that the natural map

$$g_*F(Y) \to \mathrm{Hom}_{\mathscr{C}'^{\wedge}}(h_{\mathscr{S}}, g_*F)$$

is bijective, for every $Y \in \mathrm{Ob}(\mathscr{C}')$, and every sieve $\mathscr{S}'$ covering Y. By adjunction (and by corollary 2.1.38), this is the same as saying that the monomorphism $g^*h_{\mathscr{S}'} \to g^*h_Y$ is a covering morphism. Hence, it suffices to show that, for every morphism $\varphi : h_X \to g^*h_Y$ in $\mathscr{C}^{\wedge}$, the induced subobject $g^*h_{\mathscr{S}'} \times_{g^*h_Y} h_X$ covers X. However, by adjunction φ corresponds to a morphism $h_{gX} \to h_Y$, from which we obtain the subobject $h_{\mathscr{S}'} \times_{h_Y} h_{gX}$ covering gX. Then the assertion follows from claim 2.1.48, applied to $\mathscr{T}' := h_{\mathscr{S}'} \times_{h_Y} h_{gX}$.

Finally, the assertion concerning the left adjoint $\breve{g}^*_{\mathsf{U}}$ is immediate from the definitions. $\square$

Lemma 2.1.49. *Let* $C' := (\mathscr{C}', J')$ *be a* U*-site,* $C := (\mathscr{C}, J)$ *a small site, and* $g : \mathscr{C} \to \mathscr{C}'$ *a continuous functor. Then the following holds :*

(i) *The composition*

$$\widetilde{g}^*_{\mathsf{U}} : C^{\sim}_{\mathsf{U}} \to C'^{\sim}_{\mathsf{U}} \quad : \quad F \mapsto (g_! \circ i_C F)^a$$

provides a left adjoint to $\widetilde{g}_{\mathsf{U}*}$. *(Here* $i_C : C^{\sim}_{\mathsf{U}} \to \mathscr{C}^{\wedge}_{\mathsf{U}}$ *is the forgetful functor.)*

(ii) *The natural diagram of functors :*

$$\begin{tikzcd} \mathscr{C} \arrow[r, "g"] \arrow[d, "h^a"'] & \mathscr{C}' \arrow[d, "h^a"] \\ C^{\sim}_{\mathsf{U}} \arrow[r, "\widetilde{g}^*_{\mathsf{U}}"] & C'^{\sim}_{\mathsf{U}} \end{tikzcd}$$

is essentially commutative. (Notation of (2.1.19).*)*

(iii) *Suppose moreover that all the finite limits in* $\mathscr{C}$ *are representable, and that* g *is left exact. Then* $\widetilde{g}^*_{\mathsf{U}}$ *is exact.*

Proof. The first assertion is straightforward, and (ii) is reduced to the corresponding assertion for (1.1.36), which has already been remarked. Next, since the functor $F \mapsto F^a$ is left exact on $C'^{\wedge}$, assertion (iii) follows from proposition 2.1.3(iii). $\square$

2.1.50. Let $(\mathscr{C}, J)$ be a small site, $(\mathscr{C}', J')$ a U-site, $u, v : \mathscr{C} \to \mathscr{C}'$ two functors, such that u is continuous and v is cocontinuous. Let also V be a universe such that $\mathsf{U} \subset \mathsf{V}$. Then it follows easily from (2.1.5), lemmata 2.1.47 and 2.1.49(i), and remark 2.1.27(v) that we have essentially commutative diagrams of categories :

$$\begin{tikzcd} C^{\sim}_{\mathsf{U}} \arrow[r, "\widetilde{u}^*_{\mathsf{U}}"] \arrow[d] & C'^{\sim}_{\mathsf{U}} \arrow[d] \\ C^{\sim}_{\mathsf{V}} \arrow[r, "\widetilde{u}^*_{\mathsf{V}}"] & C'^{\sim}_{\mathsf{V}} \end{tikzcd} \qquad \begin{tikzcd} C^{\sim}_{\mathsf{U}} \arrow[r, "\breve{v}^*_{\mathsf{U}}"] \arrow[d] & C'^{\sim}_{\mathsf{U}} \arrow[d] \\ C^{\sim}_{\mathsf{V}} \arrow[r, "\breve{v}^*_{\mathsf{V}}"] & C'^{\sim}_{\mathsf{V}} \end{tikzcd}$$

whose vertical arrows are the inclusion functors. More generally, the diagram for $\breve{v}^*_{\mathsf{U}}$ is well defined and essentially commuative, whenever C is a U-site, and $\mathscr{C}'$ has small Hom-sets.

Lemma 2.1.51. *Let* $C := (\mathscr{C}, J)$ *and* $C' := (\mathscr{C}', J')$ *be two sites, and* $v : \mathscr{C} \to \mathscr{C}'$, $u : \mathscr{C}' \to \mathscr{C}$ *two functors, such that* v *is left adjoint to* u. *The following conditions are equivalent :*

(a) u *is continuous.*

(b) v *is cocontinuous.*

Moreover, when these conditions hold, then for every universe V *such that* $\mathscr{C}$ *and* $\mathscr{C}'$ *are* V-*small, we have natural isomorphisms of functors :*

$$\widetilde{u}_{\mathsf{V}*} \xrightarrow{\sim} \breve{v}_{\mathsf{V}*} \qquad \widetilde{u}^*_{\mathsf{V}} \xrightarrow{\sim} \breve{v}^*_{\mathsf{V}}.$$

Proof. In view of lemma 2.1.40, we may replace U by a larger universe, after which we may assume that C and C' are small sites. In this case, the lemma follows from lemma 2.1.6. $\square$

Lemma 2.1.52. *Let* $(\mathscr{C}, J)$ *be a small site,* $(\mathscr{C}', J')$ *a* U-*site,* $u : \mathscr{C} \to \mathscr{C}'$ *a continuous and cocontinuous functor. Then we have :*

(i) $\widetilde{u}_* = \breve{u}^*$ *and this functor admits the left adjoint* $\widetilde{u}^*$ *and the right adjoint* $\breve{u}_*$.

(ii) $\widetilde{u}^*$ *is fully faithful if and only if the same holds for* $\breve{u}_*$.

(iii) *If* u *is fully faithful, then the same holds for* $\widetilde{u}^*$. *The converse holds, provided the topologies* J *and* J' *are coarser than the canonical topologies.*

Proof. (i) is clear by inspecting the definitions. Assertion (ii) follows from (i) and proposition 1.1.11(iii). Next, suppose that u is fully faithful; then the same holds for $\breve{u}_*$ (lemma 2.1.6(i)), so the claim follows from (ii). Finally, suppose that $\widetilde{u}^*$ is fully faithful, and both J and J' are coarser than the canonical topologies on $\mathscr{C}$ and $\mathscr{C}'$. In such case, the Yoneda imbedding for

$\mathscr{C}$ (resp. $\mathscr{C}'$) realizes $\mathscr{C}$ (resp. $\mathscr{C}'$) as a full subcategory of $C^\sim$ (resp. of $C'^\sim$). Then, from (1.1.36) and the explicit expression of $\widetilde{u}^*$ provided by lemma 2.1.49(i), we deduce that u is fully faithful. □

Definition 2.1.53. Let $C := (\mathscr{C}, J)$ be a site, $\mathscr{B}$ any category, and $g : \mathscr{B} \to \mathscr{C}$ a functor. Pick a universe V such that $\mathscr{B}$ is V-small and C is a V-site. According to remark 2.1.18(iii), there is a finest topology J_g on $\mathscr{B}$ such that, for every V-sheaf F on C, the V-presheaf g^*F is a V-sheaf on $(\mathscr{B}, J_g)$. By lemma 2.1.40, the topology J_g is independent of the chosen universe V. We call J_g the *topology induced by g on $\mathscr{B}$*. Clearly g is continuous for the sites $(\mathscr{B}, J_g)$ and C.

We have the following characterization of the induced topology.

Lemma 2.1.54. *In the situation of definition* 2.1.53*, let X be any object of $\mathscr{B}$, and $R \subset h_X$ any subobject in $\mathscr{B}^\wedge$. The following conditions are equivalent :*

(i) $R \in J_g(X)$.

(ii) *For every morphism $Y \to X$ in $\mathscr{B}$, the induced morphism $g_{\mathsf{V}!}(R \times_X Y) \to h_{g(Y)}$ is a bicovering morphism in $\mathscr{C}^\wedge_{\mathsf{V}}$ (for the topology J on $\mathscr{C}$).*

Proof. (i)⇒(ii) by virtue of claim 2.1.24. The converse follows easily from remark 2.1.18(ii) and corollary 2.1.38. (Details left to the reader.) □

Example 2.1.55. (i) Resume the situation of example 2.1.7, and let $f : Y \to X$ be any object of $\mathscr{C}/X$; it is easily seen that the rule $\mathscr{S} \mapsto (\iota_X)^{-1}_{|f}\mathscr{S}$ establishes a bijection between the sieves of $(\mathscr{C}/X)/f$ and the sieves of $\mathscr{C}/Y$ (notation of definition 1.5.1(iii) and (1.1.15)). Also, for every subobject $R \subset h_f$ of the presheaf h_f on $\mathscr{C}/X$, the presheaf $\iota_{X!}R$ is a subobject of h_Y. More precisely, for a sieve $\mathscr{S}$ of $(\mathscr{C}/X)/f$ and a sieve $\mathscr{T}$ of $\mathscr{C}/Y$, we have the equivalence :

(2.1.56) $$h_{\mathscr{T}} = \iota_{X!}h_{\mathscr{S}} \quad \Leftrightarrow \quad \mathscr{S} = (\iota_X)^{-1}_{|f}\mathscr{T}.$$

(ii) Suppose now that J is a topology on $\mathscr{C}$, and set $C := (\mathscr{C}, J)$. Let us endow $\mathscr{C}/X$ with the topology J_X induced by ι_X, and let us pick a universe V such that $\mathscr{C}$ is V-small; since $(\iota_X)_{\mathsf{V}!}$ commutes with fibre products, the criterion of lemma 2.1.54 says that a subobject R of an object $f : Y \to X$ of $\mathscr{C}/X$ is a covering subobject, if and only if the induced morphism $(\iota_X)_{\mathsf{V}!}R \to h_Y$ covers Y. In view of (2.1.56), it follows easily that ι_X is both continuous and cocontinuous.

(iii) Moreover, if J is a U-topology, J_X is a U-topology as well : indeed, if $G \subset \mathrm{Ob}(\mathscr{C})$ is a small topologically generating family for C, then $G/X \subset \mathrm{Ob}(\mathscr{C}/X)$ is a small topologically generating family for $(\mathscr{C}/X, J_X)$.

(iv) Next, suppose that C is a U-site; then by example 2.1.7(ii) and remark 2.1.31(ii), it follows that we may define a functor

$$\widetilde{\iota}^*_X : (\mathscr{C}/X, J_X)^\sim_{\mathsf{U}} \to C^\sim_{\mathsf{U}}$$

by the same rule as in lemma 2.1.49(i). Moreover, say that V is a universe with $\mathsf{U} \subset \mathsf{V}$, and such that $\mathscr{C}$ is V-small; then, remark 2.1.27(v) (and again remark 2.1.31(ii)) implies that this functor is (isomorphic to) the restriction of $(\widetilde{\iota}_X)^*_{\mathsf{V}}$, and since the inclusion functor $C^\sim_{\mathsf{U}} \to C^\sim_{\mathsf{V}}$ is fully faithful, we deduce that it is also a left adjoint to $\widetilde{\iota}_{X*}$.

(v) Furthermore, recalling example 2.1.7(iii), we see that $\widetilde{\iota}^*_X$ factors through a functor

$$\widetilde{e}_X : (\mathscr{C}/X, J_X)^\sim \to C^\sim/h^a_X$$

and the functor $\iota_{h^a_X} : C^\sim/h^a_X \to C^\sim$ from (1.1.13). Indeed, a direct inspection shows that we have a natural isomorphism :

$$\widetilde{e}_X(F) \xrightarrow{\sim} (e_X \circ iF)^a$$

with e_X as in example 2.1.7(iii), and where $i : (\mathscr{C}/X, J_X)^\sim \to (\mathscr{C}/X)^\wedge$ is the forgetful functor.

(vi) Let $g : Y \to Z$ be any morphism in $\mathscr{C}$; then we have the sites $(\mathscr{C}/Y, J_Y)$ and $(\mathscr{C}/Z, J_Z)$ as in (ii), as well as the functor $g_* : \mathscr{C}/Y \to \mathscr{C}/Z$ of (1.1.14). Since $\iota_Z \circ g_* = \iota_Y$, we deduce easily from (ii) that g_* is continuous for the topologies J_Y and J_Z, and more precisely, J_Y is the topology induced by g_* on $\mathscr{C}/Y$. Notice also the natural isomorphism of categories $\mathscr{C}/Y \xrightarrow{\sim} (\mathscr{C}/Z)/g$, in terms of which, the functor g_* can be viewed as a functor ι_X of the type occuring in example 2.1.7(i) (where we take $X := g$, regarded now as an object of $\mathscr{C}/Z$). Hence, all of the foregoing applies to g_* as well; especially, g_* is cocontinuous, and if C is a U-site, then $(\widetilde{g}_*)_* : (\mathscr{C}/Z, J_Z)^\sim \to (\mathscr{C}/Y, J_Y)^\sim$ admits a left adjoint $(\widetilde{g}_*)^*$.

Proposition 2.1.57. *With the notation of example* 2.1.55(v)*, the functor* $\widetilde{e}_X$ *is an equivalence.*

Proof. To begin with, let us remark the following :

Claim 2.1.58. Let F be a presheaf on $\mathscr{C}/X$, and set $G := \iota_{X!}F$. We have :

(i) If F is a sheaf for the topology J_X, then the natural map

$$h_X \times_{h_X^{\mathrm{sep}}} G^{\mathrm{sep}} \to h_X \times_{h_X^+} G^+ \qquad (2.1.59)$$

is an isomorphism.

(ii) F is a sheaf for the topology J_X, if and only if the diagram

$$\mathscr{D} \quad : \quad \begin{array}{ccc} G & \xrightarrow{\varepsilon_G} & G^a \\ {\scriptstyle e_X(F)}\downarrow & & \downarrow{\scriptstyle e_X(F)^a} \\ h_X & \xrightarrow{\varepsilon_X} & h_X^a \end{array}$$

is cartesian in $\mathscr{C}^\wedge$. (Here ε_G and ε_X are the units of adjunction.)

Proof of the claim. (i): The map is clearly a monomorphism, hence it suffices to show that it is an epimorphism. Thus, let Y be any object of $\mathscr{C}$, and consider any pair (φ, σ) consisting of elements $\varphi \in h_X(Y)$ and $\sigma \in G^+(Y)$ whose images $\overline{\varphi}$ and $\overline{\sigma}$ in $h_X^{\mathrm{sep}}(Y)$ coincide. Therefore $\varphi : Y \to X$ is a morphism in $\mathscr{C}$, and there exists a covering subobject $R \subset h_Y$ for the topology J, such that σ is the image of an element $\sigma_R \in \mathrm{Hom}_{\mathscr{C}^\wedge}(R, G)$. Let $(\psi_i : Y_i \to Y \mid i \in I)$ be a family of morphisms in $\mathscr{C}$ that generates R; then σ_R is given by a compatible system $(\sigma_i \mid i \in I)$, where $\sigma_i \in G(Y_i)$ for every $i \in I$. According to example 2.1.7(i), σ_i is the same as a pair (a_i, φ_i), where $\varphi_i : Y_i \to X$ is an object of $\mathscr{C}/X$, and $a_i \in F(\varphi_i)$, for every $i \in I$. The identity $\overline{\sigma} = \overline{\varphi}$ means that, after replacing R by a smaller covering subobject, we have

$$\varphi \circ \psi_i = \varphi_i \qquad \text{for every } i \in I.$$

It follows that $R = \iota_{X!}R'$ for a covering subobject $R' \subset h_\varphi$ in the topology J_X (example 2.1.55(ii)), and the compatible system $(a_i \mid i \in I)$ defines an element of $\mathrm{Hom}_{(\mathscr{C}/X)^\wedge}(R', F)$. Since F is a sheaf, the latter is the image of a (unique) $a \in F(\varphi)$. The pair (a, φ) yields a section $\widetilde{\sigma} \in G(Y)$, and $(\varphi, \widetilde{\sigma})$ gives an element of $h_X \times_{h_X^{\mathrm{sep}}} G^{\mathrm{sep}}(Y)$ whose image under (2.1.59) is the original (φ, σ).

(ii): In view of lemma 2.1.1, the diagram $\mathscr{D}$ is cartesian if and only if, for every object $\varphi : Y \to X$ of $\mathscr{C}/X$, the morphism ε_G induces a bijection :

$$\mathrm{Hom}_{\mathscr{C}^\wedge/h_X}(h_\varphi, e_X(F)) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{C}^\wedge/h_X^a}(\varepsilon_X \circ h_\varphi, e_X(F)^a) \qquad (2.1.60)$$

and notice that, by example 2.1.7(iii), the source of (2.1.60) is just $F(\varphi)$. Now, suppose first that $\mathscr{D}$ is cartesian, and let $R \subset h_\varphi$ be a covering subobject for the topology J_X. We have natural isomorphisms :

$$\mathrm{Hom}_{(\mathscr{C}/X)^\wedge}(R, F) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{C}^\wedge/h_X}(e_X(R), e_X(F)) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{C}^\wedge/h_X^a}(\varepsilon_X \circ e_X(R), e_X(F)^a).$$

However, a morphism $\varepsilon_X \circ e_X(R) \to e_X(F)^a$ is just a h_X^a-morphism $\iota_{X!}R \to G^a$; since $\iota_{X!}R$ covers Y for the topology J (example 2.1.55(ii)), any such morphism extends uniquely to a h_X^a-morphism $h_\varphi \to G^a$, and in view of the bijection (2.1.60), the latter comes from a unique element of $F(\varphi)$. This shows that the natural map $F(\varphi) \to \mathrm{Hom}_{(\mathscr{C}/X)^\wedge}(R,F)$ is bijective, *i.e.* F is a sheaf, as claimed.

Conversely, suppose that F is a sheaf; we decompose $\mathscr{D}$ into two subdiagrams :

$$\begin{tikzcd} G \arrow[r] \arrow[d, "e_X(F)"'] & G^+ \arrow[r] \arrow[d, "e_X(F)^+"'] & G^a \arrow[d, "e_X(F)^a"] \\ h_X \arrow[r] & h_X^+ \arrow[r] & h_X^a \end{tikzcd}$$

and it suffices to show that both of these subdiagrams are cartesian. However, denote by $\mathscr{E}$ the left subdiagram of $\mathscr{D}$, and notice that the right subdiagram is none else that $\mathscr{E}^+$; by claim 2.1.24, we know that, if $\mathscr{E}$ is cartesian, the same holds for $\mathscr{E}^+$. Thus, we are reduced to showing that $\mathscr{E}$ is cartesian, and in view of (i), this is the same as checking that the same holds for the similar diagram

$$\begin{tikzcd} G \arrow[r, "\varepsilon'_G"] \arrow[d, "e_X(F)"'] & G^{\mathrm{sep}} \arrow[d, "e_X(F)^{\mathrm{sep}}"] \\ h_X \arrow[r, "\varepsilon'_X"] & h_X^{\mathrm{sep}}. \end{tikzcd}$$

Now, let $\varphi : Y \to X$ be any object of $\mathscr{C}/X$, and $\beta : h_Y \to G^{\mathrm{sep}}$ a given h_X^{sep}-morphism. Since ε'_G is an epimorphism, β lifts to an h_X^{sep}-morphism $\beta' : h_Y \to G$; then we may find a covering subobject $j : R' \to h_Y$, such that the restriction $\beta' \circ j : R' \to G$ is an h_X-morphism. By example 2.1.55(i,ii), $R' = \iota_{X!}R$ for some subobject $R \subset h_\varphi$ covering φ. We have natural isomorphisms :

$$\mathrm{Hom}_{\mathscr{C}^\wedge/h_X}(R',G) \xrightarrow{\sim} \mathrm{Hom}_{(\mathscr{C}/X)^\wedge}(R,F) \xrightarrow{\sim} F(\varphi) \tag{2.1.61}$$

so that $\beta' \circ j$ extends to a h_X-morphism $\gamma : h_Y \to G$. By construction, $\varepsilon'_G \circ \gamma \circ j = \varepsilon'_G \circ \beta' \circ j$, whence $\varepsilon'_G \circ \gamma = \varepsilon'_G \circ \beta' = \beta$, since G^{sep} is separated, and j is a covering morphism. This shows that the natural map

$$F(\varphi) \to \mathrm{Hom}_{\mathscr{C}^\wedge/h_X^+}(\varepsilon'_X \circ h_\varphi, e_X(F)^+)$$

is surjective. Lastly, suppose that $\varepsilon'_G \circ \beta_1 = \varepsilon'_G \circ \beta_2$ for two given h_X-morphisms $h_Y \to G$; then we may find a covering subobject $\iota_{X!}R \subset h_Y$ such that β_1 and β_2 restrict to the same h_X-morphism $\iota_{X!}R \to G$. Then again, (2.1.61) says that $\beta_1 = \beta_2$, as required. ◊

Now, let $H \to h_X^a$ be any object of $C^\sim/h_X^a$, and set $H' := H \times_{h_X^a} h_X$; by example 2.1.7(iii), we may find an object F of $(\mathscr{C}/X)^\wedge$ such that $e_X(F) = (H' \to h_X)$. Since clearly $H'^a = H$, claim 2.1.58 shows that F is a sheaf for the topology J_X, and clearly $\widetilde{e}_X(F) \simeq H$. Lastly, say that H_1 and H_2 are two h_X^a-object of $C^\sim$, define H'_1, H'_2 as in the foregoing, and pick sheaves F_1, F_2 in $(\mathscr{C}/X, J_X)^\sim$ with $\widetilde{e}_X(F_i) \simeq H'_i$ (for $i = 1, 2$); then we have natural bijections :

$$\mathrm{Hom}_{\mathscr{C}^\wedge/h_X^a}(H_1,H_2) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{C}^\wedge/h_X}(H'_1,H'_2) \xrightarrow{\sim} \mathrm{Hom}_{(\mathscr{C}/X)^\wedge}(F_1,F_2)$$

so $\widetilde{e}_X$ is also fully faithful. □

The functor $\widetilde{\iota}_X^*$ given explicitly in example 2.1.55(iv) is a special case of a construction which generalizes lemma 2.1.49 to any continuous functor between two U-sites. To explain this, we notice the following :

Proposition 2.1.62. *Let $C := (\mathscr{C}, J)$ be a* U*-site, G a small topologically generating family for C. Denote by $\mathscr{G}$ the full subcategory of $\mathscr{C}$ with $\mathrm{Ob}(\mathscr{G}) = G$, and endow $\mathscr{G}$ with the topology J' induced by the inclusion functor $u : \mathscr{G} \to \mathscr{C}$. Then :*

(i) *u is cocontinuous.*
(ii) *The induced functor $\widetilde{u}_* : C^\sim \to (\mathscr{G}, J')^\sim$ is an equivalence.*

Proof. For every presheaf F on $\mathscr{C}$, denote by $\varepsilon_F : u_! u^* F \to F$ the counit of adjunction.

Claim 2.1.63. (i) ε_F is a bicovering morphism, for every $F \in \mathrm{Ob}(\mathscr{C}^\wedge)$.

(ii) The functor $\widetilde{u}_*$ is fully faithful.

Proof of the claim. (i): First we show that ε_F is a covering morphism. To this aim, recall that both u^* and $u_!$ commute with all colimits, hence the same holds for the counit of adjunction. In view of lemma 2.1.1 and corollary 2.1.38, we may then assume that $F = h_X$ for some $X \in \mathrm{Ob}(\mathscr{C})$. In this case, by assumption there exists a family $(Y_i \mid i \in I)$ of objects of $\mathscr{G}$, and a covering morphism $f : \coprod_{i \in I} h_{uY_i} \to h_X$ in $\mathscr{C}^\wedge$. By adjunction (and by (1.1.36)), f factors (uniquely) through ε_F, so the latter must be a covering morphism as well, again by corollary 2.1.38.

The claim amounts now to the following. Let $F \in \mathrm{Ob}(\mathscr{C}^\wedge)$, $X \in \mathrm{Ob}(\mathscr{C})$, and $p, q : h_X \to u_! u^* F$ two morphisms such that $\varepsilon_F \circ p = \varepsilon_F \circ q$; then $\mathrm{Equal}(p, q)$ is a covering subobject of h_X. However, pick again a covering morphism f as in the foregoing; for every $i \in I$, the restriction $f_i : Y_i \to X$ of f satisfies the identity $\varepsilon_F \circ p \circ f_i = \varepsilon_F \circ q \circ f_i$. Again by adjunction and (1.1.36), it follows that $p \circ f_i = q \circ f_i$, *i.e.* f factors through $\mathrm{Equal}(p, q)$, whence the contention, in view of corollary 2.1.38.

(ii): In view of lemma 2.1.49(i), it is easily seen that, for every sheaf F on C, the morphism $(\varepsilon_{i_C F})^a$ is the counit of adjunction $\widetilde{u}^* \circ \widetilde{u}_* F \to F$. By (i) and corollary 2.1.38, the latter is an isomorphism, so $\widetilde{u}_*$ is fully faithful (proposition 1.1.11(ii)). ♢

(i): Let Y be an object of $\mathscr{G}$ and $\mathscr{S}'$ a sieve covering uY; we need to show that $u_{|Y}^{-1} \mathscr{S}'$ covers Y in the induced topology of $\mathscr{G}$. Since u is fully faithful, the counit of adjunction $u^* h_{uY} \to h_Y$ is an isomorphism, hence the latter amounts to showing that $u^* h_{\mathscr{S}'}$ is a covering subobject of h_Y for the induced topology (claim 2.1.48). This in turns means that, for every morphism $X \to Y$ in $\mathscr{G}$, the induced morphism $u_!(u^* h_{\mathscr{S}'} \times_{h_Y} h_X) \to h_{uX}$ is a bicovering morphism in $\mathscr{C}^\wedge$ (lemma 2.1.54). However :

$$u_!(u^* h_{\mathscr{S}'} \times_{h_Y} h_X) = u_! u^*(h_{\mathscr{S}'} \times_{h_{uY}} h_{uX})$$

hence claim 2.1.63(i) reduces to showing that the natural morphism $h_{\mathscr{S}'} \times_{h_{uY}} h_{uX} \to h_{uX}$ is bicovering, which is clear.

From (i) and lemma 2.1.52(ii) we deduce that $\widetilde{u}^*$ is fully faithful. Combining with claim 2.1.63(ii) and proposition 1.1.11(i), we deduce that $\widetilde{u}_*$ is an equivalence, as stated. □

Corollary 2.1.64. *Let $C := (\mathscr{C}, J)$ and $C' := (\mathscr{C}', J')$ be two* U*-sites, $g : \mathscr{C} \to \mathscr{C}'$ a functor. We have :*

(i) *If g is continuous, the following holds :*
   (a) *For every universe V such that $\mathsf{U} \subset \mathsf{V}$, the functor $\widetilde{g}_{\mathsf{V}*} : C'^{\sim}_{\mathsf{V}} \to C^{\sim}_{\mathsf{V}}$ admits a left adjoint $\widetilde{g}^*_{\mathsf{V}} : C^{\sim}_{\mathsf{V}} \to C'^{\sim}_{\mathsf{V}}$.*
   (b) *For every pair of universes $\mathsf{V} \subset \mathsf{V}'$ containing U, we have an essentially commutative diagram of categories :*

$$\begin{array}{ccc} C^{\sim}_{\mathsf{V}} & \xrightarrow{\widetilde{g}^*_{\mathsf{V}}} & C'^{\sim}_{\mathsf{V}} \\ \downarrow & & \downarrow \\ C^{\sim}_{\mathsf{V}'} & \xrightarrow{\widetilde{g}^*_{\mathsf{V}'}} & C'^{\sim}_{\mathsf{V}'} \end{array}$$

   whose vertical arrows are the inclusion functors.
   (c) *Suppose moreover that all the finite limits in $\mathscr{C}$ are representable, and that g is left exact. Then $\widetilde{g}^*_{\mathsf{U}}$ is exact.*

(ii) *If g is cocontinuous, the following holds :*

(a) *For every universe* V *such that* $\mathsf{U} \subset \mathsf{V}$*, the functor* $\breve{g}^*_{\mathsf{V}} : C'^{\sim}_{\mathsf{V}} \to C^{\sim}_{\mathsf{V}}$ *admits a right adjoint* $\breve{g}_{\mathsf{V}*} : C^{\sim}_{\mathsf{V}} \to C'^{\sim}_{\mathsf{V}}$*.*

(b) *For every pair of universes* $\mathsf{V} \subset \mathsf{V}'$ *containing* U*, we have an essentially commutative diagram of categories :*

$$\begin{array}{ccc} C^{\sim}_{\mathsf{V}} & \xrightarrow{\breve{g}_{\mathsf{V}*}} & C'^{\sim}_{\mathsf{V}} \\ \downarrow & & \downarrow \\ C^{\sim}_{\mathsf{V}'} & \xrightarrow{\breve{g}_{\mathsf{V}'*}} & C'^{\sim}_{\mathsf{V}'} \end{array}$$

whose vertical arrows are the inclusion functors.

(iii) *If* g *is both continuous and cocontinuous, we have a natural isomorphism :*

$$\widetilde{g}_{\mathsf{V}*} \xrightarrow{\sim} \breve{g}^*_{\mathsf{V}}$$

for every universe V *with* $\mathsf{U} \subset \mathsf{V}$.

Proof. (i.a): We choose a small topologically generating family G for C, and define the site $(\mathscr{G}, J')$ and the continuous functor $u : (\mathscr{G}, J') \to C$ as in proposition 2.1.62. By applying lemma 2.1.49(i) to the continuous functor $h := g \circ u$, we deduce that $\widetilde{h}_{\mathsf{V}*} = \widetilde{u}_{\mathsf{V}*} \circ \widetilde{g}_{\mathsf{V}*}$ admits a left adjoint. Then the assertion follows from proposition 2.1.62(ii).

(i.b): More precisely, $\widetilde{g}^*_{\mathsf{V}} = \widetilde{h}^*_{\mathsf{V}} \circ \widetilde{u}_{\mathsf{V}*}$. Thus, the assertion follows from (2.1.50).

(i.c): In view of (i.b), the assertion can be checked after enlarging the universe U, so that we may assume that $\mathscr{C}$ is small, in which case we conclude by lemma 2.1.49(iii).

(ii.a): Let u and h be as in the foregoing; from proposition 2.1.62(i) we deduce that h is cocontinuous, hence $\breve{h}^*_{\mathsf{V}} = \breve{u}^*_{\mathsf{V}} \circ \breve{g}^*_{\mathsf{V}}$ admits a right adjoint; however $\breve{u}^*_{\mathsf{V}} = \widetilde{u}_{\mathsf{V}*}$ (lemma 2.1.52(i)), and the latter is an equivalence (proposition 2.1.62(ii)), whence the contention.

(ii.b): More precisely, $\breve{g}_{\mathsf{V}*} = \breve{h}_{\mathsf{V}*} \circ \widetilde{u}^*_{\mathsf{V}}$, hence the assertion follows from (2.1.50).

(iii): In view of (i.b) and (ii.b), in order to prove the assertion, we may assume that $\mathscr{C}$ and $\mathscr{C}'$ are V-small, and this case is already covered by lemma 2.1.52(i). □

Example 2.1.65. (i) In the situation of example 2.1.55(iv), by corollary 2.1.64(ii,iii), the functor $\widetilde{\iota}_{X*} = \breve{\iota}^*_X : C^{\sim} \to (\mathscr{C}/X, J_X)^{\sim}$ admits also a right adjoint $\breve{\iota}_{X*}$. We introduce a special notation and terminology for these functors :

- The functor $\widetilde{\iota}_{X*}$ shall be also denoted j^*_X, and called the functor of *restriction to* X.
- The functor $\breve{\iota}_{X*}$ shall be denoted j_{X*}, and called the *direct image* functor.
- The functor $\widetilde{\iota}^*_X$ shall be denoted $j_{X!}$, and called the functor of *extension by empty*.

Thus, j^*_X is right adjoint to $j_{X!}$, and left adjoint to j_{X*}.

(ii) Likewise, let $g : Y \to Z$ be any morphism in $\mathscr{C}$; by example 2.1.55(vi) and corollary 2.1.64(ii,iii), the functor $(\widetilde{g}_*)_*$ admits also a right adjoint $(\breve{g}_*)_*$. In agreement with (i), we shall let :

$$j^*_g := (\widetilde{g}_*)_* \qquad j_{g*} := (\breve{g}_*)_* \qquad j_{g!} := (\widetilde{g}_*)^*.$$

Clearly $j^*_g \circ j^*_Z = j^*_Y$, and we have isomorphisms of functors : $j_{Z*} \circ j_{g*} \xrightarrow{\sim} j_{Y*}$ and $j_{Z!} \circ j_{g!} \xrightarrow{\sim} j_{Y!}$.

(iii) Moreover, under the equivalence $\widetilde{e}_X$ of proposition 2.1.57, the functor j^*_X is identified to the functor :

$$C^{\sim} \to C^{\sim}/h^a_X \qquad U \mapsto X \times U$$

and $j_{X!}$ is identified to the functor $C^{\sim}/h^a_X \to C^{\sim}$ of (1.1.13). Likewise, g^* is identified to the functor $C^{\sim}/h^a_Z \to C^{\sim}/h^a_Z$ given by the rule : $(U \to h^a_Z) \mapsto (U \times_{h^a_Z} h^a_Y)$, and $g_!$ is identified to the functor $(h^a_g)_* : C^{\sim}/h^a_Y \to C^{\sim}/h^a_Z$.

2.2. **Topoi.** A U-*topos* T is a category with small Hom-sets which is equivalent to the category of sheaves on a small site. We shall usually write “topos” instead of “U-topos”, unless this may give rise to ambiguities. Denote by C_T the canonical topology on T, and say that $T = C^\sim$, for a small site $C := (\mathscr{C}, J)$; then the set $\{h^a_X \mid X \in \mathrm{Ob}(\mathscr{C})\}$ is a small topological generating family for the site (T, C_T), especially, the latter is a U-site. Moreover, the Yoneda embedding induces an equivalence

$$T \to (T, C_T)^\sim \tag{2.2.1}$$

between T and the category of sheaves on the site (T, C_T). The Hom-sets in the category $(T, C_T)^\sim$ are not small in general; however, they are essentially small, therefore $(T, C_T)^\sim$ is isomorphic to a U-topos. In view of this, the objects of T may be thought of as sheaves on (T, C_T), and one uses often the suggestive notation :

$$X(S) := \mathrm{Hom}_T(S, X) \qquad \text{for any two objects } X, S \text{ of } T.$$

The elements of $X(S)$ are also called the S-*sections* of X. The final object of T shall be denoted 1_T, and $X(1_T)$ is also called the set of *global sections* of X.

Remark 2.2.2. (i) By proposition 2.1.62(ii), if C is a U-site, then $C^\sim$ is a U-topos.

(ii) Remark 2.1.27(i,ii) can be summarized by saying that every U-topos T is a complete and cocomplete, well-powered and co-well-powered category. Morever, every epimorphism in T is universal effective, every colimit is universal, and all filtered colimits in T commute with finite limits.

Definition 2.2.3. (i) Let $C = (\mathscr{C}, J)$ and $C' = (\mathscr{C}', J')$ be two sites. A *morphism of sites* $C' \to C$ is the datum of a continuous functor $g : \mathscr{C} \to \mathscr{C}'$, such that the left adjoint $\widetilde{g}^*_{\mathsf{U}}$ of the induced functor $\widetilde{g}_{\mathsf{U}*}$ is exact (notation of definition 2.1.39(i)).

(ii) A *morphism of topoi* $f : T \to S$ is a datum (f^*, f_*, η), where

$$f_* : T \to S \qquad f^* : S \to T$$

are two functors such that f^* is left exact and left adjoint to f_*, and $\eta : \mathbf{1}_S \Rightarrow f_*f^*$ is a unit of the adjunction (these are sometimes called *geometric morphisms* : see [12, Def.2.12.1]).

(iii) Let $f := (f^*, f_*, \eta_f) : T'' \to T'$ and $g := (g^*, g_*, \eta_g) : T' \to T$ be two morphisms of topoi. The composition $g \circ f$ is the morphism

$$(f^* \circ g^*, g_* \circ f_*, (g_* * \eta_f * g^*) \circ \eta_g) : T'' \to T$$

(see remark 1.1.10(i)). The reader may verify that this composition law is associative.

(iv) Let $f, g : T \to S$ be two morphisms of topoi. A *natural transformation* $\tau : f \Rightarrow g$ is just a natural transformation of functors $\tau_* : f_* \Rightarrow g_*$. Notice that, in view of remark 1.1.10(ii), the datum of τ_* is the same as the datum of a natural transformation $\tau^* : g^* \Rightarrow f^*$.

Remark 2.2.4. (i) By corollary 2.1.64(i.c), if C and C' are U-sites, and all finite limits of $\mathscr{C}$ are representable, every left exact continuous functor $\mathscr{C} \to \mathscr{C}'$ defines a morphism of sites $C' \to C$.

(ii) More generally, let $\mathscr{L}$ be the category whose objects are all morphisms $X \to gY$ in $\mathscr{C}'$, where X (resp. Y) ranges over the objects of $\mathscr{C}'$ (resp. of $\mathscr{C}$). The morphisms $(\beta : X \to gY) \to (\beta' : X' \to gY')$ in $\mathscr{L}$ are the pairs (φ, ψ) where $\varphi : X \to X'$ (resp. $\psi : Y \to Y'$) is a morphism in $\mathscr{C}'$ (resp. in $\mathscr{C}$) and $\beta' \circ \varphi = \varphi' \circ \beta$. There is an obvious fibration $p : \mathscr{L} \to \mathscr{C}'$, such that $p(X \to gY) := X$ for every object $(X \to gY)$ in $\mathscr{L}$, and one can prove that a continuous functor $g : \mathscr{C} \to \mathscr{C}'$ defines a morphism of sites, if and only if the fibration p is locally cofiltered (see [4, Exp.V, Déf.8.1.1]; the necessity is a special case of [4, Exp.V, lemme 8.1.11]). This shows that the definition of morphism of sites depends only on g, and not on the universe U.

(iii) Every morphism of U-sites $g : C' \to C$ induces a morphism of topoi $g^\sim : C'^\sim \to C^\sim$, and conversely, a morphism $f : T \to S$ of topoi determines a morphism of sites $C_f : (T, C_T) \to (S, C_S)$ for the corresponding canonical topologies.

(iv) The topoi form a 2-category, denoted

$$\mathbf{Topos}$$

whose objects are all the topoi, whose 1-cells are the morphisms of topoi, and whose 2-cells are the natural transformations between such morphisms, as in definition 2.2.3(ii,iv).

Proposition 2.2.5. *Let T, T' be two topoi, $f : T \to T'$ be a functor.*

(i) *If f commutes with all colimits, and all fibre products in T, the following holds :*
  (a) *f is continuous for the canonical topologies on T and T'.*
  (b) *There exists a morphism of topoi $\varphi : T' \to T$, unique up to unique isomorphism, such that $\varphi^* = f$.*

(ii) *If f is exact, the following holds :*
  (a) *f is conservative if and only if it reflects epimorphisms.*
  (b) *For every morphism φ in T, the natural morphism $f(\mathrm{Im}\,\varphi) \to \mathrm{Im}(f\varphi)$ is an isomorphism.*

Proof. (i.a): Let $S := \{g_i : X_i \to X \mid i \in I\}$ be a small covering family of morphisms in (T, C_T); by lemma 2.1.42, it suffices to show that $fS := \{f(g_i) \mid i \in I\}$ is a covering family in $(T', C_{T'})$. However, our assumption implies that the induced morphism $\amalg_{i\in I} X_i \to X$ is an epimorphism, hence the same holds for the induced morphism $\amalg_{i\in I} fX_i \to fX$ in T' (by example 1.1.24(iii)). But all epimorphisms are universal effective in T' (remark 2.2.2(ii)), hence $(f(g_i) \mid i \in I)$ is a universal effective family, as required.

(i.b): By corollary 2.1.64(i.a,i.c), for every universe V such that $\mathsf{U} \subset \mathsf{V}$, the functor f gives rise to a morphism of topoi

$$(\widetilde{f}^*_{\mathsf{V}}, \widetilde{f}_{\mathsf{V}*}, \eta_{\mathsf{V}}) : (T', C_{T'})^\sim_{\mathsf{V}} \to (T, C_T)^\sim_{\mathsf{V}}$$

(where η is any choice of unit of adjunction), and it remains only to check that $\widetilde{f}^*_{\mathsf{U}}$ is isomorphic to f, under the natural identifications $h_{\mathsf{U}} : T \xrightarrow{\sim} (T, C_T)^\sim_{\mathsf{U}}$, $h'_{\mathsf{U}} : T' \xrightarrow{\sim} (T', C_{T'})^\sim_{\mathsf{U}}$. In view of corollary 2.1.64(i.b), it suffices to show that there exists a universe V such that the diagram

$$\begin{tikzcd} T \arrow[r, "f"] \arrow[d, "h_{\mathsf{V}}"'] & T' \arrow[d, "h'_{\mathsf{V}}"] \\ (T, C_T)^\sim_{\mathsf{V}} \arrow[r, "\widetilde{f}^*_{\mathsf{V}}"] & (T', C_{T'})^\sim_{\mathsf{V}} \end{tikzcd}$$

is essentially commutative (where h_{V} and h'_{V} are the Yoneda embeddings). By lemma 2.1.49(i) and (1.1.36), the latter holds whenever T is V-small.

(ii.b) follows easily from remark 2.1.33.

(ii.a): The condition is necessary, due to example 1.1.24(iii). Conversely, assume that f reflects epimorphisms, and suppose that φ is a morphism in T such that $f(\varphi)$ is an isomorphism. It follows already that φ is an epimorphism, hence it suffices to show that φ is a monomorphism (proposition 2.1.32(i)). This is the same as showing that the natural map $\iota_\varphi : Y \to \mathrm{Equal}(\varphi, \varphi)$ is an isomorphism (again by example 1.1.24(iii)); however, ι_φ is always a monomorphism, and since f is left exact, $f(\iota_\varphi)$ is the natural morphism $fY \to \mathrm{Equal}(f\varphi, f\varphi)$, which we know to be an isomorphism, so ι_φ is an epimorphism, under our assumption. $\square$

Example 2.2.6. Let T be a topos.

(i) Say that $T = (\mathscr{C}, J)^\sim$ for some small site $(\mathscr{C}, J)$. Then the category $\mathscr{C}^\wedge$ is also a topos ([3, Exp.IV, §2.6]), and since the functor $F \mapsto F^a$ of (2.1.19) is left exact, it determines a morphism of topoi $T \to \mathscr{C}^\wedge$. (On the other hand, the category $T^\wedge$ is too large to be a U-topos.)

(ii) If $f := (f^*, f_*, \eta) : T \to S$ is any morphism of topoi, we have an essentially commutative diagram of categories :

$$\begin{tikzcd} T \arrow[r] \arrow[d, "f_*"'] & T^\wedge \arrow[d, "f_*^\wedge"] \\ S \arrow[r] & S^\wedge \end{tikzcd}$$

whose horizontal arrows are the Yoneda embeddings, and where $f_*^\wedge = \mathbf{Fun}(f^{*o}, \mathbf{Set})$.

(iii) Let U be any object of T. By applying the discussion of example 2.1.65 to the U-site (T, C_T), we deduce that the category T/U (notation of (1.1.2)) is a topos, and the natural functor

$$j_U^* : T \to T/U \qquad X \mapsto X_{|U} := (X \times U \to U)$$

induces a morphisms of U-sites $(T, C_T) \to (T/U, C_{T/U})$, whence a morphism of topoi $j_U : T/U \to T$, unique up to unique isomorphism. Moreover, we also obtain a left adjoint $j_{U!} : T/U \to T$ for j_U^*, given explicitly by the rule : $(X \to U) \mapsto X$ for any U-object X of T. In case U is a subobject of the final object, the morphism j_U is called an *open subtopos* of T. Likewise, any morphism $g : Y \to Z$ in T determines, up to unique isomorphism, a morphism of topoi $j_g : T/Y \to T/Z$, with an isomorphism $j_Z \circ j_g \xrightarrow{\sim} j_Y$ of morphisms of topoi.

(iv) Let U be a subobject of 1_T. We denote by $\mathsf{C}U$ the full subcategory of T such that

$$\mathrm{Ob}(\mathsf{C}U) = \{X \in \mathrm{Ob}(T) \mid j_U^* X = 1_{T/U}\}.$$

Then $\mathsf{C}U$ is a topos, called the *complement of U in T*, and the inclusion functor $i_* : \mathsf{C}U \to T$ admits a left adjoint $i^* : T \to \mathsf{C}U$, namely, the functor which assigns to every $X \in \mathrm{Ob}(T)$ the push-out $X_{|\mathsf{C}U}$ in the cocartesian diagram :

$$\begin{tikzcd} X \times U \arrow[r, "p_X"] \arrow[d, "p_U"'] & X \arrow[d] \\ U \arrow[r] & X_{|\mathsf{C}U} \end{tikzcd}$$

where p_X and p_U are the natural projections. Moreover, i^* is an exact functor, hence the adjoint pair (i^*, i_*) defines a morphism of topoi $\mathsf{C}U \to T$, unique up to unique isomorphism. (See [3, Exp.IV, Prop.9.3.4].)

(v) Another basic example is the *global sections functor* $\Gamma : T \to \mathbf{Set}$, defined as :

$$U \mapsto \Gamma(T, U) := U(1_T).$$

Γ admits a left adjoint :

$$\mathbf{Set} \to T \ : \ S \mapsto S_T := S \times 1_T$$

(the coproduct of S copies of 1_T) and for every set S, one calls S_T the *constant sheaf with value S*. This pair of adjoint functors defines a morphism of topoi $\Gamma : T \to \mathbf{Set}$ ([3, Exp.IV, §4.3]). One can chek that there exists a unique such morphism of topoi, up to unique isomorphism.

Example 2.2.7. Let T be a topos.

(i) We say that T is *connected* (resp. *disconnected*) if the same holds for the final object 1_T of T (see example 1.1.26(iii)). Notice that, if U is any object of T, then U is connected if and only if the same holds for the topos T/U.

(ii) T is connected if and only if the natural map

$$\mathbb{Z}/2\mathbb{Z} \to \Gamma(T, \mathbb{Z}/2\mathbb{Z}_T) \tag{2.2.8}$$

is surjective (notation of example 2.2.6(v)). Indeed, $\mathbb{Z}/2\mathbb{Z}_T$ is the coproduct $1_T \amalg 1_T$, and any decomposition $\alpha : 1_T \xrightarrow{\sim} X_0 \amalg X_1$ determines a section of $1_T \amalg 1_T$, namely the composition α' of α with the coproduct of the unique morphisms $X_0 \to 1_T$ and $X_1 \to 1_T$. Conversely, let $\beta' : 1_T \to \mathbb{Z}/2\mathbb{Z}_T$ be a global section, and denote by $j_0, j_1 : 1_T \to 1_T \amalg 1_T$ the natural morphisms (so $\{j_0, j_1\}$ is the image of the map (2.2.8)). We define X_i for $i = 0, 1$, as the fibre products in the cartesian diagrams :

$$\begin{tikzcd} X_i \arrow[r] \arrow[d] & 1_T \arrow[d, "j_i"] \\ 1_T \arrow[r, "\beta'"] & \mathbb{Z}/2\mathbb{Z}_T. \end{tikzcd}$$

Since all colimits in T are universal (remark 2.2.2(ii)), the induced morphism $\beta : X_0 \amalg X_1 \to 1_T$ is an isomorphism, and it is easily seen that the rules $\alpha \mapsto \alpha'$ and $\beta' \mapsto \beta$ establish mutually inverse bijections (details left to the reader).

Remark 2.2.9. (i) Let $f : T' \to T$ be a morphism of topoi, U an object of T, and suppose that $s : T' \to T/U$ is a morphism of topoi with an isomorphism $j_U \circ s \xrightarrow{\sim} f$. Hence, we have a functorial isomorphism $s^*(j_U^* X) \xrightarrow{\sim} f^* X$ for every $X \in \mathrm{Ob}(T)$, and especially, $s^*\mathbf{1}_U = 1_{T'}$; let $\Delta_U : \mathbf{1}_U \to j_U^* j_{U!} = j_U^*(U)$ be the unit of adjunction (*i.e.* the diagonal morphism in T/U); then $\sigma := s^*(\Delta_U) : 1_{T'} \to f^*U$ is an element of $\Gamma(T', f^*U)$. Moreover, for every object $\varphi : X \to U$ in T/U, we have a cartesian diagram in T/U :

$$\mathscr{D} \quad : \quad \begin{tikzcd} X \arrow[rr, "\Gamma_\varphi"] \arrow[dr, "\varphi"] \arrow[dd, "\varphi"'] & & X \times U \arrow[dl, "j_U^* X"'] \arrow[dd, "j_U^*\varphi"] \\ & U & \\ U \arrow[rr, "\Delta_U"'] \arrow[ur, "\mathbf{1}_U"'] & & U \times U \arrow[ul, "j_U^* U"] \end{tikzcd}$$

where Γ_φ is the graph of φ, and $s^*\mathscr{D}$ is isomorphic to the cartesian diagram (in T) :

$$\mathscr{E} \quad : \quad \begin{tikzcd} s^*\varphi \arrow[r] \arrow[d] & f^*X \arrow[d, "f^*\varphi"] \\ 1_{T'} \arrow[r, "\sigma"] & f^*U. \end{tikzcd}$$

This shows that s^* – and therefore also s – is determined, up to unique isomorphism, by σ.

(ii) Conversely, if $\sigma \in \Gamma(T', f^*U)$ is any global section, then we may define a functor $s^* : T/U \to T'$ by means of the cartesian diagram $\mathscr{E}$, and clearly $s^* \circ j_U^* = f^*$. Let us also define a functor $t_! : T' \to T'/f^*U$, by the rule :

$$t_!(Y) = \sigma \circ u_Y \qquad \text{for every } Y \in \mathrm{Ob}(T')$$

where $u_Y : Y \to 1_{T'}$ is the unique morphism in T'. By inspecting the diagram $\mathscr{E}$, we deduce natural bijections :

$$\mathrm{Hom}_{T'}(Y, s^*\varphi) \xrightarrow{\sim} \mathrm{Hom}_{T'/f^*U}(t_!Y, f^*\varphi) \qquad \text{for every } Y \in \mathrm{Ob}(T') \text{ and } \varphi \in \mathrm{Ob}(T/U).$$

Since f^* is exact, it follows easily that s^* is left exact (indeed, s^* commutes with all the limits with which f^* commutes). Moreover, since all colimits are universal in T (see (2.1.27)(i)), it is easily seen that s^* commutes with all colimits. Then, by proposition 2.2.5(i.b), the functor s^* determines a morphism of topoi $s : T' \to T/U$, unique up to unique isomorphism, and an isomorphism $j_U \circ s \xrightarrow{\sim} f$.

(iii) Summing up, the constructions of (i) and (ii) establish a natural bijection between $\Gamma(T', f^*U)$ and the set of isomorphism classes of morphisms of topoi $s : T' \to T/U$ such that $j_U \circ s$ is isomorphic to f.

Definition 2.2.10. Let T be a topos.

(i) A *point* of T (or a T*-point*) is a morphism of topoi $\mathbf{Set} \to T$. If $\xi = (\xi^*, \xi_*)$ is a point, and F is any object of T, the set $\xi^* F$ is usually denoted by F_ξ.

(ii) If ξ is a point of T, and $f : T \to S$ is any morphism of topoi, we denote by $f(\xi)$ the S-point $f \circ \xi$.

(iii) A *neighborhood* of ξ is a pair (U, a), where $U \in \mathrm{Ob}(T)$, and $a \in U_\xi$. A morphism of neighborhoods $(U, a) \to (U', a')$ is a morphism $f : U \to U'$ in T such that $f_\xi(a) = a'$. The category of all neighborhoods of ξ shall be denoted $\mathbf{Nbd}(\xi)$.

(iv) We say that a set Ω of T-points is *conservative*, if the functor

$$T \to \mathsf{U}'\text{-}\mathbf{Set} \qquad F \mapsto \prod_{\xi \in \Omega} F_\xi$$

is conservative (definition 1.1.4(ii)). (Here U' is a universe such that $\mathsf{U} \subset \mathsf{U}'$ and $\Omega \in \mathsf{U}'$.) We say that T *has enough points*, if T admits a conservative set of points.

2.2.11. It follows from remark 2.2.9(iii), that a neighborhood (U, a) of ξ is the same as the datum of an isomorphism class of a point ξ_U of the topos T/U which lifts ξ, *i.e.* such that $\xi \simeq j_U \circ \xi_U$. Moreover, say that $\xi_{U'}$ is another lifting of ξ, corresponding to a neighborhood (U', a') of ξ; then, by inspecting the constructions of remark 2.2.9(i,ii) we see that, under this identification, a morphism $(U, a) \to (U', a')$ of neighborhoods of ξ corresponds to the datum of a morphism $\varphi : U \to U'$ in T and an isomorphism of T-points :

$$j_\varphi \circ \xi_U \xrightarrow{\sim} \xi_{U'}$$

where j_φ is the morphism of topoi $T/U \to T/U'$ induced by φ. Thus, let $\mathbf{Lift}(\xi)$ denote the category whose objects are the triples (U, ξ_U, ω_U), where $U \in \mathrm{Ob}(T)$, ξ_U is a T/U-point lifting ξ, and $\omega_U : j_U \circ \xi_U \xrightarrow{\sim} \xi$ is an isomorphism of T-points (here we fix, for every object U and every morphism φ of T, a choice of the morphisms j_U and j_φ : recall that the latter are determined by U and respectively φ, up to unique isomorphism). The morphisms $(U, \xi_U, \omega_U) \to (V, \xi_V, \omega_V)$ are the pairs $(\varphi, \omega_\varphi)$, where $\varphi : U \to V$ is a morphism in T, and

$$\omega_\varphi : j_\varphi \circ \xi_U \xrightarrow{\sim} \xi_V$$

is an isomorphism of T/V-points, such that :

$$\omega_{V*} \circ (j_{V*} * \omega_{\varphi*}) = \omega_{U*}.$$

The composition of such a morphism with another one $(\psi, \omega_\psi) : (V, \xi_V, \omega_V) \to (W, \xi_W, \omega_W)$ is the pair $(\psi \circ \varphi, \omega_{\psi\circ\varphi})$ determined by the commutative diagram of T/W-points :

$$\begin{tikzcd} j_\psi \circ j_\varphi \circ \xi_U \arrow[r, "j_\psi * \omega_\varphi"] \arrow[d, "\mathsf{c}_{\psi,\varphi} * \xi_U"'] & j_\psi \circ \xi_V \arrow[d, "\omega_\psi"] \\ j_{\psi\circ\varphi} \circ \xi_U \arrow[r, "\omega_{\psi\circ\varphi}"] & \xi_W . \end{tikzcd} \tag{2.2.12}$$

where $\mathsf{c}_{\psi,\varphi} : j_\psi \circ j_\varphi \xrightarrow{\sim} j_{\psi\circ\varphi}$ is the unique isomorphism. As an exercise, the reader may verify the associativity of this composition law. The foregoing shows that we have an equivalence of categories :

$$N_\xi : \mathbf{Lift}(\xi) \xrightarrow{\sim} \mathbf{Nbd}(\xi) \qquad (U, \xi_U, \omega_U) \mapsto (U, a_U) \tag{2.2.13}$$

where $a_U \in U_\xi$ is the unique morphism which fits in the commutative diagram :

$$\begin{tikzcd} \xi^* 1_T \arrow[r, "a_U"] \arrow[d, "\omega^*_U(1_T)"'] & \xi^*(U) \arrow[d, "\omega^*_U(U)"] \\ \xi^*_U \circ j^*_U(1_T) \arrow[r, "\xi^*_U(\Delta_U)"] & \xi^*_U \circ j^*_U(U) \end{tikzcd}$$

(notation of remark 2.2.9(i)). To check the functoriality of (2.2.13), let $(\varphi, \omega_\varphi)$ be as in (2.2.11), and set $\vartheta_V := j_\varphi \circ \xi_U$; we can write :

$$\vartheta^*_V(\Delta_V) = \xi^*_U \circ j^*_\varphi(\Delta_V) = \xi^*_U(\Gamma_\varphi) = \xi^*_U(j^*_U\varphi) \circ \xi^*_U(\Delta_U) = \vartheta^*_V(\varphi) \circ \xi^*_U(\Delta_U)$$

from which it follows easily that $a_V = \varphi_\xi(a_U)$, as required.

Remark 2.2.14. (i) Let T be a topos with enough points, and U any object of T; as a consequence of the discussion in (2.2.11), we deduce that also T/U has enough points. More precisely, say that Ω is a conservative set of T-points, and denote by $j_U^{-1}\Omega$ the set of all T/U-points ξ such that $j_U \circ \xi \in \Omega$; we claim that $j_U^{-1}\Omega$ is a conservative set of points. Indeed, let $\varphi : X \to Y$ be a morphism in T/U such that φ_ξ is an epimorphism for every $\xi \in j_U^{-1}\Omega$; by proposition 2.2.5(ii.a), it suffices to show that φ is an epimorphism, and the latter will follow, if we show that the same holds for $j_{U!}\varphi$.

Thus, suppose by way of contradiction, that $j_{U!}\varphi$ is not an epimorphism; then there exists a point $\xi \in \Omega$, and $y \in (j_{U!}Y)_\xi$, such that y does not lie in the image of $(j_{U!}\varphi)_\xi$. Let $a := (j_{U!}\pi_X)_\xi(y) \in U_\xi$, where $\pi_X : X \to 1_{T/U}$ is the unique morphism in T/U. By (2.2.13), we may find a lifting (U, ξ_U, ω_U) of ξ such that

$$N_\xi(U, \xi_U, \omega_U) = (U, a). \tag{2.2.15}$$

After replacing ξ by $j_U \circ \xi_U$, we may assume that ω_U is the identity of ξ, in which case (2.2.15) means that $a = \xi^*_U \Delta_U$, where $\Delta_U : 1_{T/U} \to j^*_U j_{U!} 1_{T/U}$ is the unit of adjunction (*i.e* the diagonal $U \to U \times U$). We have a commutative diagram of sets :

$$\begin{tikzcd} \xi^*_U X \arrow[r, "\xi^*_U\varphi"] \arrow[d, "\xi^*_U\Delta_X"'] & \xi^*_U Y \arrow[d, "\xi^*_U\Delta_Y"] \\ (j_{U!}X)_\xi \arrow[r, "(j_{U!}\varphi)_\xi"] & (j_{U!}Y)_\xi \end{tikzcd}$$

where $\Delta_X : X \to j^*_U j_{U!} X$ is the unit of adjunction, and likewise for Δ_Y. More plainly, $\Delta_Y : Y \times 1_{T/U} \to Y \times (j^* j_{U!} 1_{T/U})$ is the product $\mathbf{1}_Y \times \Delta_U$, and under this identification, $\xi^*_U \Delta_Y$ is the mapping $\xi^*_U Y \to \xi^*_U Y \times (j^* j_{U!} 1_{T/U})$ given by the rule : $z \mapsto (z, a)$ for every $z \in \xi^*_U Y$. Especially, we see that y lies in the image of $\xi^*_U \Delta_Y$. But by assumption, the map $\xi^*_U \varphi$ is surjective, hence y lies in the image of $(j_{U!}\xi)_\xi$, a contradiction.

(ii) The localization morphisms j_U of example 2.2.6(iii), and the notion of point of a topos, form the basis for a technique to study the local properties of objects in a topos. Indeed, suppose that $\mathbf{P}(T, F, \varphi)$ is a property of sequences $F := (F_1, \dots, F_n)$ of objects in a topos T, and of morphisms $\varphi := (\varphi_1, \dots, \varphi_m)$ in T; we say that $\mathbf{P}$ *can be checked on stalks*, if the following two conditions hold for every topos T, every $F \in \mathrm{Ob}(T)^n$, and every $\varphi \in \mathrm{Morph}(T)^m$.

(a) $\mathbf{P}(T, F, \varphi)$ implies $\mathbf{P}(\mathbf{Set}, F_\xi, \varphi_\xi)$ for every T-point ξ. (Here $F_\xi := (F_{1\xi}, \dots, F_{n\xi})$, and likewise for φ_ξ.)
(b) If Ω is a conservative set of T-points such that $\mathbf{P}(\mathbf{Set}, F_\xi, \varphi_\xi)$ holds for every $\xi \in \Omega$, then $\mathbf{P}(T, F, \varphi)$ holds.

Then we have the following :

Lemma 2.2.16. *Let T be a topos with enough points, $(U_\lambda \to 1_T \mid \lambda \in \Lambda)$ a family of morphisms in T covering the final object (in the canonical topology C_T), and $\mathbf{P}$ a property that can be checked on stalks. Then, for any $F \in \mathrm{Ob}(T)^n$ and $\varphi \in \mathrm{Morph}(T)^m$, we have $\mathbf{P}(T, F, \varphi)$ if and only if $\mathbf{P}(T/U_\lambda, F_{|U_\lambda}, \varphi_{|U_\lambda})$ holds for every $\lambda \in \Lambda$.*

Proof. Suppose first that $\mathbf{P}(T, F, \varphi)$ holds. If η is any T/U_λ-point, then $\xi := j_{U_\lambda} \circ \eta$ is a T-point, and $(j^*_{U_\lambda} F)_\eta = F_\xi$, (and likewise for φ, with obvious notation) hence $\mathbf{P}(\mathbf{Set}, (j^*_{U_\lambda} F)_\eta, (j^*_{U_\lambda} \varphi)_\eta)$ holds. Since $\mathbf{P}$ can be checked on stalks, remark 2.2.14(i) then implies that $\mathbf{P}(T/U_\lambda, F_{|U_\lambda}, \varphi_{|U_\lambda})$ holds.

Conversely, suppose that $\mathbf{P}(T/U_\lambda, F_{|U_\lambda}, \varphi_{|U_\lambda})$ holds for every $\lambda \in \Lambda$, and let ξ be any T-point. By claim 2.1.45 we may assume that Λ is a small set; then for every T-point ξ, the set $(\amalg_{\lambda \in \Lambda} U_\lambda)_\xi = \amalg_{\lambda \in \Lambda} \xi^* U_\lambda$ covers the final object of $\mathbf{Set}$, *i.e.* is not empty. Pick $\lambda \in \Lambda$ such that $\xi^* U_\lambda \neq \varnothing$; by the discussion of (2.2.11) we see that ξ lifts to a T/U_λ-point η. Since $\mathbf{P}$ can be checked on stalks, $\mathbf{P}(\mathbf{Set}, (F_{|U_\lambda})_\eta, (\varphi_{|U_\lambda})_\eta)$ holds, and this means that $\mathbf{P}(\mathbf{Set}, F_\xi, \varphi_\xi)$ holds, so $\mathbf{P}(T, F, \varphi)$ holds. $\square$

2.2.17. Let X be any (small) set; we consider the functor

$$\mathsf{F}_X : \mathbf{Lift}(\xi)^o \to \mathbf{Set} \qquad (U, \xi_U, \omega_U) \mapsto \xi^*_U \xi_{U*} X.$$

For a given morphism $(\varphi, \omega_\varphi) : (U, \xi_U, \omega_U) \to (V, \xi_V, \omega_V)$ in $\mathbf{Lift}(\xi)$, set $\vartheta_V := j_\varphi \circ \xi_U$; by definition 2.2.3(iv), ω_φ is a natural isomorphism of functors $\omega_{\varphi*} : \vartheta_{V*} \xrightarrow{\sim} \xi_{V*}$, which determines (and is determined by) a natural isomorphism of functors $\omega^*_\varphi : \xi^*_V \xrightarrow{\sim} \vartheta^*_V$. The morphism $\mathsf{F}_X(\varphi, \omega_\varphi)$ is then defined by the commutative diagram :

$$\begin{tikzcd}
\xi^*_V \vartheta_{V*}(X) \arrow[r, "\omega^*_\varphi * \vartheta_{V*}(X)"] \arrow[d, "\xi^*_V * \omega_{\varphi*}(X)"'] & \vartheta^*_V \vartheta_{V*}(X) \arrow[d, "\xi^*_U * \varepsilon_\varphi * \xi_{U*}(X)"] \\
\xi^*_V \xi_{V*}(X) \arrow[r, "\mathsf{F}_X(\varphi, \omega_\varphi)"] & \xi^*_U \xi_{U*}(X)
\end{tikzcd}$$

where $\varepsilon_\varphi : j^*_\varphi j_{\varphi*} \Rightarrow \mathbf{1}_{\mathbf{Set}}$ is the counit of adjunction. With the notation of (2.2.11), to verify that F_X is a well defined functor, amounts to checking that $\mathsf{F}_X(\psi \circ \varphi, \omega_{\psi\circ\varphi}) = \mathsf{F}_X(\varphi, \omega_\varphi) \circ \mathsf{F}_X(\psi, \omega_\psi)$. To this aim, we may assume that either ω_φ is the identity of $j_\varphi \circ \xi_U$ (which then coincides with ξ_V), or else that φ is the identity (in which case U coincides with V, and j_φ is the identity of T/U), and likewise for ω_ψ and ψ. We have then four cases to consider separately; we will proceed somewhat briskly, since these verifications are unenlightening and rather tedious.

First, suppose that both ω_φ and ω_ψ are identities. In this case, we have :

$$\mathsf{F}_X(\varphi, \omega_\varphi) = \xi^*_U * \varepsilon_\varphi * \xi_{U*} X \qquad \mathsf{F}_X(\psi, \omega_\psi) = \xi^*_V * \varepsilon_\psi * \xi_{V*} X = \xi^*_U * j^*_\varphi * \varepsilon_\psi * j_{\varphi*} * \xi_{U*} X$$

and we remark as well that $\omega_{\psi\circ\varphi} = (\mathsf{c}_{\psi,\varphi} * \xi_U)^{-1}$. Then the sought identity translates the commutativity of the diagram $\xi^*_U * \mathscr{D} * \xi_{U*}$, where $\mathscr{D}$ is the diagram :

$$\begin{tikzcd}
j^*_\varphi \circ j^*_\psi \circ j_{\psi\circ\varphi*} \arrow[rr, "\mathsf{c}^{*-1}_{\psi,\varphi} * j_{\psi\circ\varphi*}"] \arrow[d, "j^*_\varphi * j^*_\psi * \mathsf{c}^{-1}_{\psi,\varphi*}"'] & & j^*_{\psi\circ\varphi} \circ j_{\psi\circ\varphi*} \arrow[d, "\varepsilon_{\psi\circ\varphi}"] \\
j^*_\varphi \circ j^*_\psi \circ j_{\psi*} \circ j_{\varphi*} \arrow[r, "j^*_\varphi * \varepsilon_\psi * j^*_\varphi"] & j^*_\varphi \circ j_{\varphi*} \arrow[r, "\varepsilon_\varphi"] & \mathbf{1}.
\end{tikzcd}$$

Now, notice that the bottom row is the counit of adjunction defined by the morphism $j_\psi \circ j_\varphi$; then the commutativity of $\mathscr{D}$ is just a special case of remark 1.1.10(iii).

Next, suppose that φ is the identity of $U = V$, and ω_ψ is the identity of $j_\psi \circ \xi_V = \xi_W$. In this case, $\omega_{\psi\circ\varphi} = j_\psi * \omega_\varphi$, and $\mathsf{F}_X(\psi, \omega_\psi)$ is the same as in the previous case, whereas :

$$\mathsf{F}_X(\varphi, \omega_\varphi) = (\omega^*_\varphi * \xi_{U*}) \circ (\xi^*_V * \omega_{\varphi*})^{-1}.$$

Then the sought identity translates the commutativity of the diagram :

$$\begin{tikzcd}
\xi_W^* \circ \xi_{W*} \arrow[d, "\xi_V^* * \varepsilon_\psi * \xi_{V*}"'] & \xi_W^* \circ j_{\psi *} \circ \xi_{U*} \arrow[l, "\xi_W^* * j_{\psi *} * \omega_{\psi *}"'] \arrow[r, "\omega_\varphi^* * j_\psi^* * j_{\psi *} * \xi_{U*}"] \arrow[d, "\xi_V^* * \varepsilon_\psi * \xi_{U*}"] & \xi_U^* \circ j_\psi^* \circ j_{\psi *} \circ \xi_{U*} \arrow[d, "\xi_U^* * \varepsilon_\psi * \xi_{U*}"] \\
\xi_V^* \circ \xi_{V*} & \xi_V^* \circ \xi_{U*} \arrow[l, "\xi_V^* * \omega_{\varphi *}"'] \arrow[r, "\omega_\varphi^* * \xi_{U*}"] & \xi_U^* \circ \xi_{U*}.
\end{tikzcd}$$

which is an immediate consequence of the naturality of $\omega_{\varphi *}$ and ω_φ^*.

The case where ω_ψ is the identity of ξ_W and ψ is the identity of $V = W$, is similar to the previous one. The remaining case where both φ and ψ are identities is easy, and shall be left to the reader.

Remark 2.2.18. (i) Let ξ be a point of the topos T; we remark that the category $\mathbf{Nbd}(\xi)$ is cofiltered; indeed, if (U_1, a_1) and (U_2, a_2) are two neighborhoods, the natural projections $U_{12} := U_1 \times U_2 \to U_i$ (for $i = 1, 2$) give morphisms of neighborhoods $(U_{12}, (a_1, a_2)) \to (U_i, a_i)$. (Notice that $(U_1 \times U_2)_\xi = U_{1,\xi} \times U_{2,\xi}$.) Likewise, since ξ^* commutes with equalizers, for any two morphisms $\varphi, \varphi' : (U_1, a_1) \to (U_2, a_2)$, we have a morphism of neighborhoods $\psi : (\mathrm{Equal}(\varphi, \varphi'), a_1) \to (U_1, a_1)$, such that $\varphi \circ \psi = \varphi' \circ \psi$.

(ii) Denote by $\iota : \mathbf{Nbd}(\xi) \to T$ the functor given by the rule : $(U, a) \mapsto U$. Then there is a natural transformation of functors :

$$\tau_\xi : \mathrm{Hom}_T(\iota^o, F) \Rightarrow c_{F_\xi} \quad : \quad \mathbf{Nbd}(\xi)^o \to \mathbf{Set}$$

where c_{F_ξ} denotes the constant functor with value F_ξ. Indeed, if (U, a) is any neighborhood of ξ, we define $\tau_\xi(U, a) : \mathrm{Hom}_T(U, F) \to F_\xi$ by the rule : $s \mapsto s_\xi(a)$ for every $s : U \to F$. Then we claim that τ_ξ induces a natural bijection :

$$\operatorname*{colim}_{\mathbf{Nbd}(\xi)^o} \mathrm{Hom}_T(\iota^o, F) \xrightarrow{\sim} F_\xi. \tag{2.2.19}$$

Indeed, every element of the above colimit is represented by a triple (U, a, β), where (U, a) is a neighborhood of ξ, and $\beta \in \mathrm{Hom}_T(U, F)$; two such triples (U_1, a_1, β_1) and (U_2, a_2, β_2) are identified if and only if there exist morphisms $\varphi_i : (V, b) \to (U_i, a_i)$ in $\mathbf{Nbd}(\xi)$ (for $i = 1, 2$), such that $\varphi_1 \circ \beta_1 = \varphi_2 \circ \beta_2$. It is then easily seen that every such triple is equivalent to a unique triple of the form $(F, a, \mathbf{1}_F)$, which gets mapped to a under $\tau_\xi(F, a)$, whence the assertion.

(iii) For every object (U, ξ_U, ω_U) of $\mathbf{Lift}(\xi)$, let $\varepsilon_U : \xi_U^* \xi_{U*} X \to X$ be the counit of adjunction coming from the morphism of topoi ξ_U; we claim that the rule $(U, \xi_U, \omega_U) \mapsto \varepsilon_U$ defines a natural transformation

$$\mathsf{F}_X \Rightarrow c_X \tag{2.2.20}$$

where c_X denotes the constant functor with value X. Indeed, let $(\varphi, \omega_\varphi) : (U, \xi_U, \omega_U) \to (V, \xi_V, \omega_V)$ be a morphism in $\mathbf{Lift}(\xi)$; we need to show that $\varepsilon_U \circ \mathsf{F}_X(\varphi, \omega_\varphi) = \varepsilon_V$. This amounts to checking the identity :

$$\varepsilon_U \circ (\xi_U^* * \varepsilon_\varphi * \xi_{U*}) \circ (\omega_\varphi^* * \vartheta_{V*}) = \varepsilon_V \circ (\xi_V^* * \omega_{\varphi *})$$

where $\vartheta_V := j_\varphi \circ \xi_U$. Now, notice that $\varepsilon'_V := \varepsilon_U \circ (\xi_U^* * \varepsilon_\varphi * \xi_{U*}) : \vartheta_V^* \vartheta_{V*} \to \mathbf{1}$ is the counit of adjunction given by the morphism of topoi ϑ_V. On the other hand, remark 1.1.10(ii) says that :

$$\omega_\varphi^* = (\varepsilon_V * \vartheta_V^*) \circ (\xi_V^* * \omega_{\varphi *} * \vartheta_V^*) \circ (\xi_V^* * \eta'_V)$$

where $\eta'_V : \mathbf{1} \to \vartheta_{V*} \vartheta_V^*$ is the unit of adjunction given by ϑ_V. The naturality of ε_V implies the identity :

$$\varepsilon_V \circ (\xi_V^* * \xi_{V*} * \varepsilon'_V) = \varepsilon'_V \circ (\varepsilon_V * \vartheta_V^* * \vartheta_{V*}).$$

Hence we come down to showing that

$$(\xi_V^* * \xi_{V*} * \varepsilon'_V) \circ (\xi_V^* * \omega_{\varphi *} * \vartheta_V^* * \vartheta_{V*}) \circ (\xi_V^* * \eta'_V * \vartheta_{V*}) = \xi_V^* * \omega_{\varphi *}.$$

The latter in turn will follow, once we know that

$$(\xi_{V*} * \varepsilon'_V) \circ (\omega_{\varphi*} * \vartheta^*_V * \vartheta_{V*}) \circ (\eta'_V * \vartheta_{V*}) = \omega_{\varphi*}. \tag{2.2.21}$$

However, the naturality of $\omega_{\varphi*}$ says that

$$(\omega_{\varphi*} * \vartheta^*_V * \vartheta_{V*}) \circ (\eta'_V * \vartheta_{V*}) = (\xi_{V*} * \eta'_V) \circ \omega_{\varphi*} \tag{2.2.22}$$

Then (2.2.21) follows easily from (2.2.22) and the triangular identities of (1.1.8).

Lemma 2.2.23. *The natural transformation* (2.2.20) *induces a natural isomorphism :*

$$\operatorname*{colim}_{\mathbf{Lift}(\xi)^o} \mathsf{F}_X \xrightarrow{\sim} X \qquad \textit{for every (small) set } X.$$

Proof. Set $\mathscr{C} := \mathsf{Morph}(\mathbf{Lift}(\xi)^o)$. We define yet another functor

$$\mathsf{G} : \mathscr{C} \to \mathbf{Set}$$

by the rule :

$$((U', \xi_{U'}, \omega_{U'}) \xrightarrow{(\psi,\omega_\psi)} (U, \xi_U, \omega_U)) \mapsto \mathsf{G}(\psi, \omega_\psi) := \Gamma(\psi, \xi_{U*}X)$$

(here we regard $\psi : U' \to U$ as an object of T/U). Recall that a morphism $(\beta, \omega_\beta) \to (\psi, \omega_\psi)$ in $\mathscr{C}$ is a commutative diagram of morphisms in $\mathbf{Lift}(\xi)$

$$\mathscr{D} \quad : \quad \begin{array}{ccc} (U', \xi_{U'}, \omega_{U'}) & \xrightarrow{(\varphi',\omega_{\varphi'})} & (V', \xi_{V'}, \omega_{V'}) \\ {\scriptstyle (\psi,\omega_\psi)}\downarrow & & \downarrow{\scriptstyle (\beta,\omega_\beta)} \\ (U, \xi_U, \omega_U) & \xrightarrow{(\varphi,\omega_\varphi)} & (V, \xi_V, \omega_V). \end{array}$$

To such a morphism, we assign the map $\mathsf{G}(\mathscr{D}) : \mathsf{G}(\beta, \omega_\beta) \to \mathsf{G}(\psi, \omega_\psi)$ defined as follows. First, we may regard φ' as a morphism $j_{\varphi!}(\psi) \to \beta$ in T/V. Denote by

$$\Delta_\psi : \psi \to j^*_\varphi j_{\varphi!}(\psi)$$

the unit of adjunction. Then, for every $s : \beta \to \xi_{V*}X$, the section $\mathsf{G}(\mathscr{D})(s) : \psi \to \xi_{U*}X$ is the composition :

$$\psi \xrightarrow{j^*_\varphi(s\circ\varphi'\ )\circ\Delta_\psi} j^*_\varphi \xi_{V*}X \xrightarrow{j^*_\varphi * \omega^{-1}_{\varphi*}X} j^*_\varphi j_{\varphi*}\xi_{U*}X \xrightarrow{\varepsilon_\varphi * \xi_{U*}X} \xi_{U*}X.$$

With some patience, the reader may check that G is really a well defined functor. Next, we define a natural transformation :

$$\mathsf{G} \Rightarrow \mathsf{F}_X \circ \mathsf{s} \tag{2.2.24}$$

where $\mathsf{s} : \mathscr{C} \to \mathbf{Lift}(\xi)^o$ is the source functor of (1.1.17). Indeed, notice that, for every lifting (U, ξ_U, ω_U) of ξ, we have an isomorphism of categories :

$$\mathbf{Lift}(\xi)/(U, \xi_U, \omega_U) \xrightarrow{\sim} \mathbf{Lift}(\xi_U) \qquad (\psi, \omega_\psi) \mapsto (\psi, \xi_{U'}, \omega_\psi)$$

(where (ψ, ω_ψ) is as in the foregoing). On the other hand, by composing the equivalence N_{ξ_U} of (2.2.13), and the natural transformation τ_{ξ_U} of remark 2.2.18(ii) (for $F := \xi_{U*}X$), we obtain a natural transformation :

$$\tau_{\xi_U} * N_{\xi_U}(\psi, \xi_{U'}, \omega_\psi) : \Gamma(\psi, \xi_{U*}X) \to \xi^*_U \xi_{U*}X$$

which yields (2.2.24). By inspecting the construction, and in view of (2.2.19), we get a natural isomorphism :

$$\operatorname*{colim}_{\mathscr{C}} \mathsf{G} \xrightarrow{\sim} \operatorname*{colim}_{\mathbf{Lift}(\xi)^o} \mathsf{F}.$$

Now, notice that the functor

$$\mathbf{Lift}(\xi)^o \to \mathscr{C} \qquad (U, \xi_U, \omega_U) \mapsto \mathbf{1}_{(U,\xi_U,\omega_U)} \tag{2.2.25}$$

is cofinal. A simple inspection reveals that the composition of G with (2.2.25) is the constant functor with value X, whence the contention. □

2.2.26. Suppose now that $T = C^\sim$ for some small site $C := (\mathscr{C}, J)$. For a point ξ of T, we may define another category $\mathbf{Nbd}(\xi, C)$, whose objects are the pairs (U, a) where $U \in \mathrm{Ob}(\mathscr{C})$ and $a \in (h^a_U)_\xi$; the morphisms $(U, a) \to (U', a')$ are the morphisms $f : U \to U'$ in $\mathscr{C}$ such that $(h^a_f)_\xi(a) = a'$. (Notation of remark 2.1.27(iii).) The rule $(U, a) \mapsto (h^a_U, a)$ defines a functor

$$\mathbf{Nbd}(\xi, C) \to \mathbf{Nbd}(\xi). \qquad (2.2.27)$$

Proposition 2.2.28. *In the situation of* (2.2.26)*, we have :*

(i) *The category* $\mathbf{Nbd}(\xi, C)$ *is cofiltered.*
(ii) *The functor* (2.2.27) *is cofinal.*

Proof. To begin with, since ξ^* commutes with all colimits, remark 2.1.27(iii) implies easily that, for every object (U, a) of $\mathbf{Nbd}(\xi)$ there exists an object (V, b) of $\mathbf{Nbd}(\xi, C)$ and a morphism $(h^a_V, b) \to (U, a)$ in $\mathbf{Nbd}(\xi)$. Hence, it suffices to show (i).

Thus, let (U_1, a_1) and (U_2, a_2) be two objects of $\mathbf{Nbd}(\xi, C)$; since $\mathbf{Nbd}(\xi)$ is cofiltered, we may find an object (F, b) of $\mathbf{Nbd}(\xi)$ and morphisms $\varphi_i : (F, b) \to (h^a_{U_i}, a_i)$ (for $i = 1, 2$). By the foregoing, we may also assume that $(F, b) = (h^a_V, b)$ for some object (V, b) of $\mathbf{Nbd}(\xi, C)$, in which case $\varphi_i \in h^a_{U_i}(V)$ for $i = 1, 2$. By remark 2.1.27(iv), we may find a sieve $\mathscr{S}$ covering V such that $\varphi_i \in \mathrm{Hom}_{\mathscr{C}^\wedge}(h_\mathscr{S}, h^{\mathrm{sep}}_{U_i})$ for both $i = 1, 2$.

On the other hand, (2.1.2) and proposition 2.1.34 yield a natural isomorphism :

$$\operatorname*{colim}_{\mathscr{S}} h^a \circ \mathsf{s} \xrightarrow{\sim} h^a_V.$$

Therefore, since ξ^* commutes with all colimits, we may find $(f : S \to V) \in \mathrm{Ob}(\mathscr{S})$ and $c \in (h^a_S)_\xi$ such that $h^a_f : (h^a_S, c) \to (h^a_V, b)$ is a morphism of neighborhoods of ξ.

For $i = 1, 2$, denote by $\overline{\varphi}_{S,i} \in h^{\mathrm{sep}}_U(S)$ the image of φ_i (under the map induced by the natural morphism $h_S \to h_\mathscr{S}$ coming from (2.1.2)). Pick any $\varphi_{S,i} \in \mathrm{Hom}_\mathscr{C}(S, V)$ in the preimage of $\overline{\varphi}_{S,i}$; then $\varphi_{S,i}$ defines a morphism $(S, c) \to (U_i, a_i)$ in $\mathbf{Nbd}(\xi, C)$.

Next, suppose that $\varphi_1, \varphi_2 : (U, a) \to (U', a')$ are two morphisms in $\mathbf{Nbd}(\xi, C)$; arguing as in the foregoing, we may find an object (V, b) of $\mathbf{Nbd}(\xi, C)$, and $\psi \in h^{\mathrm{sep}}_U(V) = \mathrm{Hom}_{\mathscr{C}^\wedge}(h^{\mathrm{sep}}_V, h^{\mathrm{sep}}_U)$ whose image in $h^a_U(V)$ yields a morphism $\psi^a : (h^a_V, b) \to (h^a_U, a)$ in $\mathbf{Nbd}(\xi)$, and such that $\varphi^{\mathrm{sep}}_1 \circ \psi = \varphi^{\mathrm{sep}}_2 \circ \psi$ in $h^{\mathrm{sep}}_{U'}(V)$. We may then find a covering subobject $i : R \to h_V$, such that $\varphi_1 \circ (\psi \circ i) = \varphi_2 \circ (\psi \circ i)$ in $\mathrm{Hom}_{\mathscr{C}^\wedge}(R, h'_U)$. Again, by combining (2.1.2) and proposition 2.1.34, we deduce that there exists a morphism $\beta : (V', b') \to (V, b)$ in $\mathbf{Nbd}(\xi, C)$, such that $\varphi_1 \circ (\psi \circ \beta) = \varphi_2 \circ (\psi \circ \beta)$. This completes the proof of (i). □

As a corollary of proposition 2.2.28 and of remark (2.2.18)(ii), we deduce, for every sheaf F on C, a natural isomorphism :

$$\operatorname*{colim}_{\mathbf{Nbd}(\xi, C)} F \circ \iota^o_C \xrightarrow{\sim} F_\xi$$

where $\iota_C : \mathbf{Nbd}(\xi, C) \to \mathscr{C}$ is the functor given by the rule $(U, a) \mapsto U$ on every object (U, a).

2.3. **Algebra on a topos.**

Let T be any topos, and endow T with the structure of tensor category as explained in example 1.2.10 (so, the tensor functor is given by fixed choices of products for every pairs of objects of T, and any final object 1_T can be taken for unit object of $(T, \otimes)$). We notice that $(T, \otimes)$ admits an internal Hom functor (see remark 1.2.12(ii)). Indeed, let X and X' be any two objects of T. It is easily seen that the presheaf on T :

$$U \mapsto \mathrm{Hom}_{T/U}(X'_{|U}, X_{|U}) = \mathrm{Hom}_T(X' \times U, X)$$

is actually a sheaf on (T, C_T) (notation of example 2.2.6(iii)), so it is an object of T, denoted :

$$\mathscr{H}om_T(X', X).$$

The functor :

$$T \to T \quad : \quad X \mapsto \mathscr{H}om_T(X', X)$$

is right adjoint to the functor $T \to T : Y \mapsto Y \times X'$, so it is an internal Hom functor for X'.

If $f : T \to S$ is a morphism of topoi, and $Y \in \mathrm{Ob}(S)$, we have a natural isomorphism in S :

$$\mathscr{H}om_S(Y, f_*X) \xrightarrow{\sim} f_*\mathscr{H}om_T(f^*Y, X) \tag{2.3.1}$$

which, on every $U \in \mathrm{Ob}(S)$, induces the natural bijection :

$$\mathrm{Hom}_S(Y \times U, f_*X) \xrightarrow{\sim} \mathrm{Hom}_T(f^*Y \times f^*U, X)$$

given by the adjunction (f^*, f_*). By general nonsense, from (2.3.1) we derive a natural morphism in S :

$$f_*\mathscr{H}om_T(X', X) \to \mathscr{H}om_S(f_*X', f_*X)$$

and in T :

$$f^*\mathscr{H}om_S(Y', Y) \xrightarrow{\vartheta_f} \mathscr{H}om_T(f^*Y', f^*Y) \qquad \text{for any } Y, Y' \in \mathrm{Ob}(S).$$

Moreover, if $g : U \to T$ is another morphism of topoi, the diagram :

$$\begin{tikzcd}[column sep=small] g^*f^*\mathscr{H}om_S(Y', Y) \arrow[rr, "g^*\vartheta_f"] \arrow[dr, "\vartheta_{f\circ g}"'] & & g^*\mathscr{H}om_T(f^*Y', f^*Y) \arrow[dl, "\vartheta_g"] \\ & \mathscr{H}om_U(g^*f^*Y', g^*f^*Y) & \end{tikzcd} \tag{2.3.2}$$

commutes, up to a natural isomorphism.

2.3.3. Let A be any object of T, and (X, μ_X) a left A-module for the tensor category structure on T as in (2.3); for every object U of T we obtain a left $A_{|U}$-module (on the topos T/U : see example 2.2.6(iii)), by the rule :

$$(X, \mu_X)_{|U} := (X_{|U}, \mu_X \times \mathbf{1}_U).$$

If $(X', \mu_{X'})$ is another left A-module, it is easily seen that the presheaf on T :

$$U \mapsto \mathrm{Hom}_{A_{|U}\text{-}\mathbf{Mod}_l}((X, \mu_X)_{|U}, (X', \mu_{X'})_{|U})$$

is actually a sheaf for the canonical topology, so it is an object of T, denoted :

$$\mathscr{H}om_{A_l}((X, \mu_X), (X', \mu_{X'}))$$

(or just $\mathscr{H}om_{A_l}(X, X')$, if the notation is not ambiguous). The same considerations can be repeated for the sets of morphisms of right B-modules, and of (A, B)-bimodules, so one gets objects $\mathscr{H}om_{B_r}(X, X')$ and $\mathscr{H}om_{(A,B)}(X, X')$. By a simple inspection, we see that these objects are naturally isomorphic to the objects denoted in the same way in (1.2.14), so the notation is not in conflict with *loc.cit.*; it also follows that $\mathscr{H}om_{A_l}(X, X')$ is the equalizer of two morphisms in T :

$$\mathscr{H}om_T(X, X') \rightrightarrows \mathscr{H}om_T(A \times X, X') \ .$$

In the same vein, let $A, B, C \in \mathrm{Ob}(T)$ be any three objects, S an (A, B)-bimodule, S' a (C, B)-bimodule, and S'' a (C, A)-bimodule. Then the (C, B)-bimodule $\mathscr{H}om_{B_r}(S, S')$ and the (C, B)-bimodule $S'' \otimes_A S$ (see (1.2.17)) are the sheaves on (T, C_T) associated to the presheaf given by the rules : $U \mapsto \mathrm{Hom}_{B_{|U,r}}(S_{|U}, S'_{|U})$, and respectively : $U \mapsto S'(U) \otimes_{M(U)} S(U)$ for every object U of T. Furthermore, the general theory of monoids, their modules and their tensor products, developed in section 1.2 is available in the present situation, so we have a well defined notion of T-monoid (see example 1.2.21(ii) and remark 1.2.24). Via the equivalence (2.2.1), a T-monoid $\underline{M}$ is also the same as a sheaf of monoids M on the site (T, C_T), and a left (resp.

right, resp. bi-) $\underline{M}$-module is the same as the datum of a sheaf S in (T, C_T), such that $S(U)$ is a left (resp. right, resp. bi-) $M(U)$-module, for every object U of T.

2.3.4. Let $f : T_1 \to T_2$ be a morphism of topoi, A_1 an object of T_1, and (X, μ_X) a left A_1-module. Since f_* is left exact, we have a natural isomorphism : $f_*(A_1 \times X) \xrightarrow{\sim} f_*A_1 \times f_*X$, so we obtain a left f_*A_1-module :

$$f_*(X, \mu_X) := (f_*X, f_*\mu_X)$$

which we denote just f_*X, unless the notation is ambiguous. Likewise, since f^* is left exact, from any object A_2 of T_2, and any left A_2-module (Y, μ_Y), we obtain a left f^*A_2-module :

$$f^*(Y, \mu_Y) := (f^*Y, f^*\mu_Y).$$

The same considerations apply of course, also to right modules and to bimodules. Furthermore, let A_i, B_i, C_i be three objects of T_i (for $i = 1, 2$); since f_* is left exact, for any (A_1, B_1)-bimodule X and any (C_1, A_1)-bimodule X' we have a natural morphism of (f_*C_1, f_*B_1)-bimodules :

$$f_*X' \otimes_{f_*A_1} f_*X \to f_*(X' \otimes_{A_1} X)$$

and since f^* is exact, for any (A_2, B_2)-bimodule Y and any (C_2, A_2)-bimodule Y' we have a natural isomorphism :

$$f^*Y' \otimes_{f^*A_2} f^*Y \xrightarrow{\sim} f^*(Y' \otimes_{A_2} Y)$$

of (f^*C_2, f^*B_2)-bimodules.

2.3.5. The constructions of the previous paragraphs also apply to presheaves on T : this can be seen, *e.g.* as follows. Pick a universe V such that T is V-small; then $T^\wedge_{\mathsf{V}}$ is a V-topos. Hence, if A is any V-presheaf on T, and $X, X' \in \mathrm{Ob}(T^\wedge_{\mathsf{V}})$ two left A-modules, we may construct $\mathscr{H}om_{A_l}(X, X')$ as an object in $T^\wedge_{\mathsf{V}}$. Now, if A, X, X' lie in the full subcategory $T^\wedge_{\mathsf{U}}$ of $T^\wedge_{\mathsf{V}}$, it is easily seen that also $\mathscr{H}om_{A_l}(X, X')$ lies in $T^\wedge_{\mathsf{U}}$. Likewise, if A, B, C are two U-presheaves on T, we may define $X' \otimes_A X$ in $T^\wedge_{\mathsf{U}}$, for any (A, B)-bimodule X and (C, A)-bimodule X', and this tensor product will still be left adjoint to the $\mathscr{H}om$-functor for presheaves on T.

Moreover, we have a natural morphism of topoi $i_{\mathsf{V}} : (T, C_T)^\sim_{\mathsf{V}} \to T^\wedge_{\mathsf{V}}$, given by the forgetful functor (and its left adjoint $F \mapsto F^a$) (see example 2.2.6(i)). The restriction of $i_{\mathsf{V}*}$ to the full subcategory $T^\wedge_{\mathsf{U}}$ factors through the inclusion $T \to (T, C_T)^\sim_{\mathsf{V}}$, therefore, the discussion of (2.3.4) specializes to show that, for every U-presheaf A on T, and every A-module $X \in \mathrm{Ob}(T^\wedge_{\mathsf{U}})$, the object $X^a \in \mathrm{Ob}(T)$ is naturally an A^a-module. Also, for any (A, B)-bimodule X and (C, A)-bimodule X', such that A, B, C, X, X' are U-small, we have a natural isomorphism of (C^a, B^a)-bimodules :

$$X'^a \otimes_{A^a} X^a \xrightarrow{\sim} (X' \otimes_A X)^a.$$

The following definition gathers some further notions – specific to monoids over a topos – which shall be used in this work.

Definition 2.3.6. Let T be a topos, $\underline{M}$ a T-monoid, S a left (resp. right, resp. bi-) $\underline{M}$-module.

(i) S is said to be *of finite type*, if there exists a covering family $(U_\lambda \to 1_T \mid \lambda \in \Lambda)$ of the final object of T, and for every $\lambda \in \Lambda$ an integer $n_\lambda \in \mathbb{N}$ and an epimorphism of left (resp. right, resp. bi-) $\underline{M}_{|U_\lambda}$-modules : $\underline{M}^{\oplus n_\lambda}_{|U_\lambda} \to S_{|U_\lambda}$.

(ii) S is *finitely presented*, if there exists a covering family $(U_\lambda \to 1_T \mid \lambda \in \Lambda)$ of the final object of T, and for every $\lambda \in \Lambda$ integers $m_\lambda, n_\lambda \in \mathbb{N}$ and morphisms f_λ, g_λ : $\underline{M}^{\oplus m_\lambda}_{|U_\lambda} \to \underline{M}^{\oplus n_\lambda}_{|U_\lambda}$ whose coequalizer – in the category of left (resp. right, resp. bi-) $\underline{M}_{|U_\lambda}$-modules – is isomorphic to $S_{|U_\lambda}$.

(iii) S is said to be *coherent*, if it is of finite type, and for every object U in T, every submodule of finite type of $S_{|U}$ is finitely presented.

(iv) S is said to be *invertible*, if there exists a covering family $(U_\lambda \to 1_T \mid \lambda \in \Lambda)$, and for every $\lambda \in \Lambda$, an isomorphism $\underline{M}_{|U_\lambda} \xrightarrow{\sim} S_{|U_\lambda}$ of left (resp. right, resp. bi-) $\underline{M}_{|U}$-modules. (Thus, every invertible module is finitely presented.)

(v) An ideal $I \subset \underline{M}$ is said to be *invertible*, (resp. *of finite type*, resp. *finitely presented*, resp. *coherent*) if it is such, when regarded as an $\underline{M}$-bimodule.

Example 2.3.7. (i) Take again $T = \mathbf{Set}$. Then an M-module S is of finite type if and only if there exists a finite subset $\Sigma \subset S$, such that $S = M \cdot \Sigma$, with obvious notation. In this case, we say that Σ is a *finite system of generators* of S (and we also say that S is *finitely generated*; likewise, an ideal of finite type is also called finitely generated). We say that S is *cyclic*, if $S = M \cdot s$ for some $s \in S$.

(ii) If S and S' are two M-modules, the coproduct $S \oplus S'$ is the disjoint union of S and S', with scalar multiplication given by the disjoint union of the laws μ_S and $\mu_{S'}$. The product $S \times S'$ is the cartesian product of the underlying sets, with scalar multiplication given by the rule : $x \cdot (s, s') := (x \cdot s, x \cdot s')$ for every $x \in M$, $s, \in S$, $s' \in S'$.

For future use, let us also make the :

Definition 2.3.8. Let T be any topos, $\underline{P}$ a T-monoid, and $(N, +, 0)$ any monoid.

(i) We say that $\underline{P}$ is N*-graded*, if it admits a morphism of monoids $\pi : \underline{P} \to N_T$, where N_T is the constant sheaf of monoids arising from N (the coproduct of copies of the final object 1_T indexed by N). For every $n \in N$ we let $\underline{P}_n := \pi^{-1}(n_T)$, the preimage of the global section corresponding to n. Then
$$\underline{P} = \coprod_{n \in N} \underline{P}_n$$
the coproduct of the objects $\underline{P}_n$, and the multiplication law of $\underline{P}$ restricts to a map $\underline{P}_n \times \underline{P}_m \to \underline{P}_{n+m}$, for every $n, m \in N$. Especially, each $\underline{P}_n$ is a $\underline{P}_0$-module, and $\underline{P}$ is also the direct sum of the $\underline{P}_n$, in the category of $\underline{P}_0$-modules. The morphism π is called the *grading* of $\underline{P}$.

(ii) In the situation of (i), let S be a left (resp. right, resp. bi-) $\underline{P}$-module. We say that S is N*-graded*, if it admits a morphism of $\underline{P}$-modules $\pi_S : S \to N_T$, where N_T is regarded as a $\underline{P}$-bimodule via the grading π of $\underline{P}$. Then S is the coproduct $S = \coprod_{n \in N} S_n$, where $S_n := \pi_S^{-1}(n_T)$, and the scalar multiplication of S restricts to morphisms $\underline{P}_n \times S_m \to S_{n+m}$, for every $n, m \in N$. The morphism π_S is called the *grading* of S.

(iii) A morphism $\underline{P} \to \underline{Q}$ of N-graded T-monoids is a morphism of monoids that respects the gradings, with obvious meaning. Likewise one defines morphisms of N-graded $\underline{P}$-modules.

Example 2.3.9. Take $T = \mathbf{Set}$, and let M be any commutative monoid. Then we claim that the only invertible object in the tensor category M-$\mathbf{Mod}_l$ is M; *i.e.* if S and S' are any two (M, M)-bimodules, then $S \otimes_M S' \simeq M$ if and only if S and S' are both isomorphic to M.

Indeed, let $\varphi : S \otimes_M S' \xrightarrow{\sim} M$ be an isomorphism, and choose $s_0 \in S$, $s'_0 \in S'$ such that $\varphi(s_0 \otimes s'_0) = 1$. Consider the morphisms of left M-modules :
$$M \xrightarrow{\alpha} S \xrightarrow{\beta} M \qquad M \xrightarrow{\alpha'} S' \xrightarrow{\beta'} M$$
such that :
$$\alpha(m) = m \cdot s_0 \quad \beta(s) = \varphi(s \otimes s'_0) \quad \alpha'(m) = m \cdot s'_0 \quad \beta'(s') = \varphi(s_0 \otimes s')$$
for every $m \in M$, $s \in S$, $s' \in S'$; we notice that $\beta \circ \alpha = \mathbf{1}_M = \beta' \circ \alpha'$. There follows natural morphisms :
$$S' \xrightarrow{\gamma} M \otimes_M S' \xrightarrow{\alpha \otimes_M S'} S \otimes_M S' \xrightarrow{\beta \otimes_M S'} M \otimes_M S' \xrightarrow{\gamma^{-1}} S'$$

whose composition is the identity $\mathbf{1}_{S'}$. However, it is easily seen that $\varphi \circ (\alpha \otimes_M S') \circ \gamma = \beta'$ and $\gamma^{-1} \circ (\beta \otimes_M S') \circ \varphi^{-1} = \alpha'$, thus $\alpha' \circ \beta' = \mathbf{1}_{S'}$, hence both α' and β' are isomorphisms, and the same holds for α and β.

Example 2.3.10. Let $\underline{M}$ be a T-monoid, and $\mathscr{L}$ a $\underline{M}$-bimodule. For every $n \in \mathbb{N}$, let $\mathscr{L}^{\otimes n} := \mathscr{L} \otimes_{\underline{M}} \cdots \otimes_{\underline{M}} \mathscr{L}$, the n-fold tensor power of $\mathscr{L}$. The $\mathbb{N}$-graded $\underline{M}$-bimodule

$$\mathrm{Tens}^{\bullet}_{\underline{M}} \mathscr{L} := \coprod_{n \in \mathbb{N}} \mathscr{L}^{\otimes n}$$

is naturally a $\mathbb{N}$-graded T-monoid, with composition law induced by the natural morphisms $\mathscr{L}^{\otimes n} \otimes_{\underline{M}} \mathscr{L}^{\otimes m} \xrightarrow{\sim} \mathscr{L}^{\otimes n+m}$, for every $n, m \in \mathbb{N}$. (Here we set $\mathscr{L}^{\otimes 0} := \underline{M}$.) If $\mathscr{L}$ is invertible, $\mathrm{Tens}^{\bullet}_{\underline{M}} \mathscr{L}$ is a commutative $\mathbb{N}$-graded T-monoid, which we also denote $\mathrm{Sym}^{\bullet}_{\underline{M}} \mathscr{L}$.

Remark 2.3.11. (i) Let $f : T \to S$ be a morphism of topoi, $\underline{M} := (M, \mu_M)$ a T-semigroup, and $\underline{N} := (N, \mu_N)$ a S-semigroup. Then clearly $f_*\underline{M} := (f_*M, f_*\mu_M)$ is a S-semigroup, and $f^*\underline{N} := (f^*N, f^*\mu_N)$ is a T-semigroup.

(ii) Furthermore, if $1_M : 1_T \to M$ (resp. $1_N : 1_S \to N$) is a unit for M (resp. for N), then notice that $f_*1_T = 1_S$ (resp. $f^*1_S = 1_T$), since the final object is the empty product; it follows that f_*1_M (resp. f^*1_S) is a unit for $f_*\underline{M}$ (resp. for $f^*\underline{N}$).

(iii) Obviously, if $\underline{M}$ (resp. $\underline{N}$) is commutative, the same holds for $f_*\underline{M}$ (resp. $f^*\underline{N}$).

(iv) If X is a left $\underline{M}$-module, then f_*X is a left $f_*\underline{M}$-module, and if Y is a left $\underline{N}$-module, then f^*Y is a left $f^*\underline{N}$-module. The same holds for right modules and bimodules.

(v) Moreover, let $\varepsilon_M : f^*f_*\underline{M} \to \underline{M}$ (resp. $\eta_N : \underline{N} \to f_*f^*\underline{N}$) be the counit (resp. unit) of adjunction. Then the counit (resp. unit) :

$$\varepsilon_X : f^*f_*X \to X_{(\varepsilon_M)} \qquad (\text{resp. } \eta_Y : Y \to f_*f^*Y_{(\eta_N)})$$

is a morphism of $f^*f_*\underline{M}$-modules (resp. of $\underline{N}$-modules) (notation of (1.2.26)). (Details left to the reader.)

(vi) Let $\varphi : f^*\underline{N} \to \underline{M}$ be a morphism of T-monoids. Then the functor

$$\underline{N}\text{-}\mathbf{Mod}_l \to \underline{M}\text{-}\mathbf{Mod}_l \quad : \quad Y \mapsto M \otimes_{f^*N} f^*Y$$

is left adjoint to the functor :

$$\underline{M}\text{-}\mathbf{Mod}_l \to \underline{N}\text{-}\mathbf{Mod}_l \quad : \quad X \mapsto f_*X_{(\eta_N)}.$$

(And likewise for right modules and bimodules : details left to the reader.)

(vii) The considerations of (2.3.5) also apply to monoids : we get that, for any presheaf of monoids $\underline{M} := (M, \mu_M, 1_M)$ on T, the datum $\underline{M}^a := (M^a, \mu^a_M, 1^a_M)$ is a T-monoid, and we have a well defined functor :

$$\underline{M}\text{-}\mathbf{Mod}_l \to \underline{M}^a\text{-}\mathbf{Mod}_l \qquad X \mapsto X^a.$$

(And as usual, the same applies to right modules and bimodules.)

2.3.12. Let T be a topos, U any object of T, and $\underline{M}$ a T-monoid. As a special case of remark 2.3.11(i), we have the T/U-monoid $j^*_U\underline{M} = \underline{M}_{|U}$, and if we take $\varphi := \mathbf{1}_{j^*_U\underline{M}}$ in remark 2.3.11(vi), we deduce that the functor

$$j^*_U : \underline{M}\text{-}\mathbf{Mod}_l \to \underline{M}_{|U}\text{-}\mathbf{Mod}_l \qquad Y \mapsto Y_{|U}$$

admits the right adjoint j_{U*}. Now, suppose that $X \to U$ is any left $\underline{M}_{|U}$-module. The scalar multiplication of X is a U-morphism $\mu_X : M \times X \to X$ and $j_{U!}\mu_X$ is the same morphism, seen as a morphism in T (notation of example 2.2.6(iii)). In other words, $j_{U!}$ induces a faithful functor on left modules, also denoted :

$$j_{U!} : \underline{M}_{|U}\text{-}\mathbf{Mod}_l \to \underline{M}\text{-}\mathbf{Mod}_l.$$

It is easily seen that this functor is left adjoint to the foregoing functor j_U^*. Especially, this functor is right exact; it is not generally left exact, since it does not preserve the final object (unless $U = 1_T$). However, it does commute with fibre products, and therefore transforms monomorphisms into monomorphisms. All this holds also for right modules and bimodules.

2.3.13. Let T be any category as in example 1.2.21(i), denote by 1_T a final object of T, and by $\underline{M}$ any T-monoid. A *pointed left* $\underline{M}$*-module* is a datum

$$(S, 0_S)$$

consisting of a left $\underline{M}$-module S and a morphism of $\underline{M}$-modules $0_S : 1_T \to S$, where 0 is the final object of M-$\mathbf{Mod}_l$. Often we shall write S instead of $(S, 0_S)$, unless this may give rise to ambiguities. As usual, a morphism $\varphi : S \to T$ of pointed modules is a morphism of M-modules, such that $0_T = \varphi \circ 0_S$. In other words, the resulting category is just $0/M$-$\mathbf{Mod}_l$, and shall be denoted $\underline{M}$-$\mathbf{Mod}_{l\circ}$.

Likewise one may define the category $\underline{M}$-$\mathbf{Mod}_{r\circ}$ of right $\underline{M}$-modules, and $(\underline{M}, \underline{N})$-$\mathbf{Mod}_\circ$ of pointed bimodules, for given T-monoids $\underline{M}$ and $\underline{N}$.

Remark 2.3.14. Let T be a category as in remark 1.2.24.

(i) For reasons that will become readily apparent, for many purposes the categories of pointed modules are more useful than the non-pointed variant of (1.2.22). In any case, we have a faithful functor :

$$\underline{M}\text{-}\mathbf{Mod}_l \to \underline{M}\text{-}\mathbf{Mod}_{l\circ} \qquad S \mapsto S_\circ := (S \oplus 0, 0_S) \tag{2.3.15}$$

where $0_S : 0 \to S \oplus 0$ is the obvious inclusion map. Thus, we may – and often will, without further comment – regard any $\underline{M}$-module as a pointed module, in a natural way. (The same can of course be repeated for right modules and bimodules.)

(ii) In turn, when dealing with pointed M-modules, things often work out nicer if $\underline{M}$ itself is a *pointed* T*-monoid*. The latter is the datum $(\underline{M}, 0_M)$ of a T-monoid $\underline{M}$ and a morphism of $\underline{M}$-modules $0_M : 0 \to M$. A morphism of pointed T-monoids is of course just a morphism $f : \underline{M} \to \underline{M}'$ of T-monoids, such that $f \circ 0_M = 0_{M'}$. As customary, we shall often just write $\underline{M}$ instead of $(\underline{M}, 0_M)$, unless we wish to stress that $\underline{M}$ is pointed.

(iii) Let $(\underline{M}, 0_M)$ be a pointed T-monoid; a *pointed left* $(\underline{M}, 0_M)$*-module* is a pointed left $\underline{M}$-module S, such that $0 \cdot s = 0$ for every $s \in S$. A morphism of pointed left $(\underline{M}, 0_M)$-modules is just a morphism of pointed left $\underline{M}$-modules. As usual, these gadgets form a category $(\underline{M}, 0_M)$-$\mathbf{Mod}_{l\circ}$. Similarly we have the right and bi-module variant of this definition.

(iv) The forgetful functor from the category of pointed T-monoids to the category of T-monoids, admits a left adjoint :

$$\underline{M} \mapsto (\underline{M}_\circ, 0_{M_\circ}).$$

Namely, $M_\circ$ is the $\underline{M}$-module $M \oplus 0$, the zero map $0_{M_\circ} : 0 \to M \oplus 0$ is the obvious inclusion, and the scalar multiplication $M \times M_\circ \to M_\circ$ is extended to a multiplication law $\mu : M_\circ \times M_\circ \to M_\circ$ in the unique way for which $(\underline{M}_\circ, \mu, 0_{M_0})$ is a pointed monoid. The unit of adjunction $\underline{M} \to \underline{M}_\circ$ is the obvious inclusion map.

(v) If $\underline{M}$ is a (non-pointed) monoid, the restriction of scalars

$$(\underline{M}_\circ, 0_{M_\circ})\text{-}\mathbf{Mod}_{l\circ} \to \underline{M}\text{-}\mathbf{Mod}_{l\circ}$$

is an isomorphism of categories. Namely, any pointed left $\underline{M}$-module S is naturally a pointed left $\underline{M}_\circ$-module : the given scalar multiplication $M \times S \to S$ extends to a scalar multiplication $M_\circ \times S \to S$ whose restriction $0 \times S \to S$ factors through the zero section 0_S (and likewise for right modules and bimodules).

(vi) Let T be a topos. The notions introduced thus far for non-pointed T-monoids, also admit pointed variants. Thus, a pointed module $(S, 0_S)$ is said to be *of finite type* if the same holds for

S, and S is *finitely presented* if, locally on T, it is the coequalizer of two morphisms between free $\underline{M}$-modules of finite type.

Example 2.3.16. Take $T := \mathbf{Set}$, and let M be any monoid; then a pointed left M-module is just a left M-module S endowed with a distinguished *zero element* $0 \in S$, such that $m \cdot 0 = 0$ for every $m \in M$. A morphism $\varphi : S \to S'$ of pointed left M-modules is just a morphism of left M-modules such that $\varphi(0) = 0$ (and similarly for right modules and bimodules.)

Likewise, a pointed monoid is endowed with a distinguished *zero element*, denoted 0 as usual, such that $0 \cdot x = 0$ for every $x \in M$.

Remark 2.3.17. Let T be a category as in remark 1.2.24, and $\underline{M}$ a T-monoid.

(i) Regardless of whether $\underline{M}$ is pointed or not, the category $\underline{M}\text{-}\mathbf{Mod}_{l\circ}$ is also complete and cocomplete; for instance, if $(S, 0_S)$ and $(S', 0_{S'})$ are two pointed modules, the coproduct $(S'', 0_{S''}) := (S, 0_S) \oplus (S', 0_{S'})$ is defined by the push-out (in the category $\underline{M}\text{-}\mathbf{Mod}_l$) of the cocartesian diagram :

$$\begin{tikzcd} 0 \oplus 0 \arrow[r, "0_S \oplus 0_{S'}"] \arrow[d] & S \oplus S' \arrow[d] \\ 0 \arrow[r, "0_{S''}"] & S''. \end{tikzcd}$$

Likewise, if $\varphi' : S' \to S$ and $\varphi'' : S'' \to S$ are two morphisms in $\underline{M}\text{-}\mathbf{Mod}_{l\circ}$, the fibre product $S' \times_S S''$ in the category $\underline{M}\text{-}\mathbf{Mod}_l$ is naturally pointed, and represents the fibre product in the category of pointed modules. All this holds also for right modules and bimodules.

(ii) The forgetful functor $\underline{M}\text{-}\mathbf{Mod}_{l\circ} \to T_\circ := 1_T/T$ to the category of pointed objects of T, commutes with all limits, since it is a right adjoint; it also commutes with all colimits. This forgetful functor admits a left adjoint, that assigns to any $\Sigma \in \mathrm{Ob}(T)$ the *free pointed $\underline{M}$-module* $\underline{M}^{(\Sigma)\circ}$. If $\underline{M}$ is pointed, the latter is defined as the push-out in the cocartesian diagram

$$\begin{tikzcd} 1_T \times \Sigma \arrow[r] \arrow[d] & M \times \Sigma \arrow[d] \\ 1_T \arrow[r] & \underline{M}^{(\Sigma)\circ} \end{tikzcd}$$

and if $\underline{M}$ is not pointed, one defines it via the equivalence of remark 2.3.14(v) : by a simple inspection we find that in this case $\underline{M}^{(\Sigma)\circ} = (\underline{M}^{(\Sigma)})_\circ$, where $\underline{M}^{(\Sigma)}$ is the free (unpointed) $\underline{M}$-module, as in remark 1.2.24(iii).

Notice as well that the forgetful functors $T_\circ \to T$ and $\underline{M}\text{-}\mathbf{Mod}_{l\circ} \to \underline{M}\text{-}\mathbf{Mod}_l$ both commute with all connected colimits, hence the same also holds for the forgetful functor $\underline{M}\text{-}\mathbf{Mod}_{l\circ} \to T$. (See definition 1.1.37(vii).) The same can be repeated for right modules and bimodules.

(iii) Moreover, if $\varphi : S \to S'$ is any morphism in $\underline{M}\text{-}\mathbf{Mod}_{l\circ}$, we may define $\mathrm{Ker}\,\varphi$ and $\mathrm{Coker}\,\varphi$ (in the category $\underline{M}\text{-}\mathbf{Mod}_{l\circ}$); namely, the kernel is the limit of the diagram $S \xrightarrow{\varphi} S' \leftarrow 0$ and the cokernel is the colimit of the diagram $0 \leftarrow S \xrightarrow{\varphi} S'$. Especially, if S is a submodule of S', we have a well defined quotient S'/S of pointed left $\underline{M}$-modules. Furthermore, we say that a sequence of morphisms of pointed left $\underline{M}$-modules :

$$0 \to S' \xrightarrow{\varphi} S \xrightarrow{\psi} S'' \to 0$$

is *right exact*, if ψ induces an isomorphism $\mathrm{Coker}\,\varphi \xrightarrow{\sim} S''$; we say that it is *left exact*, if φ induces an isomorphism $S' \xrightarrow{\sim} \mathrm{Ker}\,\psi$, and it is *short exact* if it is both left and right exact. (Again, all this can be repeated also for right modules and bimodules.)

Example 2.3.18. Take $T = \mathbf{Set}$, and let M be a pointed or not-pointed monoid. Then the argument from example 1.2.27 can be repeated for the free pointed M-modules : if Σ is any set, we have

$$M^{(\Sigma)_\circ} \otimes_M \{1\} \xrightarrow{\sim} \{1\}^{(\Sigma)_\circ} = \Sigma_\circ$$

where $\Sigma_\circ$ is the disjoint union of Σ and the final object of $\mathbf{Set}$ (a set with one element). Hence, the cardinality of Σ is an invariant, called the *rank* of the free pointed M-module $M^{(\Sigma)_\circ}$, and denoted $\mathrm{rk}^\circ_M M^{(\Sigma)_\circ}$.

2.3.19. Let T be a topos, $(\underline{M}, 0_M)$, $(\underline{N}, 0_N)$ and $(\underline{P}, 0_P)$ three pointed T-monoids, S, (resp. S') a pointed $(\underline{M}, \underline{N})$-bimodule (resp. $(\underline{P}, \underline{N})$-bimodule); we denote

$$\mathrm{Hom}_{(\underline{N},0_N)_r}(S, S')$$

the set of all morphisms of pointed right $\underline{N}$-modules $S \to S'$. As usual, the presheaf

$$\mathscr{H}om_{(\underline{N},0_N)_r}(S, S') \quad : \quad U \mapsto \mathrm{Hom}_{(\underline{N},0_N)_{r|U}}(S_{|U}, S'_{|U})$$

(with obvious notation) is a sheaf on (T, C_T), hence it is represented by an object of T. Indeed, this object is also the fibre product in the cartesian diagram :

$$\begin{tikzcd}
\mathscr{H}om_{(\underline{N},0_N)_r}(S, S') \arrow[r] \arrow[d] & \mathscr{H}om_{N_r}(S, S') \arrow[d, "0_S^*"] \\
\mathscr{H}om_{N_r}(0, 0) \arrow[r, "0_{S'*}"] & \mathscr{H}om_{N_r}(0, S').
\end{tikzcd}$$

Especially, $\mathscr{H}om_{(\underline{N},0_N)_r}(S, S')$ is naturally a $(\underline{P}, \underline{M})$-bimodule, and moreover, it is pointed : its zero section represents the unique morphism $S \to S'$ which factors through 0.

Notice also that, for every pointed $(\underline{P}, \underline{M})$-bimodule S'', the tensor product $S'' \otimes_M S$ is naturally pointed, and as in the non-pointed case, the functor

$$(\underline{P}, \underline{M})\text{-}\mathbf{Mod}_\circ \to (\underline{P}, \underline{N})\text{-}\mathbf{Mod}_\circ \quad : \quad S'' \mapsto S'' \otimes_M S \tag{2.3.20}$$

is left adjoint to the functor

$$(\underline{P}, \underline{N})\text{-}\mathbf{Mod}_\circ \to (\underline{P}, \underline{M})\text{-}\mathbf{Mod}_\circ \quad : \quad S' \mapsto \mathscr{H}om_{(\underline{N},0_N)_r}(S, S').$$

By general nonsense, the functor (2.3.20) is right exact; especially, for any right exact sequence $T' \to T \to T'' \to 0$ of pointed $(\underline{P}, \underline{M})$-bimodules, the induced sequence

$$T' \otimes_M S \to T \otimes_M S \to T'' \otimes_M S \to 0$$

is again right exact.

Remark 2.3.21. Suppose $\underline{M}$, $\underline{N}$ and $\underline{P}$ are non-pointed T-monoids, S is a $(\underline{M}, \underline{N})$-bimodule and S'' a $(\underline{P}, \underline{M})$-bimodule.

(i) If S and S'' are pointed, one may define a tensor product $S'' \otimes_M S$ in the category $(\underline{P}, \underline{N})$-$\mathbf{Mod}_\circ$, if one regards S as a pointed $(\underline{M}_\circ, \underline{N}_\circ)$-bimodule, and S'' as a $(\underline{P}_\circ, \underline{M}_\circ)$-bimodule as in remark 2.3.14(v); then one sets simply $S'' \otimes_M S := S'' \otimes_{M_\circ} S$, which is then viewed as a pointed $(\underline{P}, \underline{N})$-bimodule. In this way one obtains a left adjoint to the corresponding internal Hom-functor $\mathscr{H}om_N$ from pointed $(\underline{P}, \underline{N})$-bimodules to pointed $(\underline{P}, \underline{M})$-bimodules (details left to the reader).

(ii) Finally, if neither S nor S'' is pointed, notice the natural isomorphism :

$$(S'' \otimes_M S)_\circ \xrightarrow{\sim} S''_\circ \otimes_{M_\circ} S_\circ \qquad \text{in the category } (\underline{P}, \underline{N})\text{-}\mathbf{Mod}_\circ.$$

Definition 2.3.22. In the situation of (2.3.19), let $P = N := (1_T)_\circ$, and notice that – with these choices of P and N – a pointed $(\underline{P}, \underline{M})$-bimodule (resp. a pointed $(\underline{M}, \underline{N})$-bimodule) is just a right pointed $\underline{M}$-module (resp. a left pointed $\underline{M}$-module), and a pointed $(\underline{P}, \underline{N})$-module is just a pointed object of T.

(i) We say that S is a *flat* pointed left $\underline{M}$-module (or briefly, that S is $\underline{M}$-*flat*), if the functor (2.3.20) transforms short exact sequences of right pointed $\underline{M}$-modules, into short exact sequences of pointed T-objects. Likewise, we define flat pointed right $\underline{M}$-modules.

(ii) Let $\varphi : \underline{M} \to \underline{M}'$ be a morphism of pointed T-monoids. We say that φ is *flat*, if $\underline{M}'$ is a flat left $\underline{M}$-module, for the module structure induced by φ.

Remark 2.3.23. (i) In the situation of remark 2.3.11(i), suppose that $\underline{M} := (M, 0_M)$ is a pointed T-monoid and $\underline{N} := (N, 0_N)$ a pointed S-monoid. By arguing as in remark 2.3.11(ii), we see that $f^*\underline{N} := (f^*N, f^*0_N)$ is a pointed T-monoid, and $f_*\underline{M} := (f_*M, f_*0_M)$ is a pointed S-monoid.

(ii) Likewise, if $(X, 0_X)$ is a pointed left $\underline{M}$-module, and $(Y, 0_Y)$ a pointed left $\underline{N}$-module, the $f_*(X, 0_X) := (f_*X, f_*0_X)$ is a pointed $f_*\underline{M}$-module, and $f^*(Y, 0_Y) := (f^*Y, f^*0_Y)$ is a pointed $f^*\underline{N}$-module (and likewise for right modules and bimodules).

(iii) Also, just as in remark 2.3.11(vii), the associated sheaf functor $F \mapsto F^a$ transforms a presheaf $\underline{M}$ of pointed monoids on T, into a pointed T-monoid $\underline{M}^a$, and sends pointed left (resp. right, resp. bi-) $\underline{M}$-modules to pointed left (resp. right, resp. bi-) $\underline{M}^a$-modules.

(iv) Moreover, if $\varphi : f^*\underline{N} \to \underline{M}$ is a morphism of pointed T-monoids, then – in view of the discussion of (2.3.19) – the adjunction of remark 2.3.11(vi) extends to pointed modules : we leave the details to the reader.

(v) Furthermore, in the situation of (2.3.12), we may also define a functor

$$j_{U!} : \underline{M}_{|U}\text{-}\mathbf{Mod}_{l\circ} \to \underline{M}\text{-}\mathbf{Mod}_{l\circ}$$

which will be a left adjoint to j^*_U. Indeed, let $(X, 0_X)$ be a left pointed $\underline{M}_{|U}$-module; the functor from (2.3.12) yields a morphism $j_{U!}0_X$ of (non-pointed) $\underline{M}$-modules, and we define $j_{U!}(X, 0_X)$ to be the push-out (in the category $\underline{M}$-$\mathbf{Mod}_l$) of the diagram $0 \leftarrow j_{U!}0_X \xrightarrow{j_{U!}0_X} j_{U!}X$. The latter is endowed with a natural morphism $0 \to j_{U!}(X, 0_X)$, so we have a well defined pointed left $\underline{M}$-module. We leave to the reader the verification that the resulting functor, called *extension by zero*, is indeed left adjoint to the restriction functor.

(vi) It is convenient to extend definition 2.3.22 to non-pointed modules and monoids : namely, if S is a non-pointed left $\underline{M}$-module, we shall say that S is *flat*, if the same holds for the pointed left $\underline{M}_\circ$-module $S_\circ$. Likewise, we say that a morphism $\varphi : \underline{M} \to \underline{N}$ of non-pointed T-monoids is *flat*, if the same holds for $\varphi_\circ$.

Lemma 2.3.24. *Let T be a topos, U any object of T, and denote by $i_* : \complement U \to T$ the inclusion functor of the complement of U in T (see example* 2.2.6(iv)*). Let also $\underline{M}$, $\underline{N}$, $\underline{P}$ be three pointed T-monoids. Then the following holds :*

(i) *The functor $j_{U!}$ of extension by zero is faithful, and transforms exact sequences of pointed left $\underline{M}_{|U}$-modules, into exact sequences of pointed left $\underline{M}$-modules (and likewise for right modules and bimodules).*

(ii) *For every pointed $(\underline{M}, \underline{N})$-bimodule S and every pointed $(\underline{P}_{|U}, \underline{M}_{|U})$-bimodule S', the natural morphism of pointed $(\underline{P}, \underline{N})$-modules*

$$j_{U!}(S' \otimes_{M_{|U}} S_{|U}) \to j_{U!}S' \otimes_M S$$

is an isomorphism.

(iii) *If S is flat pointed left $\underline{M}_{|U}$-module, then $j_{U!}S$ is a flat pointed left $\underline{M}$-module (and likewise for right modules).*

(iv) *For every pointed $(\underline{M}, \underline{N})$-bimodule S, and every pointed $(i^*\underline{P}, i^*\underline{M})$-bimodule S', the natural morphism of pointed $(\underline{P}, \underline{N})$-bimodules*

$$i_*S' \otimes_M S \to i_*(S' \otimes_{i^*M} i^*S)$$

is an isomorphism.

(v) *If S is a flat pointed left $i^*\underline{M}$-module, then i_*S is a flat left pointed $\underline{M}$-module (and likewise for right modules).*

(vi) *If S is a flat pointed left $\underline{M}$-module, then $S_{|U}$ is a flat left pointed $\underline{M}_{|U}$-module.*

Proof. (i): Let us show first that $j_{U!}$ is faithful. Indeed, suppose that $\varphi, \psi : S \to S'$ are two morphisms of left pointed $\underline{M}_{|U}$-modules, such that $j_{U!}\varphi = j_{U!}\psi$. We need to show that $\varphi = \psi$. Let $p : S' \to S''$ be the coequalizer of φ and ψ; then $j_{U!}p$ is the coequalizer of $j_{U!}\varphi$ and $j_{U!}\psi$ (since $j_{U!}$ is right exact); hence we are reduced to showing that a morphism $p : S' \to S''$ is an isomorphism if and only if the same holds for $j_{U!}p$. This follows from remark 1.2.24(ii) and the following more general :

Claim 2.3.25. Let $\varphi : X \to X'$, $A \to X$, $A \to B$ be three morphisms in T. Then φ is a monomorphism (resp. an epimorphism) if and only if the same holds for the induced morphism $\varphi \amalg_A B : X \amalg_A B \to X' \amalg_A B$.

Proof of the claim. We may assume that $T = C^\sim$ for some small site $C := (\mathscr{C}, J)$. Then $\varphi \amalg_A B = (i\varphi \amalg_{iA} iB)^a$, where $i : C^\sim \to \mathscr{C}^\wedge$ is the forgetful functor. Since the functor $F \mapsto F^a$ is exact, we are reduced to the case where $T = \mathscr{C}^\wedge$, and in this case the assertion can be checked argumentwise, *i.e.* we may assume that $T = \mathbf{Set}$, where the claim is obvious. $\diamondsuit$

Next, we already know that $j_{U!}$ transforms right exact sequences into right exact sequences. To conclude, it suffices then to check that $j_{U!}$ transforms monomorphisms into monomorphisms.To this aim, we apply again remark 1.2.24(ii) and claim 2.3.25.

(ii) is proved by general nonsense, and (iii) is an immediate consequence of (i) and (ii) : we leave the details to the reader.

(iv): By (2.3.4), we have $j_U^*(i_*S' \otimes_M S) \simeq 0 \otimes_{j_U^*M} j_U^*S \simeq 1_{T/U}$, hence $i_*S' \otimes_M S \in \mathrm{Ob}(\mathsf{C}U)$. Notice now that, for every object X of $\mathsf{C}U$, the counit of adjunction $i^*i_*X \to X$ is an isomorphism (proposition 1.1.11(ii)); by the triangular identities of (1.1.8), it follows that the same holds for the unit of adjunction $i_*X \to i_*i^*i_*X$. Especially, the natural morphism :

$$i_*S' \otimes_M S \to i_*i^*(i_*S' \otimes_M S) \xrightarrow{\sim} i_*(i^*i_*S' \otimes_{i^*M} i^*S) \xrightarrow{\sim} i_*(S' \otimes_{i^*M} i^*S).$$

is an isomorphism. The latter is the morphism of assertion (iv).

(v) follows easily from (iv) and its proof.

(vi): In view of (i), it suffices to show that the functor $S' \mapsto j_{U!}(S' \otimes_{M_{|U}} S_{|U})$ transforms exact sequences into exact sequences. The latter follows easily from (ii). $\square$

Proposition 2.3.26. *Let $\mathbf{P}(T, \underline{M}, S)$ be the property : "S is a flat pointed left $\underline{M}$-module" (for a monoid $\underline{M}$ on a topos T). Then $\mathbf{P}$ can be checked on stalks. (See remark* 2.2.14(ii)*.)*

Proof. Suppose first that S_ξ is a flat left $\underline{M}_\xi$-module for every ξ in a conservative set of T-points; let $\varphi : X \to X'$ be a monomorphism of pointed right $\underline{M}$-modules; by (2.3.4) we have a natural isomorphism

$$(\varphi \otimes_M S)_\xi \xrightarrow{\sim} \varphi_\xi \otimes_{M_\xi} S_\xi$$

in the category of pointed sets, and our assumption implies that these morphisms are monomorphisms. Since an arbitrary product of monomorphisms is a monomorphism, remark 1.1.38(iii) shows that $\varphi \otimes_M S$ is also a monomorphism, whence the contention.

Next, suppose that S is a flat pointed left $\underline{M}$-module. We have to show that the functor

$$S' \mapsto S' \otimes_{M_\xi} S_\xi \qquad (2.3.27)$$

from pointed right $\underline{M}_\xi$-modules to pointed sets, preserves monomorphisms.

However, let (U, ξ_U, ω_U) be any lifting of ξ (see (2.2.11)); in view of (2.3.4), we have

$$P_U(S') := (\xi_U^*\xi_{U*}S') \otimes_{A_\xi} S_\xi \simeq (\xi_U^*\xi_{U*}S') \otimes_{\xi_U^*A_{|U}} \xi_U^*S_{|U} \simeq \xi_U^*(\xi_{U*}S' \otimes_{A_{|U}} S_{|U})$$

and then lemma 2.3.24(vi) implies that the functor $S' \to P_U(S')$ preserves monomorphisms. By lemma 2.2.23, remark 2.2.18(i) and (2.2.13), the functor (2.3.27) is a filtered colimit of such functors P_U, hence it preserves monomorphisms as well. $\square$

2.3.28. We wish now to introduce a few notions that pertain to the special class of commutative T-monoids. When $T = \mathbf{Set}$, these notions are well known, and we wish to explain quickly that they generalize without problems, to arbitrary topoi.

To begin with, for every category T as in example 1.2.21(i), we denote by $\mathbf{Mnd}_T$ (resp. $\mathbf{Mnd}_{T^\circ}$) the category of commutative unitary non-pointed (resp. pointed) T-monoids; in case $T = \mathbf{Set}$, we shall usually drop the subcript, and write just $\mathbf{Mnd}$ (resp. $\mathbf{Mnd}_\circ$). Notice that, if $\underline{M}$ is any (pointed or not pointed) commutative T-monoid, every left or right $\underline{M}$-module is a $\underline{M}$-bimodule in a natural way, hence we shall denote indifferently by $\underline{M}$-$\mathbf{Mod}$ (resp. $\underline{M}$-$\mathbf{Mod}_\circ$) the category of non-pointed (resp. pointed) left or right $\underline{M}$-modules.

The following lemma is a special case of a result that holds more generally, for every "algebraic theory" in the sense of [11, Def.3.3.1] (see [11, Prop.3.4.1, Prop.3.4.2]).

Lemma 2.3.29. *Let T be a topos. We have :*

(i) *The category $\mathbf{Mnd}_T$ admits arbitrary limits and colimits.*
(ii) *In the category $\mathbf{Mnd}_T$, filtered colimits commute with all finite limits.*
(iii) *The forgetful functor $\iota : \mathbf{Mnd}_T \to T$ that assigns to a monoid its underlying object of T, commutes with all limits, and with all filtered colimits.*

Proof. (iii): Commutation with limits holds because ι admits a left adjoint : namely, to an object Σ of T one assigns the *free monoid* $\mathbb{N}_T^{(\Sigma)}$ generated by Σ, defined as the sheaf associated to the presheaf of monoids

$$U \mapsto \mathbb{N}^{(\Sigma(U))} \qquad \text{for every } U \in \mathrm{Ob}(T)$$

where $\mathbb{N}$ is the additive monoid of natural numbers (see remark 2.3.11(vii)). One verifies easily that this T-monoid represents the functor

$$\underline{M} \mapsto \mathrm{Hom}_T(\Sigma, \underline{M}) \qquad \mathbf{Mnd}_T \to \mathbf{Set}.$$

Moreover, if I is any small category, and $F : I \to \mathbf{Mnd}_T$ any functor, one checks easily that the limit of $\iota \circ F$ can be endowed with a unique composition law (indeed, the limit of the composition laws of the monoids F_i), such that the resulting monoid represents the limit of F.

A similar argument also shows that $\mathbf{Mnd}_T$ admits arbitrary filtered colimits, and that ι commutes with filtered colimits. It is likewise easy to show that the product of two T-monoids $\underline{M}$ and $\underline{N}$ is also the coproduct of $\underline{M}$ and $\underline{N}$. To complete the proof of (i), it suffices therefore to show that any two maps $f, g : \underline{M} \to \underline{N}$ admit a coequalizer; the latter is obtained as the coequalizer $\underline{N}'$ (in the category T) of the two morphisms :

$$\underline{M} \times \underline{N} \overset{\mu_N \circ (f \times \mathbf{1}_N)}{\underset{\mu_N \circ (g \times \mathbf{1}_N)}{\rightrightarrows}} \underline{N}.$$

We leave to the reader the verification that the composition law of N descends to a (necessarily unique) composition law on $\underline{N}'$.

(ii) follows from (iii) and the fact that the same assertion holds in T (remark 2.2.2(iii)). $\square$

Example 2.3.30. (i) For instance, if $T = \mathbf{Set}$, the product $M_1 \times M_2$ of any two commutative monoids is representable in $\mathbf{Mnd}$; its underlying set is the cartesian product of M_1 and M_2, and the composition law is the obvious one.

(ii) As usual, the kernel $\mathrm{Ker}\,\varphi$ (resp. cokernel $\mathrm{Coker}\,\varphi$) of a map of T-monoids $\varphi : \underline{M} \to \underline{N}$ is defined as the fibre product (resp. push-out) of the diagram of T-monoids

$$\underline{M} \xrightarrow{\varphi} \underline{N} \leftarrow \underline{1}_T \qquad (\text{resp. } \underline{1}_T \leftarrow \underline{M} \xrightarrow{\varphi} \underline{N}).$$

Especially, if $\underline{M} \subset \underline{N}$, one defines in this way the quotient $\underline{N}/\underline{M}$.

(iii) Also, if $T = \mathbf{Set}$, and $\varphi_1 : M \to M_1$, $\varphi_2 : M \to M_2$ are two maps in $\mathbf{Mnd}$, the push-out $M_1 \amalg_M M_2$ can be described as follows. As a set, it is the quotient $(M_1 \times M_2)/\!\sim$, where $\sim$ denotes the minimal equivalence relation such that

$$(m_1, m_2 \cdot \varphi_2(m)) \sim (m_1 \cdot \varphi_1(m), m_2) \qquad \text{for every } m \in M,\, m_1 \in M_1,\, m_2 \in M_2$$

and the composition law is the unique one such that the projection $M_1 \times M_2 \to M_1 \amalg_M M_2$ is a map of monoids. We deduce the following :

Lemma 2.3.31. *Let G be an abelian group. The following holds :*

(i) *If $\varphi : M \to N$ and $\psi : M \to G$ are two morphisms of monoids (in the topos $T = \mathbf{Set}$), $G \amalg_M N$ is the quotient $(G \times N)/\!\approx$, where $\approx$ is the equivalence relation such that :*

$$(g, n) \approx (g', n') \quad \Leftrightarrow \quad (\psi(a) \cdot g, \varphi(b) \cdot n) = (\psi(b) \cdot g', \varphi(a) \cdot n') \text{ for some } a, b \in M.$$

(ii) *If $\varphi : G \to M$ and $\psi : G \to N$ are two morphisms of monoids, the set underlying $M \amalg_G N$ is the set-theoretic quotient $(M \times N)/G$ for the G-action defined via (φ, ψ^{-1}).*

(iii) *Especially, if M is a monoid and G is a submonoid of M, then the set underlying M/G is the set-theoretic quotient of M by the translation action of G.*

Proof. (i): One checks easily that the relation $\approx$ thus defined is transitive. Let $\sim$ be the equivlence relation defined as in example 2.3.30(iii). Clearly :

$$(g, n \cdot \psi(m)) \approx (g \cdot \varphi(m), n) \qquad \text{for every } g \in G,\, n \in N \text{ and } m \in M$$

hence $(g, n) \sim (g', n')$ implies $(g, n) \approx (g', n')$. Conversely, suppose that $(\psi(a) \cdot g, \varphi(b) \cdot n) = (\psi(b) \cdot g', \varphi(a) \cdot n')$ for some $g \in G$, $n \in N$ and $a, b \in M$. Then :

$$(g, n) = (g, \varphi(a) \cdot \varphi(a)^{-1} \cdot n) \sim (\psi(a) \cdot g, \varphi(a)^{-1} \cdot n) = (\psi(b) \cdot g', \varphi(a)^{-1} \cdot n)$$

as well as : $(g', n') = (g', \varphi(b) \cdot \varphi(a)^{-1} \cdot n) \sim (\psi(b) \cdot g', \varphi(a)^{-1} \cdot n)$. Hence $(g, n) \sim (g', n')$ and the claim follows.

(ii) follows directly from example 2.3.30(iii), and (iii) is a special case of (ii). □

2.3.32. Let T be a topos. For any T-ring $\underline{R}$, we let $\underline{R}$-$\mathbf{Mod}$ be the category of $\underline{R}$-modules (defined in the usual way); especially, we may consider the T-ring $\mathbb{Z}_T$ (the constant sheaf with value $\mathbb{Z}$: see example 2.2.6(v)). Then $\mathbb{Z}_T$-$\mathbf{Mod}$ is the category of abelian T-groups. The forgetful functor $\mathbb{Z}_T\text{-}\mathbf{Mod} \to \mathbf{Mnd}_T$ admits a right adjoint :

$$\mathbf{Mnd}_T \to \mathbb{Z}_T\text{-}\mathbf{Mod} \quad : \quad \underline{M} \mapsto \underline{M}^\times.$$

The latter can be defined as the fibre product in the cartesian diagram :

$$\begin{tikzcd} \underline{M}^\times \arrow[r] \arrow[d] & \underline{M} \times \underline{M} \arrow[d, "\mu_M"] \\ 1_T \arrow[r, "1_M"] & M. \end{tikzcd}$$

For $i = 1, 2$, let $p_i : \underline{M} \times \underline{M} \to \underline{M}$ be the projections, and $p'_i : \underline{M}^\times \to \underline{M}$ the restriction of p_i; for every $U \in \mathrm{Ob}(T)$, the image of $p'_i(U) : \underline{M}^\times(U) \to \underline{M}(U)$ consists of all sections x which are *invertible*, *i.e.* for which there exists $y \in \underline{M}(U)$ such that $\mu_M(x, y) = 1_M$. It is easily seen that such inverse is unique, hence p'_i is a monomorphism, p'_1 and p'_2 define the same subobject of $\underline{M}$, and this subobject $\underline{M}^\times$ is the largest abelian T-group contained in $\underline{M}$. We say that $\underline{M}$ is *sharp*, if $\underline{M}^\times = 1_T$. The inclusion functor, from the full subcategory of sharp T-monoids, to $\mathbf{Mnd}_T$, admits a left adjoint

$$\underline{M} \mapsto \underline{M}^\sharp := \underline{M}/\underline{M}^\times.$$

We call $\underline{M}^\sharp$ the *sharpening* of $\underline{M}$.

2.3.33. Let $\underline{S}$ be a submonoid of a commutative T-monoid $\underline{M}$, and $F_S : \mathbf{Mnd}_T \to \mathbf{Set}$ the functor that assigns to any commutative T-monoid $\underline{N}$ the set of all morphisms $f : \underline{M} \to \underline{N}$ such that $f(\underline{S}) \subset \underline{N}^\times$. We claim that F_S is representable by a T-monoid $\underline{S}^{-1}\underline{M}$.

In case $T = \mathbf{Set}$, one may realize $\underline{S}^{-1}\underline{M}$ as the quotient $(\underline{S} \times \underline{M})/\!\sim$ for the equivalence relation such that $(s_1, x_1) \sim (s_2, x_2)$ if and only if there exists $t \in \underline{S}$ such that $ts_1x_2 = ts_2x_1$. The composition law of $\underline{S}^{-1}\underline{M}$ is the obvious one; then the class of a pair (s, x) is denoted naturally by $s^{-1}x$. This construction can be repeated on a general topos : letting $X := \underline{S} \times \underline{M}$, the foregoing equivalence relation can be encoded as the equalizer R of two maps $X \times X \times S \to \underline{M}$, and the quotient under this equivalence relation shall be represented by the coequalizer of two other maps $R \to X$; the reader may spell out the details, if he wishes. Equivalently, $\underline{S}^{-1}\underline{M}$ can be realized as the sheaf on (T, C_T) associated to the presheaf :

$$T \to \mathbf{Mnd} \quad : \quad U \mapsto \underline{S}(U)^{-1}\underline{M}(U)$$

(see remark 2.3.11(vii)). The natural morphism $\underline{M} \to \underline{S}^{-1}\underline{M}$ is called the *localization map*. For $T = \mathbf{Set}$, and $f \in M$ any element, we shall also use the standard notation :

$$M_f := S_f^{-1}M \qquad \text{where } S_f := \{f^n \mid n \in \mathbb{N}\}.$$

Lemma 2.3.34. *Let $f_1 : \underline{M} \to \underline{N}_1$ and $f_2 : \underline{M} \to \underline{N}_2$ be morphisms of T-monoids, $\underline{S} \subset \underline{M}$, $\underline{S}_i \subset \underline{N}_i$ ($i = 1, 2$) three submonoids, such that $f_i(\underline{S}) \subset \underline{S}_i$ for $i = 1, 2$. Then the natural morphism :*

$$(\underline{S}_1 \cdot \underline{S}_2)^{-1}(\underline{N}_1 \amalg_{\underline{M}} \underline{N}_2) \to \underline{S}_1^{-1}\underline{N}_1 \amalg_{\underline{S}^{-1}\underline{M}} \underline{S}_2^{-1}\underline{N}_2$$

is an isomorphism.

Proof. One checks easily that both these T-monoids represent the functor $\mathbf{Mnd}_T \to \mathbf{Set}$ that assigns to any T-monoid $\underline{P}$ the pairs of morphisms (g_1, g_2) where $g_i : \underline{N}_i \to \underline{P}$ satisfies $g_i(\underline{S}_i) \subset \underline{P}^\times$, for $i = 1, 2$, and $g_1 \circ f_1 = g_2 \circ f_2$. The details are left to the reader. $\square$

2.3.35. The forgetful functor $\mathbb{Z}\text{-}\mathbf{Mod}_T \to \mathbf{Mnd}_T$ from abelian T-groups to commutative T-monoids, admits a left adjoint

$$\underline{M} \mapsto \underline{M}^{\mathrm{gp}} := \underline{M}^{-1}\underline{M}.$$

A commutative T-monoid $\underline{M}$ is said to be *integral* if the unit of adjunction $\underline{M} \to \underline{M}^{\mathrm{gp}}$ is a monomorphism. The functor $\underline{M} \mapsto \underline{M}^{\mathrm{gp}}$ commutes with all colimits, since all left adjoints do; it does not commute with arbitrary limits (see example 2.3.36(v)).

We denote by $\mathbf{Int}.\mathbf{Mnd}_T$ the full subcategory of $\mathbf{Mnd}_T$ consisting of all integral monoids; when $T = \mathbf{Set}$, we omit the subscript, and write just $\mathbf{Int}.\mathbf{Mnd}$. The natural inclusion ι : $\mathbf{Int}.\mathbf{Mnd}_T \to \mathbf{Mnd}_T$ admits a left adjoint :

$$\mathbf{Mnd}_T \to \mathbf{Int}.\mathbf{Mnd}_T \quad : \quad \underline{M} \mapsto \underline{M}^{\mathrm{int}}.$$

Namely, $\underline{M}^{\mathrm{int}}$ is the image (in the category T) of the unit of adjunction $\underline{M} \to \underline{M}^{\mathrm{gp}}$. It follows easily that the category $\mathbf{Int}.\mathbf{Mnd}_T$ is cocomplete, since the colimit of a family $(\underline{M}_\lambda \mid \lambda \in \Lambda)$ of integral monoids is represented by

$$(\operatorname*{colim}_{\lambda \in \Lambda} \iota \underline{M}_\lambda)^{\mathrm{int}}.$$

Likewise, $\mathbf{Int}.\mathbf{Mnd}_T$ is complete, and limits commute with the forgetful functor to T; to check this, it suffices to show that

$$L := \lim_{\lambda \in \Lambda} \iota(\underline{M}_\lambda)$$

is integral. However, by lemma 2.3.29(iii) we have $L \subset \prod_{\lambda \in \Lambda} \underline{M}_\lambda \subset \prod_{\lambda \in \Lambda} \underline{M}_\lambda^{\mathrm{gp}}$, whence the claim.

Example 2.3.36. (i) Take $T = \mathbf{Set}$; if M is any monoid, and $a \in M$ is any element, we say that a is *regular*, if the map $M \to M$ given by the rule : $x \mapsto a \cdot x$ is injective. It is easily seen that M is integral if and only if every element of M is regular.

(ii) For an arbitrary topos T, notice that the T-monoid $\underline{G}^a$ associated to a presheaf of groups $\underline{G}$ on T, is a T-group : indeed, the condition $\underline{G}^\times = \underline{G}$ implies $(\underline{G}^a)^\times = \underline{G}^a$, since the functor $F \mapsto F^a$ is exact. More precisely, for every presheaf $\underline{M}$ of monoids on T, we have a natural isomorphism :

$$(\underline{M}^{\mathrm{gp}})^a \xrightarrow{\sim} (\underline{M}^a)^{\mathrm{gp}} \qquad \text{for every } T\text{-monoid } \underline{M}$$

since both functors are left adjoint to the forgetful functor from T-groups to presheaves of monoids on T.

(iii) It follows from (ii) that a T-monoid $\underline{M}$ is integral if and only if $\underline{M}(U)$ is an integral monoid, for every $U \in \mathrm{Ob}(T)$. Indeed, if $\underline{M}$ is integral, then $\underline{M}(U) \subset \underline{M}^{\mathrm{gp}}(U)$ for every such U, so $\underline{M}(U)$ is integral. Conversely, by definition $\underline{M}^{\mathrm{gp}}$ is the sheaf associated to the presheaf $U \mapsto \underline{M}(U)^{\mathrm{gp}}$; now, if $\underline{M}(U)$ is integral, we have $\underline{M}(U) \subset \underline{M}(U)^{\mathrm{gp}}$, and consequently $\underline{M} \subset \underline{M}^{\mathrm{gp}}$, since the functor $F \mapsto F^a$ is exact.

(iv) We also deduce from (ii) that the functor $\underline{M} \mapsto \underline{M}^a$ sends presheaves of integral monoids, to integral T-monoids. Therefore we have a natural isomorphism :

$$(2.3.37) \qquad (\underline{M}^{\mathrm{int}})^a \xrightarrow{\sim} (\underline{M}^a)^{\mathrm{int}}$$

as both functors are left adjoint to the forgetful functor from integral T-monoids, to presheaves of monoids on T. In the same vein, it is easily seen that the forgetful functor $\mathbf{Int.Mnd}_T \to T$ commutes with filtered colimits : indeed, (2.3.37) and lemma 2.3.29(iii) reduce the assertion to showing that the colimit of a filtered system of presheaves of integral monoids is integral, which can be verified directly.

(v) Take $T = \mathbf{Set}$, and let $\varphi : M \to N$ be an injective map of monoids; if N (hence M) is integral, one sees easily that the induced map $\varphi^{\mathrm{gp}} : M^{\mathrm{gp}} \to N^{\mathrm{gp}}$ is also injective. This may fail, when N is not integral : for instance, if M is any integral monoid, and $N := M_\circ$ is the pointed monoid associated to M as in remark 2.3.14(iv), then for the natural inclusion $i : M \to M_\circ$ we have $i^{\mathrm{gp}} = 0$, since $(M_\circ)^{\mathrm{gp}} = \{1\}$.

Lemma 2.3.38. *Let T be a topos, $\underline{M}$ be an integral T-monoid, and $\underline{N} \subset \underline{M}$ a T-submonoid. Then $\underline{M}/\underline{N}$ is an integral T-monoid.*

Proof. In light of example 2.3.36(iv), we are reduced to the case where $T = \mathbf{Set}$. Moreover, since the natural morphism $\underline{M}/\underline{N} \to \underline{N}^{-1}M/\underline{N}^{\mathrm{gp}}$ is an isomorphism, we may assume that $\underline{N}$ is an abelian group. Now, notice that $(\underline{M}/\underline{N})^{\mathrm{gp}} = \underline{M}^{\mathrm{gp}}/\underline{N}$ since the functor $P \mapsto P^{\mathrm{gp}}$ commutes with colimits. On the other hand, $\underline{M}/\underline{N}$ is the set-theoretic quotient of $\underline{M}$ by the translation action of $\underline{N}$ (lemma 2.3.31(iii)). This shows that the unit of adjunction $\underline{M}/\underline{N} \to (\underline{M}/\underline{N})^{\mathrm{gp}}$ is injective, as required. $\square$

2.3.39. Let M be an integral monoid. Classically, one says that M is *saturated*, if we have :

$$M = \{a \in M^{\mathrm{gp}} \mid a^n \in M \text{ for some integer } n > 0\}.$$

In order to globalize the class of saturated monoid to arbitrary topoi, we make the following :

Definition 2.3.40. Let T be a topos, $\varphi : \underline{M} \to \underline{N}$ a morphism of integral T-monoids.

(i) We say that φ is *exact* if the diagram of commutative T-monoids

$$\mathscr{D}_\varphi \quad : \quad \begin{array}{ccc} \underline{M} & \xrightarrow{\varphi} & \underline{N} \\ \downarrow & & \downarrow \\ \underline{M}^{\mathrm{gp}} & \xrightarrow{\varphi^{\mathrm{gp}}} & \underline{N}^{\mathrm{gp}} \end{array}$$

is cartesian (where the vertical arrows are the natural morphisms).

(ii) For any integer $k > 0$, the *k-Frobenius map* of $\underline{M}$ is the endomorphism $\boldsymbol{k}_M$ of $\underline{M}$ given by the rule : $x \mapsto x^k$ for every $U \in \mathrm{Ob}(T)$ and every $x \in \underline{M}(U)$. We say that $\underline{M}$ is *k-saturated*, if $\boldsymbol{k}_M$ is an exact morphism.

(iii) We say that $\underline{M}$ is *saturated*, if $\underline{M}$ is integral and k-saturated for every integer $k > 0$.

We denote by $\mathbf{Sat}.\mathbf{Mnd}_T$ the full subcategory of $\mathbf{Int}.\mathbf{Mnd}_T$ whose objects are the saturated T-monoids. As usual, when $T = \mathbf{Set}$, we shall drop the subscript, and just write $\mathbf{Sat}.\mathbf{Mnd}$ for this category. The above definition (and several of the related results in section 3.2) is borrowed from [74].

Remark 2.3.41. (i) Clearly, when $T = \mathbf{Set}$, definition 2.3.40(iii) recovers the classical notion of saturated monoid. Again, for usual monoids, it is easily seen that the forgetful functor $\mathbf{Sat}.\mathbf{Mnd} \to \mathbf{Int}.\mathbf{Mnd}$ admits a left adjoint, that assigns to any integral monoid M its *saturation* M^{sat}. The latter is the monoid consisting of all elements $x \in M^{\mathrm{gp}}$ such that $x^k \in M$ for some integer $k > 0$; especially, the torsion subgroup of M^{gp} is always contained in M^{sat}. The easy verification is left to the reader. Clearly, M is saturated if and only if $M = M^{\mathrm{sat}}$. More generally, the unit of adjunction $M \to M^{\mathrm{sat}}$ is just the inclusion map.

(ii) For a general topos T, and a morphism φ as in definition 2.3.40(i), notice that φ is exact if and only if the induced map of monoids $\varphi(U) : \underline{M}(U) \to \underline{N}(U)$ is exact for every $U \in \mathrm{Ob}(T)$. Indeed, if $\mathscr{D}_\varphi$ is cartesian, then the same holds for the induced diagram $\mathscr{D}_\varphi(U)$ of monoids; since the natural map $\underline{M}(U)^{\mathrm{gp}} \to \underline{M}^{\mathrm{gp}}(U)$ is injective (and likewise for $\underline{N}$), it follows easily that the diagram of monoids $\mathscr{D}_{\varphi(U)}$ is cartesian, *i.e.* $\varphi(U)$ is exact. For the converse, notice that $\mathscr{D}_\varphi$ is of the form $(h\mathscr{D}_\varphi)^a$, where $h : T \to T^\wedge$ is the Yoneda embedding, and $F \mapsto F^a$ denotes the associated sheaf functor $T^\wedge \to (T, C_T)^\sim = T$; the assumption means that $h\mathscr{D}$ is a cartesian diagram in $T^\wedge$, hence $\mathscr{D}$ is exact in T, since the associated sheaf functor is exact.

(iii) Example 2.3.36(iii) and (ii) imply that a T-monoid $\underline{M}$ is saturated, if and only if $\underline{M}(U)$ is a saturated monoid, for every $U \in \mathrm{Ob}(T)$. We also remark that, in view of example 2.3.36(ii), the functor $F \mapsto F^a$ takes presheaves of k-saturated (resp. saturated) monoids, to k-saturated (resp. saturated) T-monoids : indeed, if $\eta : \underline{M} \to \underline{M}^{\mathrm{gp}}$ is the unit of adjunction for a presheaf of monoids $\underline{M}$, then $\eta^a : \underline{M}^a \to (\underline{M}^{\mathrm{gp}})^a = (\underline{M}^a)^{\mathrm{gp}}$ is the unit of adjunction for the associated T-monoid, hence it is clear the functor $F \mapsto F^a$ preserves exact morphisms.

(iv) It follows easily that the inclusion functor $\mathbf{Sat}.\mathbf{Mnd}_T \to \mathbf{Int}.\mathbf{Mnd}_T$ admits a left adjoint, namely the functor

$$\mathbf{Int}.\mathbf{Mnd}_T \to \mathbf{Sat}.\mathbf{Mnd}_T \quad : \quad \underline{M} \mapsto \underline{M}^{\mathrm{sat}}$$

that assigns to $\underline{M}$ the sheaf associated to the presheaf $U \to \underline{M}'(U) := \underline{M}(U)^{\mathrm{sat}}$ on T (notice that the functor $\underline{M} \mapsto \underline{M}'$ from presheaves of integral monoids, to presheaves of saturated monoids, is left adjoint to the inclusion functor). Just as in example 2.3.36(iv), we deduce a natural isomorphism

$$(2.3.42) \qquad (\underline{M}^{\mathrm{sat}})^a \xrightarrow{\sim} (\underline{M}^a)^{\mathrm{sat}} \qquad \text{for every } T\text{-monoid } \underline{M}$$

since both functors are left adjoint to the forgetful functor from $\mathbf{Sat}.\mathbf{Mnd}_T$, to presheaves of integral monoids on T.

(v) By the usual general nonsense, the saturation functor commutes with all colimits. Moreover, the considerations of (2.3.35) can be repeated for saturated monoids : first, the category $\mathbf{Sat}.\mathbf{Mnd}_T$ is cocomplete, and arguing as in example 2.3.36(iv), one checks that filtered colimits commute with the forgetful functor $\mathbf{Sat}.\mathbf{Mnd}_T \to T$; next, if $F : \Lambda \to \mathbf{Sat}.\mathbf{Mnd}_T$ is a functor from a small category Λ, then for each integer $k > 0$, the induced diagram of integral monoids

$$\lim_{\Lambda} \mathscr{D}_{\boldsymbol{k}_F} \quad : \quad \begin{array}{ccc} \lim\limits_{\Lambda} F & \xrightarrow{\lim_{\Lambda} \boldsymbol{k}_F} & \lim\limits_{\Lambda} F \\ \downarrow & & \downarrow \\ \lim\limits_{\Lambda} F^{\mathrm{gp}} & \xrightarrow{\lim_{\Lambda} \boldsymbol{k}_F^{\mathrm{gp}}} & \lim\limits_{\Lambda} F^{\mathrm{gp}} \end{array}$$

is cartesian; since the natural morphism

$$(\lim_{\Lambda} F)^{\mathrm{gp}} \to \lim_{\Lambda} F^{\mathrm{gp}}$$

is a monomorphism, it follows easily that the limit of F is saturated, hence $\mathbf{Sat}.\mathbf{Mnd}_T$ is complete, and furthermore all limits commute with the forgetful functor to T.

2.3.43. In view of remark 2.3.11(i,ii,iii), a morphism of topoi $f : T \to S$ induces functors :

(2.3.44) $$f_* : \mathbf{Mnd}_T \to \mathbf{Mnd}_S \qquad f^* : \mathbf{Mnd}_S \to \mathbf{Mnd}_T$$

and one verifies easily that (2.3.44) is an adjoint pair of functors.

Lemma 2.3.45. *Let $f : T \to S$ be a morphism of topoi, $\underline{M}$ an S-monoid. We have :*

(i) *If $\underline{M}$ is integral (resp. saturated), $f^*\underline{M}$ is an integral (resp. saturated) T-monoid.*

(ii) *More precisely, there is a natural isomorphism :*

$$f^*(\underline{M}^{\mathrm{int}}) \xrightarrow{\sim} (f^*\underline{M})^{\mathrm{int}} \qquad \textit{(resp. } f^*(\underline{M}^{\mathrm{sat}}) \xrightarrow{\sim} (f^*\underline{M})^{\mathrm{sat}}, \ \textit{ if } \underline{M} \textit{ is integral)}.$$

(iii) *If φ is an exact morphism of integral S-monoids, then $f^*\varphi$ is an exact morphism of integral T-monoids.*

Proof. To begin with, notice that the adjoint pair (f^*, f_*) of (2.3.44) restricts to a corresponding adjoint pair of functors between the categories of abelian T-groups and abelian S-groups (since the condition $\underline{G} = \underline{G}^\times$ for monoids, is preserved by any left exact functor).

There follows a natural isomorphism :

$$(f^*\underline{M})^{\mathrm{gp}} \xrightarrow{\sim} f^*(\underline{M}^{\mathrm{gp}}) \qquad \text{for every } S\text{-monoid } \underline{M}$$

since both functors are left adjoint to the functor f_* from abelian T-groups to S-monoids. Now, if $\underline{M}$ is an integral S-module, and $\eta : \underline{M} \to \underline{M}^{\mathrm{gp}}$ is the unit of adjunction, it is easily seen that $f^*\eta : f^*\underline{M} \to (f^*\underline{M})^{\mathrm{gp}}$ is also the unit of adjunction. From this and proposition 2.2.5(ii.b), we deduce the assertion concerning $f^*(\underline{M}^{\mathrm{int}})$.

By the same token, we get assertion (iii) of the lemma. Especially, if $\underline{M}$ is saturated, then the same holds for $f^*\underline{M}$. The assertion concerning $f^*(\underline{M}^{\mathrm{sat}})$ follows by the usual argument. $\square$

Lemma 2.3.46. (i) *The functor f^* of* (2.3.44) *commutes with all finite limits and all colimits.*

(ii) *Let $\mathbf{P}(T, M)$ be the property "M is an integral (resp. saturated) T-monoid" (for a topos T). Then $\mathbf{P}$ can be checked on stalks. (See remark* 2.2.14(ii).*)*

Proof. (i): Concerning finite limits, in light of lemma 2.3.29(iii) we are reduced to the assertion that $f^* : S \to T$ is left exact, which holds by definition. Next f^* commutes with colimits, because it is a left adjoint.

(ii): A T-monoid $\underline{M}$ is integral if and only if the unit of adjunction $\eta : \underline{M} \to \underline{M}^{\mathrm{int}}$ is an isomorphism. However, $(\underline{M}^{\mathrm{int}})_\xi \xrightarrow{\sim} (\underline{M}_\xi)^{\mathrm{int}}$, in view of lemma 2.3.45(ii), and $\eta_\xi : \underline{M}_\xi \to (\underline{M}_\xi)^{\mathrm{int}}$ is the unit of adjunction. The assertion is an immediate consequence. The same argument applies as well to saturated T-monoids. $\square$

Example 2.3.47. (i) For instance, the unique morphism of topoi $\Gamma : T \to \mathbf{Set}$ (see example 2.2.6(iii)) induces a pair of adjoint functors :

$$(2.3.48) \qquad \mathbf{Mnd}_T \to \mathbf{Mnd} \ : \ M \mapsto \Gamma(T, M) \qquad \text{and} \qquad \mathbf{Mnd} \to \mathbf{Mnd}_T \ : \ P \mapsto P_T$$

where P_T is the constant sheaf of monoids on (T, C_T) with value P.

(ii) Specializing lemma 2.3.45(ii) to this adjoint pair, we obtain natural isomorphisms :

$$(2.3.49) \qquad (M_T)^{\mathrm{int}} \xrightarrow{\sim} (M^{\mathrm{int}})_T \qquad (M_T)^{\mathrm{sat}} \xrightarrow{\sim} (M^{\mathrm{sat}})_T$$

of functors $\mathbf{Mnd} \to \mathbf{Int}.\mathbf{Mnd}_T$ and $\mathbf{Int}.\mathbf{Mnd} \to \mathbf{Sat}.\mathbf{Mnd}_T$. Especially, if M is an integral (resp. saturated) monoid, then the constant T-monoid M_T is integral (resp. saturated).

(iii) If ξ is any T-point, notice also that the stalk $M_{T,\xi}$ is isomorphic to M, since ξ is a section of $\Gamma : T \to \mathbf{Set}$.

2.3.50. Let T be a topos, $\underline{R}$ a T-ring. We have a forgetful functor $\underline{R}\text{-}\mathbf{Alg} \to \mathbf{Mnd}_T$ that assigns to a (unitary, commutative) $\underline{R}$-algebra $(\underline{A}, +, \cdot, 1_A)$ its multiplicative T-monoid $(\underline{A}, \cdot)$. If $T = \mathbf{Set}$, this functor admits a left adjoint $\mathbf{Mnd} \to R\text{-}\mathbf{Alg} : M \mapsto R[M]$. Explicitly, $R[M] = \bigoplus_{x \in M} xR$, and the multiplication law is uniquely determined by the rule :

$$xa \cdot yb := (x \cdot y)ab \qquad \text{for every } x, y \in M \text{ and } a, b \in R.$$

For a general topos T, the above construction globalizes to give a left adjoint

$$(2.3.51) \qquad \mathbf{Mnd}_T \to \underline{R}\text{-}\mathbf{Alg} \quad : \quad \underline{M} \mapsto \underline{R}[\underline{M}].$$

The latter is the sheaf on (T, C_T) associated to the presheaf $U \mapsto \underline{R}(U)[\underline{M}(U)]$, for every $U \in \mathrm{Ob}(T)$. The functor (2.3.51) commutes with arbitrary colimits (since it is a left adjoint); especially, if $\underline{M} \to \underline{M}_1$ and $\underline{M} \to \underline{M}_2$ are two morphisms of monoids, we have a natural identification :

$$(2.3.52) \qquad \underline{R}[\underline{M}_1 \amalg_{\underline{M}} \underline{M}_2] \xrightarrow{\sim} \underline{R}[\underline{M}_1] \otimes_{\underline{R}[\underline{M}]} \underline{R}[\underline{M}_2].$$

By inspecting the universal properties, we also get a natural isomorphism :

$$(2.3.53) \qquad \underline{S}^{-1}\underline{R}[\underline{M}] \xrightarrow{\sim} \underline{R}[\underline{S}^{-1}\underline{M}]$$

for every monoid $\underline{M}$ and every submonoid $\underline{S} \subset \underline{M}$.

2.3.54. Likewise, if $\underline{M}$ is any T-monoid, let $\underline{R}[\underline{M}]\text{-}\mathbf{Mod}$ denote as usual the category of modules over the T-ring $\underline{R}[\underline{M}]$; we have a forgetful functor $\underline{R}[\underline{M}]\text{-}\mathbf{Mod} \to \underline{M}\text{-}\mathbf{Mod}$. When $T = \mathbf{Set}$, this functor admits a left adjoint $M\text{-}\mathbf{Mod} \to R[M]\text{-}\mathbf{Mod} : S \mapsto R[S]$. Explicitly, $R[S]$ is the free R-module with basis given by S, and the $R[M]$-module structure on $R[S]$ is determined by the rule:

$$xa \cdot sb := \mu_S(x, s)ab \qquad \text{for every } x \in M,\ s \in S \text{ and } a, b \in R.$$

For a general topos T, this construction globalizes to give a left adjoint

$$\underline{M}\text{-}\mathbf{Mod} \to \underline{R}[\underline{M}]\text{-}\mathbf{Mod} \quad : \quad (S, \mu_S) \mapsto \underline{R}[S]$$

which is defined as the sheaf associated to the presheaf $U \mapsto \underline{R}(U)[S(U)]$ in $T^\wedge$.

2.4. **Cohomology on a topos.** In this section we introduce the cohomology with values in a sheaf of (not necessarily abelian) groups over a topos. Then we explain some basic notions concerning the points of the étale and Zariski topoi of a scheme, and we conclude with the proof of Hilbert's theorem 90 (lemma 2.4.26(iv)).

Definition 2.4.1. Let T be a topos, and G a T-group.

(i) A *left G-torsor* is a left G-module (X, μ_X), inducing an isomorphism
$$(\mu_X, p_X) : G \times X \to X \times X$$
(where $p_X : G \times X \to X$ is the natural projection) and such that there exists a covering morphism $U \to 1_T$ in T for which $X(U) \neq \varnothing$. This is the same as saying that the unique morphism $X \to 1_T$ is an epimorphism.

(ii) A *morphism of left G-torsors* is just a morphism of the underlying G-modules. Likewise, we define right G-torsors, G-bitorsors, and morphisms between them. We let:
$$H^1(T, G)$$
be the set of isomorphism classes of right G-torsors.

(iii) A (left or right or bi-) G-torsor (X, μ_X) is said to be *trivial*, if $\Gamma(T, X) \neq \varnothing$.

Remark 2.4.2. (i) In the situation of definition 2.4.1, notice that $H^1(T, G)$ always contains a distinguished element, namely the class of the trivial G-torsor (G, μ_G).

(ii) Conversely, suppose that (X, μ_X) is a trivial left G-torsor, and say that $\sigma \in \Gamma(T, X)$; then we have a cartesian diagram :

$$\begin{tikzcd} G \arrow[r, "\mu_\sigma"] \arrow[d] & X \arrow[d, "\mathbf{1}_X \times \sigma"] \\ G \times X \arrow[r, "(\mu_X, p_X)"] & X \times X \end{tikzcd}$$

which shows that (X, μ_X) is isomorphic to (G, μ_G) (and likewise for right G-torsors).

(iii) Notice that every morphism $f : (X, \mu_X) \to (X', \mu_{X'})$ of G-torsors is an isomorphism. Indeed, the assertion can be checked locally on T (*i.e.*, after pull-back by a covering morphism $U \to 1_T$). Then we may assume that X admits a global section $\sigma \in \Gamma(T, X)$, in which case $\sigma' := \sigma \circ f \in \Gamma(T, X')$. Then, arguing as in (ii), we get a commutative diagram :

$$\begin{tikzcd} & G \arrow[dl, "\mu_\sigma"'] \arrow[dr, "\mu_{\sigma'}"] & \\ X \arrow[rr, "f"] & & X' \end{tikzcd}$$

where both μ_σ and $\mu_{\sigma'}$ are isomorphisms, and then the same holds for f.

(iv) The tensor product of a G-bitorsor and a left G-bitorsor is a left G-torsor. Indeed the assertion can be checked locally on T, so we are reduced to checking that the tensor product of a trivial G-bitorsors and a trivial left G-torsor is the trivial left G-torsor, which is obvious.

(v) Likewise, if $G_1 \to G_2$ is any morphism of T-groups, and X is a left G_1-torsor, it is easily seen that the base change $G_2 \otimes_{G_1} X$ yields a left G_2-torsor (and the same holds for right torsors and bitorsors). Hence the rule $G \mapsto H^1(T, G)$ is a functor from the category of T-groups, to the category of pointed sets. One can check that $H^1(T, G)$ is an essentially small set (see [38, Chap.III, §3.6.6.1]).

(vi) Let $f : T \to S$ be a morphism of topoi; if X is a left G-torsor, the f_*G-module f_*X is not necessarily a f_*G-torsor, since we may not be able to find a covering morphism $U \to 1_S$ such that $f_*X(U) \neq \varnothing$. On the other hand, if H a S-group and Y a left H-torsor, then it is easily seen that f^*Y is a left f^*H-torsor.

2.4.3. Let $f : T' \to T$ be a morphism of topoi, and G a T'-monoid; we define a U-presheaf $R^1 f_*^\wedge G$ on T, by the rule :
$$U \mapsto H^1(T'/f^*U, G_{|f^*U}).$$
(More precisely, since this set is only essentially small, we replace it by an isomorphic small set). If $\varphi : U \to V$ is any morphism in T, and X is any $G_{|f^*V}$-torsor, then $X \times_{f^*V} f^*U$ is a $G_{|f^*U}$-torsor, whose isomorphism class depends only on the isomorphism class of X; this

defines the map $R^1f_*^\wedge G(\varphi)$, and it is clear that $R^1f_*^\wedge G(\varphi\circ\psi)=R^1f_*^\wedge G(\psi)\circ R^1f_*^\wedge G(\varphi)$, for any other morphism $\psi:W\to U$ in T. Finally, we denote by :

$$R^1f_*G$$

the sheaf on (T,C_T) associated to the presheaf $R^1f_*^\wedge G$. Notice that the object R^1f_*G is *pointed, i.e.* it is endowed with a natural global section :

$$\tau_{f,G}:1_T\to R^1f_*G$$

namely, the morphism associated to the morphism of presheaves $1_T\to R^1f_*^\wedge G$ which, for every $U\in\mathrm{Ob}(T)$, singles out the isomorphism class $\tau_{f,G}(U)\in R^1f_*^\wedge G(U)$ of the trivial $G_{|f^*U}$-torsor.

2.4.4. Let $g:T''\to T'$ be another morphism of topoi, and G a T''-group. Notice that :

$$f_*^\wedge R^1g_*^\wedge G=R^1(f\circ g)_*^\wedge G$$

hence the natural morphism (in $T'^\wedge$) $R^1f_*^\wedge G\to R^1f_*G$ induces a morphism $R^1(f\circ g)_*^\wedge G\to f_*^\wedge R^1g_*G$ in $T^\wedge$, which yields, after taking associated sheaves, a morphism in T :

$$R^1(f\circ g)_*G\to f_*R^1g_*G. \tag{2.4.5}$$

One sees easily that this is a *morphism of pointed objects* of T, *i.e.* the image of the global section $\tau_{f\circ g,G}$ under this map, is the global section $f_*\tau_{g,G}$.

Next, suppose that $U\in\mathrm{Ob}(T)$ and X is any $g_*G_{|f^*U}$-torsor (on T'/f^*U); we may form the $g^*g_*G_{|g^*f^*U}$-torsor g^*X, and then base change along the natural morphism $g^*g_*G\to G$, to obtain the $G_{|g^*f^*U}$-torsor $G\otimes_{g^*g_*G}g^*X$. This rule yields a map $R^1f_*^\wedge(g_*G)\to R^1(f\circ g)_*^\wedge G$, and after taking associated sheaves, a natural morphism of pointed objects :

$$R^1f_*(g_*G)\to R^1(f\circ g)_*G. \tag{2.4.6}$$

Remark 2.4.7. As a special case, let $h:S'\to S$ be a morphism of topoi, H a S'-group. If we take $T'':=S'$, $T':=S$, $g:=h$ and $f:S\to\mathbf{Set}$ the (essentially) unique morphism of topoi, (2.4.6) and (2.4.5) boil down to maps of pointed sets :

$$H^1(S,h_*H)\to H^1(S',H)\to\Gamma(S,R^1h_*H). \tag{2.4.8}$$

These considerations are summarized in the following :

Theorem 2.4.9. *In the situation of* (2.4.4)*, there exists a natural* exact sequence of pointed objects *of T :*

$$1_T\to R^1f_*(g_*G)\to R^1(f\circ g)_*G\to f_*R^1g_*G.$$

Proof. The assertion means that (2.4.6) identifies $R^1f_*(g_*G)$ with the subobject :

$$R^1(f\circ g)_*G\times_{f_*R^1g_*G}f_*\tau_{g,G}$$

(briefly : the preimage of the trivial global section). We begin with the following :

Claim 2.4.10. In the situation of remark 2.4.7, the sequence of maps (2.4.8) identifies the pointed set $H^1(S,h_*H)$ with the preimage of the trivial global section $\tau_{h,H}$ of R^1h_*H.

Proof of the claim. Notice first that a global section Y of $R^1h_*^\wedge H$ maps to the trivial section $\tau_{h,H}$ of R^1h_*H if and only if there exists a covering morphism $U\to 1_S$ in (S,C_S), such that $Y(h^*U)\neq\varnothing$. Thus, let X be a right h_*H-torsor; the image in $H^1(S',H)$ of its isomorphism class is the class of the H-torsor $Y:=h^*X\otimes_{h^*h_*H}H$. The latter defines a global section of R^1h_*H. However, by definition there exists a covering morphism $U\to 1_S$ such that $X(U)\neq\varnothing$, hence also $h^*X(h^*U)\neq\varnothing$, and therefore $Y(h^*U)\neq\varnothing$. This shows that the image of $H^1(S,h_*H)$ lies in the preimage of $\tau_{h,H}$.

Moreover, notice that $h_*Y(U) \neq \varnothing$, hence h_*Y is a h_*H-torsor. Now, let $\varepsilon_H : h^*h_*H \to H$ (resp. $\eta_{h_*H} : h_*H \to h_*h^*h_*H$) be the counit (resp. unit of adjunction); we have a natural morphism $\alpha : h^*X \to Y_{(\varepsilon_H)}$ of h^*h_*H-modules, whence a morphism :

$$h_*\alpha : h_*h^*X \to h_*Y_{(h_*\varepsilon_H)}$$

of $h_*h^*h_*H$-modules. On the other hand, the unit of adjunction $\eta_X : X \to h_*h^*X_{(\eta_{h_*H})}$ is a morphism of h_*H-modules (remark 2.3.11(v)). Since $h_*\varepsilon_H \circ \eta_{h_*H} = \mathbf{1}_{h_*H}$ (see (1.1.8)), the composition $h_*\alpha \circ \eta_X$ is a morphism of h_*H-modules, hence it is an isomorphism, by remark 2.4.2(iii). This implies that the first map of (2.4.8) is injective.

Conversely, suppose that the class of a H-torsor X' gets mapped to $\tau_{h,H}$; we need to show that the class of X' lies in the image of $H^1(S, h_*H)$. However, the assumption means that there exists a covering morphism $U \to 1_S$ such that $X'(h^*U) \neq \varnothing$; by adjunction we deduce that $h_*X'(U) \neq \varnothing$, hence h_*X' is a h_*H-torsor. In order to conclude, it suffices to show that the image in $H^1(S', H)$ of the class of h_*X' is the class of X.

Now, the counit of adjunction $h^*h_*X' \to X'$ is a morphism of h^*h_*H-modules (remark 2.3.11(v)); by adjunction it induces a map $h^*h_*X' \otimes_{h^*h_*H} H \to H$ of H-torsors, which must be an isomorphism, according to remark 2.4.2(iii). $\Diamond$

If we apply claim 2.4.10 with $S := T'/f^*U$, $S' := T''/(g^*f^*U)$ and $h := g/(g^*f^*U)$, for U ranging over the objects of T, we deduce an exact sequence of presheaves of pointed sets :

$$1_T \to R^1f_*^\wedge(g_*H) \to R^1(f \circ g)_*^\wedge G \to f_*^\wedge R^1g_*G$$

from which the theorem follows, after taking associated sheaves. $\square$

2.4.11. Let $f : T' \to T$ be a morphism of topoi, U an object of T, G a T'-group, $p : X \to U$ a right $G_{|U}$-torsor. Then, for every object V of T we have an induced sequence of maps of sets

$$X(f^*V) \xrightarrow{p_*} U(f^*V) \xrightarrow{\partial} R^1f_*^\wedge G(V)$$

where p_* is deduced from p, and for every $\sigma \in U(f^*V)$, we let $\partial(\sigma)$ be the isomorphism class of the right $G_{|f^*V}$-torsor $(X \times_U f^*V \to f^*V)$. Clearly the image of p_* is precisely the preimage of (the isomorphism class of) the trivial $G_{|f^*V}$-torsor. After taking associated sheaves, we deduce a natural sequence of morphisms in T :

$$f_*X \xrightarrow{p} f_*U \xrightarrow{\partial} R^1f_*G \qquad (2.4.12)$$

such that the preimage of the global section $\tau_{f,G}$ is precisely the image in $\Gamma(T', U)$ of the set of global sections of X.

2.4.13. A *ringed topos* is a pair $(T, \mathscr{O}_T)$ consisting of a topos T and a (unitary, associative) T-ring $\mathscr{O}_T$, called the *structure ring* of T. A morphism $f : (T, \mathscr{O}_T) \to (S, \mathscr{O}_S)$ of ringed topoi is the datum of a morphism of topoi $f : T \to S$ and a morphism of T-rings :

$$f^\natural : f^*\mathscr{O}_S \to \mathscr{O}_T.$$

We denote, as usual, by $\mathscr{O}_T^\times \subset \mathscr{O}_T$ the subobject representing the invertible sections of $\mathscr{O}_T$. For every object U of T, and every $s \in \mathscr{O}_T(U)$, let $D(s) \subset U$ be the subobject such that :

$$\mathrm{Hom}_T(V, D(s)) := \{\varphi \in U(V) \mid \varphi^*s \in \mathscr{O}_T^\times(V)\}.$$

We say that $(T, \mathscr{O}_T)$ is *locally ringed*, if $D(0) = \varnothing_T$ (the initial object of T), and moreover

$$D(s) \cup D(1-s) = U \qquad \text{for every } U \in \mathrm{Ob}(T), \text{ and every } s \in \mathscr{O}_T(f).$$

A morphism of locally ringed topoi $f : (T, \mathscr{O}_T) \to (S, \mathscr{O}_S)$ is a morphism of ringed topoi such that

$$f^*D(s) = D(f^\natural(U)(f^*s)) \qquad \text{for every } U \in \mathrm{Ob}(S) \text{ and every } s \in \mathscr{O}_S(U).$$

If T has enough points, then $(T, \mathscr{O}_T)$ is locally ringed if and only if the stalks $\mathscr{O}_{T,\xi}$ of the structure ring at all the points ξ of T are local rings. Likewise, a morphism $f : (T, \mathscr{O}_T) \to (S, \mathscr{O}_S)$ of ringed topoi is locally ringed if and only if, for every T-point ξ, the induced map $\mathscr{O}_{S,f(\xi)} \to \mathscr{O}_{T,\xi}$ is a local ring homomorphism.

2.4.14. In the rest of this section we present a few results concerning the special case of topologies on a scheme. Hence, for any scheme X, we shall denote by $X_{\text{ét}}$ (resp. by X_{Zar}) the small étale (resp. the small Zariski) site on X. It is clear that X_{Zar} is a small site, and it is not hard to show that $X_{\text{ét}}$ is a U-site ([4, Exp.VII, §1.7]). The inclusion of underlying categories :

$$u_X : X_{\text{Zar}} \to X_{\text{ét}}$$

is a continuous functor (see definition 2.1.39(i)) commuting with finite limits, whence a morphism of topoi :

$$\widetilde{u}_X := (\widetilde{u}_X^*, \widetilde{u}_{X*}) : X_{\text{ét}}^\sim \to X_{\text{Zar}}^\sim.$$

such that the diagram of functors :

$$\begin{tikzcd} X_{\text{Zar}} \arrow[r, "u_X"] \arrow[d] & X_{\text{ét}} \arrow[d] \\ X_{\text{Zar}}^\sim \arrow[r, "\widetilde{u}_X^*"] & X_{\text{ét}}^\sim \end{tikzcd}$$

commutes, where the vertical arrows are the Yoneda embeddings (lemma 2.1.49).

The topoi $X_{\text{Zar}}^\sim$ and $X_{\text{ét}}^\sim$ are locally ringed in a natural way, and by faithfully flat descent, we see easily that $\widetilde{u}_{X*}\mathscr{O}_{X_{\text{ét}}} = \mathscr{O}_{X_{\text{Zar}}}$. By inspection, $\widetilde{u}_X$ is a morphism of locally ringed topoi.

2.4.15. For any ring R (of our fixed universe U), denote by $\mathbf{Sch}/R$ the category of R-schemes, and by $\mathbf{Sch}/R_{\text{Zar}}$ (resp. $\mathbf{Sch}/R_{\text{ét}}$) the big Zariski (resp. étale) site on $\mathbf{Sch}/R$. For $R = \mathbb{Z}$, we shall usually just write $\mathbf{Sch}_{\text{Zar}}$ and $\mathbf{Sch}_{\text{ét}}$ for these sites. The morphisms u_X of (2.4.13) are actually restrictions of a single morphism of sites :

$$u : \mathbf{Sch}_{\text{Zar}} \to \mathbf{Sch}_{\text{ét}}$$

which, for every universe V such that $\mathsf{U} \in \mathsf{V}$, induces a morphism of V-topoi :

$$\widetilde{u}_{\mathsf{V}} : (\mathbf{Sch}_{\text{ét}})^\sim_{\mathsf{V}} \to (\mathbf{Sch}_{\text{Zar}})^\sim_{\mathsf{V}}.$$

2.4.16. Let X be scheme; a *geometric point* of X is a morphism of schemes $\xi : \operatorname{Spec} \kappa \to X$, where κ is an arbitrary separably closed field. Notice that both the Zariski and étale topoi of $\operatorname{Spec} \kappa$ are equivalent to the category $\mathbf{Set}$, so ξ induces a topos-theoretic point $\xi_{\text{ét}}^\sim : (\operatorname{Spec} \kappa)_{\text{ét}}^\sim \to X_{\text{ét}}^\sim$ of $X_{\text{ét}}^\sim$ (and likewise for $X_{\text{Zar}}^\sim$). A basic feature of both the Zariski and étale topologies, is that every point of $X_{\text{Zar}}^\sim$ and $X_{\text{ét}}^\sim$ arise in this way.

More precisely, we say that two geometric points ξ and ξ' of X are *equivalent*, if there exists a third such point ξ'' which factors through both ξ and ξ'. It is easily seen that this is an equivalence relation on the set of geometric points of X, and two topos-theoretic points $\xi_{\text{ét}}^\sim$ and $\xi'^\sim_{\text{ét}}$ are isomorphic if and only if the same holds for the points $\xi_{\text{Zar}}^\sim$ and $\xi'^\sim_{\text{Zar}}$, if and only if ξ is equivalent to ξ'.

Definition 2.4.17. Let X be a scheme, x a point of X, and $\overline{x} : \operatorname{Spec} \kappa \to X$ a geometric point.

(i) We let $\kappa(x)$ be the residue field of the local ring $\mathscr{O}_{X,x}$, and set

$$|x| := \operatorname{Spec} \kappa(x) \qquad \kappa(\overline{x}) := \kappa \qquad |\overline{x}| := \operatorname{Spec} \kappa(\overline{x})$$

If $\{x\} \subset X$ is the image of $\overline{x}$, we say that $\overline{x}$ is *localized at* x, and that x is the *support* of $\overline{x}$.

(ii) The *localization of X at x* is the local scheme

$$X(x) := \operatorname{Spec} \mathscr{O}_{X,x}.$$

The *strict henselization of X at $\overline{x}$* is the strictly local scheme

$$X(\overline{x}) := \mathrm{Spec}\, \mathscr{O}_{X,\overline{x}}$$

where $\mathscr{O}_{X,\overline{x}}$ denotes the strict henselization of $\mathscr{O}_{X,x}$ relative to the geometric point $\overline{x}$ ([33, Ch.IV, Déf.18.8.7]) (recall that a local ring is called *strictly local*, if it is henselian with separably closed residue field; a scheme is called *strictly local*, if it is the spectrum of a strictly local ring: see [33, Ch.IV, Déf.18.8.2]). By definition, the geometric point $\overline{x}$ lifts to a unique geometric point of $X(\overline{x})$, which shall be denoted again by $\overline{x}$.

(iii) Moreover, we shall denote by

$$i_x : X(x) \to X \qquad i_{\overline{x}} : X(\overline{x}) \to X(x)$$

the natural morphisms of schemes, and if $\mathscr{F}$ is any sheaf on X_{Zar} (resp. $X_{\mathrm{ét}}$), we let

$$\mathscr{F}(x) := i_x^*\mathscr{F} \qquad \mathscr{F}(\overline{x}) := i_{\overline{x}}^*\mathscr{F}(x)$$

and $\mathscr{F}(\overline{x})$ is a sheaf on $X(\overline{x})_{\mathrm{Zar}}$ (resp. on $X(\overline{x})_{\mathrm{ét}}$).

(iv) If $f : Y \to X$ is any morphism of schemes, we let

$$f^{-1}(x) := Y \times_X |x| \quad f^{-1}(\overline{x}) := Y \times_X |\overline{x}| \qquad Y(x) := Y \times_X X(x) \quad Y(\overline{x}) := Y \times_X X(\overline{x}).$$

Also, if ξ is any geometric point of Y, we define $f(\xi)$ as the geometric point $f \circ \xi$ of X.

2.4.18. Many discussions concerning the Zariski or étale site of a scheme, only make appeal to general properties of these two topologies, and therefore apply indifferently to either of them, with only minor verbal changes. For this reason, to avoid tiresome repetitions, the following notational device is often useful. Namely, instead of referring each time to X_{Zar} and $X_{\mathrm{ét}}$ in the course of an argument, we shall write just X_τ, with the convention that $\tau \in \{\mathrm{Zar}, \mathrm{ét}\}$ has been chosen arbitrarily at the beginning of the discussion. In the same manner, a τ-point of X will mean a point of the topos $X_\tau^\sim$, and a τ-open subset of X will be any object of the site X_τ. With this convention, a Zar-point is a usual point of X, whereas an $\mathrm{ét}$-point shall be a geometric point. Likewise, if ξ is a given τ-point of X, the localization $X(\xi)$ makes sense for both topologies : if $\tau = \mathrm{ét}$, then $X(\xi)$ is the strict henselization as in definition 2.4.17(ii); if $\tau = \mathrm{Zar}$, then $X(\xi)$ is the usual localization of X at the (Zariski) point ξ. If $\tau = \mathrm{ét}$, the support of ξ is given by definition 2.4.17(i); if $\tau = \mathrm{Zar}$, then the support of ξ is just ξ itself (and correspondingly, in this case ξ is localized at ξ). Furthermore, $\mathscr{O}_{X,\xi}$ is a local ring if $\tau = \mathrm{Zar}$, and it is a strictly local ring, in case $\tau = \mathrm{ét}$.

2.4.19. Let $f : X \to Y$ be a morphism of schemes, $\overline{x}$ a geometric point of X, and set $\overline{y} := f(\overline{x})$. The natural morphism $f_x : X(x) \to Y(y)$ induces a unique local morphism of strictly local schemes

$$f_{\overline{x}} : X(\overline{x}) \to Y(\overline{y})$$

([33, Ch.IV, Prop.18.8.8(ii)]) that fits in a commutative diagram :

$$\begin{tikzcd}
|\overline{x}| \arrow[r, "\overline{x}"] \arrow[d, equal] & X(\overline{x}) \arrow[r, "i_{\overline{x}}"] \arrow[d, "f_{\overline{x}}"'] & X(x) \arrow[r, "i_x"] \arrow[d, "f_x"] & X \arrow[d, "f"] \\
|\overline{y}| \arrow[r, "\overline{y}"] & Y(\overline{y}) \arrow[r, "i_{\overline{y}}"] & Y(y) \arrow[r, "i_y"] & Y.
\end{tikzcd}$$

Let now $\mathscr{F}$ be any sheaf on $Y_{\mathrm{ét}}$; there follows a natural isomorphism :

$$f_{\overline{x}}^*\mathscr{F}(\overline{y}) \stackrel{\sim}{\to} (f^*\mathscr{F})(\overline{x}).$$

Notice also the natural bijections :

$$\begin{aligned} \mathscr{F}_{\overline{y}} &\stackrel{\sim}{\to} \Gamma(Y(\overline{y}), \mathscr{F}(\overline{y})) \stackrel{\sim}{\to} \Gamma(|\overline{y}|, \overline{y}^*\mathscr{F}(\overline{y})) \\ f^*\mathscr{F}_{\overline{x}} &\stackrel{\sim}{\to} \Gamma(X(\overline{x}), f^*\mathscr{F}(\overline{x})) \stackrel{\sim}{\to} \Gamma(|\overline{x}|, \overline{x}^* f^*\mathscr{F}(\overline{x})) \end{aligned} \tag{2.4.20}$$

which induce a natural identification :

$$\mathscr{F}_{\overline{y}} \stackrel{\sim}{\to} f^*\mathscr{F}_{\overline{x}} \quad : \quad \sigma \mapsto f^*_{\overline{x}}(\sigma). \tag{2.4.21}$$

2.4.22. Let X be a scheme, $x, x' \in X$ any two points, such that x is a specialization of x'. Choose a geometric point $\overline{x}$ localized at x. The localization map $\mathscr{O}_{X,x} \to \mathscr{O}_{X,x'}$ induces a natural *specialization morphism* of X-schemes :

$$X(x') \to X(x).$$

Set $W := X(\overline{x}) \times_{X(x)} X(x')$. The natural map $g : W \to X(x')$ is faithfully flat, and is the limit of a cofiltered system of étale morphisms; hence we may find $w \in W$ lying over the closed point of $X(x')$, and the induced map $\kappa(x') \to \kappa(w)$ is algebraic and separable. Choose also a geometric point $\overline{w}$ of W localized at w, and set $\overline{x}' := g(\overline{w})$. Then g induces an isomorphism $g_{\overline{w}} : W(\overline{w}) \stackrel{\sim}{\to} X(\overline{x}')$, whence a unique morphism

$$X(\overline{x}') \to X(\overline{x}) \tag{2.4.23}$$

which makes commute the diagram :

$$\begin{tikzcd} W(\overline{w}) \arrow[r, "g_{\overline{w}}"] \arrow[d, "i_{\overline{w}}"'] & X(\overline{x}') \arrow[r, "i_{\overline{x}'}"] \arrow[d] & X(x') \arrow[d] \\ W(w) \arrow[r] & X(\overline{x}) \arrow[r, "i_{\overline{x}}"] & X(x) \end{tikzcd}$$

where the left bottom arrow is the natural projection, and the right-most vertical arrow is the specialization map. In this situation, we say that $\overline{x}$ is a *specialization* of $\overline{x}'$ (and that $\overline{x}'$ is a *generization* of $\overline{x}$), and we call (2.4.23) a *strict specialization morphism*. Combining with (2.4.20), we obtain the *strict specialization map induced by* (2.4.23)

$$\mathscr{G}_{\overline{x}} \to \mathscr{G}_{\overline{x}'} \tag{2.4.24}$$

for every sheaf $\mathscr{G}$ on $X_{\text{ét}}$.

Remark 2.4.25. (i) In the situation of (2.4.19), suppose that $\mathscr{G} = f^*\mathscr{F}$ for a sheaf $\mathscr{F}$ on $Y_{\text{ét}}$. Then (2.4.24) is a map $\mathscr{F}_{f(\overline{x})} \to \mathscr{F}_{f(\overline{x}')}$. By inspecting the definition, it is easily seen that the latter agrees with the strict specialization map for $\mathscr{F}$ induced by a unique strict specialization morphism $Y(f(\overline{x}')) \to Y(f(\overline{x}))$.

(ii) Notice that (2.4.23) and (2.4.24) depend not only on the choice of w (which may not be unique, when $X(x)$ is not unibranch) but also on the geometric point $\overline{w}$. Indeed, the group of automorphisms of the $X(x')$-scheme $X(\overline{x}')$ is naturally isomorphic to the Galois group $\mathrm{Gal}(\kappa(x')^{\mathrm{s}}/\kappa(x'))$ ([33, Ch.IV, (18.8.8.1)]).

Lemma 2.4.26. *Let X be a scheme, $\mathscr{F}$ a sheaf on $X_{\text{ét}}$. We have :*

(i) *The counit of the adjunction $\varepsilon_{\mathscr{F}} : \widetilde{u}^*_X \circ \widetilde{u}_{X*}\mathscr{F} \to \mathscr{F}$ is a monomorphism.*

(ii) *Suppose there exists a sheaf $\mathscr{G}$ on X_{Zar}, and an epimorphism $f : \widetilde{u}^*\mathscr{G} \to \mathscr{F}$ (resp. a monomorphism $f : \mathscr{F} \to \widetilde{u}^*\mathscr{G}$). Then $\varepsilon_{\mathscr{F}}$ is an isomorphism.*

(iii) *The functor $\widetilde{u}^*_X$ is fully faithful.*

(iv) (Hilbert 90) $R^1\widetilde{u}_{X*}\mathscr{O}^{\times}_{X_{\text{ét}}} = 1_{X_{\mathrm{Zar}}}$.

Proof. (i): The assertion can be checked on the stalks. Hence, let ξ be any geometric point of X; we have to show that the natural map $(\widetilde{u}_{X*}\mathscr{F})_\xi \to \mathscr{F}_\xi$ is injective. To this aim, say that $s, s' \in (\widetilde{u}_{X*}\mathscr{F})_\xi$, and suppose that the image of s in $\mathscr{F}_\xi$ agrees with the image of s'; we may find an open neighborhood U of ξ in X_{Zar}, such that s and s' lie in the image of $\mathscr{F}(U)$, and by assumption, there exists an étale morphism $f : V \to U$ such that the images of s and s' coincide in $\mathscr{F}(V)$. However, $f(V) \subset U$ is an open subset ([31, Ch.IV, Th.2.4.6]), and the induced map

$V \to f(V)$ is a covering morphism in $X_{\text{ét}}$; it follows that the images of s and s' agree already in $\mathscr{F}(f(V))$, therefore also in $(\widetilde{u}_{X*}\mathscr{F})_\xi$.

(iii): According to proposition 1.1.11(ii), it suffices to show that the unit of the adjunction $\eta_{\mathscr{G}} : \mathscr{G} \to \widetilde{u}_{X*} \circ \widetilde{u}^*_X\mathscr{G}$ is an isomorphism, for every $\mathscr{G} \in \text{Ob}(X^{\sim}_{\text{Zar}})$. However, we have morphisms :

$$\widetilde{u}^*_X\mathscr{G} \xrightarrow{\widetilde{u}^*_X(\eta_{\mathscr{G}})} \widetilde{u}^*_X \circ \widetilde{u}_{X*} \circ \widetilde{u}^*_X\mathscr{G} \xrightarrow{\varepsilon_{\widetilde{u}^*_X\mathscr{G}}} \widetilde{u}^*_X\mathscr{G}.$$

whose composition is the identity of $\widetilde{u}^*_X\mathscr{G}$ (see (1.1.8)); also, (i) says that $\varepsilon_{\widetilde{u}^*_X\mathscr{G}}$ is a monomorphism, and then it follows formally that it is actually an isomorphism (*e.g.* from the dual of [10, Prop.1.9.3]). Hence the same holds for $\widetilde{u}^*_X(\eta_{\mathscr{G}})$, and by considering the stalks of the latter, we conclude that also $\eta_{\mathscr{G}}$ is an isomorphism, as required.

(ii): Suppose first that $f : \widetilde{u}^*\mathscr{G} \to \mathscr{F}$ is an epimorphism. We have just seen that $\eta_{\mathscr{G}}$ is an isomorphism, therefore we have a morphism $\widetilde{u}^* \circ \widetilde{u}_* f : \widetilde{u}^*\mathscr{G} \to \widetilde{u}^* \circ \widetilde{u}_*\mathscr{F}$ whose composition with $\varepsilon_{\mathscr{F}}$ is f; especially, $\varepsilon_{\mathscr{F}}$ is an epimorphism, so the assertion follows from (i).

In the case of a monomorphism $f : \mathscr{F} \to \widetilde{u}^*\mathscr{G}$, set $\mathscr{H} := \widetilde{u}^*\mathscr{G} \amalg_{\mathscr{F}} \widetilde{u}^*\mathscr{G}$; we may represent f as the equalizer of the two natural maps $j_1, j_2 : \widetilde{u}^*\mathscr{G} \to \mathscr{H}$. However, the natural morphism $\widetilde{u}^*(\mathscr{G} \amalg \mathscr{G}) \to \mathscr{H}$ is an epimorphism, hence the counit $\varepsilon_{\mathscr{H}}$ is an isomorphism, by the previous case. Then (iii) implies that $j_i = \widetilde{u}^* j'_i$ for morphisms $j'_i : \mathscr{G} \to \widetilde{u}_*\mathscr{H}$ $(i = 1, 2)$. Let $\mathscr{F}'$ be the equalizer of j'_1 and j'_2; then $\widetilde{u}^*\mathscr{F}' \simeq \mathscr{F}$, and since we have already seen that the unit of adjunction is an isomorphism, the assertion follows from the triangular identitities of (1.1.8).

(iv): The assertion can be checked on the stalks. To ease notation, set $\mathscr{F} := R^1\widetilde{u}_{X*}\mathscr{O}^\times_{X_{\text{ét}}}$. Let ξ be any geometric point of X, and say that $s \in \mathscr{F}_\xi$; pick a (Zariski) open neighborhood $U \subset X$ of ξ such that s lies in the image of $\mathscr{F}(U)$. We may then find a Zariski open covering $(U_\lambda \to U \mid \lambda \in \Lambda)$ of U, such that the image of s in $\mathscr{F}(U_i)$ is represented by a $\mathscr{O}^\times_{U_\lambda,\text{ét}}$-torsor on $U_{\lambda,\text{ét}}$, for every $\lambda \in \Lambda$. After replacing U by any U_λ containing the support of ξ, we may assume that s is the image of the isomorphism class of some $\mathscr{O}^\times_{U_{\text{ét}}}$-torsor $X_{\text{ét}}$ on $U_{\text{ét}}$. By faithfully flat descent, there exists a $\mathscr{O}^\times_{U_{\text{Zar}}}$-torsor X on U_{Zar}, and an isomorphism of $\mathscr{O}^\times_{U_{\text{ét}}}$-torsors :

$$X_{\text{ét}} \xrightarrow{\sim} \mathscr{O}^\times_{U_{\text{ét}}} \otimes_{\widetilde{u}^*_U\mathscr{O}^\times_{U_{\text{Zar}}}} \widetilde{u}^*_U X.$$

However, after replacing U by a smaller open neighborhood of ξ, we may suppose that $X(U) \neq \varnothing$, therefore $X_{\text{ét}}(U) \neq \varnothing$ as well, *i.e.* s is the image of the trivial section of $\mathscr{F}(U)$. $\square$

3. MONOIDS AND POLYHEDRA

Unless explicitly stated otherwise, *every monoid encountered in this chapter shall be commutative*. For this reason, we shall usually economize adjectives, and write just "monoid" when referring to commutative monoids.

3.1. **Monoids.** If M is any monoid, we shall usually denote the composition law of M by multiplicative notation: $(x, y) \mapsto x \cdot y$ (so 1 is the neutral element). However, sometimes it is convenient to be able to switch to an additive notation; to allow for that, we shall denote by $(\log M, +)$ the monoid with additive composition law, whose underlying set is the same as for the given monoid $(M, \cdot)$, and such that the identity map is an isomorphism of monoids (then, the neutral element of $\log M$ is denoted by 0). For emphasis, we may sometimes denote by $\log : M \xrightarrow{\sim} \log M$ the identity map, so that one has the tautological identities :

$$\log 1 = 0 \qquad \text{and} \qquad \log(x \cdot y) = \log x + \log y \quad \text{for every } x, y \in M.$$

Conversely, if $(M, +)$ is a given monoid with additive composition law, we may switch to a multiplicative notation by writing $(\exp M, \cdot)$, in the same way.

3.1.1. For any monoid M, and any two subsets $S, S' \subset M$, we let :

$$S \cdot S' := \{s \cdot s' \mid s \in S, s' \in S'\}$$

and S^a is defined recursively for every $a \in \mathbb{N}$, by the rule :

$$S^0 := \{1\} \qquad \text{and} \qquad S^a := S \cdot S^{a-1} \quad \text{if } a > 0.$$

Notice that the pair $(\mathscr{P}(M), \cdot)$ consisting of the set of all subsets of M, together with the composition law just defined, is itself a monoid : the neutral element is the subset $\{1\}$. In the same vein, the exponential notation for subsets of M becomes a multiplicative notation in the monoid $(\log \mathscr{P}(M), +) = (\mathscr{P}(\log M), +)$, *i.e.* we have the tautological identity : $\log S^a = a \cdot \log S$, for every $S \in \mathscr{P}(M)$ and every $a \in \mathbb{N}$.

Furthermore, for any two monoids M and N, the set $\mathrm{Hom}_{\mathbf{Mnd}}(M, N)$ is naturally a monoid. The composition law assigns to any two morphisms $\varphi, \psi : M \to N$ their product $\varphi \cdot \psi$, given by the rule : $\varphi \cdot \psi(m) := \varphi(m) \cdot \psi(m)$ for every $m \in M$.

Basic examples of monoids are the set $(\mathbb{N}, +)$ of natural numbers, and the non-negative real (resp. rational) numbers $(\mathbb{R}_+, +)$ (resp. $(\mathbb{Q}_+, +)$), with their standard addition laws.

3.1.2. Given a surjection $X \to Y$ of monoids, it may not be possible to express Y as a quotient of X – a problem relevant to the construction of *presentations* for given monoids, in terms of free monoids. For instance, consider the monoid $(\mathbb{Z}, \odot)$ consisting of the set $\mathbb{Z}$ with the composition law $\odot$ such that :

$$x \odot y := \begin{cases} x + y & \text{if either } x, y \geq 0 \text{ or } x, y \leq 0 \\ \max(x, y) & \text{otherwise} \end{cases}$$

for every $x, y \in \mathbb{Z}$. Define a surjective map $\varphi : \mathbb{N}^{\oplus 2} \to (\mathbb{Z}, \odot)$ by the rule $(n, m) \mapsto n \odot -m$, for every $n \in \mathbb{N}$. Then one verifies easily that $\mathrm{Ker}\, \varphi = \{0\}$, and nevertheless φ is not an isomorphism. The right way to proceed is indicated by the following :

Lemma 3.1.3. *Every surjective map of monoids is an effective epimorphism (in the category* $\mathbf{Mnd}$*).*

Proof. (See example 1.5.16 for the notion of effective epimorphism.) Let $\pi : M \to N$ be a surjection of monoids. For every monoid X, we have a natural diagram of sets :

$$\mathrm{Hom}_{\mathbf{Mnd}}(N, X) \xrightarrow{j} \mathrm{Hom}_{\mathbf{Mnd}}(M, X) \overset{p_1^*}{\underset{p_2^*}{\rightrightarrows}} \mathrm{Hom}_{\mathbf{Mnd}}(M \times_N M, X)$$

where $p_1, p_2 : M \times_N M \to M$ are the two natural projections, and we have to show that the map j identifies $\mathrm{Hom}_{\mathbf{Mnd}}(N, X)$ with the equalizer of p_1^* and p_2^*. First of all, the surjectivity of π easily implies that j is injective. Hence, let $\varphi : M \to X$ be any map such that $\varphi \circ p_1 = \varphi \circ p_2$; we have to show that φ factors through π. To this aim, it suffices to show that the map of sets underlying φ factors as a composition $\varphi' \circ \pi$, for some map of sets $\varphi' : N \to X$, since φ' will then be necessarily a morphism of monoids. However, the forgetful functor $F : \mathbf{Mnd} \to \mathbf{Set}$ commutes with fibre products (lemma 2.3.29(iii)), and $F(\pi)$ is an effective epimorphism, since in the category $\mathbf{Set}$ all surjections are effective epimorphisms. The assertion follows. $\square$

3.1.4. Lemma 3.1.3 allows to construct presentations for an arbitrary monoid M, as follows. First, we choose a surjective map of monoids $F := \mathbb{N}^{(S)} \to M$, for some set S. Then we choose another set T and a surjection of monoids $\mathbb{N}^{(T)} \to F \times_M F$. Composing with the natural projections $p_1, p_2 : F \times_M F \to F$, we obtain a diagram :

$$\mathbb{N}^{(T)} \overset{q_1}{\underset{q_2}{\rightrightarrows}} \mathbb{N}^{(S)} \longrightarrow M \tag{3.1.5}$$

which, in view of lemma 3.1.3, identifies M to the coequalizer of q_1 and q_2.

Definition 3.1.6. Let M be a monoid, $\Sigma \subset M$ a subset.

(i) Let $(e_\sigma \mid \sigma \in \Sigma)$ be the natural basis of the free monoid $\mathbb{N}^{(\Sigma)}$. We say that Σ is a *system of generators* for M, if the map of monoids $\mathbb{N}^{(\Sigma)} \to M$ such that $e_\sigma \mapsto \sigma$ for every $\sigma \in \Sigma$, is a surjection.

(ii) M is said to be *finitely generated* if it admits a finite system of generators.

(iii) M is said to be *fine* if it is integral and finitely generated.

(iv) A *finite presentation* for M is a diagram such as (3.1.5) that identifies M to the coequalizer of q_1 and q_2, and such that, moreover, S and T are finite sets.

(v) We say that a morphism of monoids $\varphi : M \to N$ is *finite*, if N is a finitely generated M-module, for the M-module structure induced by φ.

Lemma 3.1.7. (i) *Every finitely generated monoid admits a finite presentation.*

(ii) *Let M be a finitely generated monoid, and $(N_i \mid i \in I)$ a filtered family of monoids. Then the natural map :*

$$\operatorname*{colim}_{i \in I} \mathrm{Hom}_{\mathbf{Mnd}}(M, N_i) \to \mathrm{Hom}_{\mathbf{Mnd}}(M, \operatorname*{colim}_{i \in I} N_i)$$

is a bijection.

Proof. (i): Let M be a finitely generated monoid, and choose a surjection $\pi : \mathbb{N}^{(S)} \to M$ with S a finite set. We have seen that M is the coequalizer of the two projections $p_1, p_2 : P := \mathbb{N}^{(S)} \times_M \mathbb{N}^{(S)} \to \mathbb{N}^{(S)}$. For every finitely generated submonoid $N \subset P$, let $p_{1,N}, p_{2,N} : N \to \mathbb{N}^{(S)}$ be the restrictions of p_1 and p_2, and denote by C_N the coequalizer of $p_{1,N}$ and $p_{2,N}$. By the universal property of C_N, the map π factors through a map of monoids $\pi_N : C_N \to M$, and since π is surjective, the same holds for π_N. It remains to show that π_N is an isomorphism, for N large enough. We apply the functor $M \mapsto \mathbb{Z}[M]$ of (2.3.50), and we derive that $\mathbb{Z}[M]$ is the coequalizer of the two maps $\mathbb{Z}[p_1], \mathbb{Z}[p_2] : \mathbb{Z}[P] \to \mathbb{Z}[S]$, *i.e.* $\mathbb{Z}[M] \simeq \mathbb{Z}[S]/I$, where I is the ideal generated by $\mathrm{Im}(\mathbb{Z}[p_1] - \mathbb{Z}[p_2])$. Clearly I is the colimit of the filtered system of analogous ideals I_N generated by $\mathrm{Im}(\mathbb{Z}[p_{1,N}] - \mathbb{Z}[p_{2,N}])$, for N ranging over the filtered family $\mathscr{F}$ of finitely generated submonoids of P. By noetherianity, there exists $N \in \mathscr{F}$ such that $I = I_N$, therefore $\mathbb{Z}[M]$ is the coequalizer of $\mathbb{Z}[p_{1,N}]$ and $\mathbb{Z}[p_{2,N}]$. But the latter coequalizer is also the same as $\mathbb{Z}[C_N]$, whence the contention.

(ii): This is a standard consequence of (i). Indeed, say that $f_1, f_2 : M \to N_i$ are two morphisms whose compositions with the natural map $N_i \to N := \operatorname*{colim}_{i \in I} N_i$ agree, and pick a finite set of generators $x_1, \dots, x_n$ for M. For any morphism $\varphi : i \to j$ in the filtered category I, denote by $g_\varphi : N_i \to N_j$ the corresponding morphism; then we may find such a morphism φ, so that $g_\varphi \circ f_1(x_k) = g_\varphi \circ f_2(x_k)$ for every $k \leq n$, whence the injectivity of the map in (ii). Next, let $f : M \to N$ be a given morphism, and pick a finite presentation (3.1.5); we deduce a morphism $g : \mathbb{N}^{(S)} \to N$, and since S is finite, it is clear that g factors through a morphism $g_i : \mathbb{N}^{(S)} \to N_i$ for some $i \in I$. Set $g'_i := g_i \circ q_1$ and $g''_i := g_i \circ q_2$; by assumption, after composing g'_i and of g''_i with the natural map $N_i \to N$, we obtain the same map, so by the foregoing there exists a morphism $\varphi : i \to j$ in I such that $g_\varphi \circ g'_i = g_\varphi \circ g''_i$. It follows that $g_\varphi \circ g_i$ factors through M, whence the surjectivity of the map in (ii). $\square$

Definition 3.1.8. Let M be a monoid, $I \subset M$ an ideal.

(i) We say that I is *principal*, if it is cyclic, when regarded as an M-module.

(ii) The *radical* of I is the ideal $\mathrm{rad}(I)$ consisting of all $x \in M$ such that $x^n \in I$ for every sufficiently large $n \in \mathbb{N}$. If $I = \mathrm{rad}(I)$, we also say that I is a *radical ideal*.

(iii) A *face* of M is a submonoid $F \subset M$ with the following property. If $x, y \in M$ are any two elements, and $xy \in F$, then $x, y \in F$.

(iv) Notice that the complement of a face is always an ideal. We say that I is a *prime ideal* of M, if $M \setminus I$ is a face of M.

Proposition 3.1.9. *Let M be a finitely generated monoid, and S a finitely generated M-module. Then we have :*

(i) *Every submodule of S is finitely generated.*
(ii) *Especially, every ideal of M is finitely generated.*

Proof. Of course, (ii) is a special case of (i). To show (i), let $S' \subset S$ be an M-submodule, $\Sigma \subset S'$ any system of generators. Let $\mathscr{P}'(\Sigma)$ be the set of all finite subsets of Σ, and for every $A \in \mathscr{P}'(\Sigma)$, denote by $S'_A \subset S'$ the submodule generated by A; clearly S' is the filtered union of the family $(S'_A \mid A \in \mathscr{P}'(\Sigma))$, hence $\mathbb{Z}[S']$ is the filtered union of the family of $\mathbb{Z}[M]$-submodules $(\mathbb{Z}[S'] \mid S \in \mathscr{P}'(\Sigma))$. Since $\mathbb{Z}[M]$ is noetherian and $\mathbb{Z}[S]$ is a finitely generated $\mathbb{Z}[M]$-module, it follows that $\mathbb{Z}[S'_A] = \mathbb{Z}[S']$ for some finite subset $A \subset \Sigma$, whence the contention. $\square$

Corollary 3.1.10. *Let M be any fine and sharp monoid. The set $\mathfrak{m}_M \setminus \mathfrak{m}_M^2$ is finite, and is the unique minimal system of generators of M.*

Proof. It is easily seen that any system of generators of M must contain $\Sigma := \mathfrak{m}_M \setminus \mathfrak{m}_M^2$, hence the latter must be a finite set. On the other hand, suppose that there exists an element $x_0 \in M$ which is not contained in the submonoid M' generated by Σ. Then we may write $x_0 = x_1 y_1$ for some $x_0, y_0 \in \mathfrak{m}_M$, with $x_1 \notin M'$, so x_1 admits a similar decomposition. Proceeding in this way, we obtain a sequence of elements $(x_n \mid n \in \mathbb{N})$ with the property that $Mx_n \subset Mx_{n+1}$ for every $n \in \mathbb{N}$. We claim that $Mx_n \neq Mx_{n+1}$ for every $n \in \mathbb{N}$. Indeed, if the inequality fails for some $n \in \mathbb{N}$, we may write $x_{n+1} = ax_n$ for some $a \in \mathbb{N}$, and on the other hand, we have by construction $x_n = yx_{n+1}$ for some $y \in \mathfrak{m}_M$; summing up, we get $x_n = yax_n$, whence $ya = 1$, since M is integral, therefore $y \in M^\times$, a contradiction.

Thus, from the given x_0, we have produced an infinite strictly ascending chain of ideals of M, which is ruled out by virtue of proposition 3.1.9(ii). This means that x_0 cannot exist, and the corollary follows. $\square$

3.1.11. Let M be a monoid, $(I_\lambda \mid \lambda \in \Lambda)$ any collection of ideals of M; then it is easily seen that both $\bigcup_{\lambda \in \Lambda} I_\lambda$ and $\bigcap_{\lambda \in \Lambda} I_\lambda$ are ideals of M. The *spectrum* of M is the set :

$$\operatorname{Spec} M$$

consisting of all prime ideals of M. It has a natural partial ordering, given by inclusion of prime ideals; the minimal element of $\operatorname{Spec} M$ is the empty ideal $\varnothing \subset M$, and the maximal element is:

$$\mathfrak{m}_M := M \setminus M^\times.$$

If $(\mathfrak{p}_\lambda \mid \lambda \in \Lambda)$ is any family of prime ideals of M, then $\bigcup_{\lambda \in \Lambda} \mathfrak{p}_\lambda$ is a prime ideal of M.

Any morphism $\varphi : M \to N$ of monoids induces a natural map of partially ordered sets :

$$\varphi^* : \operatorname{Spec} N \to \operatorname{Spec} M \qquad \mathfrak{p} \mapsto \varphi^{-1}\mathfrak{p}.$$

We say that φ is *local*, if $\varphi(\mathfrak{m}_M) \subset \mathfrak{m}_N$.

Lemma 3.1.12. *Let $f_1 : M \to N_1$ and $f_2 : M \to N_2$ be two local morphisms of monoids. If N_1 and N_2 are sharp, then $N_1 \amalg_M N_2$ is sharp.*

Proof. Let $(a, b) \in N_1 \times N_2$, and suppose there exists $c \in M$, $a' \in N_1$, $b' \in N_2$ such that $(a, b) = (a' f_1(c), b')$ and $(1, 1) = (a', f_2(c)b')$; since N_2 is sharp, we deduce $f_2(c) = b' = 1$, so $b = 1$. Then, since f_2 is local, we get $c \in M^\times$, hence $f_1(c) = 1$ and $a = a' = 1$. One argues symmetrically in case $(1, 1) = (a' f_1(c), b')$ and $(a, b) = (a', f_2(c)b')$. We conclude that (a, b) represents the unit class in $N_1 \amalg_M N_2$ if and only if $a = b = 1$. Now, suppose that the class of (a, b) is invertible in $N_1 \amalg_M N_2$; it follows that there exists (c, d) such that $ac = 1$ and $bd = 1$, which implies that $a = 1$ and $b = 1$, whence the contention. $\square$

Lemma 3.1.13. *Let $S \subset M$ be any submonoid. The localization $j : M \to S^{-1}M$ induces an injective map $j^* : \operatorname{Spec} S^{-1}M \to \operatorname{Spec} M$ which identifies $\operatorname{Spec} S^{-1}M$ with the subset of $\operatorname{Spec} M$ consisting of all prime ideals $\mathfrak{p}$ such that $\mathfrak{p} \cap S = \varnothing$.*

Proof. For every $\mathfrak{p} \in \operatorname{Spec} M$, denote by $S^{-1}\mathfrak{p}$ the ideal of $S^{-1}M$ generated by the image of $\mathfrak{p}$. We claim that $\mathfrak{p} = j^*(S^{-1}\mathfrak{p})$ for every $\mathfrak{p} \in \operatorname{Spec} M$ such that $\mathfrak{p} \cap S = \varnothing$. Indeed, clearly $\mathfrak{p} \subset j^*(S^{-1}\mathfrak{p})$; next, if $f \in j^*(S^{-1}\mathfrak{p})$, there exists $s \in S$ and $g \in \mathfrak{p}$ such that $s^{-1}g = f$ in $S^{-1}M$; therefore there exists $t \in S$ such that $tg = tsf$ in M, especially $tsf \in \mathfrak{p}$, hence $f \in \mathfrak{p}$, since $t, s \notin \mathfrak{p}$. Likewise, one checks easily that $S^{-1}\mathfrak{p}$ is a prime ideal if $\mathfrak{p} \cap S = \varnothing$, and $\mathfrak{q} = S^{-1}(j^*\mathfrak{q})$ for every $\mathfrak{q} \in \operatorname{Spec} S^{-1}M$, whence the contention. □

Remark 3.1.14. (i) If we take $S_\mathfrak{p} := M \setminus \mathfrak{p}$, the complement of a prime ideal $\mathfrak{p}$ of M, we obtain the monoid

$$M_\mathfrak{p} := S_\mathfrak{p}^{-1} M$$

and $\operatorname{Spec} M_\mathfrak{p} \subset \operatorname{Spec} M$ is the subset consisting of all prime ideals $\mathfrak{q}$ contained in $\mathfrak{p}$.

(ii) Likewise, if $\mathfrak{p} \subset M$ is any prime ideal, the spectrum $\operatorname{Spec}(M \setminus \mathfrak{p})$ is naturally identified with the subset of $\operatorname{Spec} M$ consisting of all prime ideals $\mathfrak{q}$ containing $\mathfrak{p}$ (details left to the reader).

(iii) Let $S \subset M$ be any submonoid. Then there exists a smallest face F of M containing S (namely, the intersection of all the faces that contain S). It is easily seen that F is the subset of all $x \in M$ such that $xM \cap S \neq \varnothing$. From this characterization, it is clear that $S^{-1}M = F^{-1}M$. In other words, every localization of M is of the type $M_\mathfrak{p}$ for some $\mathfrak{p} \in \operatorname{Spec} M$.

Lemma 3.1.15. *Let M be a monoid, and $I \subset M$ any ideal. Then $\operatorname{rad}(I)$ is the intersection of all the prime ideals of M containing I.*

Proof. It is easily seen that a prime ideal containing I also contains $\operatorname{rad}(I)$. Conversely, say that $f \in M \setminus \operatorname{rad}(I)$; let $\varphi : M \to M_f$ be the localization map. Denote by $\mathfrak{m}$ the maximal ideal of M_f. We claim that $I \subset \varphi^{-1}\mathfrak{m}$, Indeed, otherwise there exist $g \in I$, $h \in M$ and $n \in \mathbb{N}$ such that $g^{-1} = f^{-n}h$ in M_f; this means that there exist $m \in \mathbb{N}$ such that $f^{m+n} = f^m gh$ in M, hence $f^{m+n} \in I$, which contradicts the assumption on f. On the other hand, obviously $f \notin \varphi^{-1}\mathfrak{m}$. □

Lemma 3.1.16. (i) *Let M be a monoid, and $G \subset M^\times$ a subgroup.*

- (a) *The map given by the rule : $I \mapsto I/G$ establishes a natural bijection from the set of ideals of M onto the set of ideals of M/G.*
- (b) *Especially, the natural projection $\pi : M \to M/G$ induces a bijection :*

$$\pi^* : \operatorname{Spec} M/G \to \operatorname{Spec} M$$

(ii) *Let $(M_i \mid i \in I)$ be any finite family of monoids, and for each $j \in I$, denote by $\pi_j : \prod_{i \in I} M_i \to M_j$ the natural projection. The induced map*

$$\prod_{i \in I} \operatorname{Spec} M_i \to \operatorname{Spec} \prod_{i \in I} M_i \quad : \quad (\mathfrak{p}_i \mid i \in I) \mapsto \bigcup_{i \in I} \pi_i^* \mathfrak{p}_i$$

is a bijection.

(iii) *Let $(M_i \mid i \in I)$ be any filtered system of monoids. The natural map*

$$\operatorname{Spec} \operatorname*{colim}_{i \in I} M_i \to \lim_{i \in I} \operatorname{Spec} M_i$$

is a bijection.

Proof. (i): By lemma 2.3.31(iii), M/G is the set-theoretic quotient of M by the translation action of G. By definition, any ideal I of M is stable under the G-action, hence the quotient I/G is well defined, and one checks easily that it is an ideal of M/G. Moreover, if $\mathfrak{p} \subset M$

is a prime ideal, it is easily seen that $\mathfrak{p}/G$ is a prime ideal of M/G. Assertions (a) and (b) are straightforward consequences.

(ii): The assertion can be rephrased by saying that every face F of $\prod_{i\in I} M_i$ is a product of faces $F_i \subset M_i$. However, if $\underline{m} := (m_i \mid i \in I) \in F$, then, for each $i \in I$ we can write $\underline{m} = \underline{m}(i) \cdot \underline{n}(i)$, where, for each $j \in I$, the j-th-component of $\underline{m}(i)$ (resp. of $\underline{n}(i)$) equals 1 (resp. m_j), unless $j = i$, in which case it equals m_i (resp. 1). Thus, $\underline{m}(i) \in F$ for every $i \in I$, and the contention follows easily.

(iii): Denote by M the colimit of the system $(M_i \mid i \in I)$, and $\varphi_i : M_i \to M$ the natural morphisms of monoids, as well as $\varphi_f : M_i \to M_j$ the transition maps, for every morphism $f : i \to j$ in I. Recall that the set underlying M is the colimit of the system of sets $(M_i \mid i \in I)$ (lemma 2.3.29(iii)). Let now $\mathfrak{p}_\bullet := (\mathfrak{p}_i \mid \ \in I)$ be a compatible system of prime ideals, *i.e.* such that $\mathfrak{p}_i \in \mathrm{Spec}\, M_i$ for every $i \in I$, and $\varphi_f^{-1}\mathfrak{p}_j = \mathfrak{p}_i$ for every $f : i \to j$. We let $\beta(\mathfrak{p}_\bullet) := \bigcup_{i\in I} \varphi_i(\mathfrak{p}_i)$. We claim that $\beta(\mathfrak{p}_\bullet)$ is a prime ideal of M. Indeed, suppose that $x, y \in M$ and $xy \in \beta(\mathfrak{p}_\bullet)$; since I is filtered, we may find $i \in I$, $x_i, y_i \in M_i$, and $z_i \in \mathfrak{p}_i$, such that $x = \varphi_i(x_i)$, $y = \varphi_i(y_i)$, and $xy = \varphi_i(z_i)$. Especially, $\varphi_i(z_i) = \varphi_i(x_i y_i)$, so there exists a morphism $f : i \to j$ such that $\varphi_f(z_i) = \varphi_f(x_i y_i)$. But $\varphi_f(z_i) \in \mathfrak{p}_j$, so either $\varphi_f(x_i) \in \mathfrak{p}_j$ or $\varphi_f(y_i) \in \mathfrak{p}_j$, and finally either $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$, as required.

Let $\mathfrak{p} \subset M$ be any prime ideal; it is easily seen that $\beta(\varphi_i^{-1}\mathfrak{p} \mid i \in I) = \mathfrak{p}$. To conclude, it suffices to show that $\mathfrak{p}_i = \varphi_i^{-1}\beta(\mathfrak{p}_\bullet)$, for every compatible system $\mathfrak{p}_\bullet$ as above, and every $i \in I$. Hence, fix $i \in I$ and pick $x_i \in M_i$ such that $\varphi_i(x_i) \in \beta(\mathfrak{p}_\bullet)$; then there exists $j \in I$ and $x_j \in \mathfrak{p}_j$ such that $\varphi_i(x_i) = \varphi_j(x_j)$. Since I is filtered, we may find $k \in I$ and morphisms $f : i \to k$ and $g : j \to k$ such that $\varphi_f(x_i) = \varphi_g(x_j)$, so $\varphi_f(x_i) \in \mathfrak{p}_k$, and finally $x_i \in \mathfrak{p}_i$, as sought. ☐

Remark 3.1.17. In case $(M_i \mid i \in I)$ is an infinite family of monoids, the natural map of lemma 3.1.16(ii) is still injective, but it is not necessarily surjective. For instance, let I be any infinite set, and let $\mathscr{U} \subset \mathscr{P}(I)$ be a non-principal ultrafilter; denote by ${}^*\mathbb{N}$ the quotient of $\mathbb{N}^I$ under the equivalence relation $\sim_\mathscr{U}$ such that $(a_i \mid i \in I) \sim_\mathscr{U} (b_i \mid i \in I)$ if and only if there exists $U \in \mathscr{U}$ such that $a_i = b_i$ for every $i \in U$. It is clear that the composition law on $\mathbb{N}^I$ descends to ${}^*\mathbb{N}$; the resulting structure $({}^*\mathbb{N}, +)$ is called the monoid of hypernatural numbers. Denote by $\pi : \mathbb{N}^I \to {}^*\mathbb{N}$ the projection, and let $\underline{0} \in \mathbb{N}^I$ be the unit; then it is easily seen that $\{\pi(0)\}$ is a face of ${}^*\mathbb{N}$, but $\pi^{-1}(\pi(0))$ is not a product of faces $F_i \subset \mathbb{N}$.

Definition 3.1.18. Let M be a monoid.

(i) The *dimension* of M, denoted $\dim M \in \mathbb{N} \cup \{+\infty\}$, is defined as the supremum of all $r \in \mathbb{N}$ such that there exists a chain of strict inclusions of prime ideals of M :
$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_r.$$
(ii) The *height* of a prime ideal $\mathfrak{p} \in \mathrm{Spec}\, M$ is defined as $\mathrm{ht}\,\mathfrak{p} := \dim M_\mathfrak{p}$.

(iii) A *facet* of M is the complement of a prime ideal of M of height one.

Remark 3.1.19. (i) Notice that not all epimorphisms in $\mathbf{Mnd}$ are surjections on the underlying sets; for instance, every localization map $M \to S^{-1}M$ is an epimorphism.

(ii) If $\varphi : N \to M$ is a map of fine monoids (see definition 3.1.6(vi)), then it will follow from corollary 3.4.2 that $N \times_M N$ is also finitely generated. If M is not integral, then this fails in general : a counter-example is provided by the morphism φ constructed in (3.1.2).

(iii) Let Σ be any set; it is easily seen that a free monoid $M \simeq \mathbb{N}^{(\Sigma)}$ admits a unique minimal system of generators, in natural bijection with Σ. Especially, the cardinality of Σ is determined by the isomorphism class of M; this invariant is called the *rank* of the free monoid M. This is the same as the rank of M as an $\mathbb{N}$-module (see example 1.2.27).

(iv) A submonoid of a finitely generated monoid is not necessarily finitely generated. For instance, consider the submonoid $M \subset \mathbb{N}^{\oplus 2}$, with $M := \{(0,0)\} \cup \{(a,b) \mid a > 0\}$. However, the following result shows that a face of a finitely generated monoid is again finitely generated.

Lemma 3.1.20. *Let $f : M \to N$ be a map of monoids, $F \subset N$ a face of N, and $\Sigma \subset N$ a system of generators for N. Then :*

(i) *$N^\times$ is a face of N, and $f^{-1}F$ is a face of M.*
(ii) *$\Sigma \cap F$ is a system of generators for F.*
(iii) *If N is finitely generated, $\mathrm{Spec}\, N$ is a finite set, and $\dim N$ is finite.*
(iv) *If N is finitely generated (resp. fine) then the same holds for $F^{-1}N$.*

Proof. (i) and (ii) are left to the reader, and (iii) is an immediate consequence of (ii). To show (iv), notice that – in view of (ii) – the set $\Sigma \cup \{f^{-1} \mid f \in F \cap \Sigma\}$ is a system of generators of $F^{-1}N$. $\square$

Definition 3.1.21. (i) If M is a (pointed or not pointed) monoid, and S is a pointed M-module, we say that S is *integral*, if for every $x, y \in M$ and every $s \in S$ such that $xs = ys \neq 0$, we have $x = y$. The *annihilator ideal* of S is the ideal

$$\mathrm{Ann}_M(S) := \{m \in M \mid ms = 0 \text{ for every } s \in S\}.$$

If $s \in S$ is any element, we also write $\mathrm{Ass}_M(s) := \mathrm{Ass}_M(Ms)$. The *support* of S is the subset :

$$\mathrm{Supp}\, S := \{\mathfrak{p} \in \mathrm{Spec}\, M \mid S_\mathfrak{p} \neq 0\}.$$

(ii) A pointed monoid $(M, 0_M)$ is called *integral*, if it is integral when regarded as a pointed M-module; it is called *fine*, if it is finitely generated and integral in the above sense.

(iii) The forgetful functor $\mathbf{Mnd}_\circ \to \mathbf{Set}$ (notation of (2.3.28)) admits a left adjoint, which assigns to any set Σ the *free pointed monoid* $\mathbb{N}_\circ^{(\Sigma)} := (\mathbb{N}^{(\Sigma)})_\circ$.

(iv) A morphism $\varphi : M \to N$ of pointed monoids is *local*, if both $M, N \neq 0$, and φ is local when regarded as a morphism of non-pointed monoids.

Remark 3.1.22. (i) Quite generally, a (non-pointed) monoid M is finitely generated (resp. free, resp. integral, resp. fine) if and only if $M_\circ$ has the corresponding property for pointed monoids. However, there exist integral pointed monoids which are not of the form $M_\circ$ for any non-pointed monoid M.

(ii) Let $(M, 0_M)$ be a pointed monoid. An *ideal* of $(M, 0_M)$ is a pointed submodule $I \subset M$. Just as for non-pointed monoids, we say that I is a *prime ideal*, if $M \setminus I$ is a (non-pointed) submonoid of M, and a non-pointed submonoid which is the complement of a prime ideal, is called a *face* of M. Hence the smallest ideal is $\{0\}$. Notice though, that $\{0\}$ is not necessarily a prime ideal, hence the spectrum $\mathrm{Spec}\,(M, 0_M)$ does not always admit a least element. However, if $M = N_\circ$ for some non-pointed monoid N, the natural morphism of monoids $N \to N_\circ$ induces a bijection :

$$\mathrm{Spec}\,(N_\circ, 0_{N_\circ}) \to \mathrm{Spec}\, N.$$

(iii) Let $I \subset M$ be any ideal; the inclusion map $I \to M$ can be regarded as a morphism of pointed M-modules (if M is not pointed, this is achieved via the faithful imbedding (2.3.15)), whence a pointed M-module M/I, with a natural morphism $M \to M/I$. The latter map is also a morphism of monoids, for the obvious monoid structure on M/I. One checks easily that, if M is integral, M/I is an integral pointed monoid.

(iv) Let M be a (pointed or not pointed) monoid, $\mathfrak{p} \subset M$ a prime ideal. Then the natural morphism of monoids $M \to M/\mathfrak{p}$ induces a bijection :

$$\mathrm{Spec}\, M/\mathfrak{p} \xrightarrow{\sim} \{\mathfrak{q} \in \mathrm{Spec}\, M \mid \mathfrak{p} \subset \mathfrak{q}\} = \mathrm{Spec}\, M \setminus \mathfrak{p}.$$

(v) Let M be a (pointed or not pointed) monoid, and $S \neq 0$ a pointed M-module. Then the support of S contains at least the maximal ideal of M. This trivial observation shows that a pointed M-module is 0 if and only if its support is empty.

(vi) Let M be a pointed monoid, and $\Sigma \subset M$ a *non-pointed* submonoid. The localization $\Sigma^{-1}M$ (defined in the category of monoids, as in (2.3.33)) is actually a pointed monoid : its zero element $0_{\Sigma^{-1}M}$ is the image of 0_M.

(vii) If M is a (pointed or not pointed) monoid, $\Sigma \subset M$ any *non-pointed* submonoid, and S a pointed M-module, we let as usual $\Sigma^{-1}S := \Sigma^{-1}M \otimes_M S$ (see remark 2.3.21(i), if M is not pointed). The resulting functor $M\text{-}\mathbf{Mod}_\circ \to \Sigma^{-1}M\text{-}\mathbf{Mod}_\circ$ is exact. Indeed, it is right exact, since it is left adjoint to the restriction of scalars arising from the localization map $M \to \Sigma^{-1}M$ (see (1.2.26)), and one verifies directly that it commutes with finite limits. Also, it is clear that

$$\mathrm{Supp}\,\Sigma^{-1}S = \mathrm{Supp}\,S \cap \mathrm{Spec}\,\Sigma^{-1}M.$$

(viii) Let M be a pointed monoid, and $N \subset M$ a pointed submonoid. Since the final object 1 of the category of pointed monoids is not isomorphic to the initial object $1_\circ$, the push-out of the diagram $1 \leftarrow N \to M$ is not an interesting object (it is always isomorphic to 1). Even if we form the quotient M/N in the category of non-pointed monoids, we still get always 1, since $0_M \in N$, and therefore in the quotient M/N the images of 0_M and of the unit of M coincide.

The only case that may give rise to a non-trivial quotient, is when N is *non-pointed*; in this situation we may form M/N *in the category of non-pointed monoids*, and then remark that the image of 0_M yields a zero element $0_{M/N}$ for M/N, so the latter is a pointed monoid.

Example 3.1.23. (i) Let M be a (pointed or not pointed) monoid, $G \subset M^\times$ a subgroup, and S a pointed M-module. Then $M/G \otimes_M S = S/G$ is the set of orbits of S under the induced G-action.

(ii) In the situation of (i), notice that the functor

$$M\text{-}\mathbf{Mod} \to M/G\text{-}\mathbf{Mod} \quad : \quad S \mapsto S/G$$

is exact, hence M/G is a flat M-module. (See definition 2.3.22(i).)

(iii) Likewise, if $\Sigma \subset M$ is a non-pointed submonoid, then the localization $\Sigma^{-1}M$ is a flat M-module, due to remark 3.1.22(vii).

(iv) Suppose that S is an integral pointed M-module (with M either pointed or not pointed), and let $\Sigma \subset M$ be a non-pointed submonoid. Then $\Sigma^{-1}S$ is also an integral pointed $\Sigma^{-1}M$-module. Indeed, suppose that the identity

$$(s^{-1}a) \cdot (s'^{-1}b) = (s^{-1}a) \cdot (s''^{-1}c) \neq 0 \tag{3.1.24}$$

holds for some $a, b, c \in S$ and $s, s', s'' \in \Sigma$; we need to check that $s'^{-1}b = s''^{-1}c$, or equivalently, that $s''b = s'c$ in $\Sigma^{-1}S$. However, (3.1.24) is equivalent to $s''ab = s'ac$ in $\Sigma^{-1}S$, and the latter holds if and only if there exists $t \in \Sigma$ such that $ts''ab = ts'ac$ in S. The two sides in the latter identity are $\neq 0$, as the same holds for the two sides of the identity (3.1.24); therefore $s''b = s'c$ holds already in S, and the contention follows.

(v) Likewise, in the situation of (iv), $S/\Sigma := S \otimes_M M/\Sigma$ is an integral pointed M/Σ-module. Indeed, notice the natural identification $S/\Sigma = \Sigma^{-1}S \otimes_{\Sigma^{-1}M} (\Sigma^{-1}M)/\Sigma^{\mathrm{gp}}$ which – in view of (iv) – reduces the proof to the case where Σ is a subgroup of $M^\times$. Then the assertion is easily verified, taking into account (i).

Especially, if M is an integral pointed monoid, and $\Sigma \subset M$ is any non-pointed submonoid, then both $\Sigma^{-1}M$ and M/Σ are integral pointed monoids (this generalizes lemma 2.3.38).

(vi) Let G be any abelian group, $\varphi : M \to G$ a morphism of non-pointed monoids. Then $G_\circ$ is a flat $M_\circ$-module. For the proof, we may – in light of (iii) – replace M by M^{gp}, thereby reducing to the case where M is a group. Next, by (ii), we may assume that φ is injective, in which case G is a free M-module with basis G/M.

Remark 3.1.25. (i) Let $M \to N$ and $M \to N'$ be morphisms of pointed monoids; N and N' can be regarded as pointed M-modules in an obvious way, hence we may form the tensor product $N'' := N \otimes_M N'$; the latter is endowed with a unique monoid structure such that the maps $e : N \to N''$ and $e' : N' \to N''$ given by the rule $n \mapsto n \otimes 1$ for all $n \in N$ (resp. $n' \mapsto 1 \otimes n'$ for all $n' \in N'$) are morphisms of monoids. Just as for usual ring homomorphisms, the monoid N'' is a coproduct of N and N' over M, *i.e.* there is a unique isomorphism of pointed monoids:

$$N \otimes_M N' \xrightarrow{\sim} N \amalg_M N' \tag{3.1.26}$$

that identifies e and e' to the natural morphisms $N \to N \amalg_M N'$ and $N' \to N \amalg_M N'$. As usual all this extends to non-pointed monoids. (Details left to the reader.)

(ii) Especially, if we take $M = \{1\}_\circ$, the initial object in $\mathbf{Mnd}_\circ$, we obtain an explicit description of the pointed monoid $N \oplus N'$: it is the quotient $(N \times N')/\sim$, where $\sim$ denotes the minimal equivalence relation such that $(x, 0) \sim (0, x')$ for every $x \in N$, $x' \in N'$. From this, a direct calculation shows that a direct sum of pointed integral monoids is again a pointed integral monoid.

Remark 3.1.27. (i) Clearly every pointed M-module S is the colimit of the filtered family of its finitely generated submodules. Moreover, S is the colimit of a filtered family of finitely presented pointed M-modules. Recall the standard argument : pick a countable set I, and let $\mathscr{C}$ be the (small) full subcategory of the category $M\text{-}\mathbf{Mod}_\circ$ whose objects are the coequalizers of every pair of maps of pointed M-modules $p, q : M^{(I_1)} \to M^{(I_2)}$, for every finite sets $I_1, I_2 \subset I$ (this means that, for every such pair p, q we pick one representative for this coequalizer). Then there is a natural isomorphism of pointed M-modules :

$$\operatorname*{colim}_{i\mathscr{C}/S} \iota_S \xrightarrow{\sim} S$$

where $i : \mathscr{C} \to M\text{-}\mathbf{Mod}_\circ$ is the inclusion functor, and ι_S is the functor as in (1.1.16).

(ii) If S is finitely generated, we may find a finite filtration of S by submodules $0 = S_0 \subset S_1 \subset \cdots \subset S_n = S$ such that S_{i+1}/S_i is a cyclic M-module, for every $i = 1, \dots, n$.

(iii) Notice that, if S is integral and $S' \subset S$ is any submodule, then S/S' is again integral. Moreover, if S is integral and cyclic, we have a natural isomorphism of M-modules :

$$S \xrightarrow{\sim} M/\mathrm{Ann}_M(S).$$

(Details left to the reader.)

(iv) Suppose furthermore, that $M^\sharp$ is finitely generated, and S is any pointed M-module. Lemma 3.1.16(i.a) and proposition 3.1.9(ii) easily imply that every ascending chain

$$I_0 \subset I_1 \subset I_2 \subset \cdots$$

of ideals of M is stationary; especially, the set $\{\mathrm{Ann}_M(s) \mid s \in S \setminus \{0\}\}$ admits maximal elements. Let I be a maximal element in this set; a standard argument as in commutative algebra shows that I is a prime ideal : indeed, say that $xy \in I = \mathrm{Ann}_M(s)$ and $x \notin I$; then $xs \neq 0$, hence $y \in \mathrm{Ann}_M(xs) = I$, by the maximality of I. Now, if S is also finitely generated, it follows that we may find a finite filtration of S as in (ii) such that, additionally, each quotient S_{i+1}/S_i is of the form $M/\mathfrak{p}$, for some prime ideal $\mathfrak{p} \subset M$.

These properties make the class of integral pointed modules especially well behaved : essentially, the full subcategory $M\text{-}\mathbf{Int}.\mathbf{Mod}_\circ$ of $M\text{-}\mathbf{Mod}_o$ consisting of these modules mimics closely a category of A-modules for a ring A, familiar from standard linear algebra. This shall be amply demonstrated henceforth. For instance, we point out the following combinatorial version of Nakayama's lemma :

Proposition 3.1.28. *Let M be a (pointed or not pointed) monoid, S a finitely generated integral pointed M-module, and $S' \subset S$ a pointed submodule, such that*

$$S = S' \cup \mathfrak{m}_M \cdot S.$$

Then $S = S'$.

Proof. After replacing S by S/S', we may assume that $\mathfrak{m}_M S = S$, in which case we have to check that $S = 0$. Suppose then, that $S \neq 0$; from remark 3.1.27(ii,iii) it follows that S admits a (pointed) submodule $T \subset S$ such that $S/T \simeq M/\mathfrak{m}_M$. Especially, $\mathfrak{m}_M \cdot (S/T) = 0$, *i.e.* $\mathfrak{m}_M S \subset T$, and therefore $S = T$, which contradicts the choice of T. The contention follows. □

Remark 3.1.29. (i) The integrality assumption cannot be omitted in proposition 3.1.28. Indeed, take $M := \mathbb{N}$ and $S := 0_\circ$, where 0 denotes the final $\mathbb{N}$-module. Then $S \neq 0$, but $\mathfrak{m}_M S = S$.

(ii) Let us say that an element of the M-module S is *primitive*, if it does not lie in $\mathfrak{m}_M S$. We deduce from proposition 3.1.28, the following :

Corollary 3.1.30. *Let M be a sharp (pointed or not pointed) monoid, S a finitely generated integral pointed monoid. Then the set $S \setminus \mathfrak{m}_M S$ of primitive elements of S is finite, and is the unique minimal system of generators of S.*

Proof. Indeed, it is easily seen that every system of generators of S must contain all the primitive elements, so $S \setminus \mathfrak{m}_M S$ must be finite. On the other hand, let $S' \subset S$ be the submodule generated by the primitive elements; clearly $S' \cup \mathfrak{m}_M S = S$, hence $S' = S$, by proposition 3.1.28. □

3.1.31. Let R be any ring, M a non-pointed monoid. Notice that the M-module underlying any $R[M]$-module is naturally pointed, whence a forgetful functor $R[M]\text{-}\mathbf{Mod} \to M\text{-}\mathbf{Mod}_\circ$. The latter admits a left adjoint

$$M\text{-}\mathbf{Mod}_\circ \to R[M]\text{-}\mathbf{Mod} \quad : \quad (S, 0_S) \mapsto R\langle S\rangle := \operatorname{Coker} R[0_S].$$

Likewise, the monoid $(A, \cdot)$ underlying any (commutative unitary) R-algebra A is naturally pointed, whence a forgetful functor $R\text{-}\mathbf{Alg} \to \mathbf{Mnd}_\circ$, which again admits a left adjoint

$$\mathbf{Mnd}_\circ \to R\text{-}\mathbf{Alg} \quad : \quad (M, 0_M) \mapsto R\langle M\rangle := R[M]/(0_M)$$

where $(0_M) \subset R[M]$ denotes the ideal generated by the image of 0_M.

If $(M, 0_M)$ is a pointed monoid, and S is a pointed $(M, 0_M)$-module, then notice that $R\langle S\rangle$ is actually a $R\langle M\rangle$-module, so we have as well a natural functor

$$(M, 0_M)\text{-}\mathbf{Mod}_\circ \to R\langle M\rangle\text{-}\mathbf{Mod} \qquad S \mapsto R\langle S\rangle$$

which is again left adjoint to the forgetful functor.

For instance, let $I \subset M$ be an ideal; from the foregoing, it follows that $R\langle M/I\rangle$ is naturally an $R[M]$-algebra, and we have a natural isomorphism :

$$R\langle M/I\rangle \xrightarrow{\sim} R[M]/IR[M].$$

Explicitly, for any $x \in F := M \setminus I$, let $\overline{x} \in R\langle M/I\rangle$ be the image of x; then $R\langle M/I\rangle$ is a free R-module, with basis $(\overline{x} \mid x \in F)$. The multiplication law of $R\langle M/I\rangle$ is determined as follows. Given $x, y \in F$, then $\overline{x} \cdot \overline{y} = \overline{xy}$ if $xy \in F$, and otherwise it equals zero.

Notice that, if I_1 and I_2 are two ideals of M, we have a natural identification :

$$R\langle M/(I_1 \cap I_2)\rangle \xrightarrow{\sim} R\langle M/I_1\rangle \times_{R\langle M/(I_1 \cup I_2)\rangle} R\langle M/I_2\rangle.$$

These algebras will play an important role in section 6.5. As a special case, suppose that $\mathfrak{p} \subset M$ is a prime ideal; then the inclusion $M \setminus \mathfrak{p} \subset M$ induces an isomorphism of R-algebras :

$$R[M \setminus \mathfrak{p}] \xrightarrow{\sim} R\langle M/\mathfrak{p}\rangle.$$

Furthermore, general nonsense yields a natural isomorphism of R-modules :

(3.1.32) $\quad R\langle S \otimes_M S'\rangle \xrightarrow{\sim} R\langle S\rangle \otimes_{R\langle M\rangle} R\langle S'\rangle \qquad$ for all pointed M-modules S and S'.

3.1.33. Let A be a commutative ring with unit, and $f : M \to (A, \cdot)$ a morphism of pointed monoids. Then f induces (forgetful) functors :

$$A\text{-}\mathbf{Mod} \to M\text{-}\mathbf{Mod}_\circ \qquad A\text{-}\mathbf{Alg} \to M/\mathbf{Mnd}_\circ$$

(notation of (1.1.12)) which admit left adjoints :

$$\begin{array}{ll} M\text{-}\mathbf{Mod}_\circ \to A\text{-}\mathbf{Mod} & S \mapsto S \otimes_M A := \mathbb{Z}\langle S\rangle \otimes_{\mathbb{Z}\langle M\rangle} A \\ M/\mathbf{Mnd}_\circ \to A\text{-}\mathbf{Alg} & N \mapsto N \otimes_M A := \mathbb{Z}\langle N\rangle \otimes_{\mathbb{Z}\langle M\rangle} A. \end{array}$$

Sometimes we may also use the notation :

$$S \otimes_M N := \mathbb{Z}\langle S\rangle \otimes_{\mathbb{Z}\langle M\rangle} N \quad \text{and} \quad S \overset{\mathbf{L}}{\otimes}_M K_\bullet := \mathbb{Z}\langle S\rangle \overset{\mathbf{L}}{\otimes}_{\mathbb{Z}\langle M\rangle} K_\bullet$$

for any pointed M-module S, any A-module N and any object $K_\bullet$ of $\mathsf{D}^-(A\text{-}\mathbf{Mod})$. The latter derived tensor product is obtained by tensoring with a flat $\mathbb{Z}\langle M\rangle$-flat resolution of $\mathbb{Z}\langle S\rangle$. (Such resolutions can be constructed combinatorially, starting from a simplicial resolution of S.) All the verifications are standard, and shall be left to the reader.

Definition 3.1.34. Let M be a pointed monoid, A a commutative ring with unit, $\varphi : M \to (A, \cdot)$ a morphism of pointed monoids, N an A-module. We say that N is *φ-flat* (or just M-flat, if no ambiguity is likely to arise), if the functor

$$M\text{-}\mathbf{Int}.\mathbf{Mod}_\circ \to A\text{-}\mathbf{Mod} \quad : \quad S \mapsto S \otimes_M N$$

is *exact*, in the sense that it sends exact sequences of pointed integral M-modules, to exact sequences of A-modules. We say that N is *faithfully φ-flat* if this functor is exact in the above sense, and we have $S \otimes_M N = 0$ if and only if $S = 0$.

Remark 3.1.35. (i) Notice that the functor $S \mapsto S \otimes_M N$ of definition 3.1.34 is right exact in the categorical sense (*i.e.* it commutes with finite colimits), since it is a right adjoint. However, even when N is faithfully flat, this functor is not always left exact in the categorical sense : it does not commute with finite products, nor with equalizers, in general.

(ii) Let M and S be as in definition 3.1.34, and let R be any non-zero commutative unitary ring. Denote by $\varphi : M \to (R\langle M\rangle, \cdot)$ the natural morphism of pointed monoids. In light of (3.1.32), it is clear that if $R\langle S\rangle$ is a flat $R\langle M\rangle$-module, then S is a flat pointed M-module, and the latter condition implies that $R\langle S\rangle$ is φ-flat.

Lemma 3.1.36. *Let M be a monoid, A a ring, $\varphi : M \to (A, \cdot)$ a morphism of monoids, and assume that φ is local and A is φ-flat. Then A is faithfully φ-flat.*

Proof. Let S be an integral pointed M-module, and suppose that $S \otimes_M A = \{0\}$; we have to show that $S = \{0\}$. Say that $s \in S$, and let $Ms \subset S$ be the M-submodule generated by s. Since A is φ-flat, it follows easily that $Ms \otimes_M A = \{0\}$, hence we are reduced to the case where S is cyclic. By remark 3.1.27(iii), we may then assume that $S = M/I$ for some ideal $I \subset M$. It follows that $S \otimes_M A = A/\varphi(I)A$, so that $\varphi(I)$ generates A. Since φ is local, this implies that $I = M$, whence the contention. □

Lemma 3.1.37. *Let M be an integral pointed monoid, $I, J \subset M$ two ideals, A a ring, $\alpha : M \to (A, \cdot)$ a morphism of monoids, N an α-flat A-module, and S a flat M-module. Then :*

$$IS \cap JS = (I \cap J)S$$
$$\alpha(I)N \cap \alpha(J)N = \alpha(I \cap J)N.$$

Proof. We consider the commutative ladder of pointed M-modules, with exact rows and injective vertical arrows :

$$\begin{tikzcd} 0 \arrow[r] & I \cap J \arrow[r] \arrow[d] & I \arrow[r] \arrow[d] & I/(I \cap J) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & J \arrow[r] & M \arrow[r] & M/(I \cup J) \arrow[r] & 0 \end{tikzcd} \tag{3.1.38}$$

By assumption, the ladder of A-modules (3.1.38)$\otimes_M N$ has still exact rows and injective vertical arrows. Then, the snake lemma gives the following short exact sequence involving the cokernels of the vertical arrows :

$$0 \to JN/(I \cap J)N \to N/IN \xrightarrow{p} N/(IN + JN) \to 0$$

(where we have written JN instead of $\alpha(J)N$, and likewise for the other terms). However $\mathrm{Ker}\, p = JN/(IN \cap JN)$, whence the second stated identity. The first stated identity can be deduced from the second, by virtue of remark 3.1.35(ii). $\square$

Remark 3.1.39. By inspection of the proof, we see that the first identity of lemma 3.1.37 holds, more generally, whenever $\mathbb{Z}\langle S\rangle$ is φ-flat, where $\varphi : M \to \mathbb{Z}\langle M\rangle$ is the natural morphism of pointed monoids.

Proposition 3.1.40. *Let M be a pointed integral monoid, A a ring, $\varphi : M \to (A, \cdot)$ a morphism of monoids, N an A-module. Then we have :*

(i) *The following conditions are equivalent :*
  (a) *N is φ-flat.*
  (b) $\mathrm{Tor}_i^{\mathbb{Z}\langle M\rangle}(\mathbb{Z}\langle T\rangle, N) = 0$ *for every $i > 0$ and every pointed integral M-module T.*
  (c) $\mathrm{Tor}_1^{\mathbb{Z}\langle M\rangle}(\mathbb{Z}\langle M/I\rangle, N) = 0$ *for every ideal $I \subset M$.*
  (d) *The natural map $I \otimes_M N \to N$ is injective for every ideal $I \subset M$.*

(ii) *If moreover, $M^\sharp$ is finitely generated, then the conditions* (a)-(d) *of* (i) *are equivalent to either of the following two conditions :*
  (e) $\mathrm{Tor}_1^{\mathbb{Z}\langle M\rangle}(\mathbb{Z}\langle M/\mathfrak{p}\rangle, N) = 0$ *for every prime ideal $\mathfrak{p} \subset M$.*
  (f) *The natural map $\mathfrak{p} \otimes_M N \to N$ is injective for every prime ideal $\mathfrak{p} \subset M$.*

Proof. Clearly (a)⇒(b)⇒(c)⇒(e). Next, by considering the short exact sequence of pointed integral M-modules $0 \to I \to M \to M/I \to 0$ we easily see that (c)⇔(d) and (e)⇔(f).

(c)⇒(a) : Let $\Sigma := (0 \to S' \to S \to S'' \to 0)$ be a short exact sequence of pointed integral M-modules; we need to show that the induced map $S' \otimes_M N \to S \otimes_M N$ is injective. Since the sequence $\mathbb{Z}\langle\Sigma\rangle$ is still exact, the long Tor-exact sequence reduces to showing that $\mathrm{Tor}_1^{\mathbb{Z}\langle M\rangle}(\mathbb{Z}\langle T\rangle, N) = 0$ for every pointed integral M-module T. In view of remark 3.1.27(i), we are easily reduced to the case where T is finitely generated; next, using remark 3.1.27(ii,iii), the long exact Tor-sequence, and an easy induction on the number of generators of T, we may assume that $T = M/I$, whence the contention.

Lastly, if $M^\sharp$ is finitely generated, then remark 3.1.27(iv) shows that, in the foregoing argument, we may further reduce to the case where $T = M/\mathfrak{p}$ for a prime ideal $\mathfrak{p} \subset M$; this shows that (e)⇒(a). $\square$

Lemma 3.1.41. *Let M be a pointed monoid, S a pointed M-module, and suppose that the following conditions hold for S :*

(F1) *If $s \in S$ and $a \in \mathrm{Ann}_M(s)$, then there exist $b \in \mathrm{Ann}_M(a)$ such that $s \in bS$.*

(F2) *If $a_1, a_2 \in M$ and $s_1, s_2 \in S$ satisfy the identity $a_1 s_1 = a_2 s_2 \neq 0$, then there exist $b_1, b_2 \in M$ and $t \in S$ such that $s_i = b_i t$ for $i = 1, 2$ and $a_1 b_1 = a_2 b_2$.*

Then the natural map $I \otimes_M S \to S$ is injective for every ideal $I \subset M$.

Proof. Let I be an ideal, and suppose that two elements $a_1\otimes s_1$ and $a_2\otimes s_2$ of $I\otimes_M S$ are mapped to the same element of S. If $a_is_i = 0$ for $i = 1, 2$, then (F1) says that there exist $b_1, b_2 \in M$ and $t_1, t_2 \in S$ such that $a_ib_i = 0$ and $s_i = b_it_i$ for $i = 1, 2$; thus $a_i \otimes s_i = a_i \otimes b_it_i = a_ib_i \otimes s_i = 0$ in $I \otimes_M S$. In case $a_is_i \neq 0$, pick $b_1, b_2 \in M$ and $t \in S$ as in (F2); we conclude that $a_1 \otimes s_1 = a_1 \otimes b_1t = a_1b_1 \otimes t = a_2b_2 \otimes t = a_2 \otimes s_2$ in $I \otimes_M S$, whence the contention. □

Theorem 3.1.42. *Let M be an integral pointed monoid, S a pointed M-module. The following conditions are equivalent :*

(a) *S is M-flat.*
(b) *For every morphism $M \to P$ of pointed monoids, $P \otimes_M S$ is P-flat.*
(c) *For every short exact sequence Σ of integral pointed M-modules, the sequence $\Sigma\otimes_M S$ is again short exact.*
(d) *Conditions* (F1) *and* (F2) *of lemma* 3.1.41 *hold for S.*

Proof. Clearly (b)⇒(a)⇒(c).

(c)⇒(d): To show (F1), set $I := Ma$, and denote by $i : I \to M$ the inclusion; (c) implies that the induced map $i \otimes_M S : I \otimes_M S \to S$ is injective. However, we have a natural isomorphism $I \stackrel{\sim}{\to} M/\mathrm{Ann}_M(a)$ of M-modules (remark 3.1.27(iii)), whence an isomorphism : $I \otimes_M S \stackrel{\sim}{\to} S/\mathrm{Ann}_M(a)S$, and under this identification, $i \otimes_M S$ is induced by the map $S \to S : s \mapsto as$. Thus, multiplication by a maps the subset $S \setminus \mathrm{Ann}_M(a)S$ injectively into itself, which is the claim.

For (F2), notice that $\mathbb{Z}\langle S\rangle$ is φ-flat under condition (c), for $\varphi : M \to \mathbb{Z}\langle M\rangle$ the natural morphism. Now, say that $a_1s_1 = a_2s_2 \neq 0$ in S; set $I := Ma_1$, $J := Ma_2$; the assumption means that $a_1s_1 \in IS \cap JS$, in which case remark 3.1.39 shows that there exist $t \in S$ and $b_1, b_2 \in M$ such that $a_1b_1 = a_2b_2$, and $a_1s_1 = a_1b_1t$, hence $a_2s_2 = a_2b_2t$. Since we have seen that multiplication by a_1 maps $S \setminus \mathrm{Ann}_M(a_1)S$ injectively into itself, we deduce that $s_1 = b_1t$, and likewise we get $s_2 = b_2t$.

To prove that (d)⇒(b), we observe :

Claim 3.1.43. Let P be a pointed monoid, Λ a small locally directed category (see definition 1.1.37(iv)), and $S_\bullet : \Lambda \to P\text{-}\mathbf{Mod}_\circ$ a functor, such that S_λ fulfills conditions (F1) and (F2), for every $\lambda \in \mathrm{Ob}(\Lambda)$. Then the colimit of $S_\bullet$ also fulfills conditions (F1) and (F2).

Proof of the claim. In light of remark 1.1.38(ii), we may assume that Λ is either discrete or path-connected. Suppose first that Λ is path-connected; then remark 2.3.17(ii) allows to check directly that conditions (F1) and (F2) hold for the colimit of $S_\bullet$, since they hold for every S_λ. If Λ is discrete, the assertion is that conditions (F1) and (F2) are preserved by arbitrary (small) direct sums, which we leave as an exercise for the reader. ◊

To a given pointed M-module S, we attach the small category S^*, such that :

$$\mathrm{Ob}(S^*) = S\setminus\{0\} \qquad \text{and} \qquad \mathrm{Hom}_{S^*}(s', s) = \{a \in M \mid as = s'\}.$$

The composition of morphisms is induced by the composition law of M, in the obvious way. Notice that S^* is locally directed if and only if S satisfies condition (F2).

We define a functor $F : S^* \to M\text{-}\mathbf{Mod}_\circ$ as follows. For every $s \in \mathrm{Ob}(S^*)$ we let $F(s) := M$, and for every morphism $a : s' \to s$ we let $F(a) := a \cdot \mathbf{1}_M$. We have a natural transformation $\tau : F \Rightarrow c_S$, where $c_S : S^* \to M\text{-}\mathbf{Mod}_\circ$ is the constant functor associated to S; namely, for every $s \in \mathrm{Ob}(S^*)$, we let $\tau_s : M \to S$ be the map given by the rule $a \mapsto as$ for all $a \in M$. There follows a morphism of pointed M-modules :

$$\operatorname*{colim}_{S^*} F \to S \tag{3.1.44}$$

Claim 3.1.45. If S fulfills conditions (F1), the map (3.1.44) is an isomorphism.

Proof of the claim. Indeed, we have a natural decomposition of S^* as coproduct of a family $(S^*_i \mid i \in I)$ of path-connected subcategories (for some small set I : see remark 1.1.38(ii)); especially we have $\mathrm{Hom}_{S^*}(s, s') = \varnothing$ if $s \in \mathrm{Ob}(S^*_i)$ and $s' \in \mathrm{Ob}(S^*_j)$ for some $i \neq j$ in I.

For each $i \in I$, let $F_i : S^*_i \to M\text{-}\mathbf{Mod}_\circ$ be the restriction of F. There follows a natural isomorphism :

$$\bigoplus_{i \in I} \operatorname*{colim}_{S^*_i} F_i \xrightarrow{\sim} \operatorname*{colim}_{S^*} F.$$

Since the colimit of F_i commutes with the forgetful functor to sets, an inspection of the definitions yields the following explicit description of the colimit T_i of F_i. Every element of T_i is represented by some pair (s, a) where $s \in \mathrm{Ob}(S^*_i) \subset S \setminus \{0\}$ and $a \in M$; such pair is mapped to as by (3.1.44), and two such pairs (s, a), (s', a') are identified in T_i if there exists $b \in M$ such that $bs = s'$ and $ba' = a$.

Hence, denote by S_i the image under (3.1.44) of T_i; we deduce first, that $S_i \cap S_j = \{0\}$ if $i \neq j$. Indeed, say that $t \in S_i \cap S_j$; by the foregoing, there exist $s_i \in \mathrm{Ob}(S^*_i)$, $s_j \in \mathrm{Ob}(S^*_j)$, and $a_i, a_j \in M$, such that $a_i s_i = t = a_j s_j$. If $t \neq 0$, we get morphisms $a_i : t \to s_i$ and $a_j : t \to s_j$ in S^*; say that $t \in S^*_k$ for some $k \in I$; it then follows that $S^*_i = S^*_k = S^*_j$, a contradiction. Next, it is clear that (3.1.44) is surjective. It remains therefore only to show that each T_i maps injectively onto S_i. Hence, say that (s_1, a_1) and (s_2, a_2) represent two elements of T_i with $t := a_1 s_1 = a_2 s_2$. If $t \neq 0$, we get, as before, morphisms $a_1 : t \to s_1$ and $a_2 : t \to s_2$ in S^*_i, and the two pairs are identified in T_i to the pair $(t, 1)$. Lastly, if $t = 0$, condition (F1) yields $b \in M$ and $s' \in S$ such that $a_1 b = 0$ and $s_1 = bs'$, whence a morphism $b : s_1 \to s'$ in S^*_i, and $F_i(b)(a_1, s_1) = (0, s')$ which represents the zero element of T_i. The same argument applies as well to (a_2, s_2), and the claim follows. ◊

Claim 3.1.46. Let $M \to P$ be any morphism of pointed monoids, and S a pointed M-module fulfilling conditions (F1) and (F2). Then the natural map $I \otimes_M S \to P \otimes_M S$ is injective, for every ideal $I \subset P$.

Proof of the claim. From claim 3.1.45 we deduce that $P \otimes_M S$ is the locally directed colimit of the functor $P \otimes_M F$, and notice that the pointed P-module P fulfills conditions (F1) and (F2); by claim 3.1.43 we deduce that $P \otimes_M S$ also fulfills the same conditions, so the claim follows from lemma 3.1.41. ◊

After these preliminaries, suppose that conditions (F1) and (F2) hold for S, and let $\Sigma := (0 \to T' \to T \to T'' \to 0)$ be a short exact sequence of pointed M-modules. We wish to show that $\Sigma \otimes_M S$ is still short exact. However, if $U'' \subset T''$ is any M-submodule, let $U \subset T$ be the preimage of U'', and notice that the induced sequence $0 \to T' \to U \to U'' \to 0$ is still short exact. Since a filtered colimit of short exact sequences is short exact, remark 3.1.27(i) allows to reduce to the case where T'' is finitely generated.

We shall argue by induction on the number n of generators of T''. Hence, suppose first that T'' is cyclic, and let $t \in T$ be any element whose image in T'' is a generator. Set $C := Mt \subset T$, and let $C' \subset T'$ be the preimage of C. We obtain a cocartesian (and cartesian) diagram of pointed M-modules :

$$\mathscr{D} \quad : \quad \begin{array}{ccc} C' & \longrightarrow & C \\ \downarrow & & \downarrow \\ T' & \longrightarrow & T. \end{array}$$

The induced diagram $\mathscr{D} \otimes_M S$ is still cocartesian, hence the same holds for the diagram of sets underlying $\mathscr{D} \otimes_M S$ (remark 2.3.17(ii)). Especially, if the induced map $C' \otimes_M S \to C \otimes_M S$ is injective, the same will hold for the map $T' \otimes_M S \to T \otimes_M S$. We may thus replace T' and T by respectively C' and C, which allows to assume that also T is cyclic. In this case, pick a generator $u \in T$; we claim that there exists a unique multiplication law μ_T on T, such that the surjection $p : M \to T : a \mapsto au$ is a morphism of pointed monoids. Indeed, for every $t, t' \in T$, write $t = au$ for some $a \in M$, and set $\mu_T(t, t') := at'$. Using the linearity of p we easily check that $\mu_T(t, t')$ does not depend on the choice of a, and the resulting composition law μ_T is commutative and associative. Then T' is an ideal of T, so claim 3.1.46 tells us that the map $T' \otimes_M S \to T \otimes_M S$ is injective, as required.

Lastly, suppose that $n > 1$, and the assertion is already known whenever T'' is generated by at most $n-1$ elements. Let $U'' \subset T''$ be a pointed M-submodule, such that U'' is generated by at most $n-1$ elements, and T''/U'' is cyclic. Denote by $U \subset T$ the preimage of U''; we deduce short exact sequences $\Sigma' := (0 \to T' \to U \to U'' \to 0)$ and $\Sigma'' := (0 \to U \to T \to T''/U'' \to 0)$, and by inductive assumption, both $\Sigma' \otimes_M S$ and $\Sigma'' \otimes_M S$ are short exact. Therefore the natural map $T' \otimes_M S \to T \otimes_M S$ is the composition of two injective maps, hence it is injective, as stated. $\square$

Remark 3.1.47. In the situation of remark 3.1.35(ii), suppose that M is pointed integral. Then theorem 3.1.42 implies that S is a flat pointed M-module if and only if $R\langle S\rangle$ is φ-flat.

Corollary 3.1.48. *Let M be an integral pointed monoid, S a pointed M-module. Then*

(i) *The following conditions are equivalent :*
  (a) *S is M-flat.*
  (b) *For every ideal $I \subset M$, the induced map $I \otimes_M S \to S$ is injective.*
(ii) *If moreover $M^\sharp$ is finitely generated, then these conditions are equivalent to :*
  (c) *For every prime ideal $\mathfrak{p} \subset M$, the induced map $\mathfrak{p} \otimes_M S \to S$ is injective.*

Proof. (i): Indeed, by remark 3.1.47 and proposition 3.1.40(i) (together with (3.1.32)), both (a) and (b) hold if and only if $\mathbb{Z}\langle S\rangle$ is φ-flat, for $\varphi : M \to \mathbb{Z}\langle M\rangle$ the natural morphism.

(ii): This follows likewise from proposition 3.1.40(ii). $\square$

Corollary 3.1.49. *Let $\varphi : M \to N$ be a morphism of pointed monoids, $G \subset M^\times$, $H \subset N^\times$ two subgroups such that $\varphi(G) \subset H$, and denote by $\overline{\varphi} : M/G \to N/H$ the induced morphism. Let also S be any N-module. We have :*

(i) *If $S_{(\varphi)}$ is a flat M-module, then $S/H_{(\overline{\varphi})}$ is a flat M/G-module (notation of* (1.2.26)*).*
(ii) *If moreover, M is a pointed integral monoid and S is a pointed integral H-module, then also the converse of* (i) *holds.*

Proof. (i): We have a natural isomorphism
$$(S/H)_{(\overline{\varphi})} \xrightarrow{\sim} N/H \otimes_{N/\varphi G} S_{(\varphi)}/\varphi G.$$
However, by example 3.1.23(ii), N/H is a flat $N/\varphi G$-module, and $S_{(\varphi)}/\varphi G$ is a flat M/G-module, whence the contention.

(ii): By theorem 3.1.42, it suffices to check that conditions (F1) and (F2) of lemma 3.1.41 hold in $S_{(\varphi)}$, and notice that, by the same token, both conditions hold for the M/G-module $S/H_{(\overline{\varphi})} = S_{(\varphi)}/G$, since M/G is pointed integral (example 3.1.23(v)).

Hence, say that $\varphi(a)s = 0$ for some $a \in M$ and $s \in S$; we may then find $b \in M$, $t' \in S$ and $g \in G$ such that $\varphi(gb)t = s$ and $ab = 0$. Setting $t := \varphi(g)t'$, we deduce that (F1) holds.

Next, say that $\varphi(a_1)s_1 = \varphi(a_2)s_2 \neq 0$ for some $a_1, a_2 \in M$ and $s_1, s_2 \in S$; then we may find $g \in G$, $h_1, h_2 \in H$, $b_1, b_2 \in M$ and $t' \in S$ such that $ga_1b_1 = a_2b_2$ and

$$\varphi(b_i)h_it' = s_i \qquad \text{for } i = 1, 2. \tag{3.1.50}$$

After replacing b_1 by gb_1 and h_1 by $h_1\varphi(g^{-1})$, we reduce to the case where $a_1b_1 = a_2b_2$. From (3.1.50) we deduce that
$$\varphi(a_1b_1)h_1t' = \varphi(a_1)s_1 = \varphi(a_2)s_2 = \varphi(a_2b_2)h_2t' = \varphi(a_1b_1)h_2t'$$

whence $h_1 = h_2$, since S is a pointed integral H-module. Setting $t := h_1 t'$, we see that (F2) holds. $\square$

Corollary 3.1.51. *Let $\varphi : M \to N$ be a flat morphism of pointed monoids, with M pointed integral, and let $\mathfrak{p} \subset N$ be any prime ideal. Then the morphism $M/\varphi^{-1}\mathfrak{p} \to N/\mathfrak{p}$ induced by φ is also flat.*

Proof. Let $F := N \setminus \mathfrak{p}$; by theorem 3.1.42, it suffices to check that conditions (F1) and (F2) of lemma 3.1.41 hold for the $\varphi^{-1}F$-module F. However, since $0 \notin F$, condition (F1) holds trivially. Moreover, by assumption these two conditions hold for the M-module N; hence, say that $\varphi(a_1) \cdot s_1 = \varphi(a_2) \cdot s_2$ in F, for some $a_1, a_2 \in \varphi^{-1}F$ and $s_1, s_2 \in F$. It follows that there exist $b_1, b_2 \in M$ and $t \in N$ such that $a_1 b_1 = a_2 b_2$ in M, and $\varphi(b_i) \cdot t = s_i$ for $i = 1, 2$. Since F is a face, this implies that $\varphi(b_1), \varphi(b_2), t \in F$, so (F2) holds for F, as stated. $\square$

Another corollary is the following analogue of a well known criterion due to Lazard.

Proposition 3.1.52. *Let M be an integral pointed monoid, S an integral pointed M-module, R a non-zero commutative ring with unit. The following conditions are equivalent :*

(a) *S is M-flat.*
(b) *S is the colimit of a filtered system of free pointed M-modules (see remark 2.3.17(ii)).*
(c) *$R\langle S\rangle$ is a flat $R\langle M\rangle$-module.*

Proof. Obviously (b)⇒(a) and (b)⇒(c).

(c)⇒(a) has already been observed in remark 3.1.35(ii).

(a)⇒(b): It suffices to prove that if S is flat, the category S^* attached to S as in the proof of theorem 3.1.42, is pseudo-filtered. However, a simple inspection of the construction shows that S^* is pseudo-filtered if and only if S satisfies both condition (F2) of lemma 3.1.41, and the following further condition. For every $a, b \in M$ and $s \in S$ such that $as = bs \neq 0$, there exists $t \in S$ and $c \in M$ such that $ac = bc$ and $ct = s$. This condition is satisfied by every integral pointed M-module, whence the contention. $\square$

3.1.53. Consider now, a cartesian diagram of integral pointed monoids :

$$\mathscr{D}(P_0, I, P_1) \quad : \quad \begin{array}{ccc} P_0 & \longrightarrow & P_1 \\ \downarrow & & \downarrow \\ P_2 & \longrightarrow & P_3 \end{array}$$

where P_2 (resp. P_3) is a quotient P_0/I (resp. P_1/IP_1) for some ideal $I \subset P_0$, and the vertical arrows of $\mathscr{D}(P_0, I, P_1)$ are the natural surjections. In this situation, it is easily seen that the induced map $I \to IP_1$ is bijective; especially, I is both a P_0-module and a P_1-module. Let $\varphi_i : P_i \to \mathbb{Z}\langle P_i\rangle$ be the units of adjunction, for $i = 0, 1, 2$. Let also M be any $\mathbb{Z}\langle P_0\rangle$-module.

Lemma 3.1.54. *In the situation of* (3.1.53)*, we have :*

(i) *Let $J \subset P_0$ be any ideal. Then $S := P_0/J$ admits a three-step filtration*

$$0 \subset \mathrm{Fil}_0 S \subset \mathrm{Fil}_1 S \subset \mathrm{Fil}_2 S = S$$

such that $\mathrm{Fil}_0 S$ and $\mathrm{gr}_2 S$ are P_2-modules, and $\mathrm{gr}_1 S$ is a P_1-module.

(ii) *The following conditions are equivalent :*
   (a) *M is φ_0-flat.*
   (b) *$M \otimes_{P_0} P_i$ is φ_i-flat and $\mathrm{Tor}_1^{\mathbb{Z}\langle P_0\rangle}(M, \mathbb{Z}\langle P_i\rangle) = 0$, for $i = 1, 2$.*

(iii) *The following conditions are equivalent :*
   (a) *$\mathrm{Tor}_1^{\mathbb{Z}\langle P_0\rangle}(M, \mathbb{Z}\langle P_i\rangle) = 0$ for $i = 1, 2, 3$.*
   (b) *$\mathrm{Tor}_1^{\mathbb{Z}\langle P_i\rangle}(M \otimes_{P_0} P_i, \mathbb{Z}\langle P_3\rangle) = 0$ for $i = 1, 2$.*

(iv) *Suppose moreover, that* $P_3 \neq 0$ *is a free pointed* P_2*-module. Then the* $\mathbb{Z}\langle P_0\rangle$*-module* M *is* φ_0*-flat if and only if the* $\mathbb{Z}\langle P_1\rangle$*-module* $M \otimes_{P_0} P_1$ *is* φ_1*-flat.*

Proof. (i): Define $\mathrm{Fil}_1 S := \mathrm{Ker}(S \to P_0/(I \cup J)) = (I \cup J)/J$. Then it is already clear that $S/\mathrm{Fil}_1 S$ is a P_2-module. Next, let $\mathrm{Fil}_0 S := \mathrm{Ker}(\mathrm{Fil}_1 S \to (I \cup JP_1)/JP_1) = JP_1/J$. Since $IP_1 = I$, we see that $\mathrm{Fil}_0 S$ is a P_2-module, and $(I \cup JP_1)/JP_1$ is a P_1-module.

(ii): Clearly (a)⇒(b), hence suppose that (b) holds, and let us prove (a). By proposition 3.1.40(i), it suffices to show that $\mathrm{Tor}_1^{\mathbb{Z}\langle P_0\rangle}(M, \mathbb{Z}\langle P_0/I\rangle) = 0$ for every ideal $J \subset P_0$. In view of (i), we are then reduced to showing that $\mathrm{Tor}_1^{\mathbb{Z}\langle P_0\rangle}(M, \mathbb{Z}\langle S\rangle) = 0$, whenever S is a P_i-module, for $i = 1, 2$. However, for such P_i-module S, we have a base change spectral sequence

$$E^2_{pq} : \mathrm{Tor}_p^{\mathbb{Z}\langle P_i\rangle}(\mathrm{Tor}_q^{\mathbb{Z}\langle P_0\rangle}(M, \mathbb{Z}\langle P_i\rangle), \mathbb{Z}\langle S\rangle) \Rightarrow \mathrm{Tor}_{p+q}^{\mathbb{Z}\langle P_0\rangle}(M, \mathbb{Z}\langle S\rangle).$$

Under assumption (b), we deduce : $\mathrm{Tor}_1^{\mathbb{Z}\langle P_0\rangle}(M, \mathbb{Z}\langle S\rangle) = \mathrm{Tor}_1^{\mathbb{Z}\langle P_i\rangle}(M \otimes_{P_0} P_i, \mathbb{Z}\langle S\rangle) = 0$.

(iii): Notice that the induced diagram of rings $\mathbb{Z}\langle \mathscr{D}(P_0, I, P_1)\rangle$ is still cartesian. Then, this is a special case of [36, Lemma 3.4.15].

(iv): Suppose that $M \otimes_{P_0} P_1$ is φ_1-flat, and P_3 is a free pointed P_2-module, say $P_3 \simeq P_2^{(\Lambda)\circ}$, for some set $\Lambda \neq \varnothing$; then the $\mathbb{Z}\langle P_2\rangle$-module $\mathbb{Z}\langle P_3\rangle$ is isomorphic to $\mathbb{Z}\langle P_2\rangle^{(\Lambda)}$, especially it is faithfully flat, and we deduce that $\mathrm{Tor}_1^{\mathbb{Z}\langle P_0\rangle}(M, \mathbb{Z}\langle P_i\rangle) = 0$ for $i = 1, 2$, by (iii). On the other hand, $M \otimes_{P_0} P_3$ is φ_3-flat, so it also follows that $M \otimes_{P_0} P_2$ is φ_2-flat, by proposition 3.1.52. Summing up, this shows that M fulfills condition (ii.b), hence also (ii.a), as sought. □

3.1.55. Let now $P \to Q$ be an injective morphism of integral pointed monoids, and suppose that $P^\sharp$ and $Q^\sharp$ are finitely generated monoids, and Q is a finitely generated P-module. Denote $\varphi_P : P \to \mathbb{Z}\langle P\rangle$ and $\varphi_Q : Q \to \mathbb{Z}\langle Q\rangle$ the usual units of adjunction.

Theorem 3.1.56. *In the situation of* (3.1.55), *let* M *be a* $\mathbb{Z}\langle P\rangle$*-module, and suppose that the* $\mathbb{Z}\langle Q\rangle$*-module* $M \otimes_P Q$ *is* φ_Q*-flat. Then* M *is* φ_P*-flat.*

Proof. Using lemma 3.1.54(iv), and an easy induction, it suffices to show that there exists a finite chain

$$P = Q_0 \subset Q_1 \subset \cdots \subset Q_n = Q \qquad (3.1.57)$$

of inclusions of integral pointed monoids, and for every $j = 0, \dots, n-1$ an ideal $I_j \subset Q_j$, and a cartesian diagram of integral pointed monoids $\mathscr{D}(Q_j, I_j, Q_{j+1})$ as in (3.1.53), and such that $Q_{j+1}/I_j \neq 0$ is a free pointed Q_j/I_j-module. (Notice that each $Q_i^\sharp$ is a quotient of $Q_i/P^\times$, and the latter is a submodule of $Q/P^\times$, hence $Q_i^\sharp$ is still a finitely generated monoid, by proposition 3.1.9(i).) If $P = Q$, there is nothing to prove; so we may assume that P is strictly contained in Q, and – invoking again proposition 3.1.9(i) – we further reduce to showing that there exists a monoid $Q_1 \subset Q$ strictly containing P, and an ideal $I \subset P$, such that the diagram $\mathscr{D}(P, I, Q_1)$ fulfills the above conditions.

Suppose first that the support of Q/P contains only the maximal ideal $\mathfrak{m}_P$ of P, and let $x_1, \dots, x_r$ be a finite system of generators for $\mathfrak{m}_P$ (proposition 3.1.9(ii)). For each $i = 1, \dots, r$, the localization $(Q/P)_{x_i}$ is a P_{x_i}-module with empty support, hence $(Q/P)_{x_i} = 0$ (remark 3.1.22(v)). It follows that every element of Q/P is annihilated by some power of x_i, and since Q/P is finitely generated, we may find $N \in \mathbb{N}$ large enough, such that $x_i^N Q/P = 0$ for $i = 1, \dots, r$. After replacing N by some possibly larger integer, we get $\mathfrak{m}_P^N \cdot Q/P = 0$, and we may assume that N is the least integer with this property. If $N = 0$, there is nothing to prove; hence suppose that $N > 0$, and set $Q_1 := P \cup \mathfrak{m}_P^{N-1} Q$. Notice that Q_1 is a monoid, and $Q_1/P \neq 0$ is annihilated by $\mathfrak{m}_P$. Especially, $\mathfrak{m}_P Q_1 = \mathfrak{m}_P$. Moreover, the induced map $P/\mathfrak{m}_P \to Q_1/\mathfrak{m}_P$ is injective and Q_1 is an integral pointed module, therefore the group $P^\times$ acts freely on $Q_1\setminus\mathfrak{m}_P$, *i.e.* $Q_1/\mathfrak{m}_P$ is a free pointed $P/\mathfrak{m}_P$-module. It follows easily that, if we take $I := \mathfrak{m}_P$, we do obtain a diagram $\mathscr{D}(P, \mathfrak{m}_P, Q_1)$ with the sought properties, in this case.

For the general case, let $\mathfrak{p} \subset P$ be a minimal element of $\mathrm{Supp}\, Q/P$ (for the ordering given by inclusion). Then the induced morphism $P_\mathfrak{p} \to Q_\mathfrak{p}$ still satisfies the conditions of (3.1.55) (lemma 3.1.20(iv)). Moreover, $\mathrm{Supp}\, Q_\mathfrak{p}/P_\mathfrak{p} = \{\mathfrak{p}P_\mathfrak{p}\}$ by remark 3.1.22(vii). By the previous case, we deduce that there exists a chain of inclusions of integral pointed monoids $P_\mathfrak{p} \subset Q'_1 \subset Q_\mathfrak{p}$, such that the resulting diagram $\mathscr{D}(P_\mathfrak{p}, \mathfrak{p}P_\mathfrak{p}, Q'_1)$ is cartesian, and $Q'_1/\mathfrak{p}P_\mathfrak{p} \neq 0$ is a free pointed $P_\mathfrak{p}/\mathfrak{p}P_\mathfrak{p}$-module. Let $\overline{e}_1, \dots, \overline{e}_d$ be a basis of the latter $P_\mathfrak{p}/\mathfrak{p}P_\mathfrak{p}$-module, with $\overline{e}_1 = 1$. Hence, $\overline{e}_i \in Q_\mathfrak{p} \setminus \mathfrak{p}P_\mathfrak{p}$ for every $i = 1, \dots, d$, and after multiplying $\overline{e}_2, \dots, \overline{e}_d$ by a suitable element of $P \setminus \mathfrak{p}$, we may assume that each $\overline{e}_i$ is the image in $Q_\mathfrak{p}$ of an element $e_i \in Q$. Moreover, for every $i, j \leq d$, either $\overline{e}_i\overline{e}_j = 0$, or else there exists $a_{ij} \in P_\mathfrak{p}$ and $k(i, j) \leq d$ such that $\overline{e}_i\overline{e}_j = a_{ij}\overline{e}_{k(i,j)}$. Furthermore, fix a system of generators $x_1, \dots, x_r$ for $\mathfrak{p}$; then, for every $i \leq r$ and every $j \leq d$ we have $x_i\overline{e}_j \in \mathfrak{p}P_\mathfrak{p}$. Again, after multiplying $\overline{e}_2, \dots, \overline{e}_d$ by some $c \in P\setminus\mathfrak{p}$, we may assume that $a_{ij} \in P$ for every $i, j \leq d$, and moreover that $x_i\overline{e}_j$ lies in the image of $\mathfrak{p}$ for every $i \leq r$ and $j \leq d$.

And if we multiply yet again $e_2, \dots, e_d$ by a suitable element of $P\setminus\mathfrak{p}$, we may finally reach a system of elements $e_1, \dots, e_d \in Q$ such that $e_1 = 1$ and :

- For every $i, j \leq d$, we have either $e_ie_j = 0$ or else $e_ie_j = a_{ij}e_{k(i,j)}$.
- $x_ie_j \in \mathfrak{p}$ for every $i \leq r$ and $j \leq d$.

Clearly these elements span a P-module Q_1 which is a monoid containing P and contained in Q; moreover, by construction we have $\mathfrak{p}Q_1 = \mathfrak{p}$, hence the resulting diagram $\mathscr{D}(P, \mathfrak{p}, Q_1)$ is cartesian. Notice also that $(Q_1/\mathfrak{p})_\mathfrak{p} \simeq Q'_1/\mathfrak{p}P_\mathfrak{p}$, and that $P/\mathfrak{p} = P'_\circ$, where $P' := P\setminus\mathfrak{p}$ is an integral (non-pointed) monoid. To conclude it suffices now to apply the following

Claim 3.1.58. Let P' be an integral non-pointed monoid, S a pointed $P'_\circ$-module, and $\underline{e} := (e_1, \dots, e_d)$ a system of generators for S. Suppose that $S \otimes_{P'_\circ} P'^{\mathrm{gp}}_\circ$ is a free pointed $P'^{\mathrm{gp}}_\circ$-module, and the image of $\underline{e}$ is a basis for this module. Then S is a free pointed $P'_\circ$-module, with basis $\underline{e}$.

Proof of the claim. If $\underline{e}$ is not a basis, we have a relation in S of the type $a_1e_1 = a_2e_2$, for some $a_1, a_2 \in P'$. This relation must persist in $S \otimes_{P'_\circ} P'^{\mathrm{gp}}_\circ$, and implies that $a_1 = a_2 = 0$ in $P'^{\mathrm{gp}}_\circ$. However, under the stated assumptions the localization map $P'_\circ \to P'^{\mathrm{gp}}_\circ$ is injective, a contradiction. □

3.2. **Integral monoids.**

We begin presently the study of a special class of monoids, the integral non-pointed monoids, and the subclass of saturated monoids (see definition 2.3.40(iii)). Later, we shall complement this section with further results on fine monoids (see sections 3.4 and 6.5). *Throughout this section, all the monoids under consideration shall be non-pointed.*

Definition 3.2.1. Let $\varphi : M \to N$ be a morphism of monoids.

(i) φ is said to be *integral* if, for any integral monoid M', and any morphism $M \to M'$, the push-out $N \otimes_M M'$ is integral.

(ii) φ is said to be *strongly flat* (resp. *strongly faithfully flat*) if the induced morphism $\mathbb{Z}[\varphi] : \mathbb{Z}[M] \to \mathbb{Z}[N]$ is flat (resp. faithfully flat).

Lemma 3.2.2. *Let $f : M \to N$ and $g : N \to P$ be two morphisms of monoids.*

(i) *If f and g are integral (resp. strongly flat), the same holds for $g \circ f$.*

(ii) *If f is integral (resp. strongly flat), and $M \to M'$ is any other morphism, then the morphism $\mathbf{1}_{M'} \otimes_M f : M' \to M' \otimes_M N$ is integral (resp. strongly flat).*

(iii) *If f is integral, and $S \subset M$ and $T \subset N$ are any two submonoids such that $f(S) \subset T$, then the induced morphism $S^{-1}M \to T^{-1}N$ is integral.*

(iv) *If $S \subset M$ is any submonoid, the natural map $M \to S^{-1}M$ is strongly flat.*
(v) *If f is integral, then the same holds for f^{int}, and the natural map $M^{\mathrm{int}} \otimes_M N \to N^{\mathrm{int}}$ is an isomorphism.*
(vi) *The unit of adjunction $M \to M^{\mathrm{int}}$ is an integral morphism.*

Proof. (i) is obvious. Assertion (ii) for integral maps is likewise clear, and (ii) for strongly flat morphisms follows from (2.3.52). Assertion (iv) follows immediately from (2.3.53).

(iii): Let $S^{-1}M \to M'$ be any map of integral monoids; in view of lemma 2.3.34, we have $P := M' \otimes_{S^{-1}M} T^{-1}N \simeq T^{-1}(M' \otimes_M N)$, hence P is integral, which is the claim.

(v): The second assertion follows easily by comparing the universal properties (details left to the reader); using this and and (ii), we deduce that f^{int} is integral.

(vi) is left to the reader. □

Theorem 3.2.3. *Let $\varphi : M \to N$ be a morphism of integral monoids. Consider the following conditions :*

(a) *φ is integral.*
(b) *φ is flat (see remark* 2.3.23(vi)*).*
(c) *φ is flat and injective.*
(d) *φ is strongly flat.*
(e) *For every field k, the induced map $k[\varphi] : k[M] \to k[N]$ is flat.*

Then : (e) ⇔ (d) ⇔ (c) ⇒ (b) ⇔ (a).

Proof. This result appears in [52, Prop.4.1(1)], with a different proof.

(a)⇒(b): Let $I \subset M$ be any ideal; we consider the M-module :

$$\mathsf{R}(M, I) := \bigoplus_{n \in \mathbb{N}} I^n$$

where I^n denotes, for each $n \in \mathbb{N}$, the n-th power of I in the monoid $(\mathscr{P}(M), \cdot)$ of (3.1.1). Then there exists an obvious multiplication law on $\mathsf{R}(M, I)$, such that the latter is a $\mathbb{N}$-graded integral monoid, and the inclusion $M \to \mathsf{R}(M, I)$ in degree zero is a morphism of monoids. We call $\mathsf{R}(M, I)$ the *Rees monoid* associated to M and I.

Denote also by $j : \mathsf{R}(M, I) \to M \times \mathbb{N}$ the natural inclusion map. By assumption, the monoid $\mathsf{R}(M, I) \otimes_M N$ is integral. However, the natural map $\mathsf{R}(M, I) \otimes_M N \to (\mathsf{R}(M, I) \otimes_M N)^{\mathrm{gp}}$ factors through $j \otimes_M N$. The latter means that, for every $n \in \mathbb{N}$, the induced map $I^n \otimes_M N \to N$ is injective. Then the assertion follows from proposition 3.1.40 and remark 2.3.21(ii).

(b)⇒(a): Let $M \to M'$ be any morphism of integral monoids; we need to show that the natural map $M' \otimes_M N \to M'^{\mathrm{gp}} \otimes_{M^{\mathrm{gp}}} N^{\mathrm{gp}}$ is injective (see remark 3.1.25(i)). The latter factors through the morphism $M' \otimes_M N \to M'^{\mathrm{gp}} \otimes_M N$, which is injective by theorem 3.1.42 and remark 2.3.21(ii). We are thus reduced to proving the injectivity of the natural map :

$$M'^{\mathrm{gp}} \otimes_{M^{\mathrm{gp}}} (M^{\mathrm{gp}} \otimes_M N) \to M'^{\mathrm{gp}} \otimes_{M^{\mathrm{gp}}} N^{\mathrm{gp}}.$$

By comparing the respective universal properties, it is easily seen that $M^{\mathrm{gp}} \otimes_M N$ is the localization $\varphi(M)^{-1}N$, which of course injects into N^{gp}. Then the contention follows from the following general :

Claim 3.2.4. Let G be a group, $T \to T'$ an injective morphism of monoids, $G \to P$ a morphism of monoids. Then the natural map $P \otimes_G T \to P \otimes_G T'$ is injective.

Proof of the claim. This follows easily from remark 3.1.25(i) and lemma 2.3.31(ii). ◊

(d)⇒(e) and (c)⇒(b) are trivial.

(e)⇒(c): The flatness of φ has already been noticed in remark 3.1.35(ii). To show that φ is injective, let $a_1, a_2 \in M$ and let k be any field. Under assumption (e), the image in $k[N]$ of the annihilator $\mathrm{Ann}_{k[M]}(a_1 - a_2)$ generates the ideal $\mathrm{Ann}_{k[N]}(\varphi(a_1) - \varphi(a_2))$. Set $b := a_1 a_2^{-1}$; it follows that $\mathrm{Ann}_{k[M^{\mathrm{gp}}]}(1-b)$ generates $\mathrm{Ann}_{k[N^{\mathrm{gp}}]}(1-\varphi(b))$. However, one checks easily that the annihilator of $1-b$ in $k[M^{\mathrm{gp}}]$ is either 0 if b is not a torsion element of M^{gp}, or else is generated (as an ideal) by $1 + b + \cdots + b^{n-1}$, where n is the order of b in the group M^{gp}. Now, if $\varphi(a_1) = \varphi(a_2)$, we have $\varphi(b) = 1$, hence the annihilator of $1-b$ cannot be 0, and in fact $k[N^{\mathrm{gp}}] = \mathrm{Ann}_{k[N^{\mathrm{gp}}]}(1-\varphi(b)) = nk[N^{\mathrm{gp}}]$. Since k is arbitrary, it follows that $n = 1$, *i.e.* $a_1 = a_2$.

(c)⇒(d): since φ is injective, $N_\circ$ is an integral pointed $M_\circ$-module, so the assertion is a special case of proposition 3.1.52. □

Remark 3.2.5. (i) Let G be a group; then every morphism of monoids $M \to G$ is integral. Indeed, lemma 3.2.2(i,vi) reduces the assertion to the case where M is integral, in which case it is an immediate consequence of theorem 3.2.3 and example 3.1.23(vi).

(ii) Let M be an integral monoid, and S a flat pointed $M_\circ$-module. From theorem 3.1.42 we see that $T := S \backslash \{0\}$ is an M-submodule of S, hence $S = T_\circ$.

(iii) Let $\varphi : M \to N$ be a morphism of integral monoids, and set $\Gamma := \mathrm{Coker}\,\varphi^{\mathrm{gp}}$; then the natural map $\pi : N \to \Gamma$ defines a grading on N (see definition 2.3.8(i)), which we call the *φ-grading*. As usual, we shall write N_γ instead of $\pi^{-1}(\gamma)$, for every $\gamma \in \Gamma$. We shall use the additive notation for the composition law of Γ; especially, the neutral element shall be denoted by 0. Clearly φ factors through a morphism of monoids $M \to N_0$, and each graded summand N_γ is naturally a M-module.

(iv) With the notation of (iii), we claim that the induced morphism $\overline{\varphi} : \varphi(M) \to N$ is flat (hence strongly flat, by theorem 3.2.3), if and only if N_γ is a filtered union of cyclic M-modules, for every $\gamma \in \Gamma$. Indeed, notice that a cyclic M-submodule of N_γ is a free $\varphi(M)$-module of rank one (since N is integral), hence the condition implies that N_γ is a flat $\varphi(M)$-module, hence $\overline{\varphi}$ flat. Conversely, suppose that $\overline{\varphi}$ is flat, and let $n_1, n_2 \in N_\gamma$ (for some $\gamma \in \Gamma$); this means that there exist $a_1, a_2 \in M$ such that $\varphi(a_1)n_1 = \varphi(a_2)n_2$ in N. Then, condition (F2) of theorem 3.1.42 says that there exist $n' \in N$ and $b_1, b_2 \in M$ such that $n_i = \varphi(b_i)n'$ for $i = 1, 2$; especially, $n' \in N_\gamma$, which shows that N_γ is a filtered union of cyclic M-modules.

(v) With the notation of (iii), notice as well, that a morphism $\varphi : M \to N$ of integral monoids is exact (see definition 2.3.40(i)) if and only if $\mathrm{Ker}\,\varphi^{\mathrm{gp}} \subset M$ and φ induces an isomorphism $M/\mathrm{Ker}\,\varphi^{\mathrm{gp}} \xrightarrow{\sim} N_0$. (Details left to the reader.)

Theorem 3.2.6. *Let $M \to N$ be a finite, injective morphism of integral monoids, and S a pointed $M_\circ$-module. Then S is $M_\circ$-flat if and only if $N_\circ \otimes_{M_\circ} S$ is $N_\circ$-flat.*

Proof. In light of theorem 3.1.42, we may assume that $N_\circ \otimes_{M_\circ} S$ is $N_\circ$-flat, and we shall show that S is $M_\circ$-flat. To this aim, let 1 denote the trivial monoid (the initial and final object in the category of monoids); any pointed $M_\circ$-module X is a pointed $1_\circ$-module by restriction of scalars, and if X and Y are any two pointed $M_\circ$-modules, we define a $M_\circ$-module structure on $X \otimes_{1_\circ} Y$ by the rule

$$a \cdot (x \otimes y) := ax \otimes ay \qquad \text{for every } a \in M_\circ,\ x \in X \text{ and } y \in Y.$$

With this notation, we remark :

Claim 3.2.7. Let $\varphi : M \to G$ be a morphism of monoids, where G is a group and M is integral. Then a pointed $M_\circ$-module S is $M_\circ$-flat if and only if the same holds for the $M_\circ$-module $S \otimes_{1_\circ} G_\circ$.

Proof of the claim. Suppose that S is $M_\circ$-flat; then $S = T_\circ$ for some M-module T (remark 3.2.5(ii)), and it is easily seen that $S \otimes_{1_\circ} G_\circ = (T \times G)_\circ$. By theorem 3.1.42, it suffices to check that conditions (F1) and (F2) of lemma 3.1.41 hold for $(T \times G)_\circ$.

Hence, suppose that $a \in M_\circ$ and $h \in (T \times G)_\circ$ satisfy the identity $ah = 0$; in this case, a simple inspection shows that either $a = 0$ or $h = 0$; condition (F1) follows straightforwardly. To check (F2), say that $a_1 \cdot (s_1, g) = a_2 \cdot (s_2, g_2) \neq 0$, for some $a_i \in M$, $s_i \in S$, $g_i \in G$ $(i = 1, 2)$; since (F2) holds for S, we may find $b_1, b_2 \in M$ and $t \in S$ such that $a_1 b_1 = a_2 b_2$ and $b_i t = s_i$ $(i = 1, 2)$. Notice that $g := \varphi(b_1)^{-1} g_1 = \varphi(b_2)^{-1} g_2$ in G, therefore $b_i \cdot (t, g) = (s_i, g_i)$ for $i = 1, 2$, whence the contention.

Conversely, suppose that $S \otimes_{1_\circ} G_\circ$ is $M_\circ$-flat; we wish to show that (F1) and (F2) hold for S. However, suppose that $as = 0$ for some $a \in M_\circ$ and $s \in S$ with $s \neq 0$; then $a \cdot (s \otimes e) = 0$ (where $e \in G$ is the neutral element); since (F1) holds for $S \otimes_{1_\circ} G_\circ$, we deduce that there exist $b \in M_\circ$ and $t \otimes g \in S \otimes_{1_\circ} G_\circ$ such that $ba = 0$ and $s \otimes e = b \cdot (t \otimes g) = bt \otimes \varphi(b) g$; this implies that $bt = s$, so (F1) holds for S, as sought.

Lastly, suppose that $a_1 s_1 = a_2 s_2 \neq 0$ for some $a_i \in M$ and $s_i \in S$ $(i = 1, 2)$. It follows that $a_1 \cdot (s_1 \otimes \varphi(a_2)) = a_2 \cdot (s_2 \otimes \varphi(a_1))$. By applying condition (F2) to this identity in $S \otimes_{1_\circ} G_\circ$, we deduce that the same condition holds also for S. $\diamondsuit$

Next, we observe that there is a natural isomorphism of $N_\circ$-modules :

(3.2.8) $$N_\circ \otimes_{M_\circ} (S \otimes_{1_\circ} N_\circ^{\mathrm{gp}}) \xrightarrow{\sim} (N_\circ \otimes_{M_\circ} S) \otimes_{1_\circ} N_\circ^{\mathrm{gp}} \qquad n \otimes (s \otimes g) \mapsto (n \otimes s) \otimes \varphi(n) g$$

whose inverse is given by the rule : $(n \otimes s) \otimes g \mapsto n \otimes (s \otimes \varphi(n)^{-1} g)$ for every $n \in N$, $s \in S$, $g \in N_\circ^{\mathrm{gp}}$. We leave to the reader the verification that these maps are well defined, and they are inverse to each other. In view of (3.2.8) and claim 3.2.7, we may then replace S by $S \otimes_{1_\circ} N_\circ^{\mathrm{gp}}$, which allows to assume that S is an integral pointed $M_\circ$-module and $N_\circ \otimes_{M_\circ} S$ is an integral pointed $N_\circ$-module. In this case, in view of proposition 3.1.52 we know that $\mathbb{Z}\langle N_\circ \otimes_{M_\circ} S\rangle$ is a flat $\mathbb{Z}\langle N_\circ\rangle$-module, and it suffices to show that $\mathbb{Z}\langle S\rangle$ is a flat $\mathbb{Z}\langle M_\circ\rangle$-module.

However, under our assumptions, the ring homomorphism $\mathbb{Z}\langle M_\circ\rangle \to \mathbb{Z}\langle N_\circ\rangle$ is finite and injective, so the assertion follows from [45, Part II, Th.1.2.4] and (3.1.32). $\square$

Lemma 3.2.9. *Let M be an integral monoid, and $S \subset M$ a submonoid. We have :*

(i) *The natural map $S^{-1} M^{\mathrm{sat}} \to (S^{-1} M)^{\mathrm{sat}}$ is an isomorphism.*
(ii) *If M is saturated, then M/S is saturated, and if S is a group, also the converse holds.*
(iii) *The inclusion map $M \to M^{\mathrm{sat}}$ is a local morphism.*
(iv) *The inclusion $M \subset M^{\mathrm{sat}}$ induces a natural bijection :*
$$\mathrm{Spec}\, M^{\mathrm{sat}} \xrightarrow{\sim} \mathrm{Spec}\, M.$$

Proof. (i) and (iii) are left to the reader. For (ii), the natural isomorphism $S^{-1} M / S^{\mathrm{gp}} \xrightarrow{\sim} M/S$, together with (i), reduces to the case where S is a group, in which case it suffices to remark that $(M/S)^{\mathrm{sat}} = M^{\mathrm{sat}}/S$. Lastly, to show (iv) it suffices to prove that, for any face $F \subset M$, the submonoid $F^{\mathrm{sat}} \subset M^{\mathrm{sat}}$ is a face, and $F^{\mathrm{sat}} \cap M = F$, and that every face of M^{sat} is of this form. These assertions are easy exercises, which we leave as well to the reader. $\square$

Lemma 3.2.10. *Let M be a saturated monoid such that $M^\sharp$ is fine. Then there exists an isomorphism of monoids :*
$$M \xrightarrow{\sim} M^\sharp \times M^\times$$
and if M is fine, $M^\times$ is a finitely generated abelian group. Moreover, the projection $M \to M^\sharp$ induces a bijection :
$$\mathrm{Spec}\, M^\sharp \xrightarrow{\sim} \mathrm{Spec}\, M \quad : \quad \mathfrak{p} \mapsto \mathfrak{p} \times M^\times.$$

Proof. Under the stated assumptions, $G := M^{\mathrm{gp}}/M^\times$ is a free abelian group of finite rank, hence the projection $M^{\mathrm{gp}} \to G$ admits a splitting $\sigma : G \to M^{\mathrm{gp}}$. Set $M_0 := M \cap \sigma(G)$; it is easily seen that $M = M_0 \times M^\times$, whence $M_0 \simeq M^\sharp$. If M is fine, M^{gp} is a finitely generated abelian group, hence the same holds for its direct factor $M^\times$. The last assertion can be proven directly, or can be regarded as a special case of lemma 3.1.16(i.b). $\square$

Definition 3.2.11. Let $\varphi : M \to N$ be a morphism of integral monoids.

(i) We say that φ is *k-saturated* (for some integer $k > 0$), if the push-out $P \otimes_M N$ is integral and k-saturated, for every morphism $M \to P$ with P integral and k-saturated.
(ii) We say that φ is *saturated*, if the following holds. For every morphism of monoids $M \to P$ such that P is integral and saturated, the monoid $P \otimes_M N$ is also integral and saturated.

Clearly, if φ is k-saturated for every integer $k > 0$, then φ is saturated.

Lemma 3.2.12. *Let $\varphi : M \to N$ be a morphism of integral monoids, and $S \subset M$, $T \subset N$ two submonoids such that $\varphi(S) \subset T$. The following holds :*

(i) *The localization map $M \to S^{-1}M$ is saturated.*
(ii) *If φ is saturated, the same holds for the morphisms $S^{-1}M \to T^{-1}N$ and $M/S \to N/T$ induced by φ.*
(iii) *If S and T are two groups, then φ is saturated if and only if the same holds for the induced morphism $M/S \to N/T$.*
(iv) *If φ is saturated, then the natural map $N \otimes_M M^{\mathrm{sat}} \to N^{\mathrm{sat}}$ is an isomorphism.*

Proof. (i) follows from the standard isomorphism : $S^{-1}M \otimes_M N \xrightarrow{\sim} \varphi(S)^{-1}N$, together with lemma 3.2.9(i). Next, let $M/S \to P$ and $S^{-1}M \to Q$ be morphisms of monoids. Then

$$P \otimes_{M/S} N/T \simeq (P \otimes_M N)/T \qquad \text{and} \qquad Q \otimes_{S^{-1}M} T^{-1}N \simeq T^{-1}(Q \otimes_M N)$$

(lemma 2.3.34) so assertions (ii) and (iii) follow from lemma 3.2.9(i,ii).

(iv) follows by comparing the universal properties. □

Example 3.2.13. (i) Let M be an integral monoid, $I \subset M$ an ideal, and consider again the Rees monoid $\mathsf{R}(M, I)$ of the proof of theorem 3.2.3. Clearly $\mathsf{R}(M, I)$ is an integral monoid. However, easy examples show that, even when M is saturated, $\mathsf{R}(M, I)$ is not generally saturated. More precisely, the following holds. For every ideal $J \subset M$, set

$$J^{\mathrm{sat}} := \{a \in M^{\mathrm{gp}} \mid a^n \in J^n \text{ for some integer } n > 0\}$$

where J^n denotes the n-th power of J in the monoid $(\mathscr{P}, \cdot)$ of (3.1.1). Then J^{sat} is an ideal of M^{sat}. With this notation, we have the identity :

$$\mathsf{R}(M, I)^{\mathrm{sat}} = \bigoplus_{n\in\mathbb{N}} (I^n)^{\mathrm{sat}}.$$

(Verification left to the reader.)

(ii) For instance, take $M := \mathbb{Q}_+^{\oplus 2}$, and let $I \subset M$ be the ideal consisting of all pairs (x, y) such that $x + y > 1$. Then $I^n = \{(x, y) \in \mathbb{Q}_+^{\oplus 2} \mid x + y > n\}$, and it is easily seen that $I^n = (I^n)^{\mathrm{sat}}$ for every $n \in \mathbb{N}$, hence in this case $\mathsf{R}(M, I)$ is saturated, in view of (ii). It is easily seen that $\mathsf{R}(M, I)$ does not fulfill condition (F2) of lemma 3.1.41, hence the natural inclusion map $i : M \to \mathsf{R}(M, I)$ is not flat (theorem 3.1.42), hence it is not an integral morphism, according to theorem 3.2.3. On the other hand, we have :

Lemma 3.2.14. *With the notation of example* 3.2.13(ii), *the morphism i is saturated.*

Proof. We prefer to work with the multiplicative notation, so we shall argue with the monoid $(\exp \mathbb{Q}_+^{\oplus 2}, \cdot)$ (see (3.1)). Indeed, let $\varphi : M \to P$ be any morphism of monoids, with P saturated. Clearly $\mathsf{R}(M, I)$ is the direct sum of the P-modules $P \otimes_M I^n$, for all $n \in \mathbb{N}$.

Claim 3.2.15. The natural map $P \otimes_M I^n \to I^n P$ is an isomorphism, for every $n \in \mathbb{N}$

Proof of the claim. Indeed, the map is obviously surjective. Hence, suppose that $a_1 x_1 = a_2 x_2$ for some $a_1, a_2 \in I^n$, $x_1, x_2 \in P$ such that $x_1 \otimes a_1 = x_2 \otimes a_2$. For every $\vartheta \in \mathbb{Q}$ with $0 \leq \vartheta \leq 1$, set $a_\vartheta := a_1^\vartheta \cdot a_2^{1-\vartheta}$ and notice that

$$x_\vartheta := a_1 a_\vartheta^{-1} x_1 = a_2 a_\vartheta^{-1} x_2 \in P^{\mathrm{gp}}$$

and if $N \in \mathbb{N}$ is large enough, so that $N\vartheta \in \mathbb{N}$, then $x_\vartheta^N = x_1^{N\vartheta} x_2^{N\vartheta} \in P$, hence $x_\vartheta \in P$.

Next, for any $(a, b), (a', b') \in \mathbb{Q}_+^{\oplus 2}$, set $(a, b) \vee (a', b') := (\min(a, a'), \min(b, b'))$. Choose an increasing sequence $0 := \vartheta_0 < \vartheta_1 < \cdots < \vartheta_n := 1$ of rational numbers, such that

$$b_i := a_{\vartheta_i} \vee a_{\vartheta_{i+1}} \in I^n \qquad \text{for every } i = 0, \dots, n-1.$$

Then there exist $c_i, d_i \in \mathbb{Q}_+^{\oplus 2}$ such that $a_{\vartheta_i} = b_i c_i$ and $a_{\vartheta_{i+1}} = b_i d_i$ for $i = 0, \dots, n-1$. We may then compute in $I^n \otimes_M P$:

$$a_{\vartheta_i} \otimes x_{\vartheta_i} = b_i c_i \otimes x_{\vartheta_i} = b_i \otimes c_i x_{\vartheta_i} \quad \text{and likewise} \quad a_{\vartheta_{i+1}} \otimes x_{\vartheta_{i+1}} = b_i \otimes d_i x_{\vartheta_{i+1}}.$$

By construction, we have $c_i x_{\vartheta_i} = d_i x_{\vartheta_{i+1}}$ for $i = 0, \dots, n-1$, whence the contention. ◊

In view of claim 3.2.15, we are reduced to showing that $\mathsf{R}(P, IP)$ is saturated, and by example 3.2.13, this comes down to proving that $(I^n P)^{\mathrm{sat}} = I^n P$ for every $n \in \mathbb{N}$. However, say that $x \in P^{\mathrm{gp}}$, and $x^n = a_1 x_1 \cdots a_n x_n$ for some $a_i \in I$ and $x_i \in P$; set $a := (a_1 \cdots a_n)^{1/n}$, and notice that $a \in I$. Then $x^n = a^n \cdot x_1 \cdots x_n$, so that $x a^{-1} \in P$, and finally $x \in IP$, as required. □

Lemma 3.2.14 shows that a saturated morphism is not necessarily integral. Notwithstanding, we shall see later that integrality holds for an important class of saturated morphisms (corollary 3.4.4). Now we wish to globalize the class of saturated morphisms, to an arbitrary topos. Of course, we could define the notion of saturated morphism of T-monoids, just by repeating word by word definition 3.2.11(ii). However, it is not clear that the resulting condition would be of a type which can be checked on stalks, in the sense of remark 2.2.14(ii). For this reason, we prefer to proceed as in Tsuji's work [74].

Lemma 3.2.16. *Let T be a topos, $\varphi : \underline{M} \to \underline{N}$ and $\psi : \underline{N} \to \underline{P}$ two morphisms of integral T-monoids. We have:*

(i) *If φ and ψ are exact, the same holds for $\psi \circ \varphi$.*
(ii) *If $\psi \circ \varphi$ is exact, the same holds for φ.*
(iii) *Consider a commutative diagram of integral T-monoids :*

(3.2.17)
$$\begin{array}{ccc} \underline{M} & \xrightarrow{\varphi} & \underline{N} \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\psi'} \\ \underline{M'} & \xrightarrow{\varphi'} & \underline{N'}. \end{array}$$

Then the following holds :

(a) *If* (3.2.17) *is a cartesian diagram and φ' is exact, then φ is exact.*
(b) *If* (3.2.17) *is cocartesian (in the category* $\mathbf{Int.Mnd}_T$*) and φ is exact, then φ' is exact.*

Proof. For all these assertions, remark 2.3.41(ii) easily reduces to the case where $T = \mathbf{Set}$, which therefore we assume from start. Now, (i) and (ii) are left to the reader.

(iii.a): Let $x \in M^{\mathrm{gp}}$ such that $\varphi^{\mathrm{gp}}(x) \in N$; hence $(\varphi' \circ \psi)^{\mathrm{gp}}(x) = \psi'(\varphi^{\mathrm{gp}}(x)) \in N'$, and therefore $\psi^{\mathrm{gp}}(x) \in M'$, since φ' is exact. It follows that $x \in M$, so φ is exact.

(iii.b): Let $x \in (M')^{\mathrm{gp}}$ such that $(\varphi')^{\mathrm{gp}}(x) \in N'$; then we may write $(\varphi')^{\mathrm{gp}}(x) = \varphi'(y) \cdot \psi'(z)$ for some $y \in M'$ and $z \in N$. Therefore, $(\varphi')^{\mathrm{gp}}(xy^{-1}) = \psi'(z)$; since the functor $P \mapsto P^{\mathrm{gp}}$ commutes with colimits, it follows that we may find $w \in M^{\mathrm{gp}}$ such that $\psi^{\mathrm{gp}}(w) = xy^{-1}$ and

$\varphi^{\mathrm{gp}}(w) = z$. Since φ is exact, we deduce that $w \in M$, therefore $xy^{-1} \in M'$, and finally $x \in M'$, whence the contention. $\square$

3.2.18. Let T be a topos; for two morphisms $\underline{P} \leftarrow \underline{M} \rightarrow \underline{N}$ of integral T-monoids, we set

$$\underline{N} \overset{\mathrm{int}}{\otimes}_M \underline{P} := (\underline{N} \otimes_M \underline{P})^{\mathrm{int}}$$

which is the push-out of these morphisms, in the category $\mathbf{Int}.\mathbf{Mnd}_T$. Notice that, for every morphism $f : T' \to T$ of topoi, the natural morphism of T'-monoids

$$f^*(\underline{N} \overset{\mathrm{int}}{\otimes}_M \underline{P}) \to f^*\underline{N} \overset{\mathrm{int}}{\otimes}_{f^*M} f^*\underline{P} \tag{3.2.19}$$

is an isomorphism (by lemmata 2.3.45(i) and 2.3.46(i)).

Let now $\varphi : \underline{M} \to \underline{N}$ be a morphism of integral T-monoids. For any integer $k > 0$, let $\boldsymbol{k}_M$ and $\boldsymbol{k}_N$ be the k-Frobenius maps of M and N (definition 2.3.40(ii)), and consider the cocartesian diagram :

$$\begin{tikzcd} \underline{M} \arrow[r, "\boldsymbol{k}_M"] \arrow[d, "\varphi"'] & \underline{M} \arrow[d, "\varphi'"] \\ \underline{N} \arrow[r, "\boldsymbol{k}_M \overset{\mathrm{int}}{\otimes}_M \mathbf{1}_N"] & \underline{P}. \end{tikzcd} \tag{3.2.20}$$

The endomorphism $\boldsymbol{k}_N$ factors through $\boldsymbol{k}_M \overset{\mathrm{int}}{\otimes}_M \mathbf{1}_N$ and a unique morphism $\beta : \underline{P} \to \underline{N}$ such that $\beta \circ \varphi' = \varphi$. A simple inspection shows that $\boldsymbol{k}_P = (\boldsymbol{k}_M \otimes_M \mathbf{1}_N) \circ \beta$. Now, if $\boldsymbol{k}_M$ is exact, then the same holds for $\boldsymbol{k}_M \overset{\mathrm{int}}{\otimes}_M \mathbf{1}_N$, in view of lemma 3.2.16(iii.b). Hence, if $\underline{P}$ is k-saturated, then β is exact (lemma 3.2.16(ii)), and if $\underline{M}$ is k-saturated, also the converse holds. Likewise, if $\underline{M}$ is k-saturated and β is exact, then $\underline{N}$ is k-quasi-saturated.

These considerations motivate the following :

Definition 3.2.21. Let T be a topos, $\varphi : \underline{M} \to \underline{N}$ a morphism of integral T-monoids.

(i) A commutative diagram of integral T-monoids :

$$\begin{tikzcd} \underline{M} \arrow[r, "\varphi"] \arrow[d] & \underline{N} \arrow[d] \\ \underline{M}' \arrow[r, "\varphi'"] & \underline{N}' \end{tikzcd}$$

is called an *exact square*, if the induced morphism $\underline{M}' \overset{\mathrm{int}}{\otimes}_M \underline{N} \to \underline{N}'$ is exact.

(ii) φ is said to be *k-quasi-saturated* if the commutative diagram :

$$\begin{tikzcd} \underline{M} \arrow[r, "\varphi"] \arrow[d, "\boldsymbol{k}_M"'] & \underline{N} \arrow[d, "\boldsymbol{k}_N"] \\ \underline{M} \arrow[r, "\varphi"] & \underline{N} \end{tikzcd} \tag{3.2.22}$$

is an exact square (the vertical arrows are the k-Frobenius maps).

(iii) φ is said to be *quasi-saturated* if it is k-quasi-saturated for every integer $k > 0$.

Proposition 3.2.23. *Let T be a topos.*

(i) *If* (3.2.17) *is an exact square, and $\underline{M} \to \underline{P}$ is any morphism of integral T-monoids, the square $\underline{P} \overset{\mathrm{int}}{\otimes}_M$* (3.2.17) *is exact.*

(ii) *Consider a commutative diagram of integral T-monoids :*

$$\begin{tikzcd} \underline{M} \arrow[r, "\varphi_1"] \arrow[d, "\psi"'] & \underline{N} \arrow[r, "\varphi_2"] \arrow[d] & \underline{P} \arrow[d, "\psi'"] \\ \underline{M}' \arrow[r, "\varphi'_1"] & \underline{N}' \arrow[r, "\varphi'_2"] & \underline{P}' \end{tikzcd} \tag{3.2.24}$$

and suppose that the left and right square subdiagrams of (3.2.24) *are exact. Then the same holds for the square diagram :*

$$\begin{tikzcd} \underline{M} \arrow[r, "\varphi_2\circ\varphi_1"] \arrow[d, "\psi"'] & \underline{P} \arrow[d, "\psi'"] \\ \underline{M}' \arrow[r, "\varphi'_2\circ\varphi'_1"] & \underline{P}'. \end{tikzcd}$$

Proof. (i): We have a commutative diagram :

$$\begin{tikzcd} (\underline{P} \overset{\mathrm{int}}{\otimes}_M \underline{M}') \overset{\mathrm{int}}{\otimes}_M \underline{N} \arrow[r, "\sigma"] \arrow[d, "\omega"'] & \underline{P} \overset{\mathrm{int}}{\otimes}_M (\underline{M}' \overset{\mathrm{int}}{\otimes}_M \underline{N}) \arrow[d, "\mathbf{1}_P \overset{\mathrm{int}}{\otimes}_M \alpha"] \\ (\underline{P} \overset{\mathrm{int}}{\otimes}_M \underline{M}') \overset{\mathrm{int}}{\otimes}_P (\underline{P} \overset{\mathrm{int}}{\otimes}_M \underline{N}) \arrow[r, "\beta"] & \underline{P} \overset{\mathrm{int}}{\otimes}_M \underline{N}' \end{tikzcd}$$

where ω, σ are the natural isomorphisms, and β and $\alpha : \underline{M}' \overset{\mathrm{int}}{\otimes}_M \underline{N} \to \underline{N}'$ are the natural maps. By assumption, α is exact, hence the same holds for $\mathbf{1}_P \overset{\mathrm{int}}{\otimes}_M \alpha$, by lemma 3.2.16(iii.b). So β is exact, which is the claim.

(ii): Let $\alpha : \underline{M}' \overset{\mathrm{int}}{\otimes}_M \underline{N} \to \underline{N}'$ and $\beta : \underline{N}' \overset{\mathrm{int}}{\otimes}_N \underline{P} \to \underline{P}'$ be the natural maps; by assumptions, these are exact morphisms. However, we have a natural commutative diagram :

$$\begin{tikzcd} (\underline{M}' \overset{\mathrm{int}}{\otimes}_M \underline{N}) \overset{\mathrm{int}}{\otimes}_N \underline{P} \arrow[r, "\omega"] \arrow[d, "\alpha \overset{\mathrm{int}}{\otimes}_N \mathbf{1}_P"'] & \underline{M}' \overset{\mathrm{int}}{\otimes}_M \underline{P} \arrow[d, "\gamma"] \\ \underline{N}' \overset{\mathrm{int}}{\otimes}_N \underline{P} \arrow[r, "\beta"] & \underline{P}' \end{tikzcd}$$

where ω is the natural isomorphism, and γ is the map deduced from ψ' and $\varphi'_2 \circ \varphi'_1$. Then the assertion follows from lemma 3.2.16(i,iii.b). $\square$

Corollary 3.2.25. *Let T be a topos, $\varphi : \underline{M} \to \underline{N}$, $\psi : \underline{N} \to \underline{P}$ two morphisms of integral T-monoids, and $h, k > 0$ any two integers. The following holds :*

(i) *If φ is both h-quasi-saturated and k-quasi-saturated, then φ is hk-quasi-saturated.*
(ii) *If φ and ψ are k-quasi-saturated, the same holds for $\psi \circ \varphi$.*
(iii) *If $\underline{M} \to \underline{P}$ is any map of integral T-monoids, and φ is k-quasi-saturated (resp. quasi-saturated), then the same holds for $\varphi \overset{\mathrm{int}}{\otimes}_M \mathbf{1}_P : \underline{P} \to \underline{N} \overset{\mathrm{int}}{\otimes}_M \underline{P}$.*
(iv) *Let $\underline{S} \subset \varphi^{-1}(\underline{N}^\times)$ be a T-submonoid. Then φ is k-quasi-saturated (resp. quasi-saturated) if and only if the same holds for the induced map $\varphi_S : \underline{S}^{-1}\underline{M} \to \underline{N}$.*
(v) *Let $\underline{G} \subset \operatorname{Ker}\varphi$ be a subgroup. Then φ is k-quasi-saturated (resp. quasi-saturated) if and only if the same holds for the induced map $\overline{\varphi} : \underline{M}/\underline{G} \to \underline{N}$.*
(vi) *φ is quasi-saturated if and only if it is p-quasi-saturated for every prime number p.*
(vii) *$\underline{M}$ is k-saturated (resp. saturated) if and only if the unique morphism of T-monoids $\{1\} \to \underline{M}$ is k-quasi-saturated (resp. quasi-saturated).*
(viii) *If φ is k-quasi-saturated (resp. quasi-saturated), and $\underline{M}$ is k-saturated (resp. saturated), then $\underline{N}$ is k-saturated (resp. saturated).*

(ix) *If $\underline{M}$ is integral, and $\underline{N}$ is a T-group, φ is quasi-saturated.*

Proof. (i) and (ii) are straightforward consequences of proposition 3.2.23(ii).

To show (iii), set $\underline{P}' := \underline{N} \overset{\mathrm{int}}{\otimes}_M \underline{P}$ and let us remark that we have a commutative diagram

$$\begin{tikzcd} \underline{P} \arrow[r] \arrow[d, "\varphi \overset{\mathrm{int}}{\otimes}_M \mathbf{1}_P"'] & \underline{P}_1 \arrow[r] \arrow[d] & \underline{P} \arrow[d, "\varphi \overset{\mathrm{int}}{\otimes}_M \mathbf{1}_P"] \\ \underline{P}' \arrow[r] & \underline{P}_2 \arrow[r] & \underline{P}' \end{tikzcd}$$

such that :

- the composition of the top (resp. bottom) arrows is the k-Frobenius map
- the left square subdiagram is (3.2.22) $\overset{\mathrm{int}}{\otimes}_M \underline{P}$
- the right square subdiagram is cocartesian (hence exact).

then the assertion follows from proposition 3.2.23.

(iv): Suppose that φ is k-quasi-saturated. Then the same holds for $\underline{S}^{-1}\varphi : \underline{S}^{-1}M \to \underline{S}^{-1}M \overset{\mathrm{int}}{\otimes}_M \underline{N}$, according to (iii). However, we have a natural isomorphism $\underline{S}^{-1}\underline{M} \overset{\mathrm{int}}{\otimes}_M \underline{N} \xrightarrow{\sim} \varphi(\underline{S})^{-1}\underline{N} = N$, so φ_S is k-quasi-saturated.

Conversely, suppose that φ_S is k-quasi-saturated. By (ii), in order to prove that the same holds for φ, it suffices to show that the localisation map $\underline{M} \to \underline{S}^{-1}\underline{M}$ is k-quasi-saturated. But this is clear, since (3.2.22) becomes cocartesian if we take for φ the localisation map.

(v): To begin with, $\underline{M}/\underline{G}$ is an integral monoid, by lemma 2.3.38. Suppose that φ is k-quasi-saturated. Arguing as in the proof of (iv) we see that the natural map $\underline{N} \to (\underline{M}/\underline{G}) \overset{\mathrm{int}}{\otimes}_M \underline{N}$ is an isomorphism, hence $\overline{\varphi}$ is k-quasi-saturated, by (iii).

Conversely, suppose that $\overline{\varphi}$ is k-quasi-saturated. By (ii), in order to prove that φ is saturated, it suffices to show that the same holds for the projection $\underline{M} \to \underline{M}/\underline{G}$. By (iii), we are further reduced to showing that the unique map $\underline{G} \to \{1\}$ is k-quasi-saturated, which is trivial.

(vi) is a straightforward consequence of (i). Assertion (vii) can be verified easily on the definitions. Next, suppose that φ is quasi-saturated and $\underline{M}$ is saturated. By (ii) and (vii) it follows that the unique morphism $\{1\} \to \underline{N}$ is quasi-saturated, hence (vii) implies that $\underline{N}$ is saturated, so (viii) holds.

(ix): In view of (iv), we are reduced to the case where $\underline{M}$ is also a group, in which case the assertion is obvious. ☐

Proposition 3.2.26. *Let $\varphi : M \to N$ be an integral morphism of integral monoids, $k > 0$ an integer. The following conditions are equivalent :*

(a) *φ is k-quasi-saturated.*
(b) *φ is k-saturated.*
(c) *The push-out P of the cocartesian diagram* (3.2.20), *is k-saturated.*

Proof. (a)⇒(b) by virtue of corollary 3.2.25(iii,viii), and trivially (b)⇒(c). Lastly, the implication (c)⇒(a) was already remarked in (3.2.18). ☐

Corollary 3.2.27. *Let $\varphi : M \to N$ be an integral morphism of integral monoids. Then φ is quasi-saturated if and only if it is saturated.* ☐

Proposition 3.2.26 motivates the following :

Definition 3.2.28. Let T be a topos, $\varphi : \underline{M} \to \underline{N}$ a morphism of integral T-monoids, $k > 0$ an integer. We say that φ is *k-saturated* (resp. *saturated*) if φ is integral and k-quasi-saturated (resp. and quasi-saturated).

In case $T = \mathbf{Set}$, we see that an integral morphism of integral monoids is saturated in the sense of definition 3.2.28, if and only if it is saturated in the sense of the previous definition 3.2.11. We may now state :

Corollary 3.2.29. *Let $\mathbf{P}(T,\varphi)$ be the property "φ is an integral (resp. exact, resp. k-saturated, resp. saturated) morphism of integral T-monoids" (for a topos T). Then $\mathbf{P}$ can be checked on stalks. (See remark* 2.2.14(ii).*)*

Proof. The fact that integrality of a T-monoid can be checked on stalks, has already been established in lemma 2.3.46(ii). For the property "φ is an integral morphism" (between integral T-monoids), it suffices to apply theorem 3.2.3 and proposition 2.3.26.

For the property "φ is exact" one applies lemma 2.3.46(iii). From this, and from (3.2.19) one deduces that also the properties of k-saturation and saturation can be checked on stalks. $\square$

Lemma 3.2.30. *Let $f : M \to N$ be a morphism of integral monoids. We have :*

(i) *If f is exact, then f is local.*
(ii) *Conversely, if f is an integral and local morphism, then f is exact.*

Proof. (i) is left to the reader as an exercise.

(ii): Suppose $x \in M^{\mathrm{gp}}$ is an element such that $b := f_S(x) \in N$. Write $x = z^{-1}y$ for certain $y, z \in M$; therefore $b \cdot f(z) = f(y)$ holds in N, and theorems 3.2.3, 3.1.42 imply that there exist $c \in N$ and $a_1, a_2 \in M$, such that $1 = c \cdot f(a_1)$, $b = c \cdot f(a_2)$ and $ya_1 = za_2$. Since f is local, we deduce that $a_1 \in M^\times$, hence $x = a_1^{-1}a_2$ lies in M. $\square$

Proposition 3.2.31. *Let $f : M \to N$ be an integral map of integral monoids, $k > 0$ an integer, and $N = \bigoplus_{\gamma\in\Gamma} N_\gamma$ the f-grading of N (see remark* 3.2.5(iii)*). Then the following conditions are equivalent :*

(a) *f is k-quasi-saturated.*
(b) *$N_{k\gamma} = N_\gamma^k$ for every $\gamma \in \Gamma$. (Here the k-th power of subsets of N is taken in the monoid $\mathscr{P}(N)$, as in* (3.1.1).*)*

Proof. (a)⇒(b): Let $\pi : N^{\mathrm{gp}} \to \Gamma$ be the projection, suppose that $y \in N_{k\gamma}$ for some $\gamma \in \Gamma$, and pick $x \in N^{\mathrm{gp}}$ such that $\pi(x) = \gamma$. This means that $y = x^k \cdot f^{\mathrm{gp}}(z)$ for some $z \in M^{\mathrm{gp}}$. By (a), it follows that we may find a pair $(a, b) \in M \times N$ and an element $w \in M^{\mathrm{gp}}$, such that $(aw^{-k}, bw) = (z, x)$ in $M^{\mathrm{gp}} \times N^{\mathrm{gp}}$. Especially, $b, b \cdot f(a) \in N_\gamma$, and consequently $y = b^{k-1} \cdot (b \cdot f(a)) \in N_{k\gamma}$, as stated.

(b)⇒(a): Let $S := f^{-1}(N^\times)$; by lemmata 3.2.30(ii) and 3.2.2(iii), the induced map $f_S : S^{-1}M \to N$ is exact and integral. Moreover, by corollary 3.2.25(iv), f is k-quasi-saturated if and only if the same holds for f_S, and clearly the f-grading of N agrees with the f_S-grading. Hence we may replace f by f_S and assume from start that f is exact. In this case, $G := \mathrm{Ker}\, f^{\mathrm{gp}} \subset M$, and corollary 3.2.25(v) says that f is k-quasi-saturated if and only if the same holds for the induced map $\overline{f} : M/G \to N$; moreover $\overline{f}$ is still integral, since it is deduced from f by push-out along the map $M \to M/G$. Hence we may replace f by $\overline{f}$, thereby reducing to the case where f is injective. Also, f is flat, by theorem 3.2.3. The assertion boils down to the following. Suppose that $(x, y) \in M^{\mathrm{gp}} \times N^{\mathrm{gp}}$ is a pair such that $a := f^{\mathrm{gp}}(x) \cdot y^k \in N$; we have to show that there exists a pair $(m, n) \in M \times N$ whose class in the push-out P of the diagram

$$N \xleftarrow{f} M \xrightarrow{\boldsymbol{k}_M} M$$

agrees with the image of (x, y) in P^{gp}. However, set $\gamma := \pi(y)$, and notice that $a \in N_{k\gamma}$, hence we may write $a = n_1 \cdots n_k$ for certain $n_1, \dots, n_k \in N_\gamma$. Then, according to remark 3.2.5(iv), we may find $n \in N_\gamma$ and $x_1, \dots, x_k \in M$ such that $n_i = n \cdot f(x_i)$ for every $i = 1, \dots, k$. It follows that $y \cdot n^{-1} \in f^{\mathrm{gp}}(M^{\mathrm{gp}})$, say $y = n \cdot f(z)$ for some $z \in M^{\mathrm{gp}}$. Set $m := x_1 \cdots x_k$; then

$(x, y) = (x, n \cdot f(z))$ represents the same class as $(x \cdot z^k, n)$ in P^{gp}. Especially, $f(x \cdot z^k) \cdot n^k = a = f(m) \cdot n^k$, hence $m = x \cdot n^k$, since f is injective. The claim follows. $\square$

Corollary 3.2.32. *Let $f : M \to N$ be an integral and local morphism of integral and sharp monoids. Then :*

(i) *f is exact and injective.*
(ii) *If furthermore, f is saturated, then $\mathrm{Coker}\, f^{\mathrm{gp}}$ is a torsion-free abelian group.*

Proof. (i): It follows from lemma 3.2.31 that f is exact. Next, suppose that $f(a_1) = f(a_2)$ for some $a_1, a_2 \in M$. By theorem 3.2.3 and 3.1.42, it follows that there exist $b_1, b_2 \in M$ and $t \in N$ such that $1 = f(b_1)t = f(b_2)t$ and $a_1 b_1 = a_2 b_2$. Since N is sharp, we deduce that $f(b_1) = f(b_2) = 1$, and since f is local and M is sharp, we get $b_1 = b_2 = 1$, thus $a_1 = a_2$, whence $x = 1$, which is the contention.

(ii): Let us endow N with its f-grading. Suppose now that $g \in G$ is a torsion element, and say that $g^k = 1$ in G for some integer $k > 0$; by propositions 3.2.31 and 3.2.26, we then have $N_1 = N_g^k$. Especially, there exist $a_1, \dots, a_k \in N_g$, such that $a_1 \cdots a_k = 1$. Since N is sharp, we must have $a_i = 1$ for $i = 1, \dots, k$, hence $g = 1$. $\square$

Corollary 3.2.33. *Let $\varphi : M \to N$ be a morphism of integral monoids, $F \subset N$ any face, and $\varphi_F : \varphi^{-1}F \to F$ the restriction of φ.*

(i) *If φ is flat, then the same holds for φ_F, and the induced map $\mathrm{Coker}\, \varphi_F^{\mathrm{gp}} \to \mathrm{Coker}\, \varphi^{\mathrm{gp}}$ is injective.*
(ii) *If φ is saturated, the same holds for φ_F.*

Proof. (i): The fact that φ_F is flat, is a special case of corollary 3.1.51. The assertion about cokernels boils down to showing that the induced diagram of abelian groups :

$$\begin{tikzcd} (\varphi^{-1}F)^{\mathrm{gp}} \arrow[r, "\varphi_F^{\mathrm{gp}}"] \arrow[d] & F^{\mathrm{gp}} \arrow[d] \\ M^{\mathrm{gp}} \arrow[r, "\varphi^{\mathrm{gp}}"] & N^{\mathrm{gp}} \end{tikzcd}$$

is cartesian. However, say that $\varphi^{\mathrm{gp}}(a_1 a_2^{-1}) = f_1 f_2^{-1}$ for some $a_1, a_2 \in M$ and $f_1, f_2 \in F$. This means that $\varphi(a_1) f_2 = \varphi(a_2) f_1$ in N; by condition (F2) of theorem 3.1.42, we deduce that there exist $b_1, b_2 \in M$ and $t \in N$ such that $b_2 a_1 = b_1 a_2$ and $f_i = t \cdot \varphi(b_i)$ $(i = 1, 2)$. Since F is a face, it follows that $b_1, b_2 \in \varphi^{-1}F$, hence $a_1 a_2^{-1} = b_1 b_2^{-1} \in (\varphi^{-1}F)^{\mathrm{gp}}$, as required.

(ii): By (i), the morphism φ_F is integral, hence it suffices to show that φ_F quasi-saturated (proposition 3.2.26), and to this aim we shall apply the criterion of proposition 3.2.31. Indeed, let $N = \bigoplus_{\gamma \in \Gamma} N_\gamma$ (resp. $F = \bigoplus_{\gamma \in \Gamma_F} F_\gamma$) be the φ-grading (resp. the φ_F-grading); according to (i), the induced map $j : \Gamma_F \to \Gamma$ is injective. This means that $F_\gamma = F \cap N_{j(\gamma)}$ for every $\gamma \in \Gamma_F$. Hence, for every integer $k > 0$ we may compute

$$F_{k\gamma} = F \cap N_{j(\gamma)}^k = (F \cap N_{j(\gamma)})^k = F_\gamma^k$$

where the first identity follows by applying proposition 3.2.31 (and proposition 3.2.26) to the saturated map φ, and the second identity holds because F is a face of N. $\square$

3.3. **Polyhedral cones.** Fine monoids can be studied by means of certain combinatorial objects, which we wish to describe. Part of the material that follows is borrowed from [35]. Again, *all the monoids in this section are non-pointed.*

3.3.1. Quite generally, a *convex cone* is a pair (V,σ), where V is a finite dimensional $\mathbb{R}$-vector space, and $\sigma \subset V$ is a non-empty subset such that :

$$\mathbb{R}_+ \cdot \sigma = \sigma = \sigma + \sigma$$

where the addition is formed in the monoid $(\mathscr{P}(V), +)$ as in (3.1.1), and scalar multiplication by the set $\mathbb{R}_+$ is given by the rule :

$$\mathbb{R}_+ \cdot S := \{r \cdot s \mid r \in \mathbb{R}_+, s \in S\} \qquad \text{for every } S \in \mathscr{P}(V).$$

A subset $S \subset \sigma$ is called a *ray* of σ, if it is of the form $\mathbb{R}_+ \cdot \{s\}$, for some $s \in \sigma \setminus \{0\}$.

We say that (V,σ) is a *closed convex cone* if σ is closed as a subset of V (of course, V is here regarded as a topological space via any choice of isomorphism $V \simeq \mathbb{R}^n$). We denote by $\langle\sigma\rangle \subset V$ the $\mathbb{R}$-vector space generated by σ. To a convex cone (V,σ) one assigns the *dual cone* $(V^\vee, \sigma^\vee)$, where $V^\vee := \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$, the dual of V, and :

$$\sigma^\vee := \{u \in V^\vee \mid u(v) \geq 0 \text{ for every } v \in \sigma\}.$$

Also, the *opposite cone* of σ is the cone

$$-\sigma := \{-v \in V \mid v \in \sigma\}.$$

Notice that $\sigma^\vee$ and $-\sigma$ are always closed cones. Notice as well that the restriction of the addition law of V determines a monoid structure $(\sigma, +)$ on the set σ. A map of cones

$$\varphi : (W, \sigma_W) \to (V, \sigma_V)$$

is an $\mathbb{R}$-linear map $\varphi : W \to V$ such that $\varphi(\sigma_W) \subset \sigma_V$. Clearly, the restriction of φ yields a map of monoids $(\sigma_W, +) \to (\sigma_V, +)$. If $S \subset V$ is any subset, we set :

$$S^\perp := \{u \in V^\vee \mid u(s) = 0 \text{ for every } s \in S\} \subset V^\vee.$$

Lemma 3.3.2. *Let (V,σ) be a closed convex cone, Then, under the natural identification $V \xrightarrow{\sim} (V^\vee)^\vee$, we have $(\sigma^\vee)^\vee = \sigma$.*

Proof. This follows from [19, Ch.II, §5, n.3, Cor.5]. □

3.3.3. A *convex polyhedral cone* is a cone (V,σ) such that σ is of the form :

$$\sigma := \{r_1 v_1 + \cdots + r_s v_s \in V \mid r_i \geq 0 \text{ for every } i \leq s\}$$

for a given set of vectors $v_1, \dots, v_s \in V$, called *generators* for the cone σ. One also says that $\{v_1, \dots, v_s\}$ is a *generating set* for σ, and that $\mathbb{R}_+ \cdot v_1, \dots, \mathbb{R}_+ \cdot v_s$ are *generating rays* for σ. We say that σ is a *simplicial cone*, if it is generated by a system of linearly independent vectors.

Lemma 3.3.4. *Let (V,σ) be a convex polyhedral cone, S a finite generating set for σ. Then :*

(i) *For every $v \in \sigma$ there is a subset $T \subset S$ consisting of linearly independent vectors, such that v is contained in the convex polyhedral cone generated by T.*

(ii) *(V,σ) is a closed convex cone.*

Proof. (i): Let $T \subset S$ be a subset such that v is contained in the cone generated by T; up to replacing T by a subset, we may assume that T is minimal, *i.e.* no proper subset of T generates a cone containing v. We claim that T consists of linearly independent vectors. Otherwise, we may find a linear relation of the form $\sum_{w \in T} a_w \cdot w = 0$, for certain $a_w \in \mathbb{R}$, at least one of which is non-zero; we may then assume that :

$$a_w > 0 \qquad \text{for at least one vector } w \in T. \tag{3.3.5}$$

Say also that $v = \sum_{w \in T} b_w \cdot w$ with $b_w \in \mathbb{R}_+$; by the minimality assumption on T, we must actually have $b_w > 0$ for every $w \in T$. We deduce the identity : $v = \sum_{w \in T} (b_w - ta_w) \cdot w$ for every $t \in \mathbb{R}$; let t_0 be the supremum of the set of positive real numbers t such that $b_w - ta_w \geq 0$

for every $w \in T$. In view of (3.3.5) we have $t_0 \in \mathbb{R}_+$; moreover $b_w - t_0 a_w \geq 0$ for every $w \in T$, and $b_w - t_0 a_w = 0$ for at least one vector w, which contradicts the minimality of T.

(ii): In view of (i), we are reduced to the case where σ is generated by finitely many linearly independent vectors, and for such cones the assertion is clear (details left to the reader). $\square$

3.3.6. A *face* of a convex cone σ is a subset of the form $\sigma \cap \mathrm{Ker}\, u$, for some $u \in \sigma^\vee$. The *dimension* (resp. *codimension*) of a face τ of σ is the dimension of the $\mathbb{R}$-vector space $\langle\tau\rangle$ (resp. of the $\mathbb{R}$-vector space $\langle\sigma\rangle/\langle\tau\rangle$). A *facet* of σ is a face of codimension one. Notice that if σ is a polyhedral cone, and τ is a face of σ, then (V, τ) is also a convex polyhedral cone; indeed if $S \subset V$ is a generating set for σ, then $S \cap \tau$ is a generating set for τ.

Lemma 3.3.7. *Let (V, σ) be a convex polyhedral cone. Then the faces of (V, σ) are the same as the faces of the monoid $(\sigma, +)$.*

Proof. Clearly we may assume that $\sigma \neq \{0\}$. Let F be a face of the polyhedral cone σ, and pick $u \in \sigma^\vee$ such that $F = \sigma \cap \mathrm{Ker}\, u$. Then $\sigma \setminus F = \{x \in \sigma \mid u(x) > 0\}$, and this is clearly an ideal of the monoid $(\sigma, +)$; hence F is a face of $(\sigma, +)$.

Conversely, suppose that F is a face of $(\sigma, +)$. First, we wish to show that F is a cone in V. Indeed, let $f \in F$, and $r > 0$ any real number; we have to prove that $r \cdot f \in F$. To this aim, it suffices to show that $(r/N) \cdot f \in F$ for some integer $N > 0$, so that, after replacing r by r/N for N large enough, we may assume that $0 < r < 1$. In this case, $(1 - r) \cdot f \in \sigma$, and we have $f = r \cdot f + (1 - r) \cdot f$, so that $r \cdot f \in F$, since F is a face of $(\sigma, +)$.

Next, denote by $W \subset V$ the $\mathbb{R}$-vector space spanned by F. Suppose first that $V = W$, and consider any $m \in \sigma$; then $m = f_1 - f_2$ for some $f_1, f_2 \in F$, hence $f_1 = f_2 + m$. Since F is a face of $(\sigma, +)$, this implies that m lies in F, so $F = \sigma$ in this case, especially F is a face of the convex polyhedral cone σ.

So finally, we may assume that W is a proper subspace of V. In this case, let $N := \sigma + (-F)$. We notice that $N \neq V$. Indeed, if $m \in \sigma$ and $-m \in N$, we may write $-m = m' - f$ for some $m' \in \sigma$ and $f \in F$; hence $f = m + m'$, and therefore $m \in F$; in other words, N avoids the whole of $-(\sigma \setminus F)$, which is not empty for $\sigma \neq \{0\}$.

Thus, N is a proper convex cone of V. Now, let $u_1, \dots, u_k \in N^\vee$ be a system of generators of the $\mathbb{R}$-subspace of $V^\vee$ spanned by $N^\vee$, and set $u := u_1 + \cdots + u_k$. Suppose that $x \in \sigma \cap \mathrm{Ker}\, u$; then $-x \in \mathrm{Ker}\, u_i$ for every $i = 1, \dots, k$, and therefore $-x \in N^{\vee\vee} = N$ (lemma 3.3.2). Hence, we may write $-x = m - f$ for some $m \in \sigma$ and $f \in F$, or equivalently, $f = m + x$, which shows that $x \in F$. Summarizing, we have proved that $F = \sigma \cap \mathrm{Ker}\, u$, *i.e.* F is a face of the convex cone σ. $\square$

Proposition 3.3.8. *Let (V, σ) be a convex polyhedral cone. The following holds :*

(i) *Any intersection of faces of σ is still a face of σ.*
(ii) *If τ is a face of σ, and γ is a face of (V, τ), then γ is a face of σ.*
(iii) *Every proper face of σ is the intersection of the facets that contain it.*

Proof. (i): Say that $\tau_i = \sigma \cap \mathrm{Ker}\, u_i$, where $u_1, \dots, u_n \in \sigma^\vee$. Then $\bigcap_{i=1}^n \tau_i = \sigma \cap \mathrm{Ker} \sum_{i=1}^n u_i$.

(ii): Say that $\tau = \sigma \cap \mathrm{Ker}\, u$ and $\gamma = \tau \cap \mathrm{Ker}\, v$, where $u \in \sigma^\vee$ and $v \in \tau^\vee$. Then, for large $r \in \mathbb{R}_+$, the linear form $v' := v + ru$ is non-negative on any given finite generating set of σ, hence it lies in $\sigma^\vee$, and $\gamma = \sigma \cap \mathrm{Ker}\, v'$.

(iii): To begin with, we prove the following :

Claim 3.3.9. (i) Every proper face of σ is contained in a facet.

(ii) Every face of σ of codimension 2 is the intersection of exactly two facets.

Proof of the claim. We may assume that $\langle\sigma\rangle = V$. Let τ be a face of σ of codimension at least two, and denote by $\overline{\sigma}$ be the image of σ in the quotient $\overline{V} := V/\langle\tau\rangle$; clearly $(\overline{V}, \overline{\sigma})$ is again a polyhedral cone. Moreover, choose $u \in \sigma^\vee$ such that $\tau = \sigma \cap \operatorname{Ker} u$; the linear form u descends to $\overline{u} \in \overline{\sigma}^\vee$, therefore $\overline{\sigma} \cap \operatorname{Ker} \overline{u} = \{0\}$ is a face of $\overline{\sigma}$.

(ii): If τ has codimension two, $\overline{V}$ has dimension two. Suppose that the assertion is known for $(\overline{V}, \overline{\sigma})$; then we find exactly two facets $\overline{\gamma}_1, \overline{\gamma}_2$ of $\overline{\sigma}$ whose intersection is $\{0\}$. Their preimages in V intersect σ in facets γ_1, γ_2 that satisfy $\gamma_1 \cap \gamma_2 = \tau$. Hence, we may assume from start that $\tau = \{0\}$ and V has dimension two, in which case the verification is easy, and shall be left to the reader.

(i): Arguing by induction on the codimension, it suffices to show that τ is contained in a proper face spanning a larger subspace. To this aim, suppose that the claim is known for $\overline{\sigma}$; since $\{0\}$ is a face of $\overline{\sigma}$ of codimension at least two, it is contained in a proper face $\overline{\gamma}$; the preimage γ of the latter intersects σ in a proper face containing τ. Thus again, we are reduced to the case where $\tau = \{0\}$. Pick $u_0 \in \sigma^\vee$ such that $\sigma \cap \operatorname{Ker} u_0 = \{0\}$; choose also any other $u_1 \in V^\vee$ such that $\sigma \cap \operatorname{Ker} u_1 \neq \{0\}$. Since $\dim_{\mathbb{R}} V^\vee \geq 2$, we may find a continuous map $f : [0,1] \to V^\vee \setminus \{0\}$ with $f(0) = u_0$ and $f(1) = u_1$. Let $\mathbb{P}_+(V)$ be the topological space of rays of V (*i.e.* the topological quotient $V \setminus \{0\}/\sim$ by the equivalence relation such that $v \sim v'$ if and only if v and v' generate the same ray), and define likewise $\mathbb{P}_+(V^\vee)$; let also $Z \subset P := \mathbb{P}_+(V) \times \mathbb{P}_+(V^\vee)$ be the incidence correspondence, *i.e.* the subset of all pairs $(\overline{v}, \overline{u})$ such that $u(v) = 0$, for any representative u of the class $\overline{u}$ and v of the class $\overline{v}$. Finally, let $\mathbb{P}_+(\sigma) \subset \mathbb{P}_+(V)$ be the image of $\sigma \setminus \{0\}$. Then Z (resp. $\mathbb{P}_+(\sigma)$) is a closed subset of P (resp. of $\mathbb{P}_+(V)$), hence $Y := Z \cap (\mathbb{P}_+(\sigma) \times \mathbb{P}_+(V^\vee))$ is a closed subset of P. Since the projection $\pi : P \to \mathbb{P}_+(V^\vee)$ is proper, $\pi(Y)$ is closed in $\mathbb{P}_+(V^\vee)$. Let $\overline{f} : [0,1] \to \mathbb{P}_+(V^\vee)$ be the composition of f and the natural projection $V^\vee \setminus \{0\} \to \mathbb{P}_+(V^\vee)$; then $f^{-1}(\pi(Y))$ is a closed subset of $[0,1]$, hence it admits a smallest element, say a (notice that $a > 0$). Moreover, $u_a \in \sigma^\vee$; indeed, otherwise we may find $v \in \sigma \setminus \{0\}$ such that $u_a(v) < 0$, and since $u_0(v) > 0$, we would have $u_b(v) = 0$ for some $b \in (0,a)$. The latter means that $f(b) \in \pi(Y)$, which contradicts the definition of a. Since by construction, $\sigma \cap \operatorname{Ker} u_a \neq \{0\}$, the claim follows. $\Diamond$

Let τ be any face of σ; to show that (iii) holds for τ, we argue by induction on the codimension of τ. The case of codimension 2 is covered by claim 3.3.9(ii). If τ has codimension > 2, we apply claim 3.3.9(i) to find a proper face γ containing τ; by induction, τ is the intersection of facets of γ, and each of these is the intersection of two facets in σ (again by claim 3.3.9(ii)), whence the contention. $\square$

3.3.10. Suppose σ spans V (*i.e.* $\langle\sigma\rangle = V$) and let τ be a facet of σ; by definition there exists an element $u_\tau \in \sigma^\vee$ such that $\tau = \sigma \cap \operatorname{Ker} u_\tau$, and one sees easily that the ray $R_\tau := \mathbb{R}_+ \cdot u_\tau \subset \sigma^\vee$ is well-defined, independently of the choice of u_τ. Hence the half-space :

$$H_\tau := \{v \in V \mid u_\tau(v) \geq 0\}$$

depends only on τ. Recall that the *interior* (resp. the *topological closure*) of a subset $E \subset V$ is the largest open subset (resp. the smallest closed subset) of V contained in E (resp. containing E). The *topological boundary* of E is the intersection of the topological closures of E and of its complement $V \setminus E$.

Proposition 3.3.11. *Let (V, σ) be a convex polyhedral cone, such that σ spans V. We have:*

(i) *The topological boundary of σ is the union of its facets.*
(ii) *If $\sigma \neq V$, then $\sigma = \bigcap_{\tau \subset \sigma} H_\tau$, where τ ranges over the facets of σ.*
(iii) *The rays R_τ, where τ ranges over the facets of σ, generate the cone $\sigma^\vee$.*

Proof. (i): Notice that σ spans V if and only if the interior of σ is not empty. A proper face τ is the intersection of σ with a hyperplane $\operatorname{Ker} u \subset V$ with $u \in \sigma^\vee \setminus \{0\}$; therefore, every neighborhood $U \subset V$ of any point $v \in \tau$ intersects $V \setminus \sigma$. This shows that τ lies in the topological boundary of σ.

Conversely, if v is in the boundary of σ, choose a sequence $(v_i \mid i \in \mathbb{N})$ of points of $V \setminus \sigma$, converging to the point v; by lemma 3.3.2, for every $i \in \mathbb{N}$ there exists $u_i \in \sigma^\vee$ such that $u_i(v_i) < 0$. Up to multiplication by scalars, we may assume that the vectors u_i lie on some sphere in $V^\vee$ (choose any norm on $V^\vee$); hence we may find a convergent subsequence, and we may then assume that the sequence $(u_i \mid i \in \mathbb{N})$ converges to an element $u \in V^\vee$. Necessarily $u \in \sigma^\vee$, therefore v lies on the face $\sigma \cap \mathrm{Ker}\, u$, and the assertion follows from proposition 3.3.8(iii).

(ii): Suppose, by way of contradiction, that v lies in every half-space H_τ, but $v \notin \sigma$. Choose any point v' in the interior of σ, and let $t \in [0,1]$ be the largest value such that $w := tv + (1-t)v' \in \sigma$. Clearly w lies on the boundary of σ, hence on some facet τ, by (i). Say that $\tau = \sigma \cap \mathrm{Ker}\, u$; then $u(v') > 0$ and $u(w) = 0$, so $u(v) < 0$, a contradiction.

(iii): When $\sigma = V$, there is nothing to prove, hence we may assume that $\sigma \neq V$. In this case, suppose that $u \in \sigma^\vee$, and u is not in the cone C generated by the rays R_τ. Applying lemma 3.3.2 to the cone $(V^\vee, C)$, we deduce that there exists a vector $v \in V$ with $v \in H_\tau$ for all the facets τ of σ, and $u(v) < 0$, which contradicts (ii). ☐

Corollary 3.3.12. *Let (V, σ) and (V, σ') be two convex polyhedral cones. Then :*

(i) (Farkas' theorem) *The dual $(V^\vee, \sigma^\vee)$ is also a convex polyhedral cone.*
(ii) *If τ is a face of σ, then $\tau^* := \sigma^\vee \cap \tau^\perp$ is a face of $\sigma^\vee$ such that $\langle \tau^* \rangle = \langle \tau \rangle^\perp$. Especially:*

$$\dim_{\mathbb{R}} \langle \tau \rangle + \dim_{\mathbb{R}} \langle \tau^* \rangle = \dim_{\mathbb{R}} V. \tag{3.3.13}$$

The rule $\tau \mapsto \tau^*$ is a bijection from the set of faces of σ to those of $\sigma^\vee$. The smallest face of σ is $\sigma \cap (-\sigma)$.
(iii) *$(V, \sigma \cap \sigma')$ is a convex polyhedral cone, and every face of $\sigma \cap \sigma'$ is of the form $\tau \cap \tau'$, for some faces τ of σ and τ' of σ'.*

Proof. (i): Set $W := \langle \sigma \rangle \subset V$, and pick a basis $u_1, \dots, u_k$ of $W^\perp$; by proposition 3.3.11(iii), the assertion holds for the dual $(W^\vee, \sigma^\vee)$ of the cone (W, σ). However, $W^\vee \simeq V^\vee / W^\perp$, hence the dual cone $(V^\vee, \sigma^\vee)$ is generated by lifts of generators of $(W^\vee, \sigma^\vee)$, together with the vectors u_i and $-u_i$, for $i = 1, \dots, k$.

(ii): Notice first that the faces of $\sigma^\vee$ are exactly the cones $\sigma^\vee \cap \{u\}^\perp$, for $u \in \sigma = (\sigma^\vee)^\vee$. For a given $v \in \sigma$, let τ be the smallest face of σ such that $v \in \tau$; this means that $\tau^\vee \cap \{v\}^\perp = \tau^\perp$ (where $(V^\vee, \tau^\vee)$ is the dual of (V, τ)). Hence $\sigma^\vee \cap \{v\}^\perp = \sigma^\vee \cap (\tau^\vee \cap \{v\}^\perp) = \tau^*$, so every face of $\sigma^\vee$ has the asserted form. The rule $\tau \mapsto \tau^*$ is clearly order-reversing, and from the obvious inclusion $\tau \subset (\tau^*)^*$ it follows that $\tau^* = ((\tau^*)^*)^*$, hence this map is a bijection. It follows from this, that the smallest face of σ is $(\sigma^\vee)^* = \sigma \cap (\sigma^\vee)^\perp = (\sigma^\vee)^\perp = \sigma \cap (-\sigma)$. In particular, we see that $(\sigma \cap (-\sigma))^* = \sigma^\vee$, and furthermore, (3.3.13) holds for $\tau := \sigma \cap (-\sigma)$. Identity (3.3.13) for a general face τ can be deduced by inserting τ in a maximal chain of faces of σ, and comparing with the dual chain of faces of $\sigma^\vee$ (details left to the reader). Finally, it is clear that $\langle \tau \rangle \subset \langle \tau^* \rangle^\perp$; since these spaces have the same dimension, we deduce that $\langle \tau \rangle^\perp = \langle \tau^* \rangle$.

(iii): Indeed, lemma 3.3.2 implies that $\sigma^\vee + \sigma'^\vee$ is the dual of $\sigma \cap \sigma'$, hence (i) implies that $\sigma \cap \sigma'$ is polyhedral. It also follows that every face τ of $\sigma \cap \sigma'$ is the intersection of $\sigma \cap \sigma'$ with the kernel of a linear form $u + u'$, for some $u \in \sigma^\vee$ and $u' \in \sigma'^\vee$. Consequently, $\tau = (\sigma \cap \mathrm{Ker}\, u) \cap (\sigma' \cap \mathrm{Ker}\, u')$. ☐

Corollary 3.3.14. *For a convex polyhedral cone (V, σ), the following conditions are equivalent:*

(a) $\sigma \cap (-\sigma) = \{0\}$.
(b) *σ contains no non-zero linear subspaces.*
(c) *There exists $u \in \sigma^\vee$ such that $\sigma \cap \mathrm{Ker}\, u = \{0\}$.*
(d) *$\sigma^\vee$ spans $V^\vee$.*

Proof. (a) ⇔ (b) since $\sigma \cap (-\sigma)$ is the largest subspace contained in σ. Next, (a) ⇔ (c) since $\sigma \cap (-\sigma)$ is the smallest face of σ. Finally, (a) ⇔ (d) since $\dim_{\mathbb{R}}\langle \sigma \cap (-\sigma)\rangle + \dim_{\mathbb{R}}\langle \sigma^\vee \rangle = n$ (corollary 3.3.12(ii)). □

3.3.15. A convex polyhedral cone fulfilling the equivalent conditions of corollary 3.3.14 is said to be *strongly convex*. Suppose that (V, σ) is strongly convex; then proposition 3.3.11(iii) says that σ is generated by the rays R_τ, where τ ranges over the facets of $\sigma^\vee$. The rays R_τ are uniquely determined by σ, and are called the *extremal rays* of σ. Moreover, these R_τ form *the unique minimal set of generating rays* for σ. Indeed, concerning the minimality : for each facet τ of $\sigma^\vee$, pick $v_\tau \in \sigma$ with $\mathbb{R}_+ \cdot v_\tau = R_\tau$; suppose that $v_{\tau_0} = \sum_{\tau \in S} t_\tau \cdot v_\tau$ for some subset S of the set of facets of $\sigma^\vee$, and appropriate $t_\tau > 0$, for every $\tau \in S$. It follows easily that $u(v_\tau) = 0$ for every $u \in \tau_0$, and every $\tau \in S$. But by definition of R_τ, this implies that $S = \{\tau_0\}$, which is the claim. Concerning uniqueness : suppose that Σ is another system of generating rays; especially, for any facet $\tau \subset \sigma^\vee$, the ray R_τ is in the convex cone generated by Σ; it follows easily that there exists $\rho \in \Sigma$ such that $u(\rho) = 0$ for every $u \in \tau$, in which case $\rho = R_\tau$. This shows that Σ must contain all the extremal rays.

Example 3.3.16. (i) Suppose that $\dim_{\mathbb{R}} V = 2$, and (V, σ) is a strongly convex polyhedral cone, and assume that σ generates V. Then the only face of codimension two of σ is $\{0\}$, so it follows easily from claim 3.3.9(ii) that σ admits exactly two facets, and these are also the extremal rays of σ, especially σ is simplicial. Of course, these assertions are rather obvious; in dimension > 2, the general situation is much more complicated.

(ii) Let (V, σ) be a convex polyhedral cone, and suppose that σ spans V. Let τ be a face of σ. Notice that $(\sigma, +)^{\mathrm{gp}} = (V, +)$, and $(\tau, +)$ is a face of the monoid $(\sigma, +)$, by lemma 3.3.7. Hence we may view the localization $\tau^{-1}\sigma$ naturally as a submonoid of $(V, +)$, and it is easily seen that $\tau^{-1}\sigma$ is a convex cone. By proposition 3.3.11(iii), the polyhedral cone $\sigma^\vee$ is generated by its extremal rays $\mathbb{R}l_1, \dots, \mathbb{R}l_n$, and by proposition 3.3.8(iii), we may assume that $\tau = \sigma \cap \mathrm{Ker}\,(l_1 + \cdots + l_k)$ for some $k \leq n$. Clearly $l_i(v) \geq 0$ for every $v \in \tau^{-1}\sigma$ and every $i \leq k$. Conversely, if $l \in (\tau^{-1}\sigma)^\vee$, we must have $\tau \subset \mathrm{Ker}\, l$ and $l \in \sigma^\vee$; if we write $l = \sum_{i=1}^n a_i l_i$ for some $a_i \geq 0$, it follows easily that $a_i = 0$ for every $i > k$. On the other hand, suppose that $v \in V$ satisfies the inequalities $l_i(v) \geq 0$ for $i = 1, \dots, k$; then, for every $i = k+1, \dots, n$ we may find $u_i \in \tau$ such that $l_i(v + u_i) \geq 0$, hence $v + u_{k+1} + \cdots + u_n \in \sigma$, and therefore $v \in \tau^{-1}\sigma$. This shows that $\tau^{-1}\sigma$ is a closed convex cone, and its dual $(\tau^{-1}\sigma)^\vee$ is the convex cone generated by $l_1, \dots, l_k$; especially, it is a convex polyhedral cone, and then the same holds for $\tau^{-1}\sigma$, by virtue of lemma 3.3.2 and corollary 3.3.12(i).

(iii) In the situation of (ii), let $v \in \tau$ be any element that lies in the *relative interior* of τ, *i.e.* in the complement of the union of the facets of τ. Denote by $S_v \subset \tau$ the submonoid generated by v. Then we claim that

$$\tau^{-1}\sigma = S_v^{-1}\sigma.$$

Indeed, let $s \in \sigma$ and $t \in \tau$ be any two element; in view of proposition 3.3.11(i), it is easily seen that there exists an integer $N > 0$ such that $v - N^{-1}t \in \tau$, hence $t' := Nv - t \in \tau$. Therefore $s - t = (s + t') - Nv \in S^{-1}\sigma$, and the assertion follows.

Lemma 3.3.17. *Let $f : V \to W$ be a linear map of finite dimensional $\mathbb{R}$-vector spaces, (V, σ) a convex polyhedral cone. The following holds :*

(i) *$(W, f(\sigma))$ is a convex polyhedral cone.*
(ii) *Suppose moreover, that $\sigma \cap \mathrm{Ker}\, f$ does not contain non-zero linear subspaces of V. Then, for every face τ of $f(\sigma)$ there exists a face τ' of σ such that $f(\tau') \subset \tau$, and f restricts to an isomorphism : $\langle \tau' \rangle \xrightarrow{\sim} \langle \tau \rangle$.*

Proof. (i) is obvious. To show (ii) we argue by induction on $n := \dim_{\mathbb{R}} \mathrm{Ker}\, f$. The assertion is obvious when $n = 0$, hence suppose that $n > 0$ and that the claim is already known whenever

$\operatorname{Ker} f$ has dimension $< n$. Let τ be a face of $f(\sigma)$; then $f^{-1}\tau$ is a face of $f^{-1}f(\sigma)$, hence $\sigma \cap f^{-1}\tau$ is a face of $\sigma = \sigma \cap f^{-1}f(\sigma)$. In view of proposition 3.3.8(ii), we may then replace σ by $\sigma \cap f^{-1}\tau$, and therefore assume from start that $\tau = f(\sigma)$. We may as well assume that $V = \langle\sigma\rangle$ and $W = \langle\tau\rangle$. The assumption on σ implies especially that $\sigma \neq V$, hence σ is the intersection of the half-spaces H_γ corresponding to its facets γ (proposition 3.3.11(ii)). For each facet γ of σ, let u_γ be a chosen generator of the ray R_γ (notation of (3.3.10)). Since $\sigma \cap \operatorname{Ker} f$ does not contain non-zero linear subspaces, we may find a facet γ such that $V' := \operatorname{Ker} u_\gamma$ does not contain $\operatorname{Ker} f$. Then, the inductive assumption applies to the restriction $f_{|V'} : V' \to W$ and the convex polyhedral cone $\sigma \cap V'$, and yields a face τ' of the latter, such that f induces an isomorphism $\langle\tau'\rangle \xrightarrow{\sim} \langle f(\sigma \cap V')\rangle$. Finally, $\langle f(\sigma \cap V')\rangle = W$, since V' does not contain $\operatorname{Ker} f$. □

Lemma 3.3.18. *Let $f : V \to W$ be a linear map of finite dimensional $\mathbb{R}$-vector spaces, $f^\vee : W^\vee \to V^\vee$ the transpose of f, and (W, σ) a convex polyhedral cone. Then :*

(i) $(V, f^{-1}\sigma)$ *is a convex polyhedral cone and* $(f^{-1}\sigma)^\vee = f^\vee(\sigma^\vee)$.

(ii) *For every face δ of $f^{-1}\sigma$, there exists a face τ of σ such that $\delta = f^{-1}\tau$. If furthermore, $\langle\sigma\rangle + f(V) = W$, we may find such a τ so that additionally, f induces an isomorphism:*
$$V/\langle\delta\rangle \xrightarrow{\sim} W/\langle\tau\rangle.$$

(iii) *Conversely, for every face τ of σ, the cone $f^{-1}\tau$ is a face of $f^{-1}\sigma$, and $(f^{-1}\tau)^*$ is the smallest face of $(f^{-1}\sigma)^\vee$ containing $f^\vee(\tau^*)$ (notation of corollary* 3.3.12(ii)*).*

Proof. (i): By corollary 3.3.12(i), $(W^\vee, \sigma^\vee)$ is a convex polyhedral cone, hence we may find $u_1, \dots, u_s \in \sigma^\vee$ such that $\sigma = \bigcap_{i=1}^s u_i^{-1}(\mathbb{R}_+)$. Therefore $f^{-1}\sigma = \bigcap_{i=1}^s (u_i \circ f)^{-1}(\mathbb{R}_+)$. Let $\gamma \subset V^\vee$ be the cone generated by the set $\{u_1 \circ f, \dots, u_s \circ f\}$; then $f^{-1}\sigma = \gamma^\vee$, and the assertion results from lemma 3.3.2 and a second application of (i).

(iii): For every $u \in \sigma^\vee$, we have : $f^{-1}(\sigma \cap \operatorname{Ker} u) = (f^{-1}\sigma) \cap \operatorname{Ker} u \circ f$. Since we already know that $(f^{-1}\sigma)^\vee = f^\vee(\sigma^\vee)$, we see that the faces of $f^{-1}\sigma$ are exactly the subsets of the form $f^{-1}\tau$, where τ ranges over the faces of σ. Next, for any such τ, the set $f^\vee(\tau^*)$ consists of all $u \in V^\vee$ of the form $u = w \circ f$ for some $w \in \sigma^\vee$ such that $w(\tau) = 0$. From this description it is clear that $f^\vee(\tau^*) \subset (f^{-1}\tau)^*$. To show that $(f^{-1}\tau)^*$ is the smallest face containing $f^\vee(\tau^*)$, it then suffices to prove that $f^\vee(\tau^*)^\perp \cap f^{-1}\sigma \subset f^{-1}\tau$. However, let $v \in f^\vee(\tau^*)^\perp \cap f^{-1}\sigma$; then $w \circ f(v) = 0$ for every $w \in \tau^*$, *i.e.* $f(v) \in (\tau^*)^* = \tau$, whence the contention.

(ii): The first assertion has already been shown; hence, suppose that $\langle\sigma\rangle + f(V) = W$. We deduce that $\sigma^\vee \cap \operatorname{Ker} f^\vee$ does not contain non-zero linear subspaces of $W^\vee$; indeed, if $u \in W^\vee$, and both u and $-u$ lie in $\sigma^\vee$, then u vanishes on $\langle\sigma\rangle$, and if $u \in \operatorname{Ker} f^\vee$, then u vanishes as well on $f(V)$, hence $u = 0$. We may then apply lemma 3.3.17(ii) to find a face γ of $\sigma^\vee$ such that $f^\vee(\gamma) \subset \delta^*$ and $f^\vee$ restricts to an isomorphism : $\langle\gamma\rangle \xrightarrow{\sim} \langle\delta^*\rangle$. Especially, δ^* is the smallest face of $(f^{-1}\sigma)^\vee$ containing $f^\vee(\gamma)$, hence $\delta^* = (f^{-1}\gamma^*)^*$ by (iii), *i.e.* $\delta = f^{-1}\gamma^*$. We also deduce that f induces an isomorphism : $V/\langle\delta^*\rangle^\perp \xrightarrow{\sim} W/\langle\gamma\rangle^\perp$. Since $\langle\delta^*\rangle^\perp = \langle\delta\rangle$ and $\langle\gamma\rangle^\perp = \langle\gamma^*\rangle$, the second assertion holds with $\tau := \gamma^*$. □

Lemma 3.3.19. *Let (V, σ) and (V', σ') be two convex polyhedral cones. Then :*

(i) $(V \oplus V', \sigma \times \sigma')$ *is a convex polyhedral cone.*

(ii) *Every face of $\sigma \times \sigma'$ is of the form $\tau \times \tau'$, for some faces τ of σ and τ' of σ'.*

Proof. Indeed, $\sigma \times \sigma' = (p_1^{-1}\sigma) \cap (p_2^{-1}\sigma')$, where p_1 and p_2 are the natural projections of $V \oplus V'$ onto V and V'. Hence, assertions (i) and (ii) follow from corollary 3.3.12(iii) and lemma 3.3.18(i). □

3.3.20. Let $(L, +)$ be a free abelian group of finite rank, $\sigma \subset L_{\mathbb{R}} := L \otimes_{\mathbb{Z}} \mathbb{R}$ a convex polyhedral cone. We say that σ is *L-rational* (or briefly : *rational*, when there is no danger of ambiguity) if it admits a generating set consisting of elements of L. Then it is clear that every face of a rational convex polyhedral cone is again rational (see (3.3.6)). On the other hand, let

$$(M, +) \subset (L, +)$$

be a submonoid of L; we shall denote by $(L_{\mathbb{R}}, M_{\mathbb{R}})$ the convex cone generated by M (*i.e.* the smallest convex cone in $L_{\mathbb{R}}$ containing the image of M). If M is fine, $M_{\mathbb{R}}$ is a convex polyhedral cone. Later we shall also find useful to consider the subset :

$$M_{\mathbb{Q}} := \{m \otimes q \mid m \in M, q \in \mathbb{Q}_+\} \subset L_{\mathbb{Q}} := L \otimes_{\mathbb{Z}} \mathbb{Q}$$

which is a submonoid of $L_{\mathbb{Q}}$.

Proposition 3.3.21. *Let $(L, +)$ be a free abelian group of finite rank, with dual*

$$L^{\vee} := \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z}).$$

Let also $(L_{\mathbb{R}}, \sigma)$ and $(L_{\mathbb{R}}, \sigma')$ be two L-rational convex polyhedral cones. We have :

(i) *The dual $(L^{\vee}_{\mathbb{R}}, \sigma^{\vee})$ is an $L^{\vee}$-rational convex polyhedral cone.*
(ii) *$(L_{\mathbb{R}}, \sigma \cap \sigma')$ is also an L-rational convex polyhedral cone.*
(iii) *Let $g : L' \to L$ (resp. $h : L \to L'$) be a map of free abelian groups of finite rank, and denote by $g_{\mathbb{R}} : L'_{\mathbb{R}} \to L_{\mathbb{R}}$ (resp. $h_{\mathbb{R}} : L_{\mathbb{R}} \to L'_{\mathbb{R}}$) the induced $\mathbb{R}$-linear map. Then, $(L'_{\mathbb{R}}, g^{-1}_{\mathbb{R}}\sigma)$ and $(L'_{\mathbb{R}}, h_{\mathbb{R}}(\sigma))$ are L'-rational.*
(iv) *Let L' be another free abelian group of finite rank, and $(L'_{\mathbb{R}}, \sigma')$ an L'-rational convex polyhedral cone. Then $(L_{\mathbb{R}} \oplus L'_{\mathbb{R}}, \sigma \times \sigma')$ is $L \oplus L'$-rational.*

Proof. (i) and (ii) follow easily, by inspecting the proof of corollary 3.3.12(i),(iii) : the details shall be left to the reader.

(iii): The assertion concerning $h_{\mathbb{R}}(\sigma)$ is obvious. To show the assertion for $g^{-1}_{\mathbb{R}}\sigma$, one argues as in the proof of lemma 3.3.18(i) : by (i), we may find $u_1, \dots, u_s \in L^{\vee}$ such that $\sigma = \bigcap_{i=1}^{s} u^{-1}_{i,\mathbb{R}}(\mathbb{R}_+)$. Therefore $g^{-1}_{\mathbb{R}}\sigma = \bigcap_{i=1}^{s}(u_i \circ g)^{-1}_{\mathbb{R}}(\mathbb{R}_+)$. Let $\gamma \subset V^{\vee}$ be the cone generated by the set $\{(u_1 \circ g)_{\mathbb{R}}, \dots, (u_s \circ g)_{\mathbb{R}}\}$; then $g^{-1}_{\mathbb{R}}\sigma = \gamma^{\vee}$, and the assertion results from lemma 3.3.2 and a second application of (i).

Lastly, arguing as in the proof of lemma 3.3.19(i), one derives (iii) from (ii) and (iii). $\square$

Parts (i) and (iii) of the following proposition provide the bridge connecting convex polyhedral cones to fine monoids.

Proposition 3.3.22. *Let $(L, +)$ be a free abelian group of finite rank, $(L_{\mathbb{R}}, \sigma)$ an L-rational convex polyhedral cone, and set $\sigma_L := L \cap \sigma$. Then :*

(i) (Gordan's lemma) *σ_L is a fine and saturated submonoid of L, and $L \cap \langle\sigma\rangle = \sigma^{\mathrm{gp}}_L$.*
(ii) *For every $v \in L_{\mathbb{R}}$, the subset $L \cap (\sigma - v)$ is a finitely generated σ_L-module.*
(iii) *For any submonoid $M \subset L$, we have : $M_{\mathbb{Q}} = M_{\mathbb{R}} \cap L_{\mathbb{Q}}$ and $M^{\mathrm{sat}} = M_{\mathbb{R}} \cap L$.*

Proof. (Here $\sigma - v \subset L_{\mathbb{R}}$ denotes the translate of σ by the vector $-v$, *i.e.* the subset of all $w \in L_{\mathbb{R}}$ such that $w + v \in \sigma$.) Choose $v_1, \dots, v_s \in L$ that generate σ, and set

$$C_{\varepsilon} := \left\{ \sum_{i=1}^{s} t_i v_i \mid t_i \in [0, \varepsilon] \text{ for } i = 1, \dots, s \right\} \qquad \text{for every } \varepsilon > 0.$$

(i): Clearly $L \cap \sigma$ is saturated. Since C_1 is compact and L is discrete, $C_1 \cap L$ is a finite set. We claim that $C_1 \cap L$ generates the monoid σ_L. Indeed, if $v \in \sigma_L$, write $v = \sum_{i=1}^{s} r_i v_i$, with $r_i \geq 0$ for every $i = 1, \dots, s$; hence $r_i = m_i + t_i$ for some $m_i \in \mathbb{N}$ and $t_i \in [0, 1[$, and therefore $v = v' + \sum_{i=1}^{s} m_i v_i$, where $v', v_1, \dots, v_s \in C_1 \cap L$. Next, it is clear that $\sigma^{\mathrm{gp}}_L \subset L \cap \langle\sigma\rangle$; for the converse, say that $w \in L \cap \langle\sigma\rangle$, and write $w = w_1 - w_2$, for some $w_1, w_2 \in \sigma$. Then

$w_1 = \sum_{i=1}^s t_i v_i$ for some $t_i \geq 0$; we pick $t'_i \in \mathbb{N}$ such that $t'_i \geq t_i$ for every $i \leq s$, and we set $w'_i := \sum_{i=1}^s t'_i v_i$. It follows that $w = w'_1 - w'_2$, where $w'_2 := w_2 + (w'_1 - w_1))$, and notice that $w'_1 \in \sigma_L$ and $w'_2 \in \sigma$; then we must have $w'_2 \in \sigma_L$ as well, and therefore $w \in \sigma_L^{\mathrm{gp}}$.

(ii) is similar : from the compactness of C_1 one sees that $L \cap (C_1 - v)$ is a finite set; on the other hand, arguing as in the proof of (i), one checks easily that the latter set generates the σ_L-module $L \cap (\sigma - v)$.

(iii): Let $x \in M_{\mathbb{R}} \cap L_{\mathbb{Q}}$; then we may write

$$x = \sum_{i=1}^n r_i \otimes m_i \qquad \text{where } m_i \in M,\ r_i > 0 \text{ for every } i \leq n. \tag{3.3.23}$$

Claim 3.3.24. In the situation of proposition 3.3.22(iii), let $x \in M_{\mathbb{R}} \cap L_{\mathbb{Q}}$, and write x as in (3.3.23). Then, for every $\varepsilon > 0$ there exist $q_1, \dots, q_n \in \mathbb{Q}_+$ with $|r_i - q_i| < \varepsilon$ for every $i = 1, \dots, n$, and such that $x = \sum_{i=1}^n q_i \otimes m_i$.

Proof of the claim. Up to a reordering, we may assume that $m_1, \dots, m_k$ form a basis of the $\mathbb{Q}$-vector space generated by $m_1, \dots, m_n$, therefore $m_{k+i} = \sum_{j=1}^k q_{ij} m_j$ for a matrix

$$A := (q_{ij} \mid i = 1, \dots, n-k;\ j = 1, \dots, k)$$

with entries in $\mathbb{Q}$. Let $\underline{r} := (r_1, \dots, r_k)$ and $\underline{r}' := (r_{k+1}, \dots, r_n)$; since $x \in L_{\mathbb{Q}}$, we deduce that $\underline{b} := \underline{r} + \underline{r}' \cdot A \in \mathbb{Q}^{\oplus k}$. Moreover, if $\underline{s} := (s_1, \dots, s_k) \in \mathbb{Q}^{\oplus k}$ and $\underline{s}' := (s_{k+1}, \dots, s_n) \in \mathbb{Q}^{\oplus n-k}$ satisfy the identity $\underline{b} = \underline{s} + \underline{s}' \cdot A$, then $\sum_{i=1}^n s_i \otimes m_i = x$. If we choose $\underline{s}'$ very close to $\underline{r}'$, then $\underline{s}$ shall be very close to $\underline{r}$; especially, we can achieve that both $\underline{s}$ and $\underline{s}'$ are vectors with positive coordinates. ◊

Claim 3.3.24 shows that $x \in M_{\mathbb{Q}}$, whence the first stated identity; for the second identity, we are reduced to showing that $M^{\mathrm{sat}} = M_{\mathbb{Q}} \cap L$, which is immediate. □

For various algebraic and geometric applications of the theory of polyhedral cones, one is led to study subdivisions of a given cone, in the sense of the following definition 3.3.25. Later we shall see a more abstract notion of subdivision, in the context of general fans, which however finds its roots and motivation in the intuitive manipulations of polyhedra that we formalize hereafter.

Definition 3.3.25. Let V be a finite dimensional $\mathbb{R}$-vector space.

(i) A *fan* in V is a finite set Δ consisting of convex polyhedral cones of V, such that :
  - for every $\sigma \in \Delta$, and every face τ of σ, also $\tau \in \Delta$;
  - for every $\sigma, \tau \in \Delta$, the intersection $\sigma \cap \tau$ is also an element of Δ, and is a face of both σ and τ.

(ii) We say that Δ is a *simplicial fan* if all the elements of Δ are simplicial cones.

(iii) Suppose that $V = L \otimes_{\mathbb{Z}} \mathbb{R}$ for some free abelian group L; then we say that Δ is *L-rational* if the same holds for every $\tau \in \Delta$.

(iv) A *refinement* of the fan Δ is a fan Δ' in V with $\bigcup_{\sigma \in \Delta} \sigma = \bigcup_{\tau \in \Delta'} \tau$, and such that every $\tau \in \Delta$ is the union of the $\gamma \in \Delta'$ contained in τ.

(v) A *subdivision* of a convex polyhedral cone (V, σ) is a refinement of the fan Δ_σ consisting of σ and its faces.

Lemma 3.3.26. *Let (V, σ) be any convex polyhedral cone, Δ a subdivision of (V, σ). We let*

$$\Delta^s := \{\tau \in \Delta \mid \langle \tau \rangle = \langle \sigma \rangle\}.$$

Then $\bigcup_{\tau \in \Delta^s} \tau = \sigma$.

Proof. Let $\tau_0 \in \Delta$ be any element. Then $\sigma' := \bigcup_{\tau \neq \tau_0} \tau$ is a closed subset of σ. If $\langle \tau_0 \rangle \neq \langle \sigma \rangle$, then $\sigma \setminus \tau_0$ is a dense open subset of σ contained in σ'; it follows that $\sigma' = \sigma$ in this case.

Especially, $\tau_0 = \bigcup_{\tau\neq\tau_0}(\tau \cap \tau_0)$; since each $\tau \cap \tau_0$ is a face of τ_0, we see that there must exist $\tau \neq \tau_0$ such that τ_0 is a face of τ. The lemma follows immediately. □

Example 3.3.27. (i) Certain useful subdivisions of a polyhedral cone σ are produced by means of auxiliary real-valued functions defined on σ. Namely, let us say that a continuous function $f : \sigma \to \mathbb{R}$ is a *roof*, if the following holds. There exist finitely many $\mathbb{R}$-linear forms $l_1, \dots, l_n$ on V, such that $f(v) = \min(l_1(v), \dots, l_n(v))$ for every $v \in \sigma$. The concept of roof shall be reintroduced in section 3.6, in a more abstract and general guise; however, in order to grasp the latter, it is useful to keep in mind its more concrete polyhedral incarnation. We attach to f a subdivision of σ, as follows. For every $i, j = 1, \dots, n$ define $l_{ij} := l_i - l_j$, and let $\tau_i \subset V^\vee$ be the polyhedral cone $\sigma^\vee + \mathbb{R}l_{i1} + \cdots + \mathbb{R}l_{in}$. From the identity $l_{ik} = l_{ij} + l_{jk}$ we easily deduce that $\tau_i^\vee \cap \tau_j^\vee$ is a face of both $\tau_i^\vee$ and $\tau_j^\vee$, for every $i, j = 1, \dots, n$. Denote by Θ the smallest subdivision of σ containing all the $\tau_i^\vee$; it is easily seen that

$$\sigma = \bigcup_{i=1}^{n} \tau_i^\vee$$

and the restriction of f to each $\tau_i^\vee$ agrees with l_i.

(ii) Conversely, let $f : \sigma \to \mathbb{R}$ a continuous function; suppose there exists a subdivision Θ of σ, and a system $(l_\tau \mid \tau \in \Theta)$ of $\mathbb{R}$-linear forms on V such that

- $f(v) = l_\tau(v)$ for every $\tau \in \Theta$ and every $v \in \tau$.
- $f(u + v) \geq f(u) + f(v)$ for every $u, v \in \sigma$.

Then we claim that f is a roof on σ. Indeed, let $\Theta^s \subset \Theta$ be the subset of all τ that span $\langle\sigma\rangle$. Notice fist that the system $(l_\tau \mid \tau \in \Theta^s)$ already determines f uniquely, by virtue of lemma 3.3.26. Next, let $\tau, \tau' \in \Theta^s$ be any two elements, and pick an element v of the interior of τ. For any $u \in \tau'$ and any $\varepsilon > 0$ we have, by assumption : $f(v + \varepsilon u) \geq f(v) + f(\varepsilon u)$. If ε is small enough, we have as well $v + \varepsilon u \in \tau$, in which case the foregoing inequality can be written as :

$$l_\tau(v) + \varepsilon \cdot l_\tau(u) = l_\tau(v + \varepsilon u) \geq l_\tau(v) + \varepsilon \cdot l_{\tau'}(u)$$

whence $l_\tau(u) \geq l_{\tau'}(u) = f(u)$ and the assertion follows.

Proposition 3.3.28. *Let $f : V \to W$ be a linear map of finite dimensional $\mathbb{R}$-vector spaces, (V, σ) a convex polyhedral cone, and $h : \sigma \to f(\sigma)$ the restriction of f. Then :*

(i) *There exists a subdivision Δ of $(W, f(\sigma))$ such that :*

$$h^{-1}(a + b) = h^{-1}(a) + h^{-1}(b) \qquad \textit{for every } \tau \in \Delta \textit{ and every } a, b \in \tau.$$

(ii) *Suppose moreover that $V = L \otimes_{\mathbb{Z}} \mathbb{R}$, $W = L' \otimes_{\mathbb{Z}} \mathbb{R}$ and $f = g \otimes_{\mathbb{Z}} \mathbf{1}_{\mathbb{R}}$ for a map $g : L \to L'$ of free abelian groups. If σ is L-rational, then we may find an L-rational subdivision Δ such that* (i) *holds.*

Proof. Let V_0 be the largest linear subspace contained in $\sigma \cap \mathrm{Ker}\, f$. Notice that, under the assumptions of (ii), we have : $V_0 = \mathbb{R} \otimes_{\mathbb{Z}} \mathrm{Ker}\, g$. One verifies easily that the proposition holds for the given map f and the cone (V, σ), if and only if it holds for the induced map $\overline{f} : V/V_0 \to W/f(V_0)$ and the cone $(V/V_0, \overline{\sigma})$ (where $\overline{\sigma}$ is the image of σ in V/V_0). Hence, we may replace f by $\overline{f}$, and assume from start that $\sigma \cap \mathrm{Ker}\, f$ contains no non-zero linear subspaces. Moreover, we may assume that σ spans V and $f(\sigma)$ spans W.

(i): Let S be the set of faces τ of σ such that f restricts to an isomorphism $\langle\tau\rangle \xrightarrow{\sim} W$.

Claim 3.3.29. Let $\lambda \subset f(\sigma)$ be any ray. Then :

(i) $\lambda' := h^{-1}\lambda$ is a strongly convex polyhedral cone. Especially, λ' is generated by its extremal rays (see (3.3.15)).

(ii) For every extremal ray ρ of λ' with $\rho \not\subset \operatorname{Ker} f$, there exists $\tau \in S$ such that $\rho = \tau \cap f^{-1}(\lambda)$.

Proof of the claim. λ' is a convex polyhedral cone by lemma 3.3.18(i) and corollary 3.3.12(iii). To see that λ' is strongly convex, notice that any subspace $L \subset f^{-1}(\lambda)$ lies already in $\operatorname{Ker} f$, and if $L \subset \sigma$, we must have $L = \{0\}$ by assumption. Let ρ be an extremal ray of λ' which is not contained in $\operatorname{Ker} f$; notice that λ' is the intersection of the polyhedral cones $\lambda_1 := h^{-1}\langle\lambda\rangle$ and $\lambda_2 := f^{-1}\lambda$, hence we can find faces δ_i of λ_i ($i = 1, 2$) such that $\rho = \delta_1 \cap \delta_2$ (corollary 3.3.12(iii) and lemma 3.3.18(i)). However, the only proper face of λ_2 is $\operatorname{Ker} f$ (lemma 3.3.18(ii)), hence $\delta_2 = \lambda_2$. Likewise, $f^{-1}\langle\lambda\rangle$ has no proper faces, hence $\delta_1 = \gamma \cap f^{-1}\langle\lambda\rangle$ for some face γ of σ (again by corollary 3.3.12(iii)). Since λ_2 is a half-space in $f^{-1}\langle\lambda\rangle$, we deduce easily that either $\delta_1 = \rho$ or $\delta_1 = \langle\rho\rangle$. Especially, $\dim_{\mathbb{R}}(f^{-1}\langle\lambda\rangle)/\langle\delta_1\rangle = \dim_{\mathbb{R}} \operatorname{Ker} f$. We may then apply lemma 3.3.18(ii) to the imbedding $f^{-1}\langle\lambda\rangle \subset V$, to find a face τ of σ such that :

$$\tau \cap f^{-1}\langle\lambda\rangle = \delta_1 \qquad \langle\tau\rangle \cap f^{-1}\langle\lambda\rangle = \langle\rho\rangle \qquad \dim_{\mathbb{R}} V/\langle\tau\rangle = \dim_{\mathbb{R}} \operatorname{Ker} f.$$

It follows that $\langle\tau\rangle \cap \operatorname{Ker} f = \{0\}$, and therefore $\tau \in S$, as required. $\diamondsuit$

We construct as follows a subdivision of $(W, f(\sigma))$. For every $\tau \in S$, let $F(\tau)$ be the set consisting of the facets of the polyhedral cone $f(\tau)$; set also $F := \bigcup_{\tau\in S} F(\tau)$. Notice that, for every $\gamma \in F$, the subspace $\langle\gamma\rangle$ is a hyperplane of W; we let :

$$U := f(\sigma) \setminus \bigcup_{\gamma\in F} \langle\gamma\rangle.$$

Then U is an open subset of $f(\sigma)$, and the topological closure $\overline{C}$ of every connected component C of U is a convex polyhedral cone. Moreover, if C and D are any two such connected components, the intersection $\overline{C} \cap \overline{D}$ is a face of both $\overline{C}$ and $\overline{D}$. We let Δ be the subdivision of $f(\sigma)$ consisting of the cones $\overline{C}$ – where C ranges over all the connected components of U – together with all their faces.

Claim 3.3.30. For every $\delta \in \Delta$ and every $\tau \in S$, the intersection $\delta \cap f(\tau)$ is a face of δ.

Proof of the claim. Due to proposition 3.3.8(iii), we may assume that δ is the topological closure of a connected component C of U. We may also assume that $f(\tau) \neq W$, otherwise there is nothing to prove; in that case, we have $f(\tau) = \bigcap_{\gamma\in F(\tau)} H_\gamma$, where, for each $\gamma \in F(\tau)$, the half-space H_γ is the unique one that contains both $f(\tau)$ and γ (proposition 3.3.11(ii)). It then suffices to show that $\delta \cap H_\gamma$ is a face of δ for each such H_γ. We may assume that $\delta \not\subset H_\gamma$. Since C is connected and $C \subset W \setminus \langle\gamma\rangle$, it follows that $\delta \subset -H_\gamma$, the topological closure of the complement of H_γ. Hence $(-H_\gamma)^\vee \subset \delta^\vee$ (where $(-H_\gamma)^\vee$ is the dual of the polyhedral cone $(W, -H_\gamma)$), and therefore $\delta \cap H_\gamma = \delta \cap H_\gamma \cap (-H_\gamma) = \delta \cap \langle\gamma\rangle$ is indeed a face of δ. $\diamondsuit$

Next, for every $w \in f(\sigma)$, let $I(w) := \{\tau \in S \mid w \in f(\tau)\}$.

Claim 3.3.31. Let $\delta \in \Delta$, and $w_1, w_2 \in \delta$. Then $I(w_1 + w_2) \subset I(w_1) \cap I(w_2)$.

Proof of the claim. Suppose first that w_1+w_2 is contained in a face δ' of δ; say that $\delta' = \delta \cap \operatorname{Ker} u$, for some $u \in \delta^\vee$. This means that $u(w_1 + w_2) = 0$, hence $u(w_1) = u(w_2) = 0$, *i.e.* $w_1, w_2 \in \delta'$. Hence, we may replace δ by δ', and assume that δ is the smallest element of Δ containing $w_1 + w_2$. Thus, suppose that $\tau \in I(w_1 + w_2)$; therefore $w_1 + w_2 \in f(\tau) \cap \delta$. From claim 3.3.30 we deduce that $\delta \subset f(\tau)$, hence $\tau \in I(w_1) \cap I(w_2)$, as claimed. $\diamondsuit$

Finally, we are ready to prove assertion (i). Hence, let $a, b \in f(\sigma)$ be any two vectors that lie in the same element of Δ. Clearly :

$$h^{-1}(a) + h^{-1}(b) \subset h^{-1}(a + b)$$

hence it suffices to show the converse inclusion. However, directly from claim 3.3.29(ii) we derive the identity :

$$h^{-1}(\mathbb{R}_+ \cdot w) = (\sigma \cap \operatorname{Ker} f) + \sum_{\tau \in I(w)} (\tau \cap f^{-1}(\mathbb{R}_+ \cdot w)) \qquad \text{for every } w \in f(\sigma).$$

Taking into account claim 3.3.31, we are then reduced to showing that :

$$\tau \cap f^{-1}(\mathbb{R}_+ \cdot (a+b)) \subset (\tau \cap f^{-1}(\mathbb{R}_+ \cdot a)) + (\tau \cap f^{-1}(\mathbb{R}_+ \cdot b)) \qquad \text{for every } \tau \in I(a+b).$$

The latter assertion is obvious, since f restricts to an isomorphism $\langle\tau\rangle \xrightarrow{\sim} W$.

(ii): By inspecting the construction, one verifies easily that the subdivision Δ thus exhibited shall be L-rational, whenever σ is. $\square$

3.3.32. Later we shall also be interested in rational variants of the identities of proposition 3.3.28(i). Namely, consider the following situation. Let $g : L \to L'$ be a map of free abelian groups of finite rank, $g_\mathbb{R} : L_\mathbb{R} \to L'_\mathbb{R}$ the induced $\mathbb{R}$-linear map, and $(L_\mathbb{R}, \sigma)$ an L-rational convex polyhedral cone; set $\tau := g_\mathbb{R}(\sigma)$, and denote by $h_\mathbb{R} : \sigma \to \tau$ (resp. $h_\mathbb{Q} : \sigma \cap L_\mathbb{Q} \to \tau \cap L'_\mathbb{Q}$) the restriction of $g_\mathbb{R}$. We point out, for later reference, the following observation :

Lemma 3.3.33. *In the situation of* (3.3.32)*, suppose that :*

$$h_\mathbb{R}^{-1}(x_1) + h_\mathbb{R}^{-1}(x_2) = h_\mathbb{R}^{-1}(x_1 + x_2) \qquad \textit{for every } x_1, x_2 \in \tau$$

(where the sum is taken in the monoid $(\mathscr{P}(\sigma), +)$). Then we have as well :

$$h_\mathbb{Q}^{-1}(x_1) + h_\mathbb{Q}^{-1}(x_2) = h_\mathbb{Q}^{-1}(x_1 + x_2) \qquad \textit{for every } x_1, x_2 \in \tau \cap L'_\mathbb{Q}.$$

Proof. Let $x_1, x_2 \in \tau \cap L'_\mathbb{Q}$ be any two elements, and $v \in h_\mathbb{Q}^{-1}(x_1 + x_2)$, so we may write $v = v_1 + v_2$ for some $v_i \in h_\mathbb{R}^{-1}(x_i)$ ($i = 1, 2$). Let also $u_1, \dots, u_k$ be a finite system of generators for $\sigma^\vee$, and set

$$J_i := \{j \leq k \mid u_j(v_i) = 0\} \quad E_i := g_\mathbb{R}^{-1}(x_i) \cap \bigcap_{j \in J_i} \operatorname{Ker} u_j \qquad (i = 1, 2).$$

Clearly $L_\mathbb{Q} \cap E_i$ is a dense subset of E_i for $i = 1, 2$, hence, in any neighborhood of (x_1, x_2) in $L_\mathbb{R}^{\oplus 2}$ we may find a solution $(y_1, y_2) \in L_\mathbb{Q}^{\oplus 2}$ for the system of equations

$$g_\mathbb{R}(y_i) = x_i \qquad u_j(y_i) = 0 \qquad \text{for } i = 1, 2 \text{ and every } j \in J_i.$$

Since $u_j(x_i) > 0$ for every $j \notin J_i$, we will also have $u_j(y_i) > 0$ for every $j \notin J_i$, provided y_i is sufficiently close to x_i. The lemma follows. $\square$

3.3.34. We conclude this section with some considerations that shall be useful later, in our discussion of normalized lengths for model algebras (see (9.3.41)). Keep the notation of proposition 3.3.22, and for every subset $U \subset L_\mathbb{R}$, let

$$\mathscr{S}_{L,\sigma}(U) := \{L \cap (\sigma - v) \mid v \in U\} \qquad \text{and set} \qquad \mathscr{S}_{L,\sigma} := \mathscr{S}_{L,\sigma}(L_\mathbb{R}).$$

There is a natural L-module structure on $\mathscr{S}_{L,\sigma}$; namely, notice that

$$(L \cap (\sigma - v)) + l = L \cap (\sigma - (v - l)) \qquad \text{for every } v \in L_\mathbb{R} \text{ and } l \in L$$

hence the rule $\tau_l : S \mapsto S + l$ defines a bijection of $\mathscr{S}_{L,\sigma}$ onto itself, for every $l \in L$, and clearly $\tau_l \circ \tau_{l'} = \tau_{l+l'}$ for every $l, l' \in L$. Also, for every $S \in \mathscr{S}_{L,\sigma}$ define

$$\Omega(\sigma, S) := \{v \in L_\mathbb{R} \mid L \cap (\sigma - v) = S\}$$

and denote by $\overline{\Omega}(\sigma, S)$ the topological closure of $\Omega(\sigma, S)$ in $L_\mathbb{R}$. For given $u \in L_\mathbb{R}^\vee$ and $r \in \mathbb{R}$, set $H_{u,r} := \{v \in L_\mathbb{R} \mid u(v) \geq r\}$. We shall say that a subset of $L_\mathbb{R}$ is $\mathbb{Q}$-*linearly constructible*, if it lies in the boolean subalgebra of $\mathscr{P}(L_\mathbb{R})$ generated by the subsets $H_{u \otimes_\mathbb{Q} \mathbf{1}_\mathbb{R}, r}$, for u ranging over all the $\mathbb{Q}$-linear forms $L_\mathbb{Q} \to \mathbb{Q}$, and r ranging over all rational numbers.

Proposition 3.3.35. *With the notation of* (3.3.34), *the following holds :*

(i) $\mathscr{S}_{L,\sigma}(U)$ *is a finite set, for every bounded subset* $U \subset L_{\mathbb{R}}$.

(ii) $\mathscr{S}_{L,\sigma}$ *is a finitely generated* L*-module.*

(iii) *For every non-empty* $S \in \mathscr{S}_{L,\sigma}$*, the subset* $\Omega(\sigma, S)$ *is* $\mathbb{Q}$*-linearly constructible.*

(iv) *Suppose moreover, that* σ *spans* $L_{\mathbb{R}}$*. Then, for every* $S \in \mathscr{S}_{L,\sigma}$*, the subset* $\Omega(\sigma, S)$ *is contained in the topological closure of its interior (see* (3.3.10)*).*

(v) *For every* $S \in \mathscr{S}_{L,\sigma}$*, and every* $v \in \overline{\Omega}(\sigma, S)$*, we have* $S \subset L \cap (\sigma - v)$*.*

Proof. (i): Define C_ε as in the proof of proposition 3.3.22; since U is bounded, it is contained in the union of finitely many subsets of $L_{\mathbb{R}}$ of the form $C_1 + l$, for l ranging over a finite subset of L. On the other hand, τ_l induces a bijection

$$\mathscr{S}_{L,\sigma}(C_1) \xrightarrow{\sim} \mathscr{L}_{L,\sigma}(C_1 - l) \qquad \text{for every } l \in L.$$

Hence, it suffices to check the assertion for $U = C_1$. However, the proof of proposition 3.3.22(ii) shows that $L \cap (\sigma - v)$ is generated by $L \cap (C_1 - v)$; if $v \in C_1$, the latter subset is contained in $C' := C_1 \cup (-C_1)$, which is a compact subset of $L_{\mathbb{R}}$. Therefore $L \cap C'$ is a finite set, and the claim follows.

(ii): We have already observed that the L-module $\mathscr{S}_{L,\sigma}$ is generated by $\mathscr{S}_{L,\sigma}(C_1)$, and this is a finite set, by (i).

(iii): Fix a minimal system $S_1, \dots, S_n$ of generators of the L-module $\mathscr{S}_{L,\sigma}$ (*i.e.* the S_i are chosen representatives for the orbits of the L-action on $\mathscr{S}_{L,\sigma}$). After replacing S_i by some translates $S_i + l$ (for an appropriate $l \in L$) we may also assume that either $S_i = \varnothing$, or else $0 \in S_i$, and notice that this implies :

$$S_i \subset \langle \sigma \rangle \cap L = \sigma_L^{\mathrm{gp}} \qquad \text{for every } i = 1, \dots, n \tag{3.3.36}$$

(proposition 3.3.22(i)). Set

$$A_{ij} := \{l \in L \mid S_i \subset S_j - a\} \qquad \text{for every } i, j \leq n$$

and notice that A_{ij} is a σ_L-module, for every $i, j \leq n$.

Claim 3.3.37. If $S_i, S_j \neq \varnothing$, the σ_L-module A_{ij} is finitely generated.

Proof of the claim. Fix $l \in L$ such that $\sigma_L + l \subset S_i$. Next, say that $x_1, \dots, x_t$ is a finite system of generators for the σ_L-module S_j (proposition 3.3.22(ii)); by virtue of (3.3.36), for every $s = 1, \dots, t$, we may write $x_s = a_s - b_s$ for certain $a_s, b_s \in \sigma_L$. Set $l' := b_1 + \dots + b_t$, and notice that $S_j \subset \sigma_L - l'$. Now, if $S_i \subset S_j - a$, we deduce that $\sigma_L + l \subset \sigma_L - a - l'$, especially $l \in \sigma_L - a - l'$, *i.e.* $a \in \sigma_L - (l + l')$. This shows that A_{ij} is isomorphic to an ideal of σ_L, and then the claim follows from proposition 3.1.9(ii). $\Diamond$

Now, let $i, j \leq n$ such that $S_i, S_j \neq \varnothing$. Suppose first that $i \neq j$, and let $A'_{ij} \subset A_{ij}$ be any finite generating system for the σ_L-module A_{ij}. From the construction, it is clear that every element of LS_j that contains S_i, must contain $S_j - l$, for some $l \in A'_{ij}$. To deal with the case where $i = j$, we remark, more generally :

Claim 3.3.38. Let P be any fine and saturated monoid, $M \subset P^{\mathrm{gp}}$ a non-empty finitely generated P-submodule, and $a \in P^{\mathrm{gp}}$ an element such that $aM \subset M$. Then $a \in P$.

Proof of the claim. Pick any $m \in M$, and denote by $M' \subset M$ the submodule generated by $(a^k m \mid k \in \mathbb{N})$. According to proposition 3.1.9(i), there exists $N \geq 0$ such that M' is generated by the finite system $(a^k m \mid k = 0, \dots, N)$. Especially, $a^{N+1}m \in M'$, and therefore there exists $x \in P$ and $i \leq N$ such that $a^{N+1}m = a^i mx$ in M; it follows that $a^{N+1-i} \in P$, and finally $a \in P$, since P is saturated. $\Diamond$

From (3.3.36) we see that $A_{ii} \subset \sigma_L^{\mathrm{gp}}$, if $S_i \neq \varnothing$; combining with claim 3.3.38, we deduce that $A_{ii} = \sigma_L$. Moreover, notice as well that if $S_i = S_i - a$ for some $a \in \sigma_L^{\mathrm{gp}}$, then both a

and $-a \in A_{ii}$, so that $a \in \sigma_L^\times$. Thus, let A'_{ii} be any set of representatives of $\mathfrak{m}_\sigma \setminus \mathfrak{m}_\sigma^2$, where $\mathfrak{m}_\sigma$ denotes the maximal ideal of $\sigma_L^\sharp$. If $a \in L$, and $S_i - a$ contains strictly S_i, then a is a non-invertible element of σ_L, and taking into account corollary 3.1.10, we see that A'_{ii} is finite, and there exists $l \in A'_{ii}$ such that $S_i - l \subset S_i - a$. Next, for every $i \leq n$ such that $S_i \neq \varnothing$, set

$$\mathscr{S}^i := \bigcup_j \{S_j + l \mid l \in A'_{ij}\}$$

where $j \leq n$ runs over the indices such that $S_j \neq \varnothing$. Summing up, we conclude that $\mathscr{S}^i$ is a finite set for every $i \leq n$ with $S_i \neq \varnothing$, and if an element of $\mathscr{S}_{L,\sigma}$ contains strictly S_i, then it contains some element of $\mathscr{S}^i$. Lastly, in order to prove assertion (iii), we may assume that $S = S_i$ for some $i \leq n$, and notice that :

(3.3.39) $$\Omega(\sigma, S_i) = \{v \in L_\mathbb{R} \mid S \subset \sigma - v\} \setminus \bigcup_{S' \in \mathscr{S}^i} \{v \in L_\mathbb{R} \mid S' \subset \sigma - v\}.$$

Since S is finitely generated (proposition 3.3.22(ii)), we reduce to showing that, for every $a \in L$, the subset $\Omega(\sigma, a) := \{v \in L_\mathbb{R} \mid a \in \sigma - v\}$ is $\mathbb{Q}$-linearly constructible, which follows easily from proposition 3.3.21(i) and lemma 3.3.2.

(iv): We remark :

Claim 3.3.40. Let C_ε be as in the proof of proposition 3.3.22; For every $a \in \Omega(\sigma, S)$ there exists $\varepsilon > 0$ such that $a + C_\varepsilon \subset \Omega(\sigma, S)$.

Proof of the claim. Since σ is closed in $L_\mathbb{R}$, for every $a \in L_\mathbb{R}$ and every $b \in L_\mathbb{R} \setminus \Omega(\sigma, a)$ there exists $\varepsilon > 0$ such that $(b + C_\varepsilon) \cap \Omega(\sigma, a) = \varnothing$. Taking into account (3.3.39), the claim follows easily. ◊

If σ span $L_\mathbb{R}$, the subset C_ε has non-empty interior U_ε, for every $\varepsilon > 0$, and the topological closure of U_ε equals C_ε. The assertion is then an immediate consequence of claim 3.3.40.

(v): The assertion follows easily from proposition 3.3.22(ii) : the details shall be left to the reader. □

3.3.41. Let L be as in (3.3.34), and for all integers $n, m > 0$ set

$$\frac{1}{m}L := \{v \in L_\mathbb{Q} \mid mv \in L\} \qquad \frac{1}{m}L[1/n] := \bigcup_{k \geq 0} \frac{1}{n^k m}L.$$

For future reference, let us also point out :

Lemma 3.3.42. *With the notation of* (3.3.41), *let* $\Omega \subset L_\mathbb{R}$ *be a* $\mathbb{Q}$*-linearly constructible subset. Then we have :*

(i) *The topological closure of* Ω *in* $L_\mathbb{R}$ *is again* $\mathbb{Q}$*-linearly constructible.*
(ii) *There exists an integer* $m > 0$ *such that* $\frac{1}{m}L[1/n] \cap \Omega$ *is dense in* Ω*, for every* $n > 1$.

Proof. (i): Ω is a finite union of non-empty subsets of the form $H_1 \cap \cdots \cap H_k$, where each H_i is either of the form $H_{u \otimes \mathbf{1}_\mathbb{R}, r}$ for some non-zero $\mathbb{Q}$-linear form u of $L_\mathbb{Q}$ and some $r \in \mathbb{Q}$ (and this is a closed subset of $L_\mathbb{R}$), or else is the complement in $L_\mathbb{R}$ of a subset of this type (and then its closure is a half-space $H_{-u \otimes \mathbf{1}_\mathbb{R}, r}$). One verifies that the closure of $H_1 \cap \cdots \cap H_k$ is the intersection of the closures of $H_1, \dots, H_k$, whence the assertion.

(ii): We may assume that $\Omega = \Omega_1 \cap \Omega_2$, where Ω_1 is a finite intersection of rational hyplerplanes, and Ω_2 is a finite intersection of open half-spaces (*i.e.* of complements of closed half-spaces). Suppose that $\frac{1}{m}L[1/n] \cap \Omega_1$ is dense in Ω_1; then clearly $\frac{1}{m}L[1/n] \cap \Omega$ is dense in Ω. Hence, we may further assume that Ω is a non-empty intersection of rational hyperplanes. In this case, Ω is of the form $V_\mathbb{R} + v_0$, where $v_0 \in L_\mathbb{Q}$, and $V_\mathbb{R} = V \otimes_\mathbb{Z} \mathbb{R}$ for some subgroup $V \subset L$. Notice that $L[1/n] \cap V_\mathbb{R}$ is dense in $V_\mathbb{R}$ for every integer $n > 1$. Then, any integer $m > 0$ such that $v_0 \in \frac{1}{m}L$ will do. □

3.4. **Fine and saturated monoids.** This section presents the more refined theory of fine and saturated monoids. Again, all the monoids in this section are non-pointed. We begin with a few corollaries of proposition 3.3.22(i,iii).

Corollary 3.4.1. *Let M be an integral monoid, such that $M^\sharp$ is fine. We have :*

(i) *The inclusion map $M \to M^{\mathrm{sat}}$ is a finite morphism of monoids.*
(ii) *Especially, if M is fine, any monoid N with $M \subset N \subset M^{\mathrm{sat}}$, is fine.*

Proof. (i): From lemma 3.2.9(ii) we deduce that M^{sat} is a finitely generated M-module if and only if $(M^\sharp)^{\mathrm{sat}}$ is a finitely generated $M^\sharp$-module. Hence, we may replace M by $M^\sharp$, and assume that M is fine. Pick a surjective group homomorphism $\varphi : \mathbb{Z}^{\oplus n} \to M^{\mathrm{gp}}$; it is easily seen that :

$$\varphi^{-1}(M^{\mathrm{sat}}) = (\varphi^{-1}M)^{\mathrm{sat}}$$

and clearly it suffices to show that $\varphi^{-1}N$ is finitely generated, hence we may replace M by $\varphi^{-1}M$, and assume throughout that M^{gp} is a free abelian group of finite rank. In this case, proposition 3.3.22(i,iv) already implies that M^{sat} is finitely generated. Let $a_1, \dots, a_k \in M^{\mathrm{sat}}$ be a finite system of generators, and pick integers $n_1, \dots, n_k > 0$ such that $a_i^{n_i} \in M$ for $i = 1, \dots, k$. For every $i = 1, \dots, k$ let $\Sigma_i := \{a_i^j \mid j = 0, \dots, n_i - 1\}$; it is easily seen $\Sigma_1 \cdots \Sigma_k \subset M^{\mathrm{sat}}$ is a system of generators for the M-module M^{sat} (where the product of the sets Σ_i is formed in the monoid $\mathscr{P}(M^{\mathrm{sat}})$ of (3.1.1)).

(ii) follows from (i), in view of proposition 3.1.9(i). □

Corollary 3.4.2. *Let $f : M_1 \to M$ and $g : M_2 \to M$ be two morphisms of monoids, such that M_1 and M_2 are finitely generated, and M is integral. Then the fibre product $M_1 \times_M M_2$ is a finitely generated monoid, and if M_1 and M_2 are fine, the same holds for $M_1 \times_M M_2$.*

Proof. If the monoids M, M_1 and M_2 are integral, $M_1 \times_M M_2$ injects in $M_1^{\mathrm{gp}} \times_{M^{\mathrm{gp}}} M_2^{\mathrm{gp}}$ (lemma 2.3.29(iii)), hence it is integral. To show that the fibre product is finitely generated, choose surjective morphisms $\mathbb{N}^{\oplus a} \to M_1$ and $\mathbb{N}^{\oplus b} \to M_2$, for some $a, b \in \mathbb{N}$; by composition we get maps of monoids $\varphi : \mathbb{N}^{\oplus a} \to M$, $\psi : \mathbb{N}^{\oplus b} \to M$, such that the induced morphism $P := \mathbb{N}^{\oplus a} \times_M \mathbb{N}^{\oplus b} \to M_1 \times_M M_2$ is surjective. Hence it suffices to show that P is finitely generated. To this aim, let $L := \mathrm{Ker}(\varphi^{\mathrm{gp}} - \psi^{\mathrm{gp}} : \mathbb{Z}^{\oplus a+b} \to M^{\mathrm{gp}})$; for every $i = 1, \dots, a + b$, denote also by $\pi_i : \mathbb{Z}^{\oplus a+b} \to \mathbb{Z}$ the projection onto the i-th direct summand. The system $\{\pi_i \mid i = 1, \dots, a+b\}$ generates a rational convex polyhedral cone $\sigma \subset L^\vee \otimes_{\mathbb{Z}} \mathbb{R}$, and one verifies easily that $P = L \cap \sigma^\vee$, so the assertion follows from propositions 3.3.21(i) and 3.3.22(i). □

Corollary 3.4.3. *Let $(\Gamma, +, 0)$ be an integral monoid, M a finitely generated Γ-graded monoid. Then M_0 is a finitely generated monoid, and M_γ is a finitely generated M_0-module, for every $\gamma \in \Gamma$.*

Proof. We have $M_0 = M \times_\Gamma \{0\}$, hence M_0 is finitely generated, by corollary 3.4.2. The given element $\gamma \in \Gamma$ determines a unique morphism of monoids $\mathbb{N} \to \Gamma$ such that $1 \mapsto \gamma$. Let $p_1 : M' := M \times_\Gamma \mathbb{N} \to \mathbb{N}$ and $p_2 : M' \to M$ be the two natural projections; by lemma 2.3.29(iii), we have $M_\gamma = p_2(p_1^{-1}(1))$. In light of corollary 3.4.2, M' is still finitely generated, hence we are reduced to the case where $\Gamma = \mathbb{N}$ and $\gamma = 1$. In this case, pick a finite set of generators S for M. One checks easily that $M_1 \cap S$ generates the M_0-module M_1. □

Corollary 3.4.4. *Let M be an integral monoid, such that $M^\sharp$ is fine, and $\varphi : M \to N$ a saturated morphism of monoids. Then φ is flat.*

Proof. In view of corollary 3.4.1(i) and theorem 3.2.6, it suffices to show that the M^{sat}-module $M^{\mathrm{sat}} \otimes_M N$ is flat. Hence we may replace M by M^{sat}, and assume that M is saturated. Let

$I \subset M$ be any ideal, and define $\mathsf{R}(M, I)$ as in the proof of theorem 3.2.3; by assumption, $\mathsf{R}(M, I)^{\mathrm{sat}} \otimes_M N$ is a saturated – especially, integral – monoid, *i.e.* the natural map

$$\mathsf{R}(M, I) \otimes_M N \to \mathsf{R}(M, I)^{\mathrm{gp}} \otimes_{M^{\mathrm{gp}}} N^{\mathrm{gp}}$$

is injective. The latter factors through the morphism $j \otimes_M N$, where $j : \mathsf{R}(M, I) \to M \times \mathbb{N}$ is the obvious inclusion. In light of example 3.2.13(i), we deduce that the induced map $I^{\mathrm{sat}} \otimes_M N \to N$ is injective. Now, if I is a prime ideal, then $I^{\mathrm{sat}} = I$, hence the contention follows from corollary 3.1.48(ii). □

The following corollary generalizes lemma 3.2.10.

Corollary 3.4.5. *Let $f : M \to N$ be a local, flat and saturated morphism of fine monoids, with M sharp. Then there exists an isomorphism of monoids*

$$g : N \xrightarrow{\sim} N^\sharp \times N^\times$$

that fits into a commutative diagram

$$\begin{tikzcd} M \arrow[r, "f^\sharp"] \arrow[d, "f"'] & N^\sharp \arrow[d] \\ N \arrow[r, "g"] & N^\sharp \times N^\times \end{tikzcd}$$

whose right vertical arrow is the natural inclusion map.

Proof. From lemma 3.2.30(ii), we know that f is exact, and since M is sharp, we easily deduce that $f(M)^{\mathrm{gp}} \cap N^\times = \{1\}$. Hence, the induced group homomorphism $M^{\mathrm{gp}} \oplus N^\times \to N^{\mathrm{gp}}$ is injective. On the other hand, since f is flat, local and saturated, the same holds for $f^\sharp : M \to N^\sharp$ (lemma 3.2.12(iii) and corollary 3.4.4); then corollary 3.2.32(ii) says that the cokernel of the induced group homomorphism $M^{\mathrm{gp}} \to (N^\sharp)^{\mathrm{gp}} = N^{\mathrm{gp}}/N^\times$ is a free abelian group G (of finite rank). Summing up, we obtain an isomorphism of abelian groups :

$$h : M^{\mathrm{gp}} \oplus N^\times \oplus G \xrightarrow{\sim} N^{\mathrm{gp}}$$

extending the map f^{gp}. Set $N_0 := N \cap h(M^{\mathrm{gp}} \oplus G)$; it follows easily that the natural map $N_0 \times N^\times \to N$ is an isomorphism; especially, the projection $N \to N^\sharp$ maps N_0 isomorphically onto $N^\sharp$, and the contention follows. □

3.4.6. Let $(M, \cdot)$ be a fine (non-pointed) monoid, so that M^{gp} is a finitely generated abelian group. We set $M^{\mathrm{gp}}_{\mathbb{R}} := \log M^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{R}$, and we let $M_{\mathbb{R}}$ be the convex polyhedral cone generated by the image of $\log M$. Then $(M_{\mathbb{R}}, +)$ is a monoid, and we have a natural morphism of monoids

$$\varphi : \log M \to (M_{\mathbb{R}}, +).$$

Proposition 3.4.7. *With the notation of* (3.4.6)*, we have :*

(i) *Every face of the polyhedral cone $M_{\mathbb{R}}$ is of the form $F_{\mathbb{R}}$, for a unique face F of M.*

(ii) *The induced map :*

$$\varphi^* : \mathrm{Spec}\, M_{\mathbb{R}} \to \mathrm{Spec}\, M$$

is a bijection.

Proof. Clearly, we may assume that $M \neq \{1\}$. Let $\mathfrak{p} \subset M$ be a prime ideal; we denote by $\mathfrak{p}_{\mathbb{R}}$ the ideal of $(M_{\mathbb{R}}, +)$ generated by all elements of the form $r \cdot \varphi(x)$, where r is any strictly positive real number, and x is any element of $\mathfrak{p}$. We also denote by $(M \backslash \mathfrak{p})_{\mathbb{R}}$ the convex cone of $M^{\mathrm{gp}}_{\mathbb{R}}$ generated by the image of $M \backslash \mathfrak{p}$.

Claim 3.4.8. $M_{\mathbb{R}}$ is the disjoint union of $(M \backslash \mathfrak{p})_{\mathbb{R}}$ and $\mathfrak{p}_{\mathbb{R}}$.

Proof of the claim. To begin with, we show that $M_{\mathbb{R}} = (M\backslash\mathfrak{p})_{\mathbb{R}} \cup \mathfrak{p}_{\mathbb{R}}$. Indeed, let $x \in M_{\mathbb{R}}$; then we may write $x = \sum_{i=1}^{h} m_i \otimes a_i$ for certain $a_1, \dots, a_h \in \mathbb{R}_+$ and $m_1, \dots, m_h \in M$. We may assume that $a_1, \dots, a_k \in \mathfrak{p}$ and $a_{k+1}, \dots, a_h \in M\backslash\mathfrak{p}$. Now, if $k = 0$ we have $x \in (M\backslash\mathfrak{p})_{\mathbb{R}}$, and otherwise $x \in \mathfrak{p}_{\mathbb{R}}$, which shows the assertion.

It remains to show that $(M\backslash\mathfrak{p})_{\mathbb{R}} \cap \mathfrak{p}_{\mathbb{R}} = \varnothing$. To this aim, suppose by way of contradiction, that this intersection contains an element x; this means that we have finite subsets $S_0 \subset M\backslash\mathfrak{p}$ and $S_1 \subset M$ such that $S_1 \cap \mathfrak{p} \neq \varnothing$, and an identity of the form :

$$x = \sum_{\sigma \in S_0} \sigma \otimes a_\sigma = \sum_{\sigma \in S_1} \sigma \otimes b_\sigma \tag{3.4.9}$$

where $a_\sigma > 0$ for every $\sigma \in S_0$ and $b_\sigma > 0$ for every $\sigma \in S_1$. For every $\sigma \in S_0$, choose a rational number $a'_\sigma \geq a_\sigma$; after adding the summand $\sum_{\sigma \in S_0} \sigma \otimes (a'_\sigma - a_\sigma)$ to both sides of (3.4.9), we may assume that $a_\sigma \in \mathbb{Q}_+$ for every $\sigma \in S_0$. Let $N \subset M$ be the submonoid generated by S_1; it follows that $x \in N_{\mathbb{R}} \cap M_{\mathbb{Q}} = N_{\mathbb{Q}}$ (proposition 3.3.22(iii)), hence we may assume that all the coefficients a_σ and b_σ are rational and strictly positive (see remark 3.3.24). We may further multiply both sides of (3.4.9) by a large integer, to obtain that these coefficients are actually integers. Then, up to further multiplication by some integer, the identity of (3.4.9) lifts to an identity between elements of $\log M$, of the form : $\sum_{\sigma \in S_0} a_\sigma \cdot \sigma = \sum_{\sigma \in S_1} b_\sigma \cdot \sigma$. The latter is absurd, since $S_1 \cap \mathfrak{p} \neq \varnothing$ and $S_0 \cap \mathfrak{p} = \varnothing$. ◊

Claim 3.4.8 implies that $\mathfrak{p}_{\mathbb{R}}$ is a prime ideal of $M_{\mathbb{R}}$, and clearly $\mathfrak{p} \subset \varphi^*(\mathfrak{p}_{\mathbb{R}})$. Since we have as well $M\backslash\mathfrak{p} \subset \varphi^{-1}(M\backslash\mathfrak{p})_{\mathbb{R}}$, we deduce that $\mathfrak{p} = \varphi^*(\mathfrak{p}_{\mathbb{R}})$. Hence the rule $\mathfrak{p} \mapsto \mathfrak{p}_{\mathbb{R}}$ yields a right inverse $\varphi_* : \operatorname{Spec} M \to \operatorname{Spec} M_{\mathbb{R}}$ for the natural map φ^*. To show that φ_* is also a left inverse, let $\mathfrak{q} \subset M_{\mathbb{R}}$ be a prime ideal; by lemma 3.3.7 and proposition 3.3.21(i), the face $M_{\mathbb{R}}\backslash\mathfrak{q}$ is of the form $M_{\mathbb{R}} \cap \operatorname{Ker} u$, for some $u \in M_{\mathbb{R}}^\vee \cap (\log M^{\mathrm{gp}})^\vee$. Then it is easily seen that $M_{\mathbb{R}}\backslash\mathfrak{q}$ is the convex cone generated by $\varphi(M) \cap \operatorname{Ker} u$, in other words, $M_{\mathbb{R}} \setminus \mathfrak{q} = \varphi^{-1}(\operatorname{Ker} u)_{\mathbb{R}}$. Again by claim 3.4.8, it follows that $\mathfrak{q} = (M\backslash\varphi^{-1}(\operatorname{Ker} u))_{\mathbb{R}} = (\varphi^*\mathfrak{q})_{\mathbb{R}}$, as stated. The argument also shows that every face of $M_{\mathbb{R}}$ is of the from $(M\backslash\mathfrak{p})_{\mathbb{R}}$ for a unique prime ideal $\mathfrak{p}$, which settles assertion (i). □

Corollary 3.4.10. *Let M be a fine monoid. We have :*

(i) $\dim M = \operatorname{rk}_{\mathbb{Z}}(M^{\mathrm{gp}}/M^\times)$.

(ii) $\dim(M\backslash\mathfrak{p}) + \operatorname{ht}\mathfrak{p} = \dim M$ *for every* $\mathfrak{p} \in \operatorname{Spec} M$.

(iii) *If $M \neq \{1\}$ is sharp (see* (2.3.32)*), there exists a local morphism $M \to \mathbb{N}$.*

(iv) *If M is sharp and M^{gp} is a torsion-free abelian group of rank r, there exists an injective morphism of monoids $M \to \mathbb{N}^{\oplus r}$.*

Proof. (i): By proposition 3.4.7, the dimension of M can be computed as the length of the longest chain $F_0 \subset F_1 \subset \cdots \subset F_d$ of strict inclusions of faces of $M_{\mathbb{R}}$. On the other hand, given such a maximal chain, denote by r_i the dimension of the $\mathbb{R}$-vector space spanned by F_i; in view proposition 3.3.8(ii),(iii), it is easily seen that $r_{i+1} - r_i = 1$ for every $i = 0, \dots, d-1$. Since $M_{\mathbb{R}} \cap (-M_{\mathbb{R}})$ is the minimal face of $M_{\mathbb{R}}$, we deduce that

$$\dim M = \dim_{\mathbb{R}} M_{\mathbb{R}}^{\mathrm{gp}} - \dim_{\mathbb{R}} M_{\mathbb{R}} \cap (-M_{\mathbb{R}}).$$

Clearly $\dim_{\mathbb{R}} M_{\mathbb{R}}^{\mathrm{gp}} = \operatorname{rk}_{\mathbb{Z}} M^{\mathrm{gp}}$; moreover, by proposition 3.4.7, the face $M_{\mathbb{R}} \cap (-M_{\mathbb{R}})$ is spanned by the image of the face $M^\times$ of M. whence the assertion.

(ii) is similar : again proposition 3.3.8(ii),(iii) implies that, every face F of $M_{\mathbb{R}}$ fits into a maximal strictly ascending chain of faces of $M_{\mathbb{R}}$, and the length of any such maximal chain is $\dim M$, by (i).

(iii): Notice that $\operatorname{rk}_{\mathbb{Z}} M^{\mathrm{gp}} > 0$, by (i). By proposition 3.4.7(i), $M_{\mathbb{R}}$ is strongly convex, therefore, by proposition 3.3.21(i), we may find a non-zero linear map $\varphi : M^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{Q}$, such that $M_{\mathbb{R}} \cap \operatorname{Ker} \varphi \otimes_{\mathbb{Q}} \mathbb{R} = \{0\}$ and $\varphi(M) \subset \mathbb{Q}_+$. A suitable positive integer of φ will do.

(iv): Under the stated assumption, we may regard M as a submonoid of $M_\mathbb{R}$, and the latter contains no non-zero linear subspaces. By corollary 3.3.14 and proposition 3.3.21(i), we may then find r linearly independent forms $u_1, \dots, u_r : M^{\mathrm{gp}} \otimes_\mathbb{Z} \mathbb{Q} \to \mathbb{Q}$ which are positive on M. It follows that $u_1 \otimes_\mathbb{Q} \mathbb{R}, \dots, u_r \otimes_\mathbb{Q} \mathbb{R}$ generate a polyhedral cone $\sigma^\vee \subset M_\mathbb{R}^\vee$, so its dual cone $\sigma \subset M_\mathbb{R}^{\mathrm{gp}}$ contains $M_\mathbb{R}$. By construction, σ admits precisely r extremal rays, say the rays generated by the vectors $v_1, \dots, v_r$, which we can pick in $M_\mathbb{Q}^{\mathrm{gp}}$, in which case they form a basis of the latter $\mathbb{Q}$-vector space. Now, every $x \in M_\mathbb{R}$ can be written uniquely in the form $x = \sum_{i=1}^r a_i v_i$ for certain $a_1, \dots, a_r \in \mathbb{Q}_+$; since M is finitely generated, we may find an integer $N > 0$ independent of x, such that $Na_i \in \mathbb{N}$ for every $i = 1, \dots, r$. In other words, M is contained in the monoid generated by $N^{-1}v_1, \dots, N^{-1}v_r$; the latter is isomorphic to $\mathbb{N}^{\oplus r}$. □

3.4.11. For any monoid M, the *dual* of M is the monoid $M^\vee := \mathrm{Hom}_{\mathbf{Mnd}}(M, \mathbb{N})$ (see (3.1.1)). As usual, there is a natural morphism

$$M \to M^{\vee\vee} \quad : \quad m \mapsto (\varphi \mapsto \varphi(m)) \quad \text{for every } m \in M \text{ and } \varphi \in M^\vee.$$

We say that M is *reflexive*, if this morphism is an isomorphism.

Proposition 3.4.12. *Let M be a monoid. We have :*

(i) *$M^\vee$ is integral, saturated and sharp.*

(ii) *If M is finitely generated, $M^\vee$ is fine, and we have a natural identification :*

$$(M^\vee)_\mathbb{R} \xrightarrow{\sim} (M_\mathbb{R})^\vee.$$

Moreover, $\dim M = \dim M^\vee$.

(iii) *If M is finitely generated and sharp, we have a natural identification :*

$$(M^\vee)^{\mathrm{gp}} \xrightarrow{\sim} (M^{\mathrm{gp}})^\vee.$$

(iv) *If M is fine, sharp and saturated, then M is reflexive.*

Proof. (i): It is easily seen that the natural group homomorphism

$$(M^\vee)^{\mathrm{gp}} \to (M^{\mathrm{gp}})^\vee := \mathrm{Hom}_\mathbb{Z}(M^{\mathrm{gp}}, \mathbb{Z}) \tag{3.4.13}$$

is injective. Now, say that $\varphi \in (M^\vee)^{\mathrm{gp}}$ and $N\varphi \in M^\vee$ for some $N \in \mathbb{N}$; we may view φ as group homomorphism $\varphi : M^{\mathrm{gp}} \to \mathbb{Z}$, and the assumption implies that $\varphi(M) \subset \mathbb{Z} \cap \mathbb{Q}_+ = \mathbb{N}$, whence the contention.

(ii): Indeed, let $x_1, \dots, x_n$ be a system of generators of M. Define a group homomorphism $f : (M^{\mathrm{gp}})^\vee \to \mathbb{Z}^{\oplus n}$ by the rule : $\varphi \mapsto (\varphi(x_1), \dots, \varphi(x_n))$ for every $f : M^{\mathrm{gp}} \to \mathbb{Z}$. Then $M^\vee = \varphi^{-1}(\mathbb{N}^{\oplus n})$, and since $(M^{\mathrm{gp}})^\vee$ is fine, corollary 3.4.2 implies that $M^\vee$ is fine as well. Next, the injectivity of (3.4.13) implies especially that $(M^\vee)^{\mathrm{gp}}$ is torsion-free, hence (3.4.13) $\otimes_\mathbb{Z} \mathbb{R}$ is still injective; its restriction to $(M^\vee)_\mathbb{R}$ factors therefore through an injective map $f : (M^\vee)_\mathbb{R} \to (M_\mathbb{R})^\vee$. The latter map is determined by the image of $M^\vee$, and by inspecting the definitions, we see that $f(\varphi) := \varphi^{\mathrm{gp}} \otimes 1$ for every $\varphi \in M^\vee$. To prove that f is an isomorphism, it suffices to show that it has dense image. However, say that $\varphi \in (M_\mathbb{R})^\vee$; then $\varphi : M^{\mathrm{gp}} \to \mathbb{R}$ is a group homomorphism such that $\varphi(M) \subset \mathbb{R}_+$. Since M is finitely generated, in any neighborhood of φ in $M_\mathbb{R}^{\mathrm{gp}}$ we may find some $\varphi' : M^{\mathrm{gp}} \to \mathbb{Q}_+$, and then $N\varphi' \in M^\vee$ for some integer $N \in \mathbb{N}$ large enough. It follows that φ' is in the image of f, whence the contention.

The stated equality follows from the chain of identities :

$$\dim M = \dim M_\mathbb{R} = \dim(M_\mathbb{R})^\vee = \dim(M^\vee)_\mathbb{R} = \dim M^\vee$$

where the first and the last follow from proposition 3.4.7(ii), and the second follows from corollary 3.3.12(ii).

(iii): Let us show first that, under these assumptions, (3.4.13) $\otimes_\mathbb{Z} \mathbb{R}$ is an isomorphism. Indeed, if M is sharp, $(M_\mathbb{R})^\vee$ spans $(M_\mathbb{R}^{\mathrm{gp}})^\vee$ (corollary 3.3.14 and proposition 3.4.7(i)); then the assertion follows from (ii). We deduce that $(M^\vee)^{\mathrm{gp}}$ and $(M^{\mathrm{gp}})^\vee$ are free abelian groups of the same rank, hence we may find a basis $\varphi_1, \dots, \varphi_r$ of $(M^\vee)^{\mathrm{gp}}$ (resp. $\psi_1, \dots, \psi_r$ of $(M^{\mathrm{gp}})^\vee$), and positive integers $N_1, \dots, N_r$ such that (3.4.13) is given by the rule : $\varphi_i \mapsto N_i \psi_i$ for every $i = 1, \dots, r$. But then necessarily we have $N_i = 1$ for every $i \leq r$, and (iii) follows.

(iv): It is easily seen that $M^\vee = (M_{\mathbb{R}})^\vee \cap (M^{\mathrm{gp}})^\vee$ (notation of (3.4.6)). After dualizing again we find : $M^{\vee\vee} = ((M^\vee)_{\mathbb{R}})^\vee \cap (M^{\vee\mathrm{gp}})^\vee$. From (ii) we deduce that $((M^\vee)_{\mathbb{R}})^\vee = (M_{\mathbb{R}})^{\vee\vee} = M_{\mathbb{R}}$ (lemma 3.3.2), and from (iii) we get : $(M^{\vee\mathrm{gp}})^\vee = (M^{\mathrm{gp}})^{\vee\vee} = M^{\mathrm{gp}}$. Hence $M^{\vee\vee} = M_{\mathbb{R}} \cap M^{\mathrm{gp}} = M$ (proposition 3.3.22(iii)). $\square$

Remark 3.4.14. (i) Let M be a sharp and fine monoid. Proposition 3.4.12(iii) implies that the natural map

$$\mathrm{Hom}_{\mathbf{Mnd}}(M, \mathbb{Q}_+)^{\mathrm{gp}} \to \mathrm{Hom}_{\mathbf{Mnd}}(M, \mathbb{Q})$$

is an isomorphism. Indeed, it is easily seen that this map is injective. For the surjectivity, one uses the identification $\mathrm{Hom}_{\mathbf{Mnd}}(M, \mathbb{Q}) \xrightarrow{\sim} (M^\vee)^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$, which follows from *loc.cit.* (Details left to the reader.)

(ii) For $i = 1, 2$, let $N_i \to N$ be two morphisms of monoids. By general nonsense, we have a natural isomorphism :

$$(N_1 \otimes_N N_2)^\vee \xrightarrow{\sim} N_1^\vee \times_{N^\vee} N_2^\vee.$$

(iii) If $f_i : M_i \to M$ ($i = 1, 2$) are morphisms of fine, saturated and sharp monoids, there exists a natural surjection :

$$M_1^\vee \otimes_{M^\vee} M_2^\vee \to (M_1 \times_M M_2)^\vee \tag{3.4.15}$$

whose kernel is the subgroup of invertible elements. Indeed, set $P := M_1^\vee \otimes_{M^\vee} M_2^\vee$; in view of (ii) and proposition 3.4.12(iv), we have a natural identification $P^{\vee\vee} \xrightarrow{\sim} (M_1 \times_M M_2)^\vee$, and the sought map is its composition with the double duality map $P \to P^{\vee\vee}$. Moreover, clearly P is finitely generated, and it is also integral and saturated, since saturation commutes with colimits. Hence – again by proposition 3.4.12(iv) – the double duality map induces an isomorphism $P/P^\times \xrightarrow{\sim} P^{\vee\vee}$.

(iv) In the situation of (iii), if $f_i : M_i \to M$ ($i = 1, 2$) are epimorphisms, then (3.4.15) is an isomorphism. Indeed, in this case the dual morphisms $f_i^\vee : M^\vee \to M_i^\vee$ are injective, so that P is sharp (lemma 3.1.12), whence the claim.

Theorem 3.4.16. *Let M be a saturated monoid, such that $M^\sharp$ is fine. We have :*

(i) $M = \bigcap_{\mathrm{ht}\,\mathfrak{p}=1} M_{\mathfrak{p}}$ *(where the intersection runs over the prime ideals of M of height one).*

(ii) *If moreover,* $\dim M = 1$*, then there is an isomorphism of monoids :*

$$M^\times \times \mathbb{N} \xrightarrow{\sim} M.$$

(iii) *Suppose that M^{gp} is a torsion-free abelian group, and let R be any normal domain. Then the group algebra $R[M]$ is a normal domain as well.*

Proof. (i): Pick a decomposition $M = M^\sharp \times M^\times$ as in lemma 3.2.10, and notice that $M^\sharp$ is fine, sharp and saturated. The prime ideals of M are of the form $\mathfrak{p} = \mathfrak{p}_0 \times M^\times$, where $\mathfrak{p}_0$ is a prime ideal of $M^\sharp$. Then it is easily seen that $M_{\mathfrak{p}} = M^\sharp_{\mathfrak{p}_0} \times M^\times$. Therefore, the sought assertion holds for M if and only if it holds for $M^\sharp$, and therefore we may replace M by $M^\sharp$, which reduces to the case where M is sharp, hence the natural morphism $\varphi : \log M \to M_{\mathbb{R}}$ is injective. In such situation, we have $M = M_{\mathbb{R}} \cap M^{\mathrm{gp}}$ and $M_{\mathfrak{p}} = M_{\mathfrak{p},\mathbb{R}} \cap M^{\mathrm{gp}}$ for every prime ideal $\mathfrak{p} \subset M$ (proposition 3.3.22(iii) and lemma 3.2.9(i)). Thus, we are reduced to showing that

$$M_{\mathbb{R}} = \bigcap_{\mathrm{ht}\,\mathfrak{p}=1} M_{\mathfrak{p},\mathbb{R}}.$$

However, set $\tau := (M\setminus\mathfrak p)_{\mathbb R}$; by inspecting the definitions, one sees that $M_{\mathfrak p,\mathbb R} = M_{\mathbb R} + (-\tau)$, and proposition 3.4.7 shows that τ is a facet of $M_{\mathbb R}$, hence $M_{\mathfrak p,\mathbb R}$ is the half-space denoted H_τ in (3.3.10). Then the assertion is a rephrasing of proposition 3.3.11(ii).

(ii): Arguing as in the proof of (i), we may reduce again to the case where M is sharp, in which case $M = M_{\mathbb R} \cap M^{\mathrm{gp}}$. The foregoing shows that, in case $\dim M = 1$, the cone $M_{\mathbb R}$ is a half-space, whose boundary hyperplane is the only non-trivial face σ of $M_{\mathbb R}$. However, σ is generated by the image of the unique non-trivial face of M, *i.e.* by $M^\times = \{1\}$ (proposition 3.4.7(i)), hence $\sigma = \{0\}$, so $M_{\mathbb R}$ is a half-line. Now, let $u : M^{\mathrm{gp}}_{\mathbb R} \to \mathbb R$ be a non-zero linear form, such that $u(M) \geq 0$, and $x_1,\dots,x_n$ a system of non-zero generators for M; say that $u(x_1)$ is the least of the values $u(x_i)$, for $i = 1,\dots,n$. Since M is saturated, it follows easily that every value $u(x_i)$ is an integer multiple of $u(x_1)$ (proposition 3.3.22(iii)), and then x_1 is a generator for M, so $M \simeq \mathbb N$.

(iii): To begin with, $R[M] \subset R[M^{\mathrm{gp}}]$, and since M^{gp} is torsion-free, it is clear that $[M^{\mathrm{gp}}]$ is a domain, hence the same holds for $R[M]$. Furthermore, from (i) we derive : $R[M] = \bigcap_{\mathrm{ht}\mathfrak p=1} R[M_{\mathfrak p}]$, hence it suffices to show that $R[M_{\mathfrak p}]$ is normal whenever $\mathfrak p$ has height one. However, we have $R[M_{\mathfrak p}] \simeq R[M_{\mathfrak p}^\times] \otimes_R R[\mathbb N]$ in light of (ii), and since $M_{\mathfrak p}^\times$ is torsion free, it is a filtered colimit of a family of free abelian groups of finite rank, so everything is clear. □

Example 3.4.17. Let M be a fine, sharp and saturated monoid of dimension 2.

(i) By corollary 3.4.10(i) and example 3.3.16, we see that M admits exactly two facets, which are fine saturated monoids of dimension one; by theorem 3.4.16(ii) each of these facets is generated by an element, say e_i (for $i = 1,2$). From proposition 3.3.22(iii) it follows that $\mathbb Q_+e_1 \oplus \mathbb Q_+e_2 = M_{\mathbb Q}$. Especially, we may find an integer $N > 0$ large enough, such that :

$$\mathbb N e_1 \oplus \mathbb N e_2 \subset M \subset \mathbb N\frac{e_1}{N} \oplus \mathbb N\frac{e_2}{N}.$$

(ii) Moreover, clearly e_1 and e_2 are *unimodular* elements of M^{gp} (*i.e.* they generate direct summands of the latter free abelian group of rank 2). We may then find a basis f_1, f_2 of M^{gp} with $e_1 = f_1$, and $e_2 = af_1 + bf_2$, where $a, b \in \mathbb Z$ and $(a,b) = 1$. After replacing f_2 by some element of the form $cf_2 + df_1$ with $c \in \{1,-1\}$ and $d \in \mathbb Z$, we may assume that $b > 0$ and $0 \leq a < b$. Clearly, such a normalized pair (a,b) determines the isomorphism class of M, since $M_{\mathbb R}$ is the strictly convex cone of $M^{\mathrm{gp}}_{\mathbb R}$ whose extremal rays are generated by e_1 and e_2, and $M = M^{\mathrm{gp}} \cap M_{\mathbb R}$.

(iii) More precisely, suppose that M' is another fine, sharp and saturated monoid of dimension 2, and $\varphi : M \to M'$ an isomorphism. Pick a basis f'_1, f'_2 of M'^{gp} and a normalized pair (a',b') as in (ii), such that $e'_1 := f_1$ and $e'_2 := a'f'_1 + b'f'_2$ generate the two facets of M'. Clearly, φ must send a facet of M onto a facet of M'; we distinguish two possibilities :

- either $\varphi(e_1) = e'_1$ and $\varphi(e_2) = e'_2$, in which case we get $\varphi(f_2) = b^{-1}(a'-a)f'_1 + b^{-1}b'f'_2$; especially, $b', a - a' \in b\mathbb Z$. By considering φ^{-1}, we get symmetrically that $b \in b'\mathbb Z$, so $b = b'$ and therefore $(a',b) = 1 = (a,b)$ and $0 \leq a, a' < b$, whence $a = a'$
- or else $\varphi(e_1) = e'_2$ and $\varphi(e_2) = e'_1$, in which case we get $\varphi(f_2) = b^{-1}(1 - aa')f_1 + b^{-1}b'af_2$. It follows again that $b' \in b\mathbb Z$, so $b = b'$, arguing as in the previous case. Moreover, $0 \leq a' < b$, and $1 - aa' \in b\mathbb Z$. In other words, the class of a' in the group $(\mathbb Z/b\mathbb Z)^\times$ is the inverse of the class of a.

Conversely, it is easily seen that, if M' is as above, f'_1, f'_2 is a basis of M'^{gp}, and the two facets of M' are generated by f'_1 and $a'f'_1 + b'f'_2$, for a pair (a',b') normalized as in (ii), and such that $aa' \equiv 1 \pmod b$, then there exists an isomorphism $M \stackrel{\sim}{\to} M'$ of monoids (details left to the reader). Hence, set $(\mathbb Z/b\mathbb Z)^\dagger := (\mathbb Z/b\mathbb Z)^\times/{\sim}$, where $\sim$ denotes the smallest equivalence relation such that $[a] \sim [a]^{-1}$ for every $[a] \in (\mathbb Z/b\mathbb Z)^\times$. We conclude that there exists a natural bijection between the set of isomorphism classes of fine, sharp and saturated monoids of dimension 2, and the set of pairs $(b,[a])$, where $b > 0$ is an integer, and $[a] \in (\mathbb Z/b\mathbb Z)^\dagger$.

3.4.18. Let P be an integral monoid. A *fractional ideal* of P is a P-submodule $I \subset P^{\mathrm{gp}}$ such that $I \neq \varnothing$ and $x \cdot I \subset P$ for some $x \in P$. Clearly the union and the intersection of finitely many fractional ideals, are again fractional ideals. We may also define the product of two fractional ideals $I_1, I_2 \subset P^{\mathrm{gp}}$: namely, the subset

$$I_1 I_2 := \{xy \mid x \in I_1, y \in I_2\} \subset P^{\mathrm{gp}}$$

which is again a fractional ideal, by an easy inspection. If I is a fractional ideal of P, we say that I is *finitely generated*, if it is such, when regarded as a P-module. For any two fractional ideals I_1, I_2, we let

$$(I_1 : I_2) := \{x \in P^{\mathrm{gp}} \mid x \cdot I_2 \subset I_1\}.$$

It is easily seen that $(I_1 : I_2)$ is a fractional ideal of P (if $x \in I_2$ and $yI_1 \subset P$, then clearly $xy(I_1 : I_2) \subset P$). We set

$$I^{-1} := (P : I) \qquad \text{and} \qquad I^* := (I^{-1})^{-1} \qquad \text{for every fractional ideal } I \subset P^{\mathrm{gp}}.$$

Clearly $J^{-1} \subset I^{-1}$, whenever $I \subset J$, and $I \subset I^*$ for all fractional ideals I, J. We say that I is *reflexive* if $I = I^*$. We remark that I^{-1} is reflexive, for every fractional ideal $I \subset P^{\mathrm{gp}}$. Indeed, we have $I^{-1} \subset (I^{-1})^*$, and on the other hand $(I^{-1})^* = (I^*)^{-1} \subset I^{-1}$. It follows that I^* is reflexive, for every fractional ideal I. Moreover, $I^* \subset J^*$, whenever $I \subset J$; especially, I^* is the smallest reflexive fractional ideal containing I. Notice furthermore, that $aI^{-1} = (a^{-1}I)^{-1}$ for every $a \in P^{\mathrm{gp}}$; therefore, $aI^* = (aI)^*$, for every fractional ideal I and $a \in P^{\mathrm{gp}}$.

Lemma 3.4.19. *Let P be any integral monoid, $I, J \subset P^{\mathrm{gp}}$ two fractional ideals. Then :*

(i) $(IJ)^* = (I^*J^*)^*$.

(ii) *I^* is the intersection of the invertible fractional ideals of P that contain I (see definition* 2.3.6(iv)*).*

Proof. (i): Since $IJ \subset I^*J^*$, we have $(IJ)^* \subset (I^*J^*)^*$. To show the converse inclusion, it suffices to check that $I^*J^* \subset (IJ)^*$, since $(IJ)^*$ is reflexive, and $(I^*J^*)^*$ is the smallest reflaxive fractional ideal containing I^*J^*. Now, let $a \in I$ be eny element; we get $aJ^* = (aJ)^* \subset (IJ)^*$, so $IJ^* \subset (IJ)^*$, and therefore $(IJ^*)^* \subset (IJ)^*$. Lastly, let $b \in J^*$ be any element; we get $bI^* = (bI)^* \subset (IJ^*)^*$, so $I^*J^* \subset (IJ^*)^*$, whence the lemma.

(ii): It suffices to unwind the definitions. Indeed, $a \in P^{\mathrm{gp}}$ lies in I^* if and only if $aI^{-1} \subset P$, if and only if $ab \in P$, for every $b \in P^{\mathrm{gp}}$ such that $bI \subset P$. In other words, $a \in I^*$ if and only if $a \in b^{-1}P$ for every $b \in P^{\mathrm{gp}}$ such that $I \subset b^{-1}P$, which is the contention. □

3.4.20. Let P be any integral monoid. We denote by $\mathrm{Div}(P)$ the set of all reflexive fractional ideals of P. We define a composition law on $\mathrm{Div}(P)$ by the rule :

$$I \odot J := (IJ)^* \qquad \text{for every } I, J \in \mathrm{Div}(P).$$

It follows easily from lemma 3.4.19(i) that $\odot$ is an associative law; indeed we may compute :

$$(I \odot J) \odot K = ((IJ)^*K)^* = (IJK)^* = (I(JK)^*)^* = I \odot (J \odot K)$$

for every $I, J, K \in \mathrm{Div}(P)$. Clearly $I \odot J = J \odot I$ and $P \odot I = I$, for every $I, J \in \mathrm{Div}(P)$, so $(\mathrm{Div}(P), \odot)$ is a commutative monoid. Notice as well that, if $I \subset P$, then also $I^* \subset P$ (lemma 3.4.19(ii)), so the subset of all reflexive fractional ideals contained in P is a submonoid $\mathrm{Div}_+(P) \subset \mathrm{Div}(P)$.

Example 3.4.21. Let A be an integral domain, and K the field of fractions of A. Classically, one defines the notions of *fractional ideal* and of *reflexive fractional ideal* of A : see *e.g.* [61, p.80]. In our terminology, these are understood as follows. Set $A' := A \cap K^\times$, and notice that the monoid $(A, \cdot)$ is naturally isomorphic to the integral pointed monoid $A'_\circ$. Then a fractional ideal of A is an $A'_\circ$-submodule of $K^\times_\circ = K$ of the form $I_\circ$, where $I \subset K^\times$ is a fractional ideal of A'. Likewise one may define the reflexive ideals of A. The set $\mathrm{Div}(A)$ of all reflexive ideals of A is then endowed with the unique monoid structure, such that the map $\mathrm{Div}(A') \to \mathrm{Div}(A)$ given by the rule $I \mapsto I_\circ$ is an isomorphism of monoids.

Lemma 3.4.22. *Let P be an integral monoid, and $G \subset P^\times$ a subgroup. We have :*

(i) *The rule $I \mapsto I/G$ induces a bijection from the set of fractional ideals of P to the set of fractional ideals of P/G.*
(ii) *A fractional ideal I of P is reflexive if and only if the same holds for I/G.*
(iii) *The rule $I \mapsto I/G$ defines an isomorphism of monoids*
$$\mathrm{Div}(P) \xrightarrow{\sim} \mathrm{Div}(P/G).$$
(iv) *If $P^\sharp$ is fine, every fractional ideal of P is finitely generated.*

Proof. The first assertion is left to the reader. Next, we remark that $I^{-1}/G = (I/G)^{-1}$ and $(IJ)/G = (I/G) \cdot (J/G)$, for every fractional ideals I, J of P, which imply immediately assertions (ii) and (iii). Lastly, suppose that $P^\sharp$ is finitely generated, and let I be any fractional ideal of P; pick $x \in I^{-1}$; since P is integral, I is finitely generated if and only if the same holds for xI. Hence, in order to show (iv), we may assume that $I \subset P$, in which case the assertion follows from proposition 3.1.9(ii) and lemma 3.1.16(i.a). □

In order to characterize the monoids P such that $\mathrm{Div}(P)$ is a group, we make the following :

Definition 3.4.23. Let P be an integral monoid, and $a \in P^{\mathrm{gp}}$ any element.

(a) We say that a is *power-bounded*, if there exists $b \in P$ such that $a^n b \in P$ for all $n \in \mathbb{N}$.
(b) We say that P is *completely saturated*, if all power-bounded elements of P^{gp} lie in P.

Example 3.4.24. Let $(\Gamma, \leq)$ be an ordered abelian group, and set $\Gamma^+ := \{\gamma \in \Gamma \mid \gamma \leq 1\}$. Then Γ^+ is always a saturated monoid, but it is completely saturated if and only if the convex rank of Γ is ≤ 1 (see [36, Def.6.1.20]). The proof shall be left as an exercise for the reader.

Proposition 3.4.25. *Let P be an integral monoid. We have :*

(i) *$(\mathrm{Div}(P), \odot)$ is an abelian group if and only if P is completely saturated.*
(ii) *If P is fine and saturated, then P is completely saturated.*
(iii) *Let A be a Krull domain, and set $A' := A\setminus\{0\}$. Then $(A', \cdot)$ is a completely saturated monoid.*

Proof. (i): Suppose that $I \in \mathrm{Div}(I)$ admits an inverse J in the monoid $(\mathrm{Div}(P), \odot)$, and notice that $I \odot I^{-1} \subset P$; it follows easily that $I \odot (J \cup I^{-1})^* = P$, hence $I^{-1} \subset J$, by the uniqueness of the inverse. On the other hand, if J strictly contains I^{-1}, then IJ strictly contains P, which is absurd. Thus, we see that $\mathrm{Div}(P)$ is a group if and only if $I \odot I^{-1} = P$ for every $I \in \mathrm{Div}(P)$. Now, suppose first that P is completely saturated. In view of lemma 3.4.19(ii), we are reduced to showing that P is contained in every invertible fractional ideal containing $I^{-1}I$. Hence, say that $I^{-1}I \subset aP$ for some $a \in P^{\mathrm{gp}}$; equivalently, we have $a^{-1}I^{-1}I \subset P$, *i.e.* $a^{-1}I^{-1} \subset I^{-1}$, and then $a^{-k}I^{-1} \subset I^{-1}$ for every integer $k \in \mathbb{N}$. Say that $b \in I^{-1}$ and $c \in I^*$; we conclude that $a^{-k}bc \in P$ for every $k \in \mathbb{N}$, so $a^{-1} \in P$, by assumption, and finally $P \subset aP$, as required.

Conversely, suppose that $\mathrm{Div}(P)$ is a group, and let $a \in P^{\mathrm{gp}}$ be any power-bounded element. By definition, this means that the P-submodule I of P^{gp} generated by $(a^k \mid k \in \mathbb{N})$ is a fractional ideal of P. Then I^{-1} is a reflexive fractional ideal, and by assumption I^{-1} admits an inverse, which must be I^*, by the foregoing. On the other hand, by construction we have $aI \subset I$, hence $aI^* = (aI)^* \subset I^*$. We deduce that $aP = a(I^* \odot I^{-1}) = aI^* \odot I^{-1} \subset I^* \odot I^{-1} = P$, *i.e.* $a \in P$, as stated.

(ii) is a special case of claim 3.3.38.

(iii): See [61, §12] for the basic generalities on Krull domains. One is immediately reduced to the case where A is a valuation ring whose valuation group Γ has rank ≤ 1. Taking into account (i) and lemma 3.4.22, it then suffices to show that the monoid $A'/A^\times$ is completely sataurated. However, the latter is isomorphic to the submonoid Γ^+ of elements ≤ 1 in Γ, so the assertion follows from example 3.4.24. $\square$

3.4.26. Let $\varphi : P \to Q$ be a morphism of integral monoids, and I any fractional ideal of P; notice that $IQ := \varphi^{\mathrm{gp}}(I)Q \subset Q^{\mathrm{gp}}$ is a fractional ideal of Q. Moreover, the identities

$$(I_1 \cup I_2)Q = I_1Q \cup I_2Q \qquad (I_1I_2)Q = (I_1Q) \cdot (I_2Q) \qquad \text{for all fractional ideals } I_1, I_2 \subset P^{\mathrm{gp}}$$

are immediate from the definitions. Likewise, if A an integral domain and $\alpha : P \to (A\backslash\{0\}, \cdot)$ a morphism of monoids, then the A-submodule $IA := \alpha^{\mathrm{gp}}(I)A$ of the field of fractions of A is a fractional ideal of the ring A (in the usual commutative algebraic meaning : see example 3.4.21), and we have corresponding identities :

$$(I_1 \cup I_2)A = I_1A + I_2A \qquad (I_1I_2)A = (I_1A) \cdot (I_2A) \qquad \text{for all fractional ideals } I_1, I_2 \subset P^{\mathrm{gp}}.$$

Lemma 3.4.27. *In the situation of* (3.4.26), *suppose that φ is flat and A is α-flat, and let $I, J, J' \subset P^{\mathrm{gp}}$ be three fractional ideals, with I finitely generated. Then we have :*

(i) $(J : I)Q = (JQ : IQ)$ *and* $(J : I)A = (J : I)A$.

(ii) *Especially, if I is reflexive, the same holds for IQ and IA (see* (5.6)*).*

(iii) *Suppose furthermore that A is local, and α is a local morphism. Then $JA = J'A$ if and only if $J = J'$.*

(iv) *If P is fine, the rule $I \mapsto IQ$ and $I \mapsto IA$ define morphisms of monoids*

$$\mathrm{Div}(\varphi) : \mathrm{Div}(P) \to \mathrm{Div}(Q) \qquad \mathrm{Div}(\alpha) : \mathrm{Div}(P) \to \mathrm{Div}(A)$$

(where $\mathrm{Div}(A)$ *is defined as in example* 3.4.21*), and* $\mathrm{Div}(\alpha)$ *is injective, if α is local and A is a local domain.*

Proof. (i): Say that $I = a_1P \cup \cdots \cup a_nP$ for elements $a_1, \dots, a_n \in P^{\mathrm{gp}}$. Then

$$(J : I) = a_1^{-1}J \cap \cdots \cap a_n^{-1}J \qquad \text{and} \qquad (JQ : IQ) = a_1^{-1}JQ \cap \cdots \cap a_n^{-1}JQ$$

and likewise for $(J : I)A$, hence the assertion follows from an easy induction, and the following

Claim 3.4.28. For any two fractional ideals $J_1, J_2 \subset P$, we have $(J_1 \cap J_2)Q = J_1Q \cap J_2Q$ and $(J_1 \cap J_2)A = J_1A \cap J_2A$.

Proof of the claim. Pick any $x \in P$ such that $xJ_1, xJ_2 \subset P$; since P is an integral monoid, and A is an integral domain, it suffices to show that $x(J_1 \cap J_2)Q = xJ_1Q \cap xJ_2Q$ and likewise for $x(J_1 \cap J_2)A$, and notice that $x(J_1 \cap J_2) = xJ_1 \cap xJ_2$. We may thus assume that J_1 and J_2 are ideals of P, in which case the assertion is lemma 3.1.37. $\diamondsuit$

(ii): Suppose that I is reflexive; from (i) we deduce that $((IA)^{-1})^{-1} = IA$. The assertion is an immediate consequence, once one remarks that, for any fractional ideal $J \subset A$, there is a natural isomorphism of A-modules : $J^{-1} \xrightarrow{\sim} J^\vee := \mathrm{Hom}_A(J, A)$. Indeed, the isomorphism assigns to any $x \in J^{-1}$ the map $\mu_x : J \to A : a \mapsto xa$ for every $a \in J$ (details left to the reader).

(iii): We may assume that $JA = J'A$, and we prove that $J = J'$, and by replacing J' by $J \cup J'$, we may assume that $J \subset J'$. Then the contention follows easily from lemma 3.1.36.

(iv): This is immediate from (i) and (iii). $\square$

Remark 3.4.29. In the situation of lemma 3.4.27(iv), obviously $\mathrm{Div}(\varphi)$ restricts to a morphism of submonoids :

$$\mathrm{Div}_+(\varphi) : \mathrm{Div}_+(P) \to \mathrm{Div}_+(Q).$$

3.4.30. Next, suppose that P is fine and saturated, and let $I \subset P^{\mathrm{gp}}$ be any fractional ideal. Then theorem 3.4.16(i) easily implies that :

$$I^{-1} = \bigcap_{\mathrm{ht}\,\mathfrak{p}=1} (I_\mathfrak{p})^{-1}$$

where the intersection – running over the prime ideals of P of height one – is taken within $\mathrm{Hom}_P(I, P^{\mathrm{gp}})$, which naturally contains all the $(I_\mathfrak{p})^{-1}$. The structure of the fractional ideals of $P_\mathfrak{p}$ when $\mathrm{ht}\,\mathfrak{p} = 1$ is very simple : quite generally, theorem 3.4.16(ii) easily implies that if $\dim P = 1$, then all fractional ideals are cyclic, and then clearly they are reflexive. On the other hand, I is finitely generated, by lemma 3.4.22(iv). We deduce that I is reflexive if and only if :

$$I = \bigcap_{\mathrm{ht}\,\mathfrak{p}=1} I_\mathfrak{p}. \tag{3.4.31}$$

Indeed, suppose that (3.4.31) holds; then we have $I^* = \bigcap_{\mathrm{ht}\,\mathfrak{p}=1}(I_\mathfrak{p}^{-1})^{-1} = \bigcap_{\mathrm{ht}\,\mathfrak{p}=1} I_\mathfrak{p}$, since we have just seen that $I_\mathfrak{p}$ is a reflexive fractional ideal of $P_\mathfrak{p}$, for every prime ideal $\mathfrak{p}$ of height one.

Proposition 3.4.32. *Let P be a fine and saturated monoid, and denote by $D \subset \mathrm{Spec}\, P$ the subset of all prime ideals of height one. Then the mapping :*

$$\mathbb{Z}^{\oplus D} \to \mathrm{Div}(P) \quad : \quad \sum_{\mathrm{ht}\,\mathfrak{p}=1} n_\mathfrak{p}[\mathfrak{p}] \mapsto \bigcap_{\mathrm{ht}\,\mathfrak{p}=1} \mathfrak{m}_{P_\mathfrak{p}}^{n_\mathfrak{p}} \tag{3.4.33}$$

is an isomorphism of abelian groups.

Proof. Here $\mathfrak{m}_{P_\mathfrak{p}} \subset P_\mathfrak{p}$ is the maximal ideal, and for $n \geq 0$, the notation $\mathfrak{m}_{P_\mathfrak{p}}^n$ means the usual n-th power operation in the monoid $\mathscr{P}(P^{\mathrm{gp}})$, which we extend to all integers n, by letting $\mathfrak{m}_{P_\mathfrak{p}}^n := \mathfrak{m}_{P_\mathfrak{p}}^{-n}$ whenever $n < 0$.

In order to show that (3.4.33) is well defined, set $I := \bigcap_{\mathrm{ht}\,\mathfrak{p}=1} \mathfrak{m}_\mathfrak{p}^{n_\mathfrak{p}}$. Pick, for every $\mathfrak{p}$ such that $n_\mathfrak{p} < 0$, an element $x_\mathfrak{p} \in \mathfrak{p}$, and set $y_\mathfrak{p} := x_\mathfrak{p}^{-n_\mathfrak{p}}$; if $n_\mathfrak{p} \geq 0$, set $y_\mathfrak{p} := 1$. Then it is easy to check (using theorem 3.4.16(i)) that $\prod_{\mathrm{ht}\,\mathfrak{p}=1} y_\mathfrak{p}$ lies in I^{-1}, hence I is a fractional ideal. Next, for given $\mathfrak{p}, \mathfrak{p}' \in D$, notice that $(P_\mathfrak{p})_{\mathfrak{p}'} = P^{\mathrm{gp}}$; it follows that

$$I_\mathfrak{p} = \mathfrak{m}_\mathfrak{p}^{n_\mathfrak{p}} \qquad \text{for every } \mathfrak{p} \in D \tag{3.4.34}$$

therefore I is reflexive. Furthermore, it is easily seen (from theorem 3.4.16(ii)), that every reflexive ideal of $P_\mathfrak{p}$ is of the form $\mathfrak{m}_P^n$ for some integer n, and moreover $\mathfrak{m}_P^n = \mathfrak{m}_P^m$ if and only if $n = m$. Then (3.4.31) implies that the mapping (3.4.33) is surjective, and the injectivity follows from (3.4.34). It remains to check that (3.4.33) is a group homomorphism, and to this aim we may assume – in view of lemma 3.4.27(iv) – that $\dim P = 1$, in which case the assertion is immediate. $\square$

3.4.35. A morphism $\varphi : I \to J$ of fractional ideals of P is, by definition, a morphism of P-modules. Let $x, y \in I$ be any two elements; we may find $a, b \in P$ such that $ax = by$ in I, and therefore $a\varphi(x) = \varphi(ax) = \varphi(by) = b\varphi(y)$; thus, $\varphi(y) = (b^{-1}a) \cdot \varphi(x) = (x^{-1}y) \cdot \varphi(x)$. This shows that, for every morphism $\varphi : I \to J$ of fractional ideals, there exists $c \in P^{\mathrm{gp}}$ such that $\varphi(x) = cx$ for every $x \in I$. Especially, $I \simeq J$ if and only if there exists $a \in P^{\mathrm{gp}}$ such that $I = aJ$. Likewise one may characterize the morphisms and isomorphisms of fractional ideals of an integral domain. We denote

$$\overline{\mathrm{Div}}(P)$$

the set of isomorphism classes of reflexive fractional ideals of P. From the foregoing, it is clear that, if $I \simeq I'$, we have $I \odot J \simeq I' \odot J$ for every $J \in \mathrm{Div}(P)$; therefore the composition law of $\mathrm{Div}(P)$ descends to a composition law for $\overline{\mathrm{Div}}(P)$, which makes it into a (commutative) monoid, and if P is completely saturated, then $\overline{\mathrm{Div}}(P)$ is an abelian group. We also deduce an exact sequence of monoids

$$1 \to P^\times \to P^{\mathrm{gp}} \xrightarrow{j_P} \mathrm{Div}(P) \to \overline{\mathrm{Div}}(P) \to 1 \tag{3.4.36}$$

where j_P is given by the rule A: $a \mapsto aP$ for every $a \in P$; especially, j_P restricts to a morphism of monoids

$$j_P^+ : P \to \mathrm{Div}_+(P).$$

Likewise, we define $\overline{\mathrm{Div}}(A)$, for any integral domain A : see example 3.4.21. Moreover, in the situation of (3.4.26), we see from lemma 3.4.27(iv) that, if α is local, P is fine, A is α-flat, and φ is flat, then $\mathrm{Div}(\varphi)$ and $\mathrm{Div}(\alpha)$ descend to well defined morphisms of monoids

$$\overline{\mathrm{Div}}(\varphi) : \overline{\mathrm{Div}}(P) \to \overline{\mathrm{Div}}(Q) \qquad \overline{\mathrm{Div}}(\alpha) : \overline{\mathrm{Div}}(P) \to \overline{\mathrm{Div}}(A).$$

Proposition 3.4.37. *Let P be a fine and saturated monoid, $I, J \subset P^{\mathrm{gp}}$ two fractional ideals, A a local integral domain, and $\alpha : P \to (A, \cdot)$ a local morphism of monoids. We have :*

(i) $(I : I) = P$.

(ii) *Suppose that A is α-flat. Then $IA \simeq JA$ if and only if $I \simeq J$. Especially, in this case $\overline{\mathrm{Div}}(\alpha)$ is an injective map.*

Proof. (i): Clearly it suffices to show that $(I : I) \subset P$. Hence, say that $x \in (I : I)$, and pick any $a \in I$; it follows that $x^n a \in P$ for every $n > 0$; in the additive group $\log P^{\mathrm{gp}}$ we have therefore the identity $n \cdot \log(x) + \log(a) \in \log P$, so $\log(x) + n^{-1} \log(a) \in (\log P)_{\mathbb{R}}$ for every $n > 0$. Since $(\log P)_{\mathbb{R}}$ is a convex polyhedral cone in $(\log P^{\mathrm{gp}})_{\mathbb{R}}$, we deduce that $x \in (\log P)_{\mathbb{R}} \cap (\log P^{\mathrm{gp}}) = \log P$ (proposition 3.3.22(iii)), as claimed.

(ii): We may assume that IA is isomorphic to JA, and we show that I is isomorphic to J. Indeed, the assumption means that $a(IA) = JA$ for some $x \in \mathrm{Frac}(A)$; therefore, $a \in (JA : IA)$ and $a^{-1} \in (IA : JA)$, so

$$A = (IA : JA) \cdot (JA : IA) = ((I : J) \cdot (J : I))A$$

by virtue of lemma 3.4.27(i). Since A is local, it follows that there exist $a \in (I : J)$ and $b \in (J : I)$ such that $\alpha(ab) \in A^\times$, whence $ab \in P^\times$, since α is local. It follows easily that $I = aJ$, as asserted. □

Example 3.4.38. (i) Let P be a fine and saturated monoid, and $D \subset \operatorname{Spec} P$ the subset of all prime ideals of height one; for every $\mathfrak{p} \in D$, denote

$$v_{\mathfrak{p}} : P \to P_{\mathfrak{p}}^\sharp \xrightarrow{\sim} \mathbb{N}$$

the composition of the localization map, and the natural isomorphism resulting from theorem 3.4.16(ii). A simple inspection shows that the isomorphism (3.4.33) identifies the map j_P of (3.4.36) with the morphism of monoids

$$v_P : P^{\mathrm{gp}} \to \mathbb{Z}^{\oplus D} \qquad x \mapsto (v_{\mathfrak{p}}^{\mathrm{gp}}(x) \mid \mathfrak{p} \in D).$$

With this notation, the isomorphism (3.4.33) is the map given by the rule :

$$k_\bullet \mapsto v_P^{-1}(k_\bullet + \mathbb{N}^{\oplus D}) \qquad \text{for every } k_\bullet \in \mathbb{Z}^{\oplus D}.$$

(ii) Suppose now that P is sharp and $\dim P = 2$, in which case $D = \{\mathfrak{p}_1, \mathfrak{p}_2\}$ contains exactly two elements. According to example 3.4.17(ii), we may find a basis f_1, f_2 of P^{gp}, such that the two facets $P \setminus \mathfrak{p}_1$ and $P \setminus \mathfrak{p}_2$ of P are generated respectively by $e_1 := f_1$ and $e_2 := af_1 + bf_2$, for some $a, b \in \mathbb{N}$, with $a < b$ and $(a, b) = 1$. It follows easily that P is a submonoid of the free monoid

$$Q := \mathbb{N}e'_1 \oplus \mathbb{N}e'_2 \qquad \text{where } e'_1 := b^{-1}e_1 \text{ and } e'_2 := b^{-1}e_2$$

and $Q^{\mathrm{gp}}/P^{\mathrm{gp}} \simeq \mathbb{Z}/b\mathbb{Z}$ (details left to the reader). The induced map $\mathrm{Spec}\, Q \to \mathrm{Spec}\, P$ is a homeomorphism; especially Q admits two prime ideals $\mathfrak{q}_1$, $\mathfrak{q}_2$ of height one, so that $\mathfrak{q}_i \cap P = \mathfrak{p}_i$ for $i = 1, 2$, whence – by proposition 3.4.32 – a natural isomorphism

$$s^* : \mathrm{Div}(Q) \xrightarrow{\sim} \mathrm{Div}(P)$$

and notice that $j_Q : Q^{\mathrm{gp}} \to \mathrm{Div}(Q)$ is the isomorphism given by the rule : $e'_i \mapsto \mathfrak{q}_i$ for $i = 1, 2$. Moreover, we have commutative diagrams of monoids :

$$\begin{tikzcd} P \arrow[r, "s"] \arrow[d, "v_{\mathfrak{p}_i}"'] & Q \arrow[d, "v_{\mathfrak{q}_i}"] \\ \mathbb{N} \arrow[r, "t_i"] & \mathbb{N} \end{tikzcd} \qquad (i = 1, 2).$$

Clearly, $Q \setminus \mathfrak{q}_i$ is the facet generated by e'_i, so $v_{\mathfrak{q}_i}$ is none else than the projection onto the direct factor $\mathbb{N}e'_{3-i}$, for $i = 1, 2$. In order to compute $v_{\mathfrak{p}_i}$, it then suffices to determine t_i, or equivalently t_i^{gp}. However, set $\tau_i := v_{\mathfrak{q}_i}^{\mathrm{gp}} \circ s^{\mathrm{gp}}$; clearly $\tau_1(f_2) = \tau_1(e'_2 - ae'_1) = 1$, so τ_1 is surjective. Also, $\tau_2(f_1) = b$ and $\tau_2(f_2) = -a$, so τ_2 is surjective as well; therefore both t_1 and t_2 are the identity endomorphism of $\mathbb{N}$. Summing up, we find that

$$j_P = s^* \circ j_Q \circ s^{\mathrm{gp}}$$

and the morphism v_P is naturally identified with $s^{\mathrm{gp}} : P^{\mathrm{gp}} \to Q^{\mathrm{gp}}$. Especially, we have obtained a natural isomorphism

$$\overline{\mathrm{Div}}(P) \xrightarrow{\sim} \mathbb{Z}/b\mathbb{Z}.$$

We may then rephrase in more intrinsic terms the classification of example 3.4.17(iii) : namely the isomorphism class of P is completely determined by the datum of $\overline{\mathrm{Div}}(P)$ and the equivalence class of the height one prime ideals of P in the quotient set $\overline{\mathrm{Div}}(P)^\dagger$ defined as in *loc.cit.*

(iii) In the situation of (ii), a simple inspection yields the following explicit description of all reflexive fractional ideals of P. Recall that such ideals are of the form

$$I_{k_1,k_2} := \mathfrak{m}_1^{k_1} \cap \mathfrak{m}_2^{k_2} = \{x \in P^{\mathrm{gp}} \mid v_{\mathfrak{p}_1}(x) \geq k_1,\ v_{\mathfrak{p}_2}(x) \geq k_2\}$$

where $\mathfrak{m}_i$ is the maximal ideal of $P_{\mathfrak{p}_i}$, and $k_i \in \mathbb{Z}$, for $i = 1, 2$. Then

$$I_{k_1,k_2} = \{x_1e_1 + x_2e_2 \mid x_1, x_2 \in b^{-1}\mathbb{Z},\ x_1 \geq b^{-1}k_2,\ x_2 \geq b^{-1}k_1\} \cap P^{\mathrm{gp}} \qquad \text{for all } k_1, k_2 \in \mathbb{Z}.$$

With this notation, the cyclic reflexive ideals are then those of the form

$$(x_1f_1 + x_2f_2)P = I_{x_2b, x_1b - x_2a} \qquad \text{with } x_1, x_2 \in \mathbb{Z}.$$

Especially, we see that the classes of $\mathfrak{p}_1 = I_{0,1}$ and $\mathfrak{p}_2 = I_{1,0}$ are both of order b in $\overline{\mathrm{Div}}(P)$.

The following estimate, special to the two-dimensional case, will be applied – in section 9.6 – to the proof of the almost purity theorem for towers of regular log schemes.

Lemma 3.4.39. *Let P be a fine and saturated monoid of dimension 2, and denote by b the order of the finite cyclic group $\overline{\mathrm{Div}}(P)$. We have :*

$$\mathfrak{m}_P^{[b/2]} \subset I \cdot I^{-1} \qquad \textit{for every } I \in \mathrm{Div}(P)$$

(where $[b/2]$ denotes the largest integer $\leq b/2$).

Proof. Notice first that the assertion holds for a given $I \in \mathrm{Div}(P)$, if and only if it holds for xI, for any $x \in P^{\mathrm{gp}}$. If $b = 1$, then $P = \mathbb{N}^{\oplus 2}$, in which case $\overline{\mathrm{Div}}(P) = 0$, so every reflexive fractional ideal of P is isomorphic to P, and the assertion is clear. Hence, assume that $b > 1$; let $\mathfrak{p}_1$, $\mathfrak{p}_2$ be the two prime ideals of height one of P, and define Q, e_1, e_2 and I_{k_1,k_2} for every $k_1, k_2 \in \mathbb{Z}$, as in example 3.4.38(ii,iii). With this notation, notice that

$$\mathfrak{m}_P \setminus \mathfrak{m}_P^2 = \{e_1, e_2\} \cup \Sigma \qquad \text{where } \Sigma \subset \{x_1e_1 + x_2e_2 \mid x_1, x_2 \in b^{-1}\mathbb{Z},\ 0 \leq x_1, x_2 < 1\}.$$

It follows easily that, for every $i \in \mathbb{N}$, every element of $\mathfrak{m}_P^i$ is of the form $x_1 e_1 + x_2 e_2$ with $x_1, x_2 \in b^{-1}\mathbb{N}$ and $\max(x_1, x_2) \geq b^{-1}i$. Hence, let $I \in \mathrm{Div}(P)$ and $x := x_1 e_1 + x_2 e_2 \in \mathfrak{m}_P^{[b/2]}$, and say that $bx_1 \geq [b/2]$. According to example 3.4.38(iii), we may assume that $I = \mathfrak{p}_2^j = I_{j,0}$ for some $j \in \{0, \dots, b-1\}$. Moreover, notice that the assertion holds for I if and only if it holds for I^{-1}, whose class in $\overline{\mathrm{Div}}(P)$ agrees with the class of $\mathfrak{p}_2^{b-j}$. Clearly, either $j \leq [b/2]$ or $b - j \leq [b/2]$; hence, we may assume that $j \in \{0, \dots, [b/2]\}$. Thus, $P \subset I^{-1}$, and $I \subset I \cdot I^{-1}$, and clearly $x \in I$, so we are done in this case. The case where $bx_2 \geq [b/2]$ is dealt with in the same way, by writing $I = \mathfrak{p}_1^j$ for some non-negative $j \leq [b/2]$: the details are left to the reader. $\square$

If $f : P \to Q$ is a general morphism of integral monoids, and I a fractional ideal of Q, the P-module $f^{\mathrm{gp}-1}(I)$ is not necessarily a fractional ideal of P (for instance, consider the natural map $P \to P^{\mathrm{gp}}$). One may obtain some positive results, by restricting to the class of morphisms introduced by the following :

Definition 3.4.40. Let $f : P \to Q$ be a morphism of monoids. We say that f is *of Kummer type*, if f is injective, and the induced map $f_{\mathbb{Q}} : P_{\mathbb{Q}} \to Q_{\mathbb{Q}}$ is surjective (notation of (3.3.20)).

Lemma 3.4.41. *Let $f : P \to Q$ be a morphism of monoids of Kummer type, $S_Q \subset Q$ a submonoid, and set $S_P := f^{-1}S_Q$. We have :*

(i) *The map* $\mathrm{Spec}\, f : \mathrm{Spec}\, Q \to \mathrm{Spec}\, P$ *is bijective; especially* $\dim P = \dim Q$.
(ii) *If $Q^\times$ is a torsion-free abelian group, P is the trivial monoid (resp. is sharp) if and only if the same holds for Q.*
(iii) *The induced morphism $S_P^{-1}P \to S_Q^{-1}Q$ is of Kummer type.*
(iv) *If P is integral, the unit of adjunction $P \to P^{\mathrm{sat}}$ is of Kummer type.*
(v) *Suppose that P is integral and saturated. Then $f^\sharp : P^\sharp \to Q^\sharp$ is of Kummer type.*
(vi) *If both P and Q are integral, and P is saturated, then f is exact.*

Proof. (ii) and (iv) are trivial, and (iii) is an exercise for the reader.

(i): Let $F, F' \subset Q$ be two faces such that $f^{-1}F = f^{-1}F'$, and say that $x \in F$. Then $x^n \in f(P)$ for some $n > 0$, so $x^n \in f(f^{-1}F')$, whence $x \in F'$, which implies that $\mathrm{Spec}\, f$ is injective. Next, for a given face F of P, let $F' \subset Q$ be the subset of all $x \in Q$ such that there exists $n > 0$ with $x^n \in f(F)$. It is easily seen that F' is a face of Q, and moreover $f^{-1}F' = F$, which shows that $\mathrm{Spec}\, f$ is also surjective.

(v): Clearly the map $(P^\sharp)_{\mathbb{Q}} \to (Q^\sharp)_{\mathbb{Q}}$ is surjective. Now, let $x, y \in P$ such that the images of $f(x)$ and $f(y)$ agree in $Q^\sharp$, *i.e.* there exists $u \in Q^\times$ with $u \cdot f(x) = f(y)$; we may find $n > 0$ such that $u^n, u^{-n} \in f(P)$. Say that $u^n = f(v)$, $u^{-n} = f(w)$; since $f(vw) = 1$, we have $vw = 1$, and moreover $f(vx^n) = f(y^n)$, so $vx^n = y^n$. Therefore $x^n y^{-n}, x^{-n}y^n \in P$, and since P is saturated we deduce that $xy^{-1}, x^{-1}y \in P$, so the images of x and y agree in $P^\sharp$.

(vi): Notice first that f^{gp} is injective, since the same holds for f. Suppose $x \in P^{\mathrm{gp}}$ and $f^{\mathrm{gp}}(x) \in Q$; we may then find an integer $k > 0$ and $y \in P$ such that $f(y) = f(x)^k$. Since P is saturated, it follows that $x \in P$, so f is exact. $\square$

3.4.42. Suppose that $\varphi : P \to Q$ is a morphism of integral monoids of Kummer type, with P saturated, and let $I \subset Q^{\mathrm{gp}}$ be a fractional ideal. Then $\varphi^* I := \varphi^{\mathrm{gp}-1}(I)$ is a fractional ideal of P. Indeed, by assumption there exists $a \in Q$ such that $aI \subset Q$; we may find $k > 0$ and $b \in P$ such that $a^k = \varphi(b)$, so $\varphi(bx) \in \varphi(P)^{\mathrm{gp}} \cap Q = \varphi(P)$ for every $x \in \varphi^* I$, since φ is exact (lemma 3.4.41(v)); therefore $b \cdot \varphi^*(I) \subset P$.

3.4.43. In the situation of (3.4.42), suppose that both P and Q are fine and saturated, and let $\mathrm{gr}_\bullet Q^{\mathrm{gp}}$ be the φ-grading of Q, indexed by $(\Gamma, +) := Q^{\mathrm{gp}}/P^{\mathrm{gp}}$ (see remark 3.2.5(iii)); for every $x \in Q^{\mathrm{gp}}$, denote $\overline{x} \in \Gamma$ the image of x. Let also $I \subset Q^{\mathrm{gp}}$ be any fractional ideal, and denote by

$\mathrm{gr}_\bullet I$ the Γ-grading on I deduced from the φ-grading of Q^{gp}; arguing as in (3.4.42), it is easily seen that, more generally, $\varphi^*(x^{-1}\mathrm{gr}_{\overline{x}}I)$ is a fractional ideal of P, for every $x \in Q$ (details left to the reader). For every prime ideal $\mathfrak{q}$ of height one in Q, we have a commutative diagram of monoids :

$$\begin{tikzcd} P \arrow[r, "\varphi"] \arrow[d, "v_{\varphi^{-1}\mathfrak{q}}"'] & Q \arrow[d, "v_{\mathfrak{q}}"] \\ \mathbb{N} \arrow[r, "e_{\mathfrak{q}}"] & \mathbb{N} \end{tikzcd} \qquad (3.4.44)$$

where $v_{\mathfrak{q}}$ and $v_{\varphi^{-1}\mathfrak{q}}$ are defined as in example 3.4.38(i), and $e_{\mathfrak{q}}$ is the multiplication by a non-zero (positive) integer, which we call the *ramification index of* φ *at* $\mathfrak{q}$, and we denote also $e_{\mathfrak{q}}$.

Lemma 3.4.45. *In the situation of* (3.4.43)*, suppose that* I *is a reflexive fractional ideal. Then* $\varphi^*(a^{-1} \cdot \mathrm{gr}_{\overline{a}}I)$ *is a reflexive fractional ideal of* P*, for every* $a \in Q^{\mathrm{gp}}$*.*

Proof. Clearly, we may replace I by $a^{-1}I$, and reduce to the case where $a = 1$, in which case we have to check that φ^*I is a reflexive fractional ideal. However, according to example 3.4.38(i), we may write $I = v_Q^{-1}(k_\bullet + \mathbb{N}^{\oplus D})$, where $D \subset \mathrm{Spec}\, Q$ is the subset of the height one prime ideals, and $k_\bullet \in \mathbb{Z}^{\oplus D}$. Set

$$k'_\bullet := ([e_{\mathfrak{q}}^{-1}k_{\mathfrak{q}}] \mid \mathfrak{q} \in D)$$

(where, for a real number x, we let $[x]$ be the smallest integer $\geq x$). Since $\mathrm{Spec}\,\varphi$ is bijective (lemma 3.4.41(i)), the commutative diagrams (3.4.44) imply that $\varphi^*I = v_P^{-1}(k'_\bullet + \mathbb{N}^{\oplus D})$, whence the contention. $\square$

Example 3.4.46. Let P be as in example 3.4.38(ii), set $Q := \mathrm{Div}_+(P)$, and take $\varphi := j_P^+ : P \to Q$ (notation of (3.4.35)). The discussion of *loc.cit.* shows that φ is a morphism of Kummer type, and notice that the φ-grading of Q is indexed by $Q^{\mathrm{gp}}/P^{\mathrm{gp}} = \overline{\mathrm{Div}}(P)$. Now, pick any $x \in Q$, and let $\overline{x} \in \overline{\mathrm{Div}}(P)$ be the equivalence class of x; by lemma 3.4.45, the P-module $\mathrm{gr}_{\overline{x}}Q$ is isomorphic to a reflexive fractional ideal of P. We claim that the isomorphism class of $\mathrm{gr}_{\overline{x}}Q$ is precisely $\overline{x}^{-1}$ (where the inverse is formed in the commutative group $\overline{\mathrm{Div}}(P)$). Indeed, let $a \in P^{\mathrm{gp}}$ be any element; by definition, we have $a \in \varphi^*(x^{-1}\mathrm{gr}_{\overline{x}}Q)$ if and only if $\varphi^{\mathrm{gp}}(a) \in x^{-1}\mathrm{gr}_{\overline{x}}Q$, if and only if $ax \in Q$, if and only if $a \in x^{-1}$, whence the claim. Thus, the family

$$(\mathrm{gr}_\gamma \mathrm{Div}_+(P) \mid \gamma \in \overline{\mathrm{Div}}(P))$$

is a complete system of representatives for the isomorphism classes of the reflexive fractional ideals of a fine, sharp and saturated monoid P of dimension 2.

Remark 3.4.47. Further results on reflexive fractional ideals for monoids, and their divisor class groups can be found in [23].

3.5. **Fans.** According to Kato ([53, §9]), a fan is to a monoid what a scheme is to a ring. More prosaically, the theory of fans is a reformulation of the older theory of rational polyhedral decompositions, developed in [54].

Definition 3.5.1. (i) A *monoidal space* is a datum $(T, \mathscr{O}_T)$ consisting of a topological space T and a sheaf of monoids $\mathscr{O}_T$ on T.

(ii) A *morphism of monoidal spaces* is a datum

$$(f, \log f) : (T, \mathscr{O}_T) \to (S, \mathscr{O}_S)$$

consisting of a continuous map $f : T \to S$, and a morphism $\log f : f^*\mathscr{O}_S \to \mathscr{O}_T$ of sheaves of monoids that is *local*, *i.e.* whose stalk $(\log f)_t : \mathscr{O}_{S,f(t)} \to \mathscr{O}_{T,t}$ is a local morphism, for every $t \in T$. The *strict locus* of $(f, \log f)$ is the subset

$$\mathrm{Str}(f, \log f) \subset T$$

consisting of all $t \in T$ such that $(\log f)_t$ is an isomorphism.

(iii) We say that a monoidal space $(T, \mathscr{O}_T)$ is *sharp*, (resp. *integral*, resp. *saturated*) if $\mathscr{O}_T$ is a sheaf of sharp (resp. integral, resp. integral and saturated) monoids.

(iv) For any monoidal space (resp. integral monoidal space) $(T, \mathscr{O}_T)$, the *sharpening* (resp. the *saturation*) of $(T, \mathscr{O}_T)$ is the sharp monoidal space $(T, \mathscr{O}_T)^\sharp := (T, \mathscr{O}_T^\sharp)$ (resp. $(T, \mathscr{O}_T)^{\mathrm{sat}} := (T, \mathscr{O}_T^{\mathrm{sat}})$).

It is easily seen that the rule $(T, \mathscr{O}_T) \mapsto (T, \mathscr{O}_T)^\sharp$ extends to a functor from the category of monoidal spaces to the full subcategory of sharp monoidal spaces. This functor is right adjoint to the corresponding fully faithful embedding of categories.

Likewise, the functor $(T, \mathscr{O}_T) \mapsto (T, \mathscr{O}_T)^{\mathrm{sat}}$ is right adjoint to the fully faithful embedding of the category of saturated monoidal spaces, into the category of integral monoidal spaces.

3.5.2. Let P be any monoid; for every $f \in P$, let us set

$$D(f) := \{\mathfrak{p} \in \operatorname{Spec} P \mid f \notin \mathfrak{p}\}.$$

Notice that $D(f) \cap D(g) = D(fg)$ for every $f, g \in P$. We endow $\operatorname{Spec} P$ with the topology whose basis of open subsets consists of the subsets $D(f)$, for every $f \in P$. Notice that $\mathfrak{m}_P$ is the only closed point of $\operatorname{Spec} P$ (especially, $\operatorname{Spec} P$ is trivially quasi-compact).

By lemma 3.1.13, the localization map $j_f : P \to P_f$ induces an identification $j_f^* : \operatorname{Spec} P_f \xrightarrow{\sim} D(f)$. It is easily seen that $(j_f^*)^{-1} D(fg) = D(j(g)) \subset \operatorname{Spec} P_f$; in other words, the topology of $\operatorname{Spec} P_f$ agrees with the topology induced from $\operatorname{Spec} P$, via j^*.

Next, for every $f \in P$ we set :

$$\mathscr{O}_{\operatorname{Spec} P}(D(f)) := P_f.$$

We claim that $\mathscr{O}_{\operatorname{Spec} P}(D(f))$ depends only on the open subset $D(f)$, up to natural isomorphism (and not on the choice of f). More precisely, say that $D(f) \subset D(g)$ for two given elements $f, g \in P$; it follows that the image of g in P_f lies outside the maximal ideal $\mathfrak{m}_{P_f}$, hence $g \in P_f^\times$, and therefore the localization map $j_f : P \to P_f$ factors uniquely through a morphism of monoids :

$$j_{f,g} : P_g \to P_f.$$

Likewise, if $D(g) \subset D(f)$ as well, the localization $j_g : P \to P_g$ factors through a unique map $j_{g,f} : P_f \to P_g$, whence the identities :

$$j_{f,g} \circ j_{g,f} \circ j_f = j_f \qquad j_{g,f} \circ j_{f,g} \circ j_g = j_g$$

and since j_f and j_g are epimorphisms, we see that $j_{f,g}$ and $j_{g,f}$ are mutually inverse isomorphisms.

3.5.3. Say that $D(f) \subset D(g) \subset D(h)$ for some $f, g, h \in P$; by direct inspection, it is clear that $j_{f,g} \circ j_{g,h} = j_{f,h}$, so the rule $D(f) \mapsto P_f$ yields a well defined presheaf of monoids on the site $\mathscr{C}_P$ of open subsets of $\operatorname{Spec} P$ of the form $D(f)$ for some $f \in P$. Then $\mathscr{O}_{\operatorname{Spec} P}$ is trivially a sheaf on $\mathscr{C}_P$ (notice that if $D(f) = \bigcup_{i \in I} D(g_i)$ is an open covering of $D(f)$, then $D(g_i) = D(f)$ for some $i \in I$). According to [26, Ch.0, §3.2.2] it follows that $\mathscr{O}_{\operatorname{Spec} P}$ extends uniquely to a well defined sheaf of monoids on $\operatorname{Spec} P$, whence a monoidal space $(\operatorname{Spec} P, \mathscr{O}_{\operatorname{Spec} P})$. By inspecting the construction, we find natural identifications :

$$(\mathscr{O}_{\operatorname{Spec} P})_{\mathfrak{p}} \xrightarrow{\sim} P_{\mathfrak{p}} \qquad \text{for every } \mathfrak{p} \in \operatorname{Spec} P \tag{3.5.4}$$

and moreover :

$$P \xrightarrow{\sim} \Gamma(\operatorname{Spec} P, \mathscr{O}_{\operatorname{Spec} P}).$$

It is also clear that the rule

$$P \mapsto (\operatorname{Spec} P, \mathscr{O}_{\operatorname{Spec} P}) \tag{3.5.5}$$

defines a functor from the category $\mathbf{Mnd}^o$ to the category of monoidal spaces.

Proposition 3.5.6. *The functor* (3.5.5) *is right adjoint to the functor :*

$$(T, \mathscr{O}_T) \mapsto \Gamma(T, \mathscr{O}_T)$$

from the category of monoidal spaces, to the category $\mathbf{Mnd}^o$*.*

Proof. Let $f : P \to \Gamma(T, \mathscr{O}_T)$ be a map of monoids. We define a morphism

$$\varphi_f := (\varphi_f, \log \varphi_f) : (T, \mathscr{O}_T) \to (\operatorname{Spec} P, \mathscr{O}_{\operatorname{Spec} P})$$

as follows. Given $t \in T$, let $f_t : P \to \mathscr{O}_{T,t}$ be the morphism deduced from f, and denote by $\mathfrak{m}_t \subset \mathscr{O}_{T,t}$ the maximal ideal. We set $\varphi_f(t) := f_t^{-1}(\mathfrak{m}_t)$. In order to show that φ_f is continuous, it suffices to prove that $U_s := \varphi_f^{-1}(D(s))$ is open in M_T, for every $s \in P$. However, $U_s = \{t \in T \mid f_t(s) \in \mathscr{O}_{T,t}^\times\}$, and it is easily seen that this condition defines an open subset (details left to the reader). Next, we define $\log \varphi_f$ on the basic open subsets $D(s)$. Indeed, let $j_s : \Gamma(T, \mathscr{O}_T) \to \mathscr{O}_T(U_s)$ be the natural map; by construction, $j_s \circ f(s)$ is invertible in $\mathscr{O}_T(U_s)$, hence $j_s \circ f$ extends to a unique map of monoids :

$$P_s = \mathscr{O}_{\operatorname{Spec} P}(D(s)) \to \varphi_{f*}\mathscr{O}_T(D(s)).$$

By [26, Ch.0, §3.2.5], the above rule extends to a unique morphism $\mathscr{O}_{\operatorname{Spec} P} \to \varphi_{f*}\mathscr{O}_T$ of sheaves of monoids, whence – by adjunction – a well defined morphism $\log \varphi_f : \varphi_f^*\mathscr{O}_{\operatorname{Spec} P} \to \mathscr{O}_T$. In order to show that $(\varphi_f, \log \varphi_f)$ is the sought morphism of monoidal spaces, it remains to check that $(\log \varphi_f)_t : P_{\varphi_f(t)} \to \mathscr{O}_{T,t}$ is a local morphism, for every $t \in T$. However, let $i_t : P \to P_{\varphi_f(t)}$ be the localization map; by construction, we have $(\log \varphi_f)_t \circ i_t = f_t$, and the contention is a straightforward consequence.

Conversely, say that $(\varphi, \log \varphi) : (T, \mathscr{O}_T) \to (\operatorname{Spec} P, \mathscr{O}_{\operatorname{Spec} P})$ is a morphism of monoidal spaces; then $\log \varphi$ corresponds to a unique morphism $\psi : \mathscr{O}_{\operatorname{Spec} P} \to \varphi_*\mathscr{O}_T$, and we set

$$f_\varphi := \Gamma(\operatorname{Spec} P, \psi) : P \to \Gamma(T, \mathscr{O}_T).$$

By inspecting the definitions, it is easily seen that $f_{\varphi_f} = f$ for every morphism of monoids f as above. To conclude, it remains only to show that the rule $(\varphi, \log \varphi) \mapsto f_\varphi$ is injective. However, for a given morphism of monoidal spaces $(\varphi, \log \varphi)$ as above, and every point $t \in T$, we have a commutative diagram of monoids :

$$\begin{tikzcd} P \arrow[r, "f_\varphi"] \arrow[d] & \Gamma(T, \mathscr{O}_T) \arrow[d, "j_t"] \\ P_{\varphi(t)} \arrow[r, "\log \varphi_t"] & \mathscr{O}_{T,t} \end{tikzcd}$$

Since $\log \varphi_t$ is local, it follows that $\varphi(t) = (f_\varphi \circ j_t)^{-1}\mathfrak{m}_t$, especially f_φ determines $\varphi : T \to \operatorname{Spec} P$. Finally, since the map $P \to P_{\varphi(t)}$ is an epimorphism, we see that $\log \varphi_t$ is determined by f_φ as well, and the proposition follows. □

Definition 3.5.7. Let $(T, \mathscr{O}_T)$ be a sharp monoidal space.

(i) We say that $(T, \mathscr{O}_T)$ is an *affine fan*, if there exists a monoid P and an isomorphism of sharp monoidal spaces $(\operatorname{Spec} P, \mathscr{O}_{\operatorname{Spec} P})^\sharp \xrightarrow{\sim} (T, \mathscr{O}_T)$.

(ii) In the situation of (i), if P can be chosen to be finitely generated (resp. fine), we say that $(T, \mathscr{O}_T)$ is a *finite* (resp. *fine*) *affine fan*.

(iii) We say that $(T, \mathscr{O}_T)$ is a *fan*, if there exists an open covering $T = \bigcup_{i \in I} U_i$, such that the induced sharp monoidal space $(U_i, \mathscr{O}_{T|U_i})$ is an affine fan, for every $i \in I$. We denote by $\mathbf{Fan}$ the full subcategory of the category of monoidal spaces, whose objects are the fans.

(iv) In the situation of (iii), if the covering $(U_i \mid i \in I)$ can be chosen, so that $(U_i, \mathscr{O}_{T|U_i})$ is a finite (resp. fine) affine fan for every $i \in I$, we say that $(T, \mathscr{O}_T)$ is *locally finite* (resp. *locally fine*).

(v) We say that the fan $(T, \mathscr{O}_T)$ is *finite* (resp. *fine*) if it is locally finite (resp. locally fine) and quasi-compact.

(vi) Let $(T, \mathscr{O}_T)$ be a fan. The *simplicial locus* $T_{\mathrm{sim}} \subset T$ is the subset of all $t \in T$ such that $\mathscr{O}_{T,t}$ is a free monoid of finite rank.

Remark 3.5.8. (i) For every monoid P, let T_P denote the affine fan $(\mathrm{Spec}\, P)^\sharp$. In light of proposition 3.5.6, it is easily seen that the functor $P \mapsto T_P$ is an equivalence from the opposite of the full subcategory of sharp monoids, to the category of affine fans.

(ii) Since the saturation functor commutes with localizations (lemma 3.2.9(i)), it is easily seen that the saturation of a fan is a fan, and more precisely, the saturation of an affine fan T_P, is naturally isomorphic to $T_{P^{\mathrm{sat}}}$.

(iii) Let Q_1 and Q_2 be two monoids; since the product $P \times Q$ is also the coproduct of P and Q in the category $\mathbf{Mnd}$ (see example 2.3.30(i)), we have a natural isomorphism in the category of fans :

$$T_{P\times Q} \stackrel{\sim}{\to} T_P \times T_Q.$$

More generally, suppose that $P \to Q_i$, for $i = 1, 2$, are two morphisms of monoids. Then we have a natural isomorphism of fans :

$$T_{Q_1 \otimes_P Q_2} \stackrel{\sim}{\to} T_{Q_1} \times_{T_P} T_{Q_2}.$$

From this, a standard argument shows that fibre products are representable in the category of fans.

(iv) Furthermore, lemma 3.1.16(ii) implies that the natural map :

$$\pi : \mathrm{Spec}\,(P \times Q) \to \mathrm{Spec}\, P \times \mathrm{Spec}\, Q$$

is a homeomorphism (where the product of $\mathrm{Spec}\, P$ and $\mathrm{Spec}\, Q$ is taken in the category of topological spaces and continuous maps).

(v) Moreover, we have natural isomorphisms of monoids :

$$\mathscr{O}_{T_{P\times Q}, \pi^{-1}(s,t)} \stackrel{\sim}{\to} \mathscr{O}_{T_P,s} \times \mathscr{O}_{T_Q,t} \qquad \text{for every } s \in \mathrm{Spec}\, P \text{ and } t \in \mathrm{Spec}\, Q.$$

(vi) For any fan $T := (T, \mathscr{O}_T)$, and any monoid M, we shall use the standard notation :

$$T(M) := \mathrm{Hom}_{\mathbf{Fan}}((\mathrm{Spec}\, M)^\sharp, T).$$

Especially, if T is an affine fan, say $T = (\mathrm{Spec}\, P)^\sharp$, then $T(M) = \mathrm{Hom}_{\mathbf{Mnd}}(P, M^\sharp)$; for instance, if T is affine, $T(\mathbb{N})$ is a monoid, and $T(\mathbb{Q}_+)^{\mathrm{gp}}$ is a $\mathbb{Q}$-vector space. Furthermore, by standard general nonsense we have natural identifications of sets :

$$(T_1 \times_T T_2)(M) \stackrel{\sim}{\to} T_1(M) \times_{T(M)} T_2(M)$$

for any pair of T-fans T_1 and T_2, and every monoid M. If T, T_1 and T_2 are affine, this identification is also an isomorphism of monoids.

Example 3.5.9. (i) The topological space underlying the affine fan $(\mathrm{Spec}\, \mathbb{N}, \mathscr{O}_{\mathrm{Spec}\, \mathbb{N}})^\sharp$ consists of two points : $\mathrm{Spec}\, \mathbb{N} = \{\varnothing, \mathfrak{m}\}$, where $\mathfrak{m} := \mathbb{N}\backslash\{0\}$ is the closed point. The structure sheaf $\mathscr{O} := \mathscr{O}_{\mathrm{Spec}\, \mathbb{N}}$ is determined as follows. The two stalks are $\mathscr{O}_\varnothing = \{1\}$ (the trivial monoid) and $\mathscr{O}_\mathfrak{m} = \mathbb{N}$; the global sections are $\Gamma(\mathrm{Spec}\, \mathbb{N}, \mathscr{O}) = \mathbb{N}$.

(ii) Let $(T, \mathscr{O}_T)$ be any fan, P any monoid, with maximal ideal $\mathfrak{m}_P$, and $\varphi : T_P := (\mathrm{Spec}\, P, \mathscr{O}_{\mathrm{Spec}\, P})^\sharp \to (T, \mathscr{O}_T)$ a morphism of fans. Say that $\varphi(\mathfrak{m}_P) \in U$ for some affine open subset $U \subset F$; then $\varphi(\mathrm{Spec}\, P) \subset U$, hence φ factors through a morphism of fans $T_P \to (U, \mathscr{O}_{T|U})$. In view of proposition 3.5.6, such a morphism corresponds to a unique morphism of monoids $\varphi^\natural : \mathscr{O}_T(U) \to P^\sharp$, and then $\varphi(\mathfrak{m}_P) = \varphi^{\natural-1}(\mathfrak{m}_P) \in \operatorname{Spec} \mathscr{O}_T(U) = U$. The map on stalks determined by φ is the local morphism

$$\mathscr{O}_{T,\varphi(\mathfrak{m}_P)} \xrightarrow{\sim} \mathscr{O}_T(U)_{\varphi(\mathfrak{m}_P)}/\mathscr{O}_T(U)^\times_{\varphi(\mathfrak{m}_P)} \to P^\sharp$$

obtained from $\varphi^\natural$ after localization at the prime ideal $\varphi^{\natural-1}(\mathfrak{m}_P)$.

(iii) For any two monoids M and N, denote by $\text{loc.Hom}_{\mathbf{Mnd}}(M, N)$ the set of local morphisms of monoids $M \to N$. The discussion in (ii) leads to a natural identification :

$$T(P) \xrightarrow{\sim} \coprod_{t\in T} \text{loc.Hom}_{\mathbf{Mnd}}(\mathscr{O}_{T,t}, P^\sharp)$$

For any monoid P. The *support* of a P-point $\varphi \in T(P)$ is the unique point $t \in T$ such that φ corresponds to a local morphism $\mathscr{O}_{T,t} \to P^\sharp$.

Example 3.5.10. (i) Let P be any monoid, $k > 0$ any integer, set $T_P := (\operatorname{Spec} P)^\sharp$, and let $\boldsymbol{k}_P : P \to P$ be the k-Frobenius map of P (definition 2.3.40(ii)). Then

$$\boldsymbol{k}_{T_P} := \operatorname{Spec} \boldsymbol{k}_P : T_P \to T_P$$

is a well defined endomorphism inducing the identity on the underlying topological space.

(ii) More generally, let F be any fan; for every integer $k > 0$ we have the *k-Frobenius endomorphism*

$$\boldsymbol{k}_F : F \to F$$

which induces the identity on the underlying topological space, and whose restriction to any affine open subfan $U \subset F$ is the endomorphism $\boldsymbol{k}_U$ defined as in (i).

3.5.11. Let P be a monoid, M a P-module, and set $T_P := (\operatorname{Spec} P)^\sharp$. We define a presheaf $M^\sim$ on the site of basic affine open subsets $D(f) \subset \operatorname{Spec} P$ (for all $f \in P$), by the rule :

$$U \mapsto M^\sim(U) := M \otimes_P \mathscr{O}_{T_P}(U)$$

(and for an inclusion $U' \subset U$ of basic open subsets, the corresponding morphism $M^\sim(U) \to M^\sim(U')$ is deduced from the restriction map $\mathscr{O}_{T_P}(U) \to \mathscr{O}_{T_P}(U')$). It is easily seen that $M^\sim$ is a sheaf, hence it extends to a well defined sheaf of $\mathscr{O}_{T_P}$-modules on T_P ([26, Ch.0, §3.2.5]). Clearly $\Gamma(T_P, M^\sim) = M$, and the rule $M \mapsto M^\sim$ yields a well defined functor P-$\mathbf{Mod} \to \mathscr{O}_{T_P}$-$\mathbf{Mod}$, which is left adjoint to the global section functor on $\mathscr{O}_{T_P}$-modules : $\mathscr{M} \mapsto \Gamma(T_P, \mathscr{M})$ (verification left to the reader).

Definition 3.5.12. Let $(T, \mathscr{O}_T)$ be a fan, $\mathscr{M}$ a $\mathscr{O}_T$-module. We say that $\mathscr{M}$ is *quasi-coherent*, if there exists an open covering $T = \bigcup_{i\in I} U_i$ of T by affine open subsets, and for each $i \in I$ a $\mathscr{O}_T(U_i)$-module M_i with an isomorphism of $\mathscr{O}_{T|U_i}$-modules $\mathscr{M}_{|U_i} \xrightarrow{\sim} M_i^\sim$.

Remark 3.5.13. (i) Let $(T, \mathscr{O}_T)$ be a fan, $\mathscr{M}$ a quasi-coherent $\mathscr{O}_T$-module, and $U \subset T$ be any open subset, such that $(U, \mathscr{O}_{T|U})$ is an affine fan (briefly : an affine open subfan of T). Then, since U admits a unique closed point $t \in U$, it is easily seen that $\mathscr{M}_{|U}$ is naturally isomorphic to $\mathscr{M}_t^\sim$ as a $\mathscr{O}_{T|U}$-module.

(ii) In the same vein, if $\mathscr{M}$ is an invertible $\mathscr{O}_T$-module (see definition 2.3.6(iv)), then the restriction $\mathscr{M}_{|U}$ of $\mathscr{M}$ to any affine open subset, is isomorphic to $\mathscr{O}_{T|U}$.

(iii) For any fan $(T, \mathscr{O}_T)$, the sheaf of abelian groups $\mathscr{O}_T^{\text{gp}}$ is quasi-coherent (exercise for the reader). Suppose that T is integral; then an $\mathscr{O}_T$-submodule $\mathscr{I} \subset \mathscr{O}_T^{\text{gp}}$ is called a *fractional ideal* (resp. a *reflexive fractional ideal*) of $\mathscr{O}_T$ if $\mathscr{I}$ is quasi-coherent, and $\mathscr{I}(U)$ is a fractional ideal (resp. a reflexive fractional ideal) of $\mathscr{O}_T(U)$, for every affine open subset $U \subset T$.

(iv) Let P be any integral monoid, $I \subset P^{\text{gp}}$ a fractional ideal of P, and set $T_P := (\operatorname{Spec} P)^\sharp$. It follows easily from lemma 3.4.22(i) that $I^\sim \subset \mathscr{O}_{T_P}^{\text{gp}}$ is a fractional ideal of $\mathscr{O}_{T_P}$, and $I^\sim$ is reflexive if and only if I is a reflexive fractional ideal of P (lemma 3.4.22(ii)).

(v) Suppose that T is locally finite; in this case, it follows easily from proposition 3.1.9(ii) that every quasi-coherent ideal of $\mathscr{O}_T$ is coherent. Likewise, if T is also integral, and $\mathscr{I} \subset \mathscr{O}_T^{\mathrm{gp}}$ is a (quasi-coherent) fractional ideal of $\mathscr{O}_T$, then $\mathscr{I}$ is coherent, provided the stalks $\mathscr{I}_t$ are finitely generated $\mathscr{O}_{T,t}$-modules, for every $t \in T$. (Details left to the reader.)

Remark 3.5.14. (i) Let T be any integral fan. We define a sheaf $\mathscr{D}iv_T$ on T, by letting $\mathscr{D}iv_T(U)$ be the set of all reflexive fraction ideals of $\mathscr{O}_U$, for every open subset $U \subset T$.

(ii) Now, suppose that T is locally fine; in this case, we can endow $\mathscr{D}iv_T$ with a natural structure of T-monoid, as follows. First, we define a presheaf of monoids on the site $\mathscr{C}_T$ of affine open subsets of T by the rule :

$$U \mapsto \mathscr{D}iv_T(U) := (\mathrm{Div}(\mathscr{O}_T(U)), \odot)$$

(notation of (3.4.20)) and for an inclusion $U' \subset U$ of affine open subset, the corresponding morphism of monoids $\mathscr{D}iv_T(U) \to \mathscr{D}iv_T(U')$ is deduced from the flat map $\mathscr{O}_T(U) \to \mathscr{O}_T(U')$, by virtue of lemma 3.4.27(iv). Arguing as in (3.5.3), we see that $\mathscr{D}iv_T$ is a sheaf on $\mathscr{C}_T$, and then [26, Ch.0, §3.2.2] implies that $\mathscr{D}iv_T$ extends uniquely to a sheaf of monoids on T. It is then clear that the sheaf of sets underlying this T-monoid is (naturally isomorphic to) the sheaf defined in (i).

(iii) In the situation of (ii), we have likewise a T-submonoid $\mathscr{D}iv_T^+ \subset \mathscr{D}iv_T$ (remark 3.4.29), and we may also define a T-monoid $\overline{\mathscr{D}iv}_T$ (see (3.4.35)). Moreover, we have the global version of (3.4.36) : namely, the sequence of T-monoids

$$1 \to \mathscr{O}_T^{\mathrm{gp}} \xrightarrow{j_T} \mathscr{D}iv_T \to \overline{\mathscr{D}iv}_T \to 1$$

is exact (recall that $\mathscr{O}_T^\times = 1_T$, the initial T-monoid), and j_T restricts to a map of T-monoids

$$\mathscr{O}_T \to \mathscr{D}iv_T^+.$$

Indeed, the assertion can be checked on the stalks over each $t \in T$, where it reduces to the exact sequence (3.4.36) for $P := \mathscr{O}_{T,t}$. Lastly, we remark that, if T is locally fine and saturated, then $\mathscr{D}iv_T$ and $\overline{\mathscr{D}iv}_T$ are abelian T-groups (proposition 3.4.25(i,ii)).

3.5.15. If $f : T' \to T$ is a morphism of fans, and $\mathscr{M}$ is any $\mathscr{O}_T$-module, then we define as usual the $\mathscr{O}_{T'}$-module :

$$f^*\mathscr{M} := f^{-1}\mathscr{M} \otimes_{f^{-1}\mathscr{O}_T} \mathscr{O}_{T'}$$

where $f^{-1}\mathscr{M}$ denotes the usual sheaf-theoretic inverse image of $\mathscr{M}$ (so $f^{-1}\mathscr{O}_T$ means here what was denoted $f^*\mathscr{O}_T$ in definition 3.5.1(ii)). The rule $\mathscr{M} \mapsto f^*\mathscr{M}$ yields a left adjoint to the functor

$$\mathscr{O}_{T'}\text{-}\mathbf{Mod} \to \mathscr{O}_T\text{-}\mathbf{Mod} \qquad \mathscr{N} \mapsto f_*\mathscr{N}$$

(verification left to the reader). Notice that, if $\mathscr{M}$ is quasi-coherent, then $f^*\mathscr{M}$ is a quasi-coherent $\mathscr{O}_{T'}$-module. Indeed, the assertion is local on T', hence we are reduced to the case where $T' = (\mathrm{Spec}\, P')$ and $T = (\mathrm{Spec}\, P)$ for some monoids P and P'. In this case, the functor $M \mapsto f^*(M^\sim) : P\text{-}\mathbf{Mod} \to \mathscr{O}_{T'}\text{-}\mathbf{Mod}$ is left adjoint to the functor $\mathscr{M} \mapsto \Gamma(T', \mathscr{M})$ on $\mathscr{O}_{T'}$-modules. The latter functor also admits the left adjoint given by the rule : $M \mapsto (M \otimes_P P')^\sim$, whence a natural isomorphism of $\mathscr{O}_{T'}$-modules :

$$f^*(M^\sim) \xrightarrow{\sim} (M \otimes_P P')^\sim.$$

3.5.16. Let $T := (T, \mathscr{O}_T)$ be a fan, $t \in T$ any point. The *height* of t is :

$$\mathrm{ht}_T(t) := \dim \mathscr{O}_{T,t} \in \mathbb{N} \cup \{+\infty\}$$

(see definition 3.1.18) and the *dimension* of T is $\dim T := \sup(\mathrm{ht}_T(t) \mid t \in T)$. (If $T = \varnothing$ is the empty fan, we let $\dim T := -\infty$.)

Suppose that T is locally finite; then it follows from (3.5.4) and lemma 3.1.20(iii),(iv) that the height of any point of T is an integer. Moreover, let $U(t) \subset T$ denote the subset of all points $x \in T$ which specialize to t (*i.e.* such that the topological closure of $\{x\}$ in T contains t); clearly $U(t)$ is the intersection of all the open neighborhoods of t in T, and we have a natural homeomorphism :

$$\text{(3.5.17)} \qquad \operatorname{Spec} \mathscr{O}_{T,t} \xrightarrow{\sim} U(t).$$

Especially, if T is locally finite, $U(t)$ is a finite set, and moreover $U(t)$ is an open subset : indeed, if $U \subset T$ is any finite affine open neighborhood of t, we have $U(t) \subset U$, hence $U(t)$ can be realized as the intersection of the finitely many open neighborhoods of t in U. In this case, (3.5.17) induces an isomorphism of fans :

$$\text{(3.5.18)} \qquad (\operatorname{Spec} \mathscr{O}_{T,t})^\sharp \xrightarrow{\sim} (U(t), \mathscr{O}_{T|U(t)}).$$

Therefore, for every $h \in \mathbb{N}$, let $T_h \subset T$ be the subset of all points of T of height $\leq h$; clearly $U(t) \subset T_h$ whenever $t \in T_h$, hence the foregoing shows that – if T is locally finite – T_h is an open subset of T for every $h \in \mathbb{N}$, and $T = \bigcup_{h \in \mathbb{N}} T_h$.

Notice also that the simplicial locus of a fan T is closed under generizations. Therefore, T_{sim} is an open subset of T, whenever T is locally finite.

3.5.19. In the situation of remark 3.5.8(v), suppose additionally that P and Q are finitely generated. The natural projection $P \times Q \to P$ induces a morphism $j : T_P \to T_{P\times Q}$ of affine fans, and it is easily seen that $j(t) = \pi^{-1}(t, \varnothing)$ for every $t \in T_P$. It follows that the restriction of j is a homeomorphism $U(t) \xrightarrow{\sim} U(j(t))$ for every $t \in T_P$, and moreover $\log j : j^*\mathscr{O}_{T_{P\times Q}} \to \mathscr{O}_{T_P}$ is an isomorphism. We conclude that j is an open immersion.

3.5.20. Let P be a fine, sharp and saturated monoid, and set $Q := P^\vee$ (notation of (3.4.11)). By proposition 3.4.12(iv), we have a natural identification $P \xrightarrow{\sim} Q^\vee$. By proposition 3.4.7(ii) and corollary 3.3.12(ii), the rule

$$\text{(3.5.21)} \qquad F \mapsto F^* := F^*_{\mathbb{R}} \cap P$$

establishes a natural bijection from the faces of Q to those of P. For every face F of Q, set $\mathfrak{p}_F := P \setminus F^*$; there follows a natural bijection $F \mapsto \mathfrak{p}_F$ between the set of all faces of Q and $\operatorname{Spec} P$, such that

$$F \subset F' \Leftrightarrow \mathfrak{p}_F \subset \mathfrak{p}_{F'}.$$

Moreover, set $T_P := (\operatorname{Spec} P)^\sharp$; we have natural identifications :

$$F^\vee \xrightarrow{\sim} \mathscr{O}_{T_P,\mathfrak{p}_F} \qquad \text{for every face } F \text{ of } Q$$

under which, the specialization maps $\mathscr{O}_{T_P,\mathfrak{p}_{F'}} \to \mathscr{O}_{T_P,\mathfrak{p}_F}$ correspond to the restriction maps $(F')^\vee \to F^\vee : \varphi \mapsto \varphi_{|F}$.

Definition 3.5.22. Let $T := (T, \mathscr{O}_T)$ be a fan.

(i) An *integral* (resp. a *rational*) *partial subdivision* of T is a morphism $f : (T', \mathscr{O}_{T'}) \to T$ of fans such that, for every $t \in T'$, the group homomorphism

$$(\log f)^{\mathrm{gp}}_t : \mathscr{O}^{\mathrm{gp}}_{T,f(t)} \to \mathscr{O}^{\mathrm{gp}}_{T',t} \qquad \text{(resp. the } \mathbb{Q}\text{-linear map } (\log f)^{\mathrm{gp}}_t \otimes_{\mathbb{Z}} \mathbf{1}_{\mathbb{Q}})$$

is surjective.

(ii) If $f : T' \to T$ is an integral (resp. rational) partial subdivision, and the induced map

$$T'(\mathbb{N}) \to T(\mathbb{N}) \qquad (\text{resp. } T'(\mathbb{Q}_+) \to T(\mathbb{Q}_+)) \quad : \quad \varphi \mapsto f \circ \varphi$$

is bijective, we say that f is an integral (resp. a rational) *subdivision* of T.

(iii) A morphism of fans $f : T' \to T$ is *finite* (resp. *proper*), if the fibre $f^{-1}(t)$ is a finite (resp. and non-empty) set, for every $t \in T$.

(iv) A subdivision $T' \to T$ of T is *simplicial*, if $T'_{\mathrm{sim}} = T'$.

Remark 3.5.23. (i) Let T be any integral fan. Then the counit of adjunction

$$T^{\mathrm{sat}} \to T$$

is an integral subdivision. This morphism is also a homeomorphism on the underlying topological spaces, in light of 3.2.9(iv).

(ii) Let $f : T' \to T$ be any integral subdivision. Then f restricts to a bijection $T'_0 \xrightarrow{\sim} T_0$ on the sets of points of height zero. Indeed, notice that, if $t' \in T'$ is a point of height zero, then $f(t') \in T_0$ since the map $(\log f)^{\mathrm{gp}}_{t'}$ must be local; moreover $\mathrm{loc.Hom}_{\mathbf{Mnd}}(\mathscr{O}_{T,t'}, \mathbb{N})$ consists of precisely one element, namely the unique map $\sigma_t : \mathbb{N} \to \{1\}$, and if $t'_1, t'_2 \in T'_0$ have the same image in T_0, the sections $\sigma_{t'_1}$ and $\sigma_{t'_2}$ have the same image in $T(\mathbb{N})$, hence they must coincide, so that $t'_1 = t'_2$, as claimed.

(iii) Let $f : T' \to T$ be an integral subdivision of locally fine and saturated fans. In general, the image of a point $t' \in T'$ of height one may have height strictly greater than one. On the other hand, for any $t \in T$ of height one, and any $t' \in f^{-1}(t)$, the map $\mathbb{Z} \xrightarrow{\sim} \mathscr{O}^{\mathrm{gp}}_{T,t} \to \mathscr{O}^{\mathrm{gp}}_{T',t'}$ must be surjective (theorem 3.4.16(ii)), therefore $\mathscr{O}^{\mathrm{gp}}_{T',t'}$ is a cyclic group; however $\mathscr{O}_{T',t'}$ is also sharp and saturated, so it must be either the trivial monoid $\{1\}$ or $\mathbb{N}$. The first case is excluded by (ii), so $\mathrm{ht}(t') = 1$, and moreover $(\log f)_{t'}$ is an isomorphism (and there exists a unique such isomorphism). Since the induced map $T'(\mathbb{N}) \to T(\mathbb{N})$ is bijective, it follows easily that $f^{-1}(t)$ consists of exactly one point, and therefore f restricts to an isomorphism $f^{-1}(T_1) \xrightarrow{\sim} T_1$.

Proposition 3.5.24. *Let $f : T' \to T$ be a morphism of fans, with T' locally finite, and consider the following conditions :*

(a) *The induced map $T'(\mathbb{N}) \to T(\mathbb{N})$ is injective.*
(b) *For every integral saturated monoid P, the induced map $T'(P) \to T(P)$ is injective.*
(c) *f is a partial rational subdivision.*

Then we have : (a) $\Leftrightarrow$ (b) $\Rightarrow$ (c).

Proof. Obviously (b) $\Rightarrow$ (a). Conversely, assume that (a) holds, let P be a saturated monoid, and suppose we have two sections in $T'(P)$ whose images in $T(P)$ agree. In light of example 3.5.9(iii), this means that we may find two points $t'_1, t'_2 \in T'$, such that $f(t'_1) = f(t'_2) = t$, and two local morphisms of monoids $\sigma_i : \mathscr{O}_{T',t'_1} \to P/P^\times$ whose compositions with $\log f_{t'_i}$ ($i = 1, 2$) yield the same morphism $\mathscr{O}_{T,t} \to P/P^\times$, and we have to show that these maps are equal. In view of lemma 3.2.9(ii), we may then replace P by $P/P^\times$, and assume that P is sharp. Since the stalks of $\mathscr{O}_{T'}$ are finitely generated, the morphisms σ_i factor through a finitely generated submonoid $M \subset P$. We may then replace P by its submonoid M^{sat}, which allows to assume additionally that P is finitely generated (corollary 3.4.1(ii)). In this case, we may find an injective map $j : P \to \mathbb{N}^{\oplus r}$ (corollary 3.4.10(iv); notice that j is trivially a local morphism), hence we may replace σ_i by $j \circ \sigma_i$ (for $i = 1, 2$), after which we may assume that $P = \mathbb{N}^{\oplus r}$ for some $r \in \mathbb{N}$. Let $\delta : P \to \mathbb{N}$ be the local morphism given by the rule : $(x_1, \dots, x_r) \mapsto x_1 + \dots + x_r$ for every $x_1, \dots, x_r \in \mathbb{N}$; the compositions $\delta \circ \sigma_i$ (for $i = 1, 2$) are two elements of $T'(\mathbb{N})$ whose images agree in $T(\mathbb{N})$, hence they must coincide by assumption. This implies already that $t'_1 = t'_2$. Next, let $\pi_k : P \to \mathbb{N}$ (for $k = 1, \dots, r$) be the natural projections, and fix $k \leq r$; the morphisms $\pi_k \circ \sigma_i$ for $i = 1, 2$ are not necessarily local, but they determine elements of $T'(\mathbb{N})$ whose images agree again in $T(\mathbb{N})$, hence they must coincide. Since k is arbitrary, we deduce that $\sigma_1 = \sigma_2$, as stated.

Next, we suppose that (b) holds, and we wish to show assertion (c); the latter is local on F', hence we may assume that both F and F' are affine, say $F = (\mathrm{Spec}\, Q)^\sharp$ and $F' = (\mathrm{Spec}\, Q')^\sharp$, with Q' finitely generated and sharp, and then we are reduced to checking that the map $Q^{\mathrm{gp}} \otimes_{\mathbb{Z}}$

$\mathbb{Q} \to Q'^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$ induced by f is surjective, or equivalently, that the dual map :

$$\mathrm{Hom}_{\mathbf{Mnd}}(Q', \mathbb{Q}) \to \mathrm{Hom}_{\mathbf{Mnd}}(Q, \mathbb{Q})$$

is injective. To this aim, we may further assume that Q' is integral, in which case, by remark 3.4.14(i) we have $\mathrm{Hom}_{\mathbf{Mnd}}(Q', \mathbb{Q}) = \mathrm{Hom}_{\mathbf{Mnd}}(Q', \mathbb{Q}_+)^{\mathrm{gp}}$; the contention is an easy consequence. □

3.5.25. In light of proposition 3.5.24, we may ask whether the surjectivity of the map on P-points induced by a morphism f of fans can be similarly characterized. This turns out to be the case, but more assumptions must be made on the morphism f, and also some additional restrictions must be imposed on the type of monoid P. Namely, we shall consider monoids of the form Γ_+, where $(\Gamma, \leq)$ is any totally ordered abelian group, and $\Gamma_+ \subset \Gamma$ is the subgroup of all elements ≤ 1 (where $1 \in \Gamma$ denotes the neutral element). With this notation, we have the following :

Proposition 3.5.26. *Let $f : T' \to T$ be a finite partial integral subdivision, with T locally finite. The following conditions are equivalent :*

(a) *The induced map $T'(\mathbb{N}) \to T(\mathbb{N})$ is surjective.*
(b) *For every totally ordered abelian group $(\Gamma, \geq)$, the induced map $T'(\Gamma_+) \to T(\Gamma_+)$ is surjective.*

Proof. Obviously, we need only to show that (a) ⇒ (b). Thus suppose, by way of contradiction, that (a) holds, but nevertheless there exists a totally ordered abelian group $(\Gamma, \leq)$, and an element of $T(\Gamma_+)$ which is not in the image of $T'(\Gamma_+)$. Such element corresponds to a local morphism of monoids $\varphi : \mathscr{O}_{T,t} \to \Gamma_+$, for some $t \in T$, and the assumption means that φ does not factor through the monoid $\mathscr{O}_{T',s}$, for any $s \in f^{-1}(t)$. Set

$$P := \mathscr{O}^{\mathrm{int}}_{T,t} \qquad Q_s := P^{\mathrm{gp}} \times_{\mathscr{O}^{\mathrm{gp}}_{T',s}} \mathscr{O}^{\mathrm{int}}_{T',s} \quad \text{for every } s \in f^{-1}(t)$$

Notice that, since the map $P^{\mathrm{gp}} \to \mathscr{O}^{\mathrm{gp}}_{T',s}$ is surjective, we have

$$Q_s/Q_s^{\times} \simeq \mathscr{O}^{\mathrm{int}}_{T',s}/(\mathscr{O}^{\mathrm{int}}_{T',s})^{\times}$$

and there is a natural injective morphism of monoids $g_s : P \to Q_s$, determined by the pair $(i, (\log f)^{\mathrm{int}}_s)$, where $i : P^{\mathrm{int}} \to P^{\mathrm{gp}}$ is the natural morphism; moreover, $P^{\mathrm{gp}} = Q_s^{\mathrm{gp}}$ for every $s \in f^{-1}(t)$. Clearly φ factors through a morphism $\overline{\varphi} : P \to \Gamma_+$; since the unit of adjunction $\mathscr{O}_{T,t} \to P$ is surjective, it follows that P is sharp and $\overline{\varphi}$ is local. Moreover, the group homomorphism $\overline{\varphi}^{\mathrm{gp}} : P^{\mathrm{gp}} \to \Gamma$ factors uniquely through each Q_s^{gp}. Our assumption then states that we may find, for each $s \in f^{-1}(t)$, an element $x_s \in Q_s$ whose image in Γ lies in the complement of Γ_+, *i.e.* the image of x_s^{-1} lies in the maximal ideal $\mathfrak{m} \subset \Gamma_+$. Let $P' \subset P^{\mathrm{gp}}$ be the submonoid generated by P and by $(x_s^{-1} \mid s \in f^{-1}(t))$. By construction, P' is finitely generated, and the morphism $\overline{\varphi}$ extends uniquely to a morphism $P' \to \Gamma_+$, which maps each x_s^{-1} into $\mathfrak{m}$. It follows that all the x_s^{-1} lie in the maximal ideal of P'. Let us now pick any local morphism $\psi' : P' \to \mathbb{N}$ (corollary 3.4.10(iii)); by restriction, ψ' induces a local morphism $\psi : P \to \mathbb{N}$, which – according to (a) – must factor through a local morphism $\psi_s : Q_s \to \mathbb{N}$, for at least one $s \in f^{-1}(t)$. However, on the one hand we have $\psi'^{\mathrm{gp}} = \psi^{\mathrm{gp}} = \psi_s^{\mathrm{gp}}$; on the other hand $\psi'(x_s^{-1}) \neq 0$, hence $\psi_s(x_s) = \psi'^{\mathrm{gp}}(x_s) \notin \mathbb{N}$, a contradiction. □

Example 3.5.27. Let T be a locally fine fan, $\varphi : F \to T$ an integral subdivision, with F locally fine and saturated, and $k > 0$ an integer. Suppose we have a commutative diagram of fans :

(3.5.28)
$$\begin{tikzcd} F \arrow[r, "\varphi"] \arrow[d, "g"'] & T \arrow[d, "\boldsymbol{k}_T"] \\ F \arrow[r, "\varphi"] & T. \end{tikzcd}$$

where $\boldsymbol{k}_T$ is the k-Frobenius endomorphism (example 3.5.10(ii)). Then we claim that necessarily $g = \boldsymbol{k}_F$. Indeed, suppose that this fails; then we may find a point $t \in F$ such that the composition of g and the open immersion $j_t : U(t) \to F$ is not equal to $j_t \circ \boldsymbol{k}_{U(t)}$. Set $P := \mathscr{O}_{F,t}$; then $g \circ j_t \neq j_t \circ \boldsymbol{k}_{U(t)}$ in $F(P)$. However, an easy computation shows that $\varphi(g \circ j_t) = \varphi(j_t \circ \boldsymbol{k}_{U(t)})$, which contradicts proposition 3.5.24.

3.5.29. Let $P := \coprod_{n\in\mathbb{N}} P_n$ be a $\mathbb{N}$-graded monoid (see definition 2.3.8); then P_0 is a submonoid of P, every P_n is a P_0-module, and $P_+ := \coprod_{n>0} P_n$ is an ideal of P. For every $a \in P$, the localization P_a is $\mathbb{Z}$-graded in an obvious way, and we denote by $P_{(a)} \subset P_a$ the submonoid of elements of degree 0. Notice that there is a natural identification of P_0-monoids

$$P_{(a^n)} \xrightarrow{\sim} P_{(a)} \qquad \text{for every integer } n > 0. \tag{3.5.30}$$

Set as well :

$$D_+(a) := (\mathrm{Spec}\, P_{(a)})^\sharp$$

and notice that the natural map $P \to P_{(a)}$ induces a morphism of fans $\pi_a : D_+(a) \to T_P := (\mathrm{Spec}\, P_0)^\sharp$. If $b \in P$ is any other element, in order to determine the fibre product $D_+(a) \times_{T_P} D_+(b)$ we may assume – in light of (3.5.30) – that $a, b \in P_n$ for the same integer n, in which case we have natural isomorphisms

$$P_{(a)} \otimes_{P_0} P_{(b)} \xrightarrow{\sim} P_{(ab)} \xleftarrow{\sim} P_{(a)}[b^{-1}a]$$

(see remark 3.1.25(i)) onto the localization of $P_{(a)}$ obtained by inverting its element $a^{-1}b$; this is of course the same as $P_{(b)}[a^{-1}b]$. In other words $D_+(a) \times_{T_P} D_+(b)$ is naturally isomorphic to $D_+(ab)$, identified to an open subfan in both $D_+(a)$ and $D_+(b)$. We may then glue the fans $D_+(a)$ for a ranging over all the elements of P, to obtain a new fan, denoted :

$$\mathrm{Proj}\, P$$

called the *projective fan* associated to P. By inspecting the construction, we see that the morphisms π_a assemble to a well defined morphism of fans $\pi_P : \mathrm{Proj}\, P \to T_P$. Each element $a \in P$ yields an open immersion $j_a : D_+(a) \to \mathrm{Proj}\, P$, and if $b \in P$ is any other element, j_{ab} factors through an open immersion $D_+(ab) \to D_+(a)$.

3.5.31. Let $\varphi : P \to P'$ be a morphism of $\mathbb{N}$-graded monoids (so $\varphi P_n \subset P'_n$ for every $n \in \mathbb{N}$). Set :

$$G(\varphi) := \bigcup_{a\in P} D_+(\varphi(a)) \subset \mathrm{Proj}\, P'.$$

Notice that, for every $a \in P$, φ induces a morphism $\varphi_{(a)} : P_{(a)} \to P'_{(a)}$, whence a morphism of affine fans $(\mathrm{Proj}\,\varphi)_a : D_+(\varphi(a)) \to D_+(a) \subset \mathrm{Proj}\, P$. Moreover, if $b \in P$ is any other element, it is easily seen that $(\mathrm{Proj}\,\varphi)_a$ and $(\mathrm{Proj}\,\varphi)_b$ agree on $D_+(\varphi(a)) \cap D_+(\varphi(b))$. Therefore, the morphisms $(\mathrm{Proj}\,\varphi)_a$ glue to a well defined morphism :

$$\mathrm{Proj}\,\varphi : G(\varphi) \to \mathrm{Proj}\, P.$$

Notice that $G(\varphi) = \mathrm{Proj}\, P'$, whenever φP generates the ideal P'_+. Moreover, we have

$$(\mathrm{Proj}\,\varphi)^{-1} D_+(a) = D_+(\varphi(a)) \qquad \text{for every } a \in P. \tag{3.5.32}$$

Indeed, say that $D_+(b) \subset G(\varphi)$ for some $b \in P'$, and $(\mathrm{Proj}\,\varphi) D_+(b) \subset D_+(a)$. In order to show that $D_+(b) \subset D(\varphi(a))$, it suffices to check that $D_+(b\varphi(c)) \subset D_+(\varphi(a))$ for every $c \in P$. However, the assumption means that the natural map

$$P_{(c)} \to P'_{(\varphi(c))} \to P'_{(b\varphi(c))} \to P'_{(b\varphi(c))}/P'^{\times}_{(b\varphi(c))}$$

factors through the localization $P_{(c)} \to P_{(ac)}$. This is equivalent to saying that $\varphi(c^{-1}a)$ is invertible in $P'_{(b\varphi(c))}$, in which case the localization $P'_{(\varphi(c))} \to P'_{(b\varphi(c))}$ factors through the localization

$P'_{(\varphi(c))} \to P'_{(\varphi(ac))}$. The latter means that the open immersion $D_+(b\varphi(c)) \subset D_+(\varphi(c))$ factors through the open immersion $D_+(\varphi(ac)) \subset D_+(\varphi(c))$, as claimed.

3.5.33. In the situation of (3.5.29), set $Y := \mathrm{Proj}\, P$ to ease notation. Let M be a $\mathbb{Z}$-graded P-module; for every $a \in P$, let $M_{(a)} \subset M_a := M \otimes_P P_a$ be the $P_{(a)}$-submodule of degree zero elements (for the natural grading on M_a). We deduce a quasi-coherent $\mathscr{O}_{D_+(a)}$-module $M^\sim_{(a)}$ (see definition 3.5.12). Moreover, if $b \in P$ is any other element, we have a natural identification

$$\widetilde{\omega}_{a,b} : M^\sim_{(a)|D_+(a)\cap D_+(b)} \xrightarrow{\sim} M^\sim_{(b)|D_+(a)\cap D_+(b)}.$$

This can be verified as follows. First, in view of (3.5.30), we may assume that $a, b \in P_n$, for some $n \in \mathbb{N}$, in which case we consider the P-linear morphism :

$$M_{(a)} \to M_{(b)} \otimes_{P_{(b)}} P_{(b)}[a^{-1}b] \quad : \quad \frac{x}{a^m} \mapsto \frac{x}{b^m} \otimes \frac{b^m}{a^m} \qquad \text{for every } x \in M_{nm}.$$

It is easily seen that this map is actually $P_{(a)}$-linear, hence it extends to a $\mathscr{O}_T(D_+(a) \cap D_+(b))$-linear morphism :

$$\omega_{a,b} : M_{(a)} \otimes_{P_{(a)}} P_{(a)}[b^{-1}a] \xrightarrow{\sim} M_{(b)} \otimes_{P_{(b)}} P_{(b)}[a^{-1}b].$$

Moreover, $\omega_{a,b} \circ \omega_{b,a}$ is the identity map, hence $\omega_{a,b}$ induces the sought isomorphism $\widetilde{\omega}_{a,b}$. Furthermore, for any $a, b, c \in P$, set $D_+(a, b, c) := D_+(a) \cap D_+(b) \cap D_+(c)$; we have the identity :

$$\widetilde{\omega}_{a,c|D_+(a,b,c)} = \widetilde{\omega}_{b,c|D_+(a,b,c)} \circ \widetilde{\omega}_{a,b|D_+(a,b,c)}$$

which shows that the locally defined sheaves $M^\sim_{(a)}$ glue to a well defined $\mathscr{O}_Y$-module, which we shall denote $M^\sim$. Especially, for every $n \in \mathbb{Z}$, let $P(n)$ be the $\mathbb{Z}$-graded P-module such that $P(n)_k := P_{n+k}$ for every $k \in \mathbb{Z}$ (with the convention that $P_n := \varnothing$ if $n < 0$); we set :

$$\mathscr{O}_Y(n) := P(n)^\sim.$$

Every element $a \in P_n$ induces a natural isomorphism :

$$\mathscr{O}_Y(n)_{|D_+(a)} \xrightarrow{\sim} \mathscr{O}_{D_+(a)} \quad : \quad x \mapsto f^{-k}x \quad \text{for every local section } x.$$

Hence on the open subset :

$$U_n(P) := \bigcup_{a \in P_n} D_+(a)$$

the sheaf $\mathscr{O}_Y(n)$ restricts to an invertible $\mathscr{O}_{U_n(P)}$-module (see definition 2.3.6(iv)). Especially, if P_1 generates P_+, the $\mathscr{O}_Y$-modules $\mathscr{O}_Y(n)$ are invertible, for every $n \in \mathbb{Z}$.

3.5.34. In the situation of (3.5.31), let M be a $\mathbb{Z}$-graded P-module. Then $M' := M \otimes_P P'$ is a $\mathbb{Z}$-graded P'-module, with the grading defined by the rule :

$$M'_n := \bigcup_{j+k=n} \mathrm{Im}(M_j \otimes_{P_0} P'_k \to M'). \tag{3.5.35}$$

There follows a $P_{(a)}$-linear morphism :

$$M_{(a)} \to M'_{(\varphi(a))} \quad : \quad \frac{x}{a^k} \mapsto \frac{x \otimes 1}{\varphi(a)^k} \qquad \text{for every } a \in P \tag{3.5.36}$$

and since both localization and tensor product commute with arbitrary colimits, it is easily seen that (3.5.36) extends an injective $P'_{(\varphi(a))}$-linear map

$$M_{(a)} \otimes_{P_{(a)}} P'_{(\varphi(a))} \to M'_{(\varphi(a))}$$

whence a map of $\mathscr{O}_{D_+(\varphi(a))}$-modules $(\mathrm{Proj}\,\varphi)^*M^\sim_{|D_+(\varphi(a))} \to (M')^\sim_{|D_+(\varphi(a))}$, and the system of such maps, for a ranging over the elements of P, is compatible with all open immersions $D_+(\varphi(ab)) \subset D_+(a)$, whence a well defined monomorphism of $\mathscr{O}_{G(\varphi)}$-modules

$$(\mathrm{Proj}\,\varphi)^*M^\sim \to (M')^\sim_{|G(\varphi)}. \tag{3.5.37}$$

Moreover, if $a \in P_1$, then for every $m \in M_j$ and $x \in P'_k$, we may write

$$\frac{m \otimes x}{\varphi(a)^{j+k}} = \frac{m}{a^j} \otimes \frac{x}{\varphi(a)^j}$$

so the above map is an isomorphism on $D_+(a)$. Thus, (3.5.37) restricts to an isomorphism on the open subset

$$G_1(\varphi) := \bigcup_{a \in P_1} D_+(\varphi(a)).$$

Especially (3.5.37) is an isomorphism whenever P_1 generates P_+. Notice as well that $G_1(\varphi) \subset U_1(P') \cap G(\varphi)$, and actually $G_1(\varphi) = U_1(P')$ if $\varphi(P)$ generates P'_+.

3.5.38. Let P be as in (3.5.29), and $f : P_0 \to Q$ a given morphism of monoids. Then $P' := P \otimes_{P_0} Q$ is naturally $\mathbb{N}$-graded, so that the natural map $f_P : P \to P'$ is a morphism of graded monoids. Every element of P' is of the form $a \otimes b = (a \otimes 1) \cdot (1 \otimes b)$, where $a \in P$ and $b \in Q$. Then lemma 2.3.34 yields a natural isomorphism of Q-monoids :

$$P_{(a)} \otimes_{P_0} Q[b^{-1}] \xrightarrow{\sim} P'_{(a\otimes b)}$$

whence an isomorphism of affine fans :

$$\beta_{a\otimes b} : D_+(a \otimes b) \xrightarrow{\sim} D_+(a) \times_{T_P} D(b)$$

such that $(\pi_a \times_{T_P} j_b^*) \circ \beta_{a\otimes b} = \pi_{a\otimes b}$, where $j_b^* : D(b) \to T_Q := (\mathrm{Spec}\,Q)^\sharp$ is the natural open immersion. Especially, it is easily seen that the isomorphisms $\beta_{a\otimes 1}$ assemble to a well defined isomorphism of fans :

$$(\mathrm{Proj}\,f_P, \pi_{P'}) : \mathrm{Proj}\,P' \xrightarrow{\sim} \mathrm{Proj}\,P \times_{T_P} T_Q$$

such that $(\pi_P \times_{T_P} \mathbf{1}_{T_Q}) \circ (\mathrm{Proj}\,f_P, \pi_{P'}) = \pi_{P'}$. Lastly, if $g : Q \to R$ is another morphism of monoids, $T_R := (\mathrm{Spec}\,R)^\sharp$, and $P'' := P' \otimes_Q R$, then we have the identity :

$$((\mathrm{Proj}\,f_P, \pi_{P'}) \times_{T_Q} \mathbf{1}_{T_R}) \circ (\mathrm{Proj}\,g_{P'}, \pi_{P''}) = (\mathrm{Proj}\,(g \circ f)_P, \pi_{P''}). \tag{3.5.39}$$

Moreover, for every $\mathbb{Z}$-graded module M, the map $(\mathrm{Proj}\,f_P)^*M^\sim \to (M \otimes_{P_0} Q)^\sim_{|G(f_P)}$ of (3.5.37) is an isomorphism, regardless of whether or not P_1 generates P_+ (verification left to the reader). Especially, we get a natural identification :

$$(\mathrm{Proj}\,f_P)^*\mathscr{O}_Y(n) \xrightarrow{\sim} \mathscr{O}_{Y'}(n) \qquad \text{for every } n \in \mathbb{Z}$$

where $Y := \mathrm{Proj}\,P$ and $Y' := \mathrm{Proj}\,P'$.

3.5.40. In the situation of (3.5.38), let $\varphi : R \to P$ be a morphism of $\mathbb{N}$-graded monoids. There follow morphisms of fans :

$$\mathrm{Proj}\,\varphi : G(\varphi) \to \mathrm{Proj}\,R \qquad \mathrm{Proj}(f_P \circ \varphi) : G(f_P \circ \varphi) \to \mathrm{Proj}\,R$$

and in view of (3.5.32), it is easily seen that :

$$G(f_P \circ \varphi) = (\mathrm{Proj}\,f_P)^{-1}(G(\varphi)). \tag{3.5.41}$$

3.5.42. Let now $(T, \mathscr{O}_T)$ be any fan. A $\mathbb{N}$-*graded* $\mathscr{O}_T$-*monoid* is a $\mathbb{N}$-graded T-monoid $\mathscr{P}$, with a morphism $\mathscr{O}_T \to \mathscr{P}$ of T-monoids. We say that such a $\mathscr{O}_T$-monoid is *quasi-coherent*, if it is such, when regarded as a $\mathscr{O}_T$-module. To a quasi-coherent $\mathbb{N}$-graded $\mathscr{O}_T$-monoid $\mathscr{P}$, we attach a morphism of fans :

$$\pi_{\mathscr{P}} : \mathrm{Proj}\,\mathscr{P} \to T$$

constructed as follows. First, for every affine open subfan $U \subset T$, the monoid $\mathscr{P}(U)$ is $\mathbb{N}$-graded, so we have the projective fan $\mathrm{Proj}\,\mathscr{P}(U)$, and the morphism of monoids $\mathscr{O}_T(U) \to \mathscr{P}(U)$ induces a morphism of fans $\mathrm{Proj}\,\mathscr{P}(U) \to U$. Next, say that $U_1, U_2 \subset T$ are two affine open subsets; for any affine open subset $V \subset U_1 \cap U_2$ we have restriction maps $\rho_{V,i} : \mathscr{P}(U_i) \to \mathscr{P}(V)$ inducing isomorphisms of graded $\mathscr{O}_T(V)$-monoids :

$$\mathscr{P}(U_i) \otimes_{\mathscr{O}_T(U_i)} \mathscr{O}_T(V) \xrightarrow{\sim} \mathscr{P}(V).$$

whence isomorphisms of V-fans :

$$\mathrm{Proj}\,\mathscr{P}(V) \xrightarrow{\sim} \mathrm{Proj}\,\mathscr{P}(U_i) \otimes_{\mathscr{O}_T(U_i)} \mathscr{O}_T(V) \xrightarrow{\mathrm{Proj}\,\rho_{V,i}} \mathrm{Proj}\,\mathscr{P}(U_i) \times_{U_i} V$$

which in turn yield natural identifications :

$$\vartheta_V : \mathrm{Proj}\,\mathscr{P}(U_1) \times_{U_1} V \xrightarrow{\sim} \mathrm{Proj}\,\mathscr{P}(U_2) \times_{U_2} V.$$

If $W \subset V$ is a smaller affine open subset, (3.5.39) implies that $\vartheta_V \times_V \mathbf{1}_W = \vartheta_W$, and therefore the isomorphisms ϑ_V glue to a single isomorphism of $U_1 \cap U_2$-fans :

$$\mathrm{Proj}\,\mathscr{P}(U_1) \times_{U_1} (U_1 \cap U_2) \xrightarrow{\sim} \mathrm{Proj}\,\mathscr{P}(U_2) \times_{U_2} (U_1 \cap U_2)$$

which is furthermore compatible with base change to any triple intersection $U_1 \cap U_2 \cap U_3$ of affine open subsets (details left to the reader). In such situation, we may glue the fans $\mathrm{Proj}\,\mathscr{P}(U)$ – with $U \subset T$ ranging over all the open affine subsets – along the above isomorphisms, to obtain the sought fan $\mathrm{Proj}\,\mathscr{P}$; the construction also comes with a well defined morphism to T, as required. Then, for every such open affine U, the induced morphism $\mathrm{Proj}\,\mathscr{P}(U) \to \mathrm{Proj}\,\mathscr{P}$ is an open immersion; finally a direct inspection shows that, for every smaller affine open subset $V \subset U$ we have :

$$U_n(\mathscr{P}(U)) \cap \mathrm{Proj}\,\mathscr{P}(V) = U_n(\mathscr{P}(V)) \qquad \text{for every } n \in \mathbb{N}$$

(where the intersection is taken in $\mathrm{Proj}\,\mathscr{P}$). Hence the union of all the open subsets $U_n(\mathscr{P}(U))$ is an open subset $U_n(\mathscr{P}) \subset \mathrm{Proj}\mathscr{P}$, intersecting each $\mathrm{Proj}\,\mathscr{P}(U)$ in its subset $U_n(\mathscr{P}(U))$.

3.5.43. To ease notation, set $Y := \mathrm{Proj}\,\mathscr{P}$, and let $\pi : Y \to T$ be the projection. Let $\mathscr{M}$ be a $\mathbb{Z}$-graded $\mathscr{P}$-module, quasi-coherent as a $\mathscr{O}_T$-module; for every affine open subset $U \subset T$, the graded $\mathscr{P}(U)$-module $\mathscr{M}(U)$ yields a quasi-coherent $\mathscr{O}_{\pi^{-1}U}$-module $\mathscr{M}_U^\sim$, and every inclusion of affine open subset $U' \subset U$ induces a natural isomorphism $\mathscr{M}^\sim_{U|U'} \xrightarrow{\sim} \mathscr{M}^\sim_{U'}$ of $\mathscr{O}_{\pi^{-1}U'}$-modules. Therefore the modules $\mathscr{M}_U^\sim$ glue to a well defined $\mathscr{O}_Y$-module $\mathscr{M}^\sim$.

For every $n \in \mathbb{Z}$, denote by $\mathscr{M}(n)$ the $\mathbb{Z}$-graded $\mathscr{P}$-module such that $\mathscr{M}(n)_k := \mathscr{M}_{n+k}$ for every $k \in \mathbb{Z}$ (especially, with the convention that $\mathscr{P}_k := 0$ whenever $k < 0$, we obtain in this way the $\mathscr{P}$-module $\mathscr{P}(n)$). We set :

$$\mathscr{O}_Y(n) := \mathscr{P}(n)^\sim \quad \text{and} \quad \mathscr{M}^\sim(n) := \mathscr{M}^\sim \otimes_{\mathscr{O}_Y} \mathscr{O}_Y(n).$$

Clearly the restriction of $\mathscr{O}_Y(n)$ to $U_n(\mathscr{P})$ is invertible, for every $n \in \mathbb{Z}$.

Moreover, for every $n \in \mathbb{Z}$, the scalar multiplication $\mathscr{P}(n) \otimes_{\mathscr{O}_T} \mathscr{M} \to \mathscr{M}(n)$ determines a well defined morphism of $\mathscr{O}_Y$-modules :

$$\mathscr{M}^\sim(n) \to \mathscr{M}(n)^\sim$$

and arguing as in (3.5.34) we see that the restriction of this map is an isomorphism on $U_1(\mathscr{P})$. Especially, we have natural morphisms of $\mathscr{O}_Y$-modules :

$$\mathscr{O}_Y(n) \otimes_{\mathscr{O}_Y} \mathscr{O}_Y(m) \to \mathscr{O}_Y(n+m) \qquad \text{for every } n, m \in \mathbb{Z}$$

whose restrictions to $U_1(\mathscr{P})$ are isomorphisms.

Example 3.5.44. Let T be a fan, $\mathscr{L}$ an invertible $\mathscr{O}_T$-module, and set $\mathscr{P}(\mathscr{L}) := \mathrm{Sym}^{\bullet}_{\mathscr{O}_T}\mathscr{L}$ (see example 2.3.10). Then the morphism

$$\pi_{\mathscr{P}(\mathscr{L})} : \mathbb{P}(\mathscr{L}) := \mathrm{Proj}\, \mathscr{P}(\mathscr{L}) \to T$$

is an isomorphism. Indeed, the assertion can be checked locally on every affine open subset $U \subset T$, hence say that $U = (\mathrm{Spec}\, P)^\sharp$ for some monoid P, and $\mathscr{L} \simeq \mathscr{O}_{T|U}$, in which case the P-monoid $\mathscr{P}(\mathscr{L})(U)$ is isomorphic to $P \times \mathbb{N}$ (with its natural morphism $P \to P \times \mathbb{N} : x \mapsto (x, 0)$ for every $x \in P$), and the sought isomorphism corresponds to the natural identification :

(3.5.45) $$P = (P \times \mathbb{N})_{(1,1)}$$

where $(1,1) \in P \times \{1\} = (P \times \mathbb{N})_1$. Likewise, $\mathscr{O}_{\mathbb{P}(\mathscr{L})}(n)$ is the $\mathscr{O}_{\mathbb{P}(\mathscr{L})}$-module associated to the graded $(P \times \mathbb{N})$-module $P \times \mathbb{N}(n) = \mathscr{L}^{\otimes n}(U) \otimes_P (P \times \mathbb{N})$, so (3.5.45) induces a natural isomorphism

$$\pi^*_{\mathscr{P}(\mathscr{L})}\mathscr{L}^{\otimes n} \xrightarrow{\sim} \mathscr{O}_{\mathbb{P}(\mathscr{L})}(n) \qquad \text{for every } n \in \mathbb{N}.$$

3.5.46. In the situation of (3.5.42), let $\varphi : \mathscr{P} \to \mathscr{P}'$ be a morphism of quasi-coherent $\mathbb{N}$-graded $\mathscr{O}_T$-monoids (defined in the obvious way). By the foregoing, for every affine open subset $U \subset T$, we have an induced morphism $\mathrm{Proj}\, \varphi(U) : G(\varphi(U)) \to \mathrm{Proj}\mathscr{P}(U)$ of U-fans, where $G(\varphi(U)) \subset \mathrm{Proj}\, \mathscr{P}(U)$ is an open subset of $\mathrm{Proj}\, \mathscr{P}'$. Let $V \subset U$ be a smaller affine open subset; in light of (3.5.41), we have

$$G(\varphi(V)) = G(\varphi(U)) \cap \mathrm{Proj}\, \mathscr{P}(V).$$

It follows that the union of all the open subsets $G(\varphi(U))$ is an open subset $G(\varphi)$ such that

$$G(\varphi) \cap \mathrm{Proj}\, \mathscr{P}(U) = G(\varphi(U)) \qquad \text{for every affine open subset } U \subset T$$

and the morphisms $\mathrm{Proj}\, \varphi(U)$ assemble to a well defined morphism

$$\mathrm{Proj}\, \varphi : G(\varphi) \to \mathrm{Proj}\, \mathscr{P}.$$

Moreover, if $\mathscr{M}$ is a $\mathbb{Z}$-graded quasi-coherent $\mathscr{P}$-module, the morphisms (3.5.37) assemble to a well defined morphism of $\mathscr{O}_{G(\varphi)}$-modules :

(3.5.47) $$(\mathrm{Proj}\, \varphi)^*\mathscr{M}^\sim \to (\mathscr{M}')^\sim_{|G(\varphi)}$$

where the grading of $\mathscr{M}' := \mathscr{M} \otimes_{\mathscr{P}} \mathscr{P}'$ is defined as in (3.5.35). Likewise, the union of all open subsets $G_1(\varphi(U))$ is an open subset $G_1(\varphi) \subset U_1(\mathscr{P}) \cap G(\varphi)$, such that the restriction of (3.5.47) to $G_1(\varphi)$ is an isomorphism. Especially, set $Y := \mathrm{Proj}\, \mathscr{P}$ and $Y' := \mathrm{Proj}\, \mathscr{P}'$; we have a natural morphism :

$$(\mathrm{Proj}\, \varphi)^*\mathscr{O}_Y(n) \xrightarrow{\sim} \mathscr{O}_{Y'}(n)_{|G(\varphi)}$$

which is an isomorphism, if $\mathscr{P}_1$ generates $\mathscr{P}_+ := \coprod_{n>0} \mathscr{P}_n$ locally on T.

3.5.48. On the other hand, let $f : T' \to T$ be a morphism of fans. The discussion in (3.5.38) implies that f induces a natural isomorphism of T'-fans :

(3.5.49) $$\mathrm{Proj}\, f^*\mathscr{P} \xrightarrow{\sim} \mathrm{Proj}\, \mathscr{P} \times_T T'.$$

Moreover, set $Y := \mathrm{Proj}\, \mathscr{P}$, $Y' := \mathrm{Proj}\, f^*\mathscr{P}$, and let $\pi_Y : Y' \to Y$ be the projection deduced from (3.5.49); then there follows a natural identification :

$$\mathscr{O}_{Y'}(n) \xrightarrow{\sim} \pi_Y^*\mathscr{O}_Y(n) \qquad \text{for every } n \in \mathbb{Z}.$$

3.5.50. Keep the notation of (3.5.42), and to ease notation, set $Y := \mathrm{Proj}\,\mathscr{P}$. Let $\mathscr{C}$ be the category whose objects are all the pairs $(\psi : X \to T, \mathscr{L})$, where ψ is a morphism of fans, and $\mathscr{L}$ is an invertible $\mathscr{O}_X$-module; the morphisms $(\psi : X \to T, \mathscr{L}) \to (\psi' : X' \to T, \mathscr{L}')$ are the pairs (β, h), where $\beta : X \to X'$ is a morphism of T-fans, and $h : \beta^*\mathscr{L}' \xrightarrow{\sim} \mathscr{L}$ is an isomorphism of $\mathscr{O}_{X'}$-modules (with composition of morphisms defined in the obvious way). Consider the functor

$$F_{\mathscr{P}} : \mathscr{C}^o \to \mathbf{Set}$$

which assigns to any object $(\psi, \mathscr{L})$ of $\mathscr{C}$, the set consisting of all morphisms of graded $\mathscr{O}_X$-monoids

$$g : \psi^*\mathscr{P} \to \mathrm{Sym}^\bullet_{\mathscr{O}_X}\mathscr{L}$$

which are epimorphisms on the underlying $\mathscr{O}_X$-modules (notation of example 2.3.10). On a morphism (β, h) as in the foregoing, and an element $g' \in F_{\mathscr{P}}(\psi', \mathscr{L}')$, the functor acts by the rule :

$$F_{\mathscr{P}}(\beta, h) := (\mathrm{Sym}^\bullet_{\mathscr{O}_X} h) \circ \beta^* g'.$$

Lemma 3.5.51. *In the situation of* (3.5.50), *the following holds :*

(i) *The object* $(\pi_{\mathscr{P}|U_1(\mathscr{P})} : U_1(\mathscr{P}) \to T, \mathscr{O}_Y(1)_{|U_1(\mathscr{P})})$ *represents the functor* $F_{\mathscr{P}}$.

(ii) *If* $\mathscr{P}$ *is an integral* T*-monoid, the* $\mathscr{O}_T$*-monoid* $\mathscr{P}^{\mathrm{sat}}$ *admits a unique grading such that the unit of adjunction* $\mathscr{P} \to \mathscr{P}^{\mathrm{sat}}$ *is a* $\mathbb{N}$*-graded morphism, and there is a natural isomorphism of* $\mathrm{Proj}\,\mathscr{P}$*-fans :*

$$\mathrm{Proj}\,(\mathscr{P}^{\mathrm{sat}}) \xrightarrow{\sim} (\mathrm{Proj}\,\mathscr{P})^{\mathrm{sat}}.$$

Proof. (i): The proof is *mutatis mutandis*, the same as that of lemma 6.4.26 (with some minor simplifications). We leave it as an exercise for the reader.

(ii): The first assertion shall be left to the reader. The second assertion is local on $\mathrm{Proj}\,\mathscr{P}$, hence we may assume that $T = (\mathrm{Spec}\,P_0)$, and $\mathscr{P} = P^\sim$ for some $\mathbb{N}$-graded integral P_0-monoid P. Let $a \in P^{\mathrm{sat}}$ be any element; by definition we have $a^n \in P$ for some $n > 0$, and we know that the open subsets $D_+(a)$ et $D_+(a^n)$ coincide in $\mathrm{Proj}(\mathscr{P}^{\mathrm{sat}})$; hence we come down to showing that $(P_{(a)})^{\mathrm{sat}} = (P^{\mathrm{sat}})_{(a)}$ for every $a \in P$, which can be left to reader. $\square$

Definition 3.5.52. Let $(T, \mathscr{O}_T)$ be a fan (resp. an integral fan), $\mathscr{I} \subset \mathscr{O}_T$ an ideal (resp. a fractional ideal) of $\mathscr{O}_T$.

(i) Let $f : X \to T$ be a morphism of fans (resp. of integral fans); then $f^{-1}\mathscr{I}$ is an ideal (resp. a fractional ideal) of $f^{-1}\mathscr{O}_T$, and we let :

$$\mathscr{I}\mathscr{O}_X := \log f(f^{-1}\mathscr{I}) \cdot \mathscr{O}_X$$

which is the smallest ideal (res. fractional ideal) $\mathscr{O}_X$ containing the image of $f^{-1}\mathscr{I}$.

(ii) A *blow up* of the ideal $\mathscr{I}$ is a morphism of fans (resp. of integral fans) $\varphi : T' \to T$ which enjoys the following universal property. The ideal (resp. fractional ideal) $\mathscr{I}\mathscr{O}_{T'}$ is invertible, and every morphism of fans (resp. of integral fans) $X \to T$ such that $\mathscr{I}\mathscr{O}_X$ is invertible, factors uniquely through φ.

3.5.53. Let T be a fan (resp. an integral fan), $\mathscr{I} \subset \mathscr{O}_T$ a quasi-coherent ideal (resp. fractional ideal), and consider the $\mathbb{N}$-graded $\mathscr{O}_T$-monoid :

$$\mathscr{B}(\mathscr{I}) := \coprod_{n \in \mathbb{N}} \mathscr{I}^n$$

where $\mathscr{I}^n \subset \mathscr{O}_T$ is the ideal (resp. fractional ideal) associated to the presheaf $U \mapsto \mathscr{I}(U)^n$ for every open subset $U \subset T$ (notation of (3.1.1), with the convention that $\mathscr{I}^0 := \mathscr{O}_T$) and the multiplication law of $\mathscr{B}(\mathscr{I})$ is defined in the obvious way.

Proposition 3.5.54. *The natural projection*

$$\mathrm{Proj}\,\mathscr{B}(\mathscr{I}) \to T$$

is a blow up of the ideal $\mathscr{I}$.

Proof. We shall consider the case where T is not necessarily integral, and $\mathscr{I} \subset \mathscr{O}_T$; the case of a fractional ideal of an integral fan is proven in the same way. Set $Y := \mathrm{Proj}\,\mathscr{B}(\mathscr{I})$; to begin with, let us show that $\mathscr{I}\mathscr{O}_Y$ is invertible. The assertion is local on T, hence we may assume that $T = (\mathrm{Spec}\,P)^\sharp$, and $\mathscr{I} = I^\sim$ for some ideal $I \subset P$, so $Y = \mathrm{Proj}\,B(I)$, where $B(I) = \coprod_{n\in\mathbb{N}} I^n$. Let $a \in B(I)_1 = I$ be any element; then the restriction of $\mathscr{I}\mathscr{O}_Y$ to $D_+(a)$ is generated by $1 = a/a \in B(I)_{(a)}$, so clearly $\mathscr{I}_{|D_+(a)} \simeq \mathscr{O}_Y$; since $U_1(B(I)) = Y$ (notation of (3.5.33)), the contention follows.

Next, let $\varphi : X \to T$ be a morphism of fans, such that $\mathscr{I}\mathscr{O}_X$ is an invertible ideal. It follows easily that $\mathscr{I}^n\mathscr{O}_X$ is invertible for every $n \in \mathbb{N}$, so the natural map of $\mathbb{N}$-graded $\mathscr{O}_X$-monoids

$$\varphi^*\mathrm{Sym}^\bullet_{\mathscr{O}_T}(\mathscr{I}) \xrightarrow{\sim} \mathrm{Sym}^\bullet_{\mathscr{O}_X}(\mathscr{I}\mathscr{O}_X) \to \mathscr{B}(\mathscr{I}\mathscr{O}_X)$$

is an isomorphism. On the other hand, the projection $\mathrm{Proj}\,\mathscr{B}(\mathscr{I}\mathscr{O}_X) \to X$ is an isomorphism (example (3.5.44)), whence – in view of (3.5.49) – a natural morphism of T-fans :

$$X \to \mathrm{Proj}\,\mathscr{B}(\mathscr{I}). \tag{3.5.55}$$

To conclude, it remains to show that (3.5.55) is the only morphism of T-fans from X to $\mathrm{Proj}\,\mathscr{B}(\mathscr{I})$. The latter assertion can be checked again locally on T, so we are reduced as above to the case where T is the spectrum of P, and $\mathscr{I}$ is associated to I. We may also assume that $X = (\mathrm{Spec}\,Q)^\sharp$, and φ is given by a morphism of monoids $f : P \to Q$. Then the hypothesis means that the ideal $f(I)Q$ is isomorphic to Q (see remark 3.5.13(ii)), hence it is generated by an element of the form $f(a)$, for some $a \in I$, and the endomorphism $x \mapsto f(a)x$ of $f(I)Q$, is an isomorphism. In such situation, it is clear that f factors uniquely through a morphism of monoids $P \to B(I)_{(a)}$; namely, one defines $g : B(I)_{(a)} \to Q$ by the rule : $a^{-k}x \mapsto f(a)^{-k}f(x)$ (for every $x \in I^k$), which is well defined, by the foregoing observations. The morphism $(\mathrm{Spec}\,g)^\sharp : X \to D_+(a)$ must then agree with (3.5.55). □

Example 3.5.56. (i) Let P be a monoid, $I \subset P$ any finitely generated ideal, $\{a_1, \dots, a_n\}$ a finite system of generators of I; set $T := (\mathrm{Spec}\,P)^\sharp$, and let $\varphi : T' \to T$ be the blow up of the ideal $I^\sim \subset \mathscr{O}_T$. Then T' admits an open covering consisting of the affine fans $D_+(f_i)$. The latter are the spectra of the monoids Q_i consisting of all fractions of the form $a \cdot f_i^{-t}$, for every $a \in I^n$; we have $a \cdot f_i^{-t} = b \cdot f_i^{-s}$ in Q_i if and only if there exists $k \in \mathbb{N}$ such that $a \cdot f_i^{s+k} = b \cdot f_i^{t+k}$, if and only if the two fractions are equal in P_{f_i}, in other word, Q_i is the submonoid of P_{f_i} generated by P and $\{f_j \cdot f_i^{-1} \mid j \leq n\}$, for every $i = 1, \dots, n$.

(ii) Consider the special case where P is fine, and the ideal $I \subset P$ is generated by two elements $f, g \in P$. Let $t \in T'$ be any point; up to swapping f and g, we may assume that t corresponds to a prime ideal $\mathfrak{p} \subset P[f/g]$, hence $\varphi(t)$ corresponds to $\mathfrak{q} := j^{-1}\mathfrak{p} \subset P$, where $j : P \to P[f/g]$ is the natural map. We have the following two possibilities :

- Either $f/g \in \mathfrak{p}$, in which case let $y \in F' := P[f/g] \setminus \mathfrak{p}$ be any element; writing $x = y\cdot(f/g)^n$ for some $n \geq 0$ and $y \in P$, we deduce that $n = 0$, so $x = y \in F := P \setminus \mathfrak{q}$, therefore $F' = j(F)$. Notice as well that in this case f/g is not invertible in $P[f/g]$, hence $P[f/g]^\times = P^\times$, whence $\dim P[f/g] = \dim P$, and , by corollary 3.4.10(i).
- Or else $f/g \notin \mathfrak{p}$, in which case the same argument yields $F' = j(F)[f/g]$. In this case, f/g is invertible in $P[f/g]$ if and only if it is invertible in the face F', hence $\mathrm{ht}\,\mathfrak{p} = \dim P - \mathrm{rk}_\mathbb{Z} F' \geq \dim P - \dim F - 1$, by corollary 3.4.10(i),(ii).

In either event, corollary 3.4.10(i),(ii) implies the inequality :

$$1 \geq \mathrm{ht}(\varphi(t)) - \mathrm{ht}(t) \geq 0 \qquad \text{for every } t \in T'.$$

3.6. **Special subdivisions.** In this section we explain how to construct – either by geometrical or combinatorial means – useful subdivisions of given fans.

3.6.1. Let T be any locally fine and saturated fan, and $t \in T$ any point. By reflexivity (proposition 3.4.12(iv)), the elements $s \in \mathscr{O}^{\mathrm{gp}}_{T,t} \otimes_{\mathbb{Z}} \mathbb{Q}$ correspond bijectively to $\mathbb{Q}$-linear forms $\rho_s : U(t)(\mathbb{Q}_+)^{\mathrm{gp}} \to \mathbb{Q}$, and $s \in \mathscr{O}^{\mathrm{gp}}_{T,t}$ if and only if ρ_s restricts to a morphism of monoids $U(t)(\mathbb{N}) \to \mathbb{Z}$. Moreover, this bijection is compatible with specialization maps : if t' is a generization of t in T, then the form $U(t')(\mathbb{Q}_+)^{\mathrm{gp}} \to \mathbb{Q}$ induced by the image of s in $\mathscr{O}^{\mathrm{gp}}_{T,t'} \otimes_{\mathbb{Z}} \mathbb{Q}$ is the restriction of ρ_s (see (3.5.20)).

Hence, any global section $\lambda \in \Gamma(T, \mathscr{O}^{\mathrm{gp}}_T)$ yields a well defined function

$$\rho_\lambda : T(\mathbb{N}) \to \mathbb{Z}$$

whose restriction to $U(t)(\mathbb{N})$ is the restriction of a $\mathbb{Z}$-linear form on $U(t)(\mathbb{N})^{\mathrm{gp}}$, for every $t \in T$; conversely, any such function arises from a unique global section of $\mathscr{O}^{\mathrm{gp}}_T$. Likewise, we have a natural isomorphism between the $\mathbb{Q}$-vector space of global sections λ of $\mathscr{O}^{\mathrm{gp}}_T \otimes_{\mathbb{Z}} \mathbb{Q}$, and the space of functions $\rho_\lambda : T(\mathbb{Q}_+) \to \mathbb{Q}$ with a corresponding linearity property.

Let now $\rho : T(\mathbb{Q}_+) \to \mathbb{Q}$ be any function; we may attach to ρ a sheaf of fractional ideals of $\mathscr{O}_{T,\mathbb{Q}}$ (notation of (3.3.20)), by the rule :

$$\mathscr{I}_{\rho,\mathbb{Q}}(U) := \{s \in \mathscr{O}_{T,\mathbb{Q}}(U) \mid \rho_s \geq \rho_{|U}\} \qquad \text{for every open subset } U \subset T$$

In this generality, not much can be said concerning $\mathscr{I}_{\rho,\mathbb{Q}}$; to advance, we restrict our attention to a special class of functions, singled out by the following :

Definition 3.6.2. Let T be a locally fine and saturated fan.

(a) A *roof* on T is a function :

$$\rho : T(\mathbb{Q}_+) \to \mathbb{Q}$$

such that, for every $t \in T$, there exist $k := k(t) \in \mathbb{N}$ and $\mathbb{Q}$-linear forms

$$\lambda_1, \dots, \lambda_k : U(t)(\mathbb{Q}_+)^{\mathrm{gp}} \to \mathbb{Q}$$

with $\rho(s) = \min(\lambda_i(s) \mid i = 1, \dots, k)$ for every $s \in U(t)(\mathbb{Q}_+)$.

(b) An *integral roof* on T is a roof ρ on T such that $\rho(s) \in \mathbb{Z}$ for every $s \in T(\mathbb{N})$.

3.6.3. The interest of the notion of roof on a fan T, is that it encodes in a geometrical way, an integral subdivision of T, together with a coherent sheaf of fractional ideals of $\mathscr{O}_T$ (see definition 2.3.6(iii)). This shall be seen in several steps. To begin with, let T and ρ be as in definition 3.6.2(a). For any $t \in T$, pick a system $\underline{\lambda} := \{\lambda_1, \dots, \lambda_k\}$ of $\mathbb{Q}$-linear forms fulfilling condition (b) of the definition; then for every $i = 1, \dots, k$ let us set :

$$U(t,i)(\mathbb{N}) := \{x \in U(t)(\mathbb{N}) \mid \rho(x) = \lambda_i(x)\}.$$

Notice that $U(t)(\mathbb{N}) = \mathscr{O}_{T,t}$, by proposition 3.4.12(iv). Moreover, say that $\underline{\lambda}$ is *irredundant for* t if no proper subsystem of $\underline{\lambda}$ fulfills condition (b) of definition 3.6.2 relative to $U(t)$.

Lemma 3.6.4. *With the notation of* (3.6.3)*, the following holds :*

(i) $U(t,i)(\mathbb{N})$ *is a saturated fine monoid for every* $i \leq k$.

(ii) *There is a unique system of* $\mathbb{Q}$*-linear forms which is irredundant for* t.

(iii) *If* $\underline{\lambda}$ *is irredundant for* t*, then* $\dim U(t,i)(\mathbb{N}) = \mathrm{ht}_T(t)$ *for every* $i = 1, \dots, k$.

Proof. (i): We leave to the reader the verification that $U(t,i)$ is a saturated monoid. Next, let $\sigma_i \subset U(t)(\mathbb{R}_+)^{\mathrm{gp}\vee}$ be the convex polyhedral cone spanned by the linear forms

$$((\lambda_j - \lambda_i) \otimes_{\mathbb{Q}} \mathbb{R} \mid j = 1, \dots, k).$$

Then σ_i is $\mathscr{O}^{\mathrm{gp}\vee}_{T,t}$-rational, so $\sigma_i^\vee$ is $\mathscr{O}^{\mathrm{gp}}_{T,t}$-rational, and $\sigma_i^\vee \cap \mathscr{O}^{\mathrm{gp}}_{T,t}$ is a fine monoid (propositions 3.3.21(i), and 3.3.22(i)), therefore the same holds for $U(t,i)(\mathbb{N}) = \sigma_i^\vee \cap \mathscr{O}_{T,t}$ (corollary 3.4.2).

(iii): Notice that $\mathrm{ht}_T(t) = \dim U(t)(\mathbb{N})$, by proposition 3.4.12(ii) and (3.5.18). In view of corollary 3.4.10(i), it follows already that $\dim U(t,i) \leq \mathrm{ht}_T(t)$. Now, let $\lambda_1, \dots, \lambda_k$ be an irredundant system, and suppose, by contradiction, that $\dim U(t,i)(\mathbb{N}) < \dim U(t)(\mathbb{N})$ for some $i \leq k$. Especially, $\sigma_i^\vee \cap U(t)(\mathbb{R}_+)$ does not span the $\mathbb{R}$-vector space $U(t)(\mathbb{R}_+)^{\mathrm{gp}}$, and therefore the dual $\sigma_i + U(t)(\mathbb{R})^\vee$ is not strongly convex (corollary 3.3.14). After relabeling, we may assume that $i = 1$. Hence there exist $a_j, b_j \in \mathbb{R}_+$ and $\varphi, \varphi' \in U(t)(\mathbb{R})^\vee$, and an identity :

$$\sum_{j=2}^{k} a_j(\lambda_j - \lambda_i) + \varphi = -\sum_{j=2}^{k} b_j(\lambda_j - \lambda_i) - \varphi'.$$

Moreover, $\sum_{j=2}^k (a_j + b_j) > 0$. It follows that there exist $\psi \in U(t)(\mathbb{R})^\vee$, and non-negative real numbers $(c_j \mid j = 2, \dots k)$ such that

$$\lambda_i = \sum_{j=2}^{k} c_j\lambda_j + \psi \quad \text{and} \quad \sum_{j=2}^{k} c_j = 1.$$

On the other hand, the irredundancy condition means that there exists $x \in U(t,1)(\mathbb{N})$ such that $\lambda_j(x) > \lambda_1(x)$ for every $j > 1$. Since $\psi(x) \geq 0$, we get a contradiction.

(ii): The assertion is clear, if $\mathrm{ht}_T(t) \leq 1$. Hence suppose that the height of t is ≥ 2, and let $\underline{\lambda} := \{\lambda_1, \dots, \lambda_k\}$ and $\underline{\mu} := \{\mu_1, \dots, \mu_r\}$ be two irredundant systems for t. Fix $i \leq k$, and pick $x \in U(t,i)(\mathbb{N})$ which does not lie on any proper face of $U(t,i)(\mathbb{N})$ (the existence of x is ensured by (iii) and proposition 3.4.7(i)); say that $\mu_1(x) = \rho(x)$. Since i is arbitrary, the assertion shall follow, once we have shown that μ_1 agrees with λ_i on the whole of $U(t,i)(\mathbb{N})$.

However, by definition we have $\mu_1(y) \geq \lambda_i(y)$ for every $y \in U(t,i)(\mathbb{N})$, and then it is easily seen that $\mathrm{Ker}(\mu_1 - \lambda_i) \cap U(t,i)(\mathbb{N})$ is a face of $U(t,i)(\mathbb{N})$; since $x \in \mathrm{Ker}(\mu_1 - \lambda_i)$, we deduce that $\mu_1 - \lambda_i$ vanishes identically on $U(t,i)(\mathbb{N})$. □

3.6.5. Henceforth, we denote by $\underline{\lambda}(t) := \{\lambda_1, \dots, \lambda_k\}$ the irredundant system of $\mathbb{Q}$-linear forms for t. Let $1 \leq i,j \leq k$; then we claim that $U(t,i,j)(\mathbb{N}) := U(t,i)(\mathbb{N}) \cap U(t,j)(\mathbb{N})$ is a face of both $U(t,i)(\mathbb{N})$ and $U(t,j)(\mathbb{N})$. Indeed say that $x, x' \in U(t,i)(\mathbb{N})$ and $x + x' \in U(t,i,j)(\mathbb{N})$; these conditions translate the identities :

$$\lambda_i(x) + \lambda_i(x') = \lambda_j(x) + \lambda_j(x') \qquad \lambda_i(x) \leq \lambda_j(x) \qquad \lambda_i(x') \leq \lambda_j(x')$$

whence $x, x' \in U(t,i,j)(\mathbb{N})$. Define :

$$U(t,i) := (\mathrm{Spec}\, U(t,i)(\mathbb{N})^\vee)^\sharp \qquad U(t,i,j) := (\mathrm{Spec}\, U(t,i,j)(\mathbb{N})^\vee)^\sharp \qquad \text{for every } i,j \leq k.$$

According to (3.5.20) and (3.5.18), the inclusion maps $U(t,i,j)(\mathbb{N}) \to U(t,l)(\mathbb{N})$ (for $l = i,j$) are dual to open immersions

$$U(t,i) \leftarrow U(t,i,j) \to U(t,j). \tag{3.6.6}$$

We may then attach to t and $\rho_{|U(t)}$ the fan $U(t,\rho)$ obtained by gluing the affine fans $U(t,i)$ along their common intersections $U(t,i,j)$. The duals of the inclusions $U(t,i)(\mathbb{N}) \to U(t)(\mathbb{N})$ determine a well defined morphism of locally fine and saturated fans :

$$U(t,\rho) \to U(t) \tag{3.6.7}$$

which, by construction, induces a bijection on $\mathbb{N}$-points : $U(t,\rho)(\mathbb{N}) \xrightarrow{\sim} U(t)(\mathbb{N})$, so it is a rational subdivision, according to proposition 3.5.24.

3.6.8. Let now $t' \in T$ be a generization of t; clearly the system $\underline{\lambda}' := \{\lambda'_1, \dots, \lambda'_k\}$ consisting of the restrictions λ'_i of the linear forms λ_i to the $\mathbb{Q}$-vector subspace $U(t')(\mathbb{Q}_+)^{\mathrm{gp}}$, fulfills condition (b) of definition 3.6.2, relative to t'. However, $\underline{\lambda}'$ may fail to be irredundant; after relabeling, we may assume that the subsystem $\{\lambda'_1, \dots, \lambda'_l\}$ is irredundant for t', for some $l \leq k$. With the foregoing notation, we have obvious identities :

$$U(t', i)(\mathbb{N}) = U(t, i)(\mathbb{N}) \cap U(t')(\mathbb{N}) \qquad U(t', i, j)(\mathbb{N}) = U(t, i, j)(\mathbb{N}) \cap U(t')(\mathbb{N})$$

for every $i, j \leq l$; whence, in light of remark 3.4.14(iii), a commutative diagram of fans :

$$\begin{tikzcd}
U(t', i) \arrow[d] & U(t', i, j) \arrow[l] \arrow[r] \arrow[d] & U(t', j) \arrow[d] \\
U(t, i) \times_{U(t)} U(t') & U(t, i, j) \times_{U(t)} U(t') \arrow[l] \arrow[r] & U(t, i) \times_{U(t)} U(t')
\end{tikzcd}$$

whose top horizontal arrows are the open immersions (3.6.6) (with t replaced by t'), whose bottom horizontal arrows are the open immersions (3.6.6)$\times_{U(t)} U(t')$, and whose vertical arrows are natural isomorphisms. Since $U(t')$ is an open subset of $U(t)$, we deduce an open immersion

$$j_{t,t'} : U(t', \rho) \to U(t, \rho).$$

If t'' is a generization of t', it is clear that $j_{t',t''} \circ j_{t,t'} = j_{t,t''}$, hence we may glue the fans $U(t, \rho)$ along these open immersions, to obtain a locally fine and saturated fan $T(\rho)$. Furthermore, the morphisms (3.6.7) glue to a single rational subdivision :

$$T(\rho) \to T. \tag{3.6.9}$$

Remark 3.6.10. (i) In the language of definition 3.3.25 the foregoing lengthy procedure translates as the following simple geometric operation. Given a fan Δ (consisting of a collection of convex polyhedral cones of a $\mathbb{R}$-vector space V), a roof on Δ is a piecewise linear function $F := \bigcup_{\sigma \in \Delta} \sigma \to \mathbb{R}$, which is concave on each $\sigma \in \Delta$ (and hence it is a roof on each such σ, in the sense of example 3.3.27). Then, such a roof determines a natural refinement Δ' of Δ; namely, Δ' is the coarsest refinement such that, for each $\sigma' \in \Delta'$, the function $\rho_{|\sigma'}$ is the restriction of a $\mathbb{R}$-linear form on V. This refinement Δ' corresponds to the present $T(\rho)$.

(ii) Moreover, let P be a fine, sharp and saturated monoid of dimension d, set $T_P := (\operatorname{Spec} P)^\sharp$, and suppose that $f : T \to T_P$ is any integral, fine, proper and saturated subdivision. Then f corresponds to a geometrical subdivision Δ of the strictly convex polyhedral cone $T_P(\mathbb{R}_+) = P^\vee_{\mathbb{R}}$, and we claim that Δ can be refined by the subdivision associated to a roof on T_P. Namely, let Δ_{d-1} be the subset of Δ consisting of all σ of dimension $d - 1$; every $\sigma \in \Delta_{d-1}$ is the intersection of a d-dimensional element of Δ and a hyperplane $H_\sigma \subset P^{\mathrm{gp}\vee}_{\mathbb{R}}$; such hyperplane is the kernel of a linear form λ_σ on $P^{\mathrm{gp}\vee}_{\mathbb{R}}$. Let us define

$$\rho(x) := \sum_{\sigma \in \Delta_{d-1}} \min(0, \lambda_\sigma(x)) \qquad \text{for every } x \in T_P(\mathbb{R}_+).$$

Then it is easily seen that the subdivision of $T_P(\mathbb{R}_+)$ associated to the roof ρ as in (i), refines the subdivision Δ. In the language of fans, this construction translates as follows. For every point $\sigma \in T$ of height $d - 1$, let $H_\sigma \subset P^{\mathrm{gp}}$ be the kernel of the surjection $P^{\mathrm{gp}} \to \mathscr{O}^{\mathrm{gp}}_{T,\sigma}$ induced by $\log f$; notice that H_σ is a free abelian group of rank one, and pick a generator s_σ of H_σ, which – as in (3.6.1) – corresponds to a function $\lambda_\sigma : T_P(\mathbb{N}) \to \mathbb{Z}$, so we may again consider the integral roof ρ on T_P defined as in the foregoing. Then it is easily seen that the rational subdivision $T(\rho) \to T_P$ associated to ρ, factors as the composition of f and a (necessarily unique) integral subdivision $g : T(\rho) \to T$. More precisely, for every mapping

$\varepsilon : \{\sigma \in T \mid \mathrm{ht}(\sigma) = d-1\} \to \{0,1\}$, let us set

$$\lambda_\varepsilon := \sum_{\mathrm{ht}(\sigma)=d-1} \varepsilon(\sigma)\cdot\lambda_\sigma \qquad \text{and} \qquad U(\varepsilon)(\mathbb{N}) := \{x \in T_P(\mathbb{N}) \mid \rho(x) = \lambda_\varepsilon(x)\}.$$

Whenever $U(\varepsilon)(\mathbb{N})$ has dimension d, let $t_\varepsilon \in T(\rho)$ be the unique point such that $U(\varepsilon)(\mathbb{N})^\vee = \mathscr{O}_{T(\rho),t_\varepsilon}$. As the reader may check, there exists a unique closed point $\tau \in T$, such that $\varepsilon(\sigma)\cdot\lambda_\sigma(x) \leq 0$ for every $x \in U(\tau)(\mathbb{N})$ and every $\sigma \in U(\tau)$ of height $d-1$. Then we have $g(t_\varepsilon) = \tau$, and the restriction $U(t_\varepsilon) \to U(\tau)$ of g is deduced from the inclusion $U(\varepsilon)(\mathbb{N}) \subset U(\tau)(\mathbb{N})$ of submonoids of $T_P(\mathbb{N})$.

(iii) Furthermore, in the situation of (ii), the roof ρ on T_P can also be viewed as a roof on T, and then it is clear from the construction that the morphism $g : T(\rho) \to T$ is also the subdivision of T attached to the roof ρ.

We wish now to establish some basic properties of the sheaf of fractional ideals

$$\mathscr{I}_\rho := \mathscr{I}_{\rho,\mathbb{Q}} \cap \mathscr{O}_T^{\mathrm{gp}}$$

attached to a given roof on T. First we remark :

Lemma 3.6.11. *Keep the notation of* (3.6.5)*, and let $s \in \mathscr{O}^{\mathrm{gp}}_{T,t} \otimes_{\mathbb{Z}} \mathbb{Q}$ be any element such that $\rho_s \geq \rho_{|U(t)}$. Then we have :*

(i) *There exist $\varphi \in (\mathscr{O}_{T,t})_{\mathbb{Q}}$ (notation of* (3.3.20)*) and $c_1, \dots, c_k \in \mathbb{R}_+$ such that :*

$$\rho_s = \rho_\varphi + \sum_{i=1}^k c_i\lambda_i \qquad \sum_{i=1}^k c_i = 1.$$

(ii) *The stalk $\mathscr{I}_{\rho,t}$ is a finitely generated $\mathscr{O}_{T,t}$-module.*

Proof. (i): Let $\sigma \subset U(t)(\mathbb{R}_+)^{\mathrm{gp}}$ be the convex polyhedral cone spanned by the linear forms $((\lambda_i - \rho_s)\otimes_{\mathbb{Q}}\mathbb{R} \mid i = 1,\dots,k)$. Then the assumption on s means that

$$U(t)(\mathbb{R}_+) \cap \sigma^\vee = U(t)(\mathbb{R}_+) \cap \mathrm{Ker}\,\rho_s \otimes_{\mathbb{Q}} \mathbb{R}.$$

Especially $\sigma^\vee \cap U(t)(\mathbb{R}_+)$ does not span $U(t)(\mathbb{R}_+)^{\mathrm{gp}}$. Then one can repeat the proof of lemma 3.6.4(iii) to derive the assertion.

(ii): By remark 3.4.14(i), we may write $\lambda_i = \rho_{s_i} - \rho_{s'_i}$, where $s_i, s'_i \in (\mathscr{O}_{T,t})_{\mathbb{Q}}$ for each $i \leq k$; pick $N \in \mathbb{N}$ large enough, so that $Ns'_i \in \mathscr{O}_{T,t}$ for every $i \leq k$, and set $\tau := N\sum_{i=1}^k s'_i$. Then $\tau + \mathscr{I}_{\rho,t} \subset \mathscr{O}^{\mathrm{gp}}_{T,t} \cap (\mathscr{O}_{T,t})_{\mathbb{Q}} = \mathscr{O}_{T,t}$. By proposition 3.1.9(ii), we deduce that $\tau + \mathscr{I}_{\rho,t}$ is a finitely generated ideal, whence the contention. □

Proposition 3.6.12. *Let T be a locally fine and saturated fan, ρ a roof on T. Then the associated fractional ideal $\mathscr{I}_\rho$ of $\mathscr{O}_T$ is coherent.*

Proof. In view of lemma 3.6.11(ii) and remark 3.5.13(v), it suffices to show that $\mathscr{I}_\rho$ is quasi-coherent, *i.e.* for every generization t' of t, the image of $\mathscr{I}_{\rho,t}$ in $\mathscr{O}_{T,t'}$ generates the $\mathscr{O}_{T,t'}$-module $\mathscr{I}_{\rho,t'}$. Fix such t'; by propositions 3.3.21(i) and 3.4.7(i), there exists $\lambda \in \mathscr{O}_{T,t} = U(t)(\mathbb{N})^\vee$ such that $U(t')(\mathbb{N}) = \mathrm{Ker}\,\lambda$; especially, we see that $U(t')(\mathbb{N})^{\mathrm{gp}}$ is a direct summand of $U(t)(\mathbb{N})^{\mathrm{gp}}$. Now, let $s' \in \mathscr{I}_{\rho,t'}$ be any local section; it follows that we may find $s \in \mathscr{O}^{\mathrm{gp}}_{T,t}$ such that $\rho_s : U(t)(\mathbb{N})^{\mathrm{gp}} \to \mathbb{Z}$ is a $\mathbb{Z}$-linear extension of the corresponding $\mathbb{Z}$-linear form $\rho_{s'} : U(t')(\mathbb{N})^{\mathrm{gp}} \to \mathbb{Z}$. Let also $\{\lambda_1,\dots,\lambda_k\}$ be the irredundant system of $\mathbb{Q}$-linear forms for t (relative to the roof ρ). For every $i \leq k$ we have the following situation :

$$\lambda_{\mathbb{R}} := \lambda \otimes_{\mathbb{Q}} \mathbb{R} \in U(t,i)(\mathbb{R}_+)^\vee \qquad (\rho_s - \lambda_i)\otimes_{\mathbb{Q}}\mathbb{R} \in U(t',i)(\mathbb{R}_+)^\vee.$$

However, $U(t',i)(\mathbb{R}_+) = U(t,i)(\mathbb{R}_+) \cap \mathrm{Ker}\,\lambda_{\mathbb{R}}$, hence $U(t',i)(\mathbb{R}_+)^\vee = U(t,i)(\mathbb{R}_+)^\vee + \mathbb{R}\lambda_{\mathbb{R}}$. Especially, there exists $r_i \in \mathbb{R}_+$ and $\varphi \in U(t)(\mathbb{R}_+)^\vee$ such that $(\rho_s - \lambda_i)\otimes_{\mathbb{Q}}\mathbb{R} = \varphi - r_i\lambda_{\mathbb{R}}$. Let

N be an integer greater than $\max(r_1, \dots, r_k)$; it follows that $s + N\lambda \in \mathscr{I}_{\rho,t}$ and its image in $\mathscr{I}_{\rho,t'}$ equals s'. □

Proposition 3.6.13. *In the situation of* (3.6.3) *suppose that ρ is an integral roof on T. Then the morphism* (3.6.9) *is the saturation of a blow up of the fractional ideal $\mathscr{I}_\rho$.*

Proof. We have to exhibit an isomorphism $f : T(\rho) \to X := \mathrm{Proj}\, \mathscr{B}(\mathscr{I})^{\mathrm{sat}}$ of T-fans. We begin with :

Claim 3.6.14. (i) The fractional ideal $\mathscr{I}_\rho \mathscr{O}_{T(\rho)}$ is invertible.

(ii) For every $n \in \mathbb{N}$, denote by $n\rho : T(\mathbb{Q}_+) \to \mathbb{Q}_+$ the function given by the rule $x \mapsto n \cdot \rho(x)$ for every $x \in T(\mathbb{Q}_+)$. Then :

$$\mathscr{B}(\mathscr{I})^{\mathrm{sat}} = \mathscr{B}' := \coprod_{n \in \mathbb{N}} \mathscr{I}_{n\rho}$$

Proof of the claim. (ii): The assertion is local on T, hence we may assume that $T = U(t)$ for some $t \in T$, in which case, denote by $\underline{\lambda} := \{\lambda_1, \dots, \lambda_k\}$ the irredundant system of $\mathbb{Q}$-linear forms for t. Let $n \in \mathbb{N}$, and $s \in \mathscr{I}_{n\rho}(U(t))$; it is easily seen that the T-monoid $\mathscr{B}'$ is saturated, hence it suffices to show that there exists an integer $k > 0$ such that $k\rho_s = \rho_{s'}$ for some $s' \in \mathscr{I}^k(U(t))$ (notation of (3.6.1)). However, lemma 3.6.11(ii) implies more precisely that we may find such k, so that the corresponding s' lies in the ideal generated by $\underline{\lambda}$.

(i): The assertion is local on $T(\rho)$, hence we consider again $t \in T$ and the corresponding $\underline{\lambda}$ as in the foregoing. It suffices to show that $\mathscr{J} := \mathscr{I}_\rho \mathscr{O}_{U(t,i)}$ is invertible for every $i = 1, \dots, k$ (notation of (3.6.5)). However, by inspecting the constructions it is easily seen that $\mathscr{J}(U(t,i))$ consists of all $s \in (U(t,i)(\mathbb{N})^\vee)^{\mathrm{gp}}$ such that $\rho_s(x) \geq \rho(x)$ for every $x \in U(t,i)(\mathbb{Q}_+)$, *i.e.* $\rho_s(x) \geq \lambda_i(x)$ for every $x \in U(t,i)(\mathbb{Q}_+)$. However, since ρ is integral, we have $\lambda_i \in \mathscr{O}_{T,t}^{\mathrm{gp}}$; if we apply lemma 3.6.11(i) with T replaced by $U(t,i)$, we conclude that $\mathscr{J}(U(t,i))$ is the fractional ideal generated by λ_i, whence the contention. ◇

In view of claim 3.6.14(i) we see that there exists a unique morphism f of T-fans from $T(\rho)$ to X. It remains to check that f is an isomorphism; the latter assertion is local on X, hence we may assume that $T = U(t)$ for some $t \in T$, and then we let again $\underline{\lambda}$ be the irredundant system for t. A direct inspection yields a natural identification of $\mathscr{O}_{T,t}$-monoids :

$$\mathscr{B}'(U(t,i))_{(\lambda_i)} \xrightarrow{\sim} U(t,i)(\mathbb{N})^\vee \qquad \text{for every } i = 1, \dots, k$$

whence an isomorphism $U(t,i) \xrightarrow{\sim} D_+(\lambda_i) \subset X$, which – by uniqueness – must coincide with the restriction of f. On the other hand, the proof of claim 3.6.14(ii) also shows that $X = D_+(\lambda_1) \cup \dots \cup D_+(\lambda_k)$, and the proposition follows. □

Example 3.6.15. Let P be a fine, sharp and saturated monoid.

(i) The simplest non-trivial roofs on $T_P := (\mathrm{Spec}\, P)^\sharp$ are the functions ρ_λ such that

$$\rho_\lambda(x) := \min(0, \lambda(x)) \qquad \text{for every } x \in T_P(\mathbb{Q}_+).$$

where λ is a given element of $\mathrm{Hom}_{\mathbb{Q}}(T_P(\mathbb{Q}_+)^{\mathrm{gp}}, \mathbb{Q}) \simeq P^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$. Such a ρ_λ is an integral roof, provided $\lambda \in P^{\mathrm{gp}}$. In the latter case, we may write $\lambda = \rho_{s_1} - \rho_{s_2}$, for some $s_1, s_2 \in P$. Set $\rho' := \rho_\lambda + \rho_{s_2}$, *i.e.* $\rho' = \min(\rho_{s_1}, \rho_{s_2})$mplest non-trivial roofs on $T_P := (\mathrm{Spec}\, P)^\sharp$ are the functions ρ_λ such that

$$\rho_\lambda(x) := \min(0, \lambda(x)) \qquad \text{for every } x \in T_P(\mathbb{Q}_+).$$

where λ is a given element of $\mathrm{Hom}_{\mathbb{Q}}(T_P(\mathbb{Q}_+)^{\mathrm{gp}}, \mathbb{Q}) \simeq P^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$. Such a ρ_λ is an integral roof, provided $\lambda \in P^{\mathrm{gp}}$. In the latter case, we may write $\lambda = \rho_{s_1} - \rho_{s_2}$, for some $s_1, s_2 \in P$. Set $\rho' := \rho_\lambda + \rho_{s_2}$, *i.e.* $\rho' = \min(\rho_{s_1}, \rho_{s_2})$; clearly $T(\rho_\lambda) = T(\rho')$, and on the other hand lemma 3.6.11(i) implies that the ideal $\mathscr{I}_{\rho'}$ is the saturation of the ideal generated by s_1 and s_2.

(ii) More generally, any system $\lambda_1, \dots, \lambda_n \in \mathrm{Hom}_{\mathbb{Q}}(T_P(\mathbb{Q}_+)^{\mathrm{gp}}, \mathbb{Q})$ of $\mathbb{Q}$-linear forms yields a roof ρ on T_P, such that $\rho(x) := \sum_{i=1}^n \min(0, \lambda_i(x))$ for every $x \in T_P(\mathbb{Q}_+)$. A simple inspection shows that the corresponding subdivision $T(\rho) \to T$ can be factored as the composition of n subdivisions $g_i : T_i \to T_{i-1}$, where $T_0 := T_P$, $T_n := T(\rho)$, and each g_i (for $i \leq n$) is the subdivision of T_{i-1} corresponding to the roof ρ_{λ_i} as defined in (i).

(iii) These subdivisions of T_P "by hyperplanes" are precisely the ones that occur in remark 3.6.10(ii),(iii). Summing up, we conclude that every proper integral and saturated subdivision $g : T \to T_P$ of T_P can be dominated by another subdivision $f : T(\rho) \to T_P$ of the type considered in (ii), so that f factors as the composition of g and a subdivision $h : T(\rho) \to T$ which is also of the type (ii). Especially, both f and h can be realized as the composition of finitely many saturated blow up of ideals generated by at most two elements of P.

3.6.16. Let P be a fine, sharp and saturated monoid. A proper, integral, fine and saturated subdivision of

$$T_P := (\mathrm{Spec}\, P)^\sharp$$

is essentially equivalent to a $(P^{\mathrm{gp}})^\vee$-rational subdivision of the polyhedral cone $\sigma := P^\vee_{\mathbb{R}}$ (see (3.4.6) and definition 3.3.25). A standard way to subdivide a polyhedron σ consists in choosing a point $x_0 \in \sigma \setminus \{0\}$, and forming all the polyhedra $x_0 * F$, where F is any proper face of σ, and $x_0 * F$ denotes the convex span of x_0 and F. We wish to describe the same operation in terms of the topological language of affine fans.

Namely, pick any non-zero $\varphi \in T_P(\mathbb{Q}_+)$ (φ corresponds to the point x_0 in the foregoing). Let $U(\varphi) \subset \mathrm{Spec}\, P$ be the set of all prime ideals $\mathfrak{p}$ such that $\varphi(P \setminus \mathfrak{p}) \neq \{0\}$; in other words, the complement of $U(\varphi)$ is the topological closure of the support of φ in T_P, especially, $U(\varphi)$ is an open subset of T_P. Denote by $j : P = \Gamma(T_P, \mathscr{O}_{T_P}) \to \Gamma(U(\varphi), \mathscr{O}_{T_P})$ the restriction map. The morphism of monoids :

$$P \to \Gamma(U(\varphi), \mathscr{O}_{T_P}) \times \mathbb{Q}_+ \qquad x \mapsto (j(x), \varphi(x))$$

determines a cocartesian diagram of fans

$$\begin{tikzcd} U(\varphi) \times (\mathrm{Spec}\, \mathbb{Q}_+)^\sharp \arrow[r, "\beta'"] \arrow[d, "\psi'"'] & U(\varphi) \times (\mathrm{Spec}\, \mathbb{N})^\sharp \arrow[d, "\psi"] \\ T_P \arrow[r, "\beta"] & T_{\varphi^{-1}\mathbb{N}} := (\mathrm{Spec}\, \varphi^{-1}\mathbb{N})^\sharp \end{tikzcd}$$

Lemma 3.6.17. *With the notation of* (3.6.16)*, the morphisms ψ and ψ' are proper rational subdivisions, which we call the* subdivisions centered at φ*.*

Proof. Notice that both β and β' are homeomorphisms on the underlying topological spaces, and moreover both $\log \beta^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbf{1}_{\mathbb{Q}}$ and $\log \beta'^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbf{1}_{\mathbb{Q}}$ are isomorphisms. Thus, it suffices to show that ψ is a proper rational subdivision, hence we may replace P by $\varphi^{-1}\mathbb{N}$, which allows to assume that φ is a morphism of monoids $P \to \mathbb{N}$, and ψ is a morphism of fans

$$T' := U(\varphi) \times (\mathrm{Spec}\, \mathbb{N})^\sharp \to T_P.$$

In this situation, by inspecting the construction, we find that ψ restricts to an isomorphism :

$$\psi^{-1}U(\varphi) \xrightarrow{\sim} U(\varphi)$$

and the preimage of the closed subset $T_P \setminus U(\varphi)$ is the preimage of the closed point $\mathfrak{m} \in (\mathrm{Spec}\, \mathbb{N})^\sharp$ under the natural projection $T' \to (\mathrm{Spec}\, \mathbb{N})^\sharp$; in view of the discussion of (3.5.19), this is naturally identified with $U(\varphi) \times \{\mathfrak{m}\}$. Moreover, the restriction $U(\varphi) \times \{\mathfrak{m}\} \to T_P \setminus U(\varphi)$ of ψ is the map $(t, \mathfrak{m}) \mapsto t \cup \varphi^{-1}\mathfrak{m}$ (recall that $t \subset P$ is a prime ideal which does not contain $\varphi^{-1}\mathfrak{m}$). Thus, the assertion will follow from :

Claim 3.6.18. The $\mathbb{Q}$-linear map

$$\log \psi^{\mathrm{gp}}_{(t,\mathfrak{m})} \otimes_{\mathbb{Z}} \mathbf{1}_{\mathbb{Q}} : P^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{Q} \to (\mathscr{O}^{\mathrm{gp}}_{T_P,t} \times \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is surjective for every $t \in U(\varphi)$, and the induced map :

$$T'(\mathbb{Q}_+) \to T_{\varphi^{-1}\mathbb{N}}(\mathbb{Q}_+) = T_P(\mathbb{Q}_+) \tag{3.6.19}$$

is bijective.

Proof of the claim. Indeed, let $F_t \subset P^\vee$ be the face of $P^\vee$ corresponding to the point $t \in U(\varphi)$, under the bijection (3.5.21); then $\varphi \notin F_t$, whence a natural isomorphism of monoids :

$$(F_t + \mathbb{N}\varphi)^\vee \xrightarrow{\sim} \mathscr{O}_{T_P,t} \times \mathbb{N} \quad : \quad \lambda \mapsto (\lambda_{|F}, \lambda(\varphi))$$

whose inverse, composed with $\log \psi_{(t,\mathfrak{m})}$, yields the restriction map $P \to (F_t + \mathbb{N}\varphi)^\vee$. This interpretation makes evident the surjectivity of $\log \psi^{\mathrm{gp}}_{(t,\mathfrak{m})} \otimes_{\mathbb{Z}} \mathbf{1}_{\mathbb{Q}}$. In view of example 3.5.9(iii), the bijectivity of (3.6.19) is also clear, if one remarks that :

$$P^\vee_{\mathbb{R}} = \bigcup_{t \in U(\varphi)} (F_t + \mathbb{N}\varphi)_{\mathbb{R}}.$$

The latter identity is obvious from the geometric interpretation in terms of polyhedral cones. A formal argument runs as follows. Let $\varphi' \in P^\vee_{\mathbb{R}}$; since P spans $P^{\mathrm{gp}}_{\mathbb{R}}$, the cone $(P_{\mathbb{R}})^\vee$ is strongly convex (corollary 3.3.14), hence the line $\varphi' + \mathbb{R}\varphi \subset P^{\mathrm{gp}}_{\mathbb{R}}$ is not contained in $P^\vee_{\mathbb{R}}$, therefore there exists a largest $r \in \mathbb{R}$ such that $\varphi' - r\varphi \in P^\vee_{\mathbb{R}}$, and necessarily $r \geq 0$. If $\varphi' - r\varphi = 0$, the assertion is clear; otherwise, let F be the minimal face of $P^\vee$ such that $\varphi' - r\varphi \in F_{\mathbb{R}}$, so that $\varphi' = (\varphi' - r\varphi) + r\varphi \in (F + \mathbb{N}\varphi)_{\mathbb{R}}$. Thus, we are reduced to showing that $\varphi \notin F$. But notice that $\varphi' - r\varphi$ lies in the relative interior of F; therefore, if $\varphi \in F$, we may find $\varepsilon > 0$ such that $\varphi' - (r + \varepsilon)\varphi$ still lies in $F_{\mathbb{R}}$, contradicting the definition of r. ◻

3.6.20. Lemma 3.6.17 is frequently used to construct subdivisions centered at an *interior point* of T_P, *i.e.* a point $\varphi \in T_P(\mathbb{N})$ which does not lie on any proper face of $T_P(\mathbb{N})$ (equivalently, the support of φ is the closed point $\mathfrak{m}_P$ of T_P). In this case $U(\varphi) = T_P \setminus \{\mathfrak{m}_P\} = (T_P)_{d-1}$, where $d := \dim P$. By lemma 3.6.17, the $\mathbb{N}$-point φ lifts to a unique $\mathbb{Q}_+$-point $\widetilde{\varphi}$ of $(T_P)_{d-1} \times (\mathrm{Spec}\, \mathbb{N})^\sharp$, and by inspecting the definitions, it is easily seen that – under the identification of remark 3.5.8(iii) – the support of $\widetilde{\varphi}$ is the point $(\varnothing, \mathfrak{m}_{\mathbb{N}})$, where $\varnothing \in T_P$ is the generic point. More precisely, we may identify $(\mathrm{Spec}\, \mathbb{N})^\sharp(\mathbb{Q}_+)$ with $\mathbb{Q}_+$, and $(T_P)_{d-1}(\mathbb{Q}_+)$ with a cone in the $\mathbb{Q}$-vector space $T_P(\mathbb{Q}_+)^{\mathrm{gp}}$, and then $\widetilde{\varphi}$ corresponds to the point $(0,1) \in T_P(\mathbb{Q}_+)^{\mathrm{gp}} \times \mathbb{Q}_+$.

Suppose now that we have an integral roof

$$\rho : (T_P)_{d-1}(\mathbb{Q}_+) \to \mathbb{Q}$$

and let $\pi : T(\rho) \to (T_P)_{d-1}$ be the associated subdivision. Let also $\psi : (T_P)_{d-1} \times (\mathrm{Spec}\, \mathbb{N})^\sharp \to T_P$ be the subdivision centered at φ; we deduce a new subdivision :

$$T^* := T(\rho) \times (\mathrm{Spec}\, \mathbb{N})^\sharp \xrightarrow{\pi \times \mathbf{1}_{(\mathrm{Spec}\, \mathbb{N})^\sharp}} (T_P)_{d-1} \times (\mathrm{Spec}\, \mathbb{N})^\sharp \xrightarrow{\psi} T_P \tag{3.6.21}$$

whose restriction to the preimage of $(T_P)_{d-1}$ is T_P-isomorphic to π.

Lemma 3.6.22. *In the situation of* (3.6.20), *there exist an integral roof*

$$\widetilde{\rho} : T_P(\mathbb{Q}_+) \to \mathbb{Q}$$

whose restriction to $(T_P)_{d-1}(\mathbb{Q}_+)$ *agrees with* ρ, *and a morphism* $T^* \to T(\widetilde{\rho})$ *of* T_P*-monoids, whose underlying continuous map is a homeomorphism.*

Proof. According to remark 3.5.8(vi), we have a natural identification :

$$T_P(\mathbb{Q}_+) = T(\rho)(\mathbb{Q}_+) \times (\mathrm{Spec}\,\mathbb{N})^\sharp(\mathbb{Q}_+) = T(\rho)(\mathbb{Q}_+) \times \mathbb{Q}_+ \subset T_P(\mathbb{Q}_+)^{\mathrm{gp}} \times \mathbb{Q}.$$

mapping the point φ to $(0,1)$. For a given $c \in \mathbb{R}$, denote by $\rho_c : T_P(\mathbb{Q}_+) \to \mathbb{Q}$ the function given by the rule : $(x,y) \mapsto \rho(x) + cy$ for every $x \in T(\rho)(\mathbb{Q}_+)$ and every $y \in \mathbb{Q}_+$.

Let $t \in T(\rho)$ be any point of height $d-1$; by assumption, there exists a $\mathbb{Q}$-linear form $\lambda_t : U(\pi(t))(\mathbb{Q}_+)^{\mathrm{gp}} \to \mathbb{Q}$ whose restriction to $U(t)(\mathbb{Q}_+)$ agrees with the restriction of ρ. Therefore, the restriction of ρ_c to $U(t,\mathfrak{m}_\mathbb{N})(\mathbb{Q}_+) = U(t)(\mathbb{Q}_+) \times \mathbb{Q}_+$ agrees with the restriction of the $\mathbb{Q}$-linear form

$$\lambda_{t,c} : U(\pi(t))(\mathbb{Q}_+)^{\mathrm{gp}} \times \mathbb{Q} = T_P(\mathbb{Q}_+)^{\mathrm{gp}} \to \mathbb{Q} \qquad (x,y) \mapsto \lambda_t(x) + cy.$$

For any two points $t, t' \in T(\rho)$ of height $d-1$, with $\pi(t) = \pi(t')$, pick a finite system of generators $\{x_1, \dots, x_n\}$ for $U(t')(\mathbb{N})$, and let $y_1, \dots, y_n \in U(t)(\mathbb{Q})^{\mathrm{gp}}$, $a_1, \dots, a_n \in \mathbb{Q}$ such that

$$x_i = y_i + a_i\varphi \qquad \text{in the vector space } U(\pi(t))(\mathbb{Q}_+)^{\mathrm{gp}}.$$

In case $x_i \in U(t)(\mathbb{N})$, we shall have $a_i = 0$, and otherwise we remark that $a_i > 0$. Indeed, if $a_i < 0$ we would have $x_i - a_i\varphi \in U(t)(\mathbb{Q}_+)^{\mathrm{gp}} \cap T_P(\mathbb{Q}_+) \subset U(\pi(t))(\mathbb{Q}_+)$; however, $U(\pi(t))(\mathbb{Q}_+)$ is a proper face of $T_P(\mathbb{Q}_+)$, hence $\varphi \in U(\pi(t))(\mathbb{Q}_+)$, a contradiction.

We may then find $c > 0$ large enough, so that $\lambda_{t'}(x_i) < \lambda_{t,c}(x_i)$ for every $x_i \notin U(t)(\mathbb{N})$. It follows easily that

(3.6.23) $$\lambda_{t',c}(x) < \lambda_{t,c}(x) \qquad \text{for all } x \in U(t',\mathfrak{m}_\mathbb{N})(\mathbb{Q}_+) \backslash U(t,\mathfrak{m}_\mathbb{N})(\mathbb{Q}_+).$$

Clearly we may choose c large enough, so that (3.6.23) holds for every pair t, t' as above, and then it is clear that ρ_c will be a roof on T_P. Notice that the points of height d of T^* are precisely those of the form $(t, \mathfrak{m}_\mathbb{N})$, for $t \in T_P$ of height $d-1$, so the points of $T(\rho_c)$ of height d are in natural bijection with those of T^*, and if $\tau \in T(\rho_c)$ corresponds to $\tau^* \in T^*$ under this bijection, we have an injective map $U(\tau^*)(\mathbb{N}) \to U(\tau)(\mathbb{N})$, commuting with the induced projections to $T_P(\mathbb{N})$. There follows a morphism of T_P-fans $U(\tau^*) \to U(\tau)$ inducing a bijection $U(\tau^*)(\mathbb{Q}_+) \xrightarrow{\sim} U(\tau)(\mathbb{Q}_+)$. Since $T(\rho_c)$ (resp. T^*) is the union of all such $U(\tau)$ (resp. $U(\tau^*)$), we deduce a morphism $T^* \to T(\rho_c)$ inducing a homeomorphism on underlying topological spaces. Lastly, say that $\tau^* = (t, \mathfrak{m})$; then $U(\tau^*)(\mathbb{N})^{\mathrm{gp}} = U(t)(\mathbb{N})^{\mathrm{gp}} \oplus \mathbb{Z}\varphi$, and on the other hand $U(t)(\mathbb{N})^{\mathrm{gp}}$ is a direct factor of $U(\tau)(\mathbb{N})^{\mathrm{gp}}$ (since the specialization map $\mathscr{O}^{\mathrm{gp}}_{T(\rho_c),\tau} \to \mathscr{O}^{\mathrm{gp}}_{T,t}$ is surjective). It follows easily that we may choose for c a suitable positive integer, in such a way that the resulting roof $\widetilde{\rho} := \rho_c$ will also be integral. $\square$

3.6.24. Let P be a fine, sharp and saturated monoid, and set as usual $T_P := (\mathrm{Spec}\,P)^\sharp$. There is a canonical choice of a point in $T_P(\mathbb{N})$ which does not lie on any proper face of $T_P(\mathbb{N})$; namely, one may take the $\mathbb{N}$-point φ_P defined as the sum of the generators of the one-dimensional faces of $T_P(\mathbb{N})$ (such faces are isomorphic to $\mathbb{N}$, by theorem 3.4.16(ii)).

Set $d := \dim P$; if ρ is a given integral roof for $(T_P)_{d-1}$, and $c \in \mathbb{N}$ is a sufficiently large, we may then attach to the datum (φ_P, ρ, c) an integral roof $\widetilde{\rho}$ of T_P extending ρ as in lemma 3.6.22, and such that $\widetilde{\rho}(\varphi_P) = c$. More generally, let T be a fine and saturated fan of dimension d, and suppose we have a given integral roof ρ_{d-1} on T_{d-1}; for every point $t \in T$ of height d we have the corresponding canonical point φ_t in the "interior" of $U(t)(\mathbb{N})$, and we may then pick an integer c_d large enough, so that ρ_{d-1} extends to an integral roof $\rho_d : T(\mathbb{Q}_+) \to \mathbb{Q}$ with $\rho_d(\varphi_t) = c_d$ for every $t \in T$ of height d, and such that the associated subdivision is T_P-homeomorphic to $T(\rho_{d-1}) \times (\mathrm{Spec}\,\mathbb{N})$.

This is the basis for the inductive construction of an integral roof on T_P which is canonical in a certain restricted sense. Indeed, fix an increasing sequence of positive integers $\underline{c} := (c_2, \dots, c_d)$; first we define $\rho_1 : T_1(\mathbb{Q}_+) \to \mathbb{Q}_+$ to be the identically zero map. This roof is extended recursively to a function ρ_h on T_h, for each $h = 2, \dots, d$, by the rule given above, in such a way that $\rho_h(\varphi_t) = c_i$ for every point t of height $i \leq h$. By the foregoing we see that the sequence $\underline{c}$ can be chosen so that ρ_d shall again be an integral roof.

3.6.25. More generally, let $\mathscr{S} := \{P_1, \dots, P_k\}$ be any finite set of fine, sharp and saturated monoids. We let $\mathscr{S}$-**Fan** be the full subcategory of **Fan** whose objects are the fans T such that, for every $t \in T$, there exists $P \in \mathscr{S}$ and an open immersion $U(t) \subset (\mathrm{Spec}\, P)^\sharp$. Then the foregoing shows that we may find a sequence of integers $\underline{c}(\mathscr{S}) := (c_2, \dots, c_d)$, with $d := \max(\dim\, P_i \mid i = 1, \dots, k)$ such that the following holds. Every object T of $\mathscr{S}$-**Fan** is endowed with an integral roof $\rho_T : T(\mathbb{Q}_+) \to \mathbb{Q}_+$ such that :

- $\rho_T(\varphi_t) = c_i$ whenever $\mathrm{ht}(t) = i \geq 2$, and ρ_T vanishes on $T_1(\mathbb{Q}_+)$.
- Every open immersion $g : T' \to T$ in $\mathscr{S}$-**Fan** determines an open immersion $\widetilde{g} : T'(\rho_{T'}) \to T(\rho_T)$ such that the diagram

$$\begin{tikzcd} T'(\rho_{T'}) \arrow[r, "\widetilde{g}"] \arrow[d, "\pi_{T'}"'] & T(\rho_T) \arrow[d, "\pi_T"] \\ T' \arrow[r, "g"] & T \end{tikzcd}$$

  commutes (where π_T and $\pi_{T'}$ are the subdivisions associated to ρ_T and $\rho_{T'}$).
- If $\dim\, T = d$, there exists a natural rational subdivision

$$T_{d-1}(\rho_{T_{d-1}}) \times (\mathrm{Spec}\, \mathbb{N})^\sharp \to T(\rho_T) \tag{3.6.26}$$

  which is a homeomorphism on the underlying topological spaces.

3.6.27. Notice as well that, by construction, $T_1(\rho_{T_1}) = T_1$; hence, by composing the morphisms (3.6.26), we obtain a rational subdivision of $T(\rho_T)$:

$$T_1 \times (\mathrm{Spec}\, \mathbb{N}^{\oplus d-1})^\sharp \to \cdots \to T_{d-2}(\rho_{T_{d-2}}) \times (\mathrm{Spec}\, \mathbb{N}^{\oplus 2})^\sharp \to T_{d-1}(\rho_{T_{d-1}}) \times (\mathrm{Spec}\, \mathbb{N})^\sharp \to T(\rho_T).$$

In view of theorem 3.4.16(ii), it is easily seen that every affine open subset of T_1 is isomorphic to $(\mathrm{Spec}\, \mathbb{N})^\sharp$, and any two such open subsets have either empty intersection, or else intersect in their generic points. In any case, we deduce a natural epimorphism

$$(\mathbb{Q}_+)_{T_1} \to \mathscr{O}_{T_1,\mathbb{Q}}$$

(notation of (3.3.20)) from the constant T_1-monoid arising from $\mathbb{Q}_+$; whence an epimorphism :

$$\vartheta_T : (\mathbb{Q}_+^{\oplus d})_{T(\rho_T)} \to \mathscr{O}_{T(\rho_T),\mathbb{Q}}$$

which is compatible with open immersions $g : T' \to T$ in $\mathscr{S}$-**Fan**, in the following sense. Set $d' := \dim\, T'$, and notice that $d' \leq d$; denote by $\pi_{dd'} : \mathbb{Q}_+^{\oplus d} \to \mathbb{Q}_+^{\oplus d'}$ the projection on the first d' direct summands; then the diagram of T'-monoids :

$$\begin{tikzcd} (\mathbb{Q}_+^{\oplus d})_{T'(\rho_{T'})} \arrow[r, "\widetilde{g}^*\vartheta_T"] \arrow[d, "(\pi_{dd'})_{T'(\rho_{T'})}"'] & \widetilde{g}^*\mathscr{O}_{T(\rho_T),\mathbb{Q}} \arrow[d, "(\log \widetilde{g})_{\mathbb{Q}}"] \\ (\mathbb{Q}_+^{\oplus d'})_{T'(\rho_{T'})} \arrow[r, "\vartheta_{T'}"] & \mathscr{O}_{T'(\rho_{T'}),\mathbb{Q}} \end{tikzcd} \tag{3.6.28}$$

commutes.

3.6.29. Let $f : T' \to T$ be a proper morphism, with T' locally fine, such that the induced map $T(\mathbb{Q}_+) \to T'(\mathbb{Q}_+)$ is injective, and let $s \in T$ be any element. For every $t \in f^{-1}(s)$, we set

$$G_t := U(t)(\mathbb{Q}_+)^{\mathrm{gp}} \cap U(s)(\mathbb{N})^{\mathrm{gp}} \qquad H_t := U(t)(\mathbb{N})^{\mathrm{gp}} \qquad \delta_t := (G_t : H_t)$$

and define $\delta(f, s) := \max(\delta_t \mid t \in f^{-1}(s))$.

Lemma 3.6.30. *For every $s \in T$ we have :*

(i) $\delta(f, s) \in \mathbb{N}$.
(ii) *If $t, t' \in f^{-1}(s)$, and t is a specialization of t' in T', then $\delta_{t'} \leq \delta_t$.*
(iii) *If f is a rational subdivision, the following conditions are equivalent :*
  (a) $\delta(f, s) = 1$.
  (b) $U(s)(\mathbb{N}) = \bigcup_{t \in f^{-1}(s)} U(t)(\mathbb{N})$.

Proof. (i): Since $f^{-1}(s)$ is a finite set, the assertion means that $\delta_t < +\infty$ for every $t \in f^{-1}(s)$. However, for such a t, let $P := \mathscr{O}_{T,s}$ and $Q := \mathscr{O}_{T',t}$; then

$$U(t)(\mathbb{Q}_+)^{\mathrm{gp}} = \mathrm{Hom}_{\mathbb{Z}}(Q^{\mathrm{gp}}, \mathbb{Q}) \qquad \text{and} \qquad U(t)(\mathbb{N})^{\mathrm{gp}} = \mathrm{Hom}_{\mathbb{Z}}(Q^{\mathrm{gp}}, \mathbb{Z})$$

(remark 3.4.14(i) and proposition 3.4.12(iii)). By proposition 3.5.24 we know that the map $(\log f)^{\mathrm{gp}}_t \otimes_{\mathbb{Z}} \mathbb{Q} : P^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{Q} \to Q^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$ is surjective. Since Q is finitely generated, it follows that the image Q' of P^{gp} in Q^{gp} is a subgroup of finite index, and then it is easily seen that $\delta_t = (Q^{\mathrm{gp}} : Q')$.

(ii): Under the stated assumptions we have :

$$G_{t'} = G_t \cap U(t')(\mathbb{Q}_+)^{\mathrm{gp}} \qquad H_{t'} = H_t \cap U(t')(\mathbb{Q}_+)^{\mathrm{gp}}$$

whence the contention.

(iii): Assume that (b) holds, and let $\varphi \in U(t)(\mathbb{Q}_+)^{\mathrm{gp}} \cap U(s)(\mathbb{N})^{\mathrm{gp}}$ for some $t \in f^{-1}(s)$. Pick any element $\varphi_0 \in U(t)(\mathbb{N})$ which does not lie on any proper face of $\mathscr{O}^\vee_{T',t}$; we may then find an integer $a > 0$ large enough, so that $\varphi + a\varphi_0 \in U(t)(\mathbb{Q}_+)$. Set $\varphi_1 := \varphi + a\varphi_0$ and $\varphi_2 := a\varphi_0$; then $\varphi_1, \varphi_2 \in U(t)(\mathbb{Q}_+) \cap U(s)(\mathbb{N}) = U(t)(\mathbb{N})$, hence $\varphi = \varphi_1 - \varphi_2 \in U(t)(\mathbb{N})^{\mathrm{gp}}$. Since φ is arbitrary, we see that $\delta_t = 1$, whence (a).

Conversely, suppose that (a) holds, and let $\varphi \in U(s)(\mathbb{N})$; then there exists $t \in f^{-1}(s)$ such that $\varphi \in U(t)(\mathbb{Q}_+)$. Thus, $\varphi \in U(t)(\mathbb{N})^{\mathrm{gp}} \cap U(t)(\mathbb{Q}_+) = U(t)(\mathbb{N})$, whence (b). ☐

Theorem 3.6.31. *Every locally fine and saturated fan T admits an integral, proper, simplicial subdivision $f : T' \to T$, whose restriction $f^{-1}T_{\mathrm{sim}} \to T_{\mathrm{sim}}$ is an isomorphism of fans.*

Proof. Let T be such a fan. By induction on $h \in \mathbb{N}$, we shall construct a system of integral, proper simplicial subdivisions $f_h : S(h) \to T_h$ of the open subsets T_h (notation of (3.5.16)), such that, for every $h \in \mathbb{N}$, the restriction $f^{-1}_{h+1}(T_h) \to T_h$ of f_{h+1} is isomorphic to f_h, and such that $f^{-1}_h T_{h,\mathrm{sim}} \to T_{h,\mathrm{sim}}$ is an isomorphism. Then, the colimit of the morphisms f_h will be the sought subdivision of T.

For $h \leq 1$, we may take $S(h) := T_h$.

Next, suppose that $h > 1$, and that $f_{h-1} : S(h-1) \to T_{h-1}$ has already been given; as a first step, we shall exhibit a rational, proper simplicial subdivision of T_h. Indeed, for every $t \in T_h \setminus T_{h-1}$, choose a $\mathbb{N}$-point $\varphi_t \in U(t)(\mathbb{N})$ in the following way. If $t \in T_{\mathrm{sim}}$, then let φ_t be the (unique) generator of an arbitrarily chosen one-dimensional face of $U(t)(\mathbb{N})$; and otherwise take any point φ_t which does not lie on any proper face of $U(t)(\mathbb{N})$.

With these choices, notice that $U(\varphi_t) \times (\mathrm{Spec}\, \mathbb{N})^\sharp = U(t)$ in case $t \in T_{\mathrm{sim}}$, and otherwise $U(\varphi_t) = U(t)_{h-1}$ (notation of (3.6.16)). By lemma 3.6.17, we obtain corresponding rational subdivisions $U(\varphi_t) \times (\mathrm{Spec}\, \mathbb{N})^\sharp \to U(t)$ of $U(t)$ centered at φ_t. Notice that if t lies in the simplicial locus, this subdivision is an isomorphism, and in any case, it restricts to an isomorphism on the preimage of $U(t)_{h-1}$.

By composing with the restriction of $f_{h-1} \times (\operatorname{Spec} \mathbb{N})^\sharp$, we get a rational subdivision:

$$g_t : T'_t := f_{h-1}^{-1} U(\varphi_t) \times (\operatorname{Spec} \mathbb{N})^\sharp \to U(t)$$

whose restriction to the preimage of $U(t)_{h-1}$ is an isomorphism. Moreover, g_t is an isomorphism if t lies in T_{sim}. Also notice that g_t is simplicial, since the same holds for f_{h-1}.

If t, t' are any two distinct points of T of height h, we deduce an isomorphism

$$g_t^{-1}(U(t) \cap U(t')) \xrightarrow{\sim} g_{t'}^{-1}(U(t) \cap U(t'))$$

hence we may glue the fans T'_t and the morphisms g_t along these isomorphism, to obtain the sought simplicial rational subdivision $g : T' \to T_h$.

For the next step, we shall refine g locally at every point s of height h; *i.e.* for such s, we shall find an integral, proper simplicial subdivision $f_s : T''_s \to U(s)$, whose restriction to $U(s)_{h-1}$ agrees with g, hence with f_{h-1}. Once this is accomplished, we shall be able to build the sought subdivision f_h by gluing the morphisms f_s and f_{h-1} along the open subsets $U(s)_{h-1}$.

Of course, if s lies in the simplicial locus of T, we will just take for f_s the restriction of g, which by construction is already an isomorphism.

Henceforth, we may assume that $T = U(s)$ is an affine fan of dimension h with $s \notin T_{\mathrm{sim}}$, and $g : T' \to T$ is a given proper rational simplicial subdivision, whose restriction to $g^{-1}T_{h-1}$ is an integral subdivision. We wish to apply the criterion of lemma 3.6.30(iii), which shows that g is an integral subdivision if and only if $\delta(g, s) = 1$. Thus, let $t_1, \dots, t_k$ be the points of T' such that $\delta_{t_i} = \delta(g, s)$ for every $i = 1, \dots, k$. Since $\delta(g, s)$ is anyway a positive integer (lemma 3.6.30(i)), a simple descending induction reduces to the following :

Claim 3.6.32. Given g as above, we may find a proper rational simplicial subdivision $g' : T'' \to T$ such that the following holds :

(i) The restriction of g' to $g'^{-1}T_{h-1}$ is isomorphic to the restriction of g.

(ii) Let $t'_1, \dots, t'_{k'} \in T''$ be the points such that $\delta_{t'_i} = \delta(g', s)$ for every $i = 1, \dots, k'$. We have $\delta(g', s) \leq \delta(g, s)$, and if $\delta(g', s) = \delta(g, s)$ then $k' < k$.

Proof of the claim. Set $t := t_1$; by definition, there exists $\varphi' \in U(s)(\mathbb{N})$ which lies in $U(t)(\mathbb{Q}_+) \setminus U(t)(\mathbb{N})$; this means that there exists a morphism φ fitting into a commutative diagram :

$$\begin{tikzcd} \mathscr{O}_{T,s} \arrow[r, "(\log g)_t"] \arrow[d, "\varphi'"'] & \mathscr{O}_{T',t} \arrow[d, "\varphi"] \\ \mathbb{N} \arrow[r] & \mathbb{Q}_+. \end{tikzcd}$$

Say that $\mathscr{O}_{T',t} \simeq \mathbb{N}^{\oplus r}$, and let $(\pi_1, \dots, \pi_r)$ be the (essentially unique) basis of $\mathscr{O}^\vee_{T'(h),t}$; then $\varphi = a_1\pi_1 + \cdots a_r\pi_r$, for some $a_1, \dots, a_r \geq 0$, and after subtracting some positive integer multiple of π, we may assume that $0 \leq a_i < 1$ for every $i = 1, \dots, r$. Moreover, the coefficients are all strictly positive if and only if φ is a local morphism; more generally, we let t' be the unique generization of t such that φ factors through a local morphism $\mathscr{O}_{T',t'} \to \mathbb{N}$. Set $e := \operatorname{ht}(t')$, and denote by $\pi_1, \dots, \pi_e$ the basis of $\mathscr{O}^\vee_{T',t'}$, so that :

$$\varphi = b_1\pi_1 + \cdots + b_e\pi_e \tag{3.6.33}$$

for unique rational coefficients $b_1, \dots, b_e$ such that $0 < b_i < 1$ for every $i = 1, \dots, e$.

Denote by $Z \subset T'$ the topological closure of $\{t'\}$; for every $u \in Z \cap T'$, the morphism φ factors through a morphism $\varphi_u : \mathscr{O}_{T,u} \to \mathbb{Q}_+$, and we may therefore consider the subdivision of $U(u)$ centered at φ_u as in (3.6.16), which fits into a commutative diagram of fans :

$$\begin{tikzcd}
(U(u)\backslash Z) \times (\mathrm{Spec}\,\mathbb{Q}_+)^\sharp \arrow[r] \arrow[d] & (U(u)\backslash Z) \times (\mathrm{Spec}\,\mathbb{N})^\sharp \arrow[d, "\psi_u"] \\
U(u) \arrow[r] & U'(u) := (\mathrm{Spec}\,\varphi_u^{-1}\mathbb{N})^\sharp.
\end{tikzcd}$$

We complete the family $(\psi_u \mid u \in Z)$, by letting $U'(u) := U(u)$ and $\psi_u := \mathbf{1}_{U(u)}$ for every $u \in T' \setminus Z$. Notice then, that the topological spaces underlying $U(u)$ and $U'(u)$ agree for every $u \in T'$, and for every $u_1, u_2 \in T'$, the restrictions of ψ_{u_1} and ψ_{u_2} :

$$\psi_{u_i}^{-1}(U(u_1) \cap U(u_2)) \to U(u_1) \cap U(u_2) \qquad (i = 1, 2)$$

are isomorphic. Furthermore, by construction, each restriction $U(u) \to T$ of g factors uniquely through a morphism $\beta_u : U'(u) \to T$, hence the family $(\beta_u \circ \psi_u \mid u \in T')$ glues to a well defined morphism of fans $g' : T'' \to T$. By a direct inspection, it is easily seen that g' is a proper rational simplicial subdivision which fulfills condition (i) of the claim.

Moreover, the map of topological spaces underlying g' factors naturally through a continuous map $p : T'' \to T'$, so that $g' = g \circ p$. The restriction $V := p^{-1}(T' \setminus Z) \to T$ of g' is isomorphic to the restriction $T' \setminus Z \to T$ of g, hence :

$$\delta_t = \delta_{p(t)} \qquad \text{for every } t \in p^{-1}(T' \setminus Z).$$

It follows that, if $k > 1$ and $T' \setminus Z$ contains at least one of the points $t_2, \dots, t_k$, then $\delta(g', s) \geq \delta(g, s)$. On the other hand, $p^{-1}(T' \setminus Z)$ contains at most $k - 1$ points u of T'' such that $\delta_u = \delta(g, s)$, and for the remaining points $u' \in p^{-1}(T' \setminus Z)$ we have $\delta_{u'} < \delta(g, s)$. Since obviously

$$\delta(g', s) = \max(\delta(g'_{|p^{-1}(T' \setminus Z)}, s), \delta(g'_{|p^{-1}Z}, s))$$

we see that condition (ii) holds provided we show :

$$\delta(g'_{|p^{-1}Z}, s) = \max(b_1, \dots, b_e) \cdot \delta(g, s). \tag{3.6.34}$$

Hence, let us fix $u \in Z$, and let $(v, x) \in (U(u)\backslash Z) \times (\mathrm{Spec}\,\mathbb{N})^\sharp$ be any point (see (3.5.19)); if $x = \varnothing$, then $(v, x) \notin p^{-1}Z$, so it suffices to consider the points of the form $(v, \mathfrak{m})$ (where $\mathfrak{m} \subset \mathbb{N}$ is the maximal ideal). Moreover, say that $\mathrm{ht}(u) = d$; in view of lemma 3.6.30(ii), it suffices to consider the points $(v, \mathfrak{m})$ such that $\mathrm{ht}(v) = d - 1$. There are exactly e such points, namely the prime ideals $v_i := (\pi_i \circ \sigma)^{-1}\mathfrak{m}$, where $\pi_1, \dots, \pi_e$ are as in (3.6.33), and $\sigma : \mathscr{O}_{T',u} \to \mathscr{O}_{T',t'}$ is the specialization map.

In order to estimate $\delta_{(v,\mathfrak{m})}$ for some $v := v_i$, we look at the transpose of the map

$$(\log g')^{\mathrm{gp}}_{(v,\mathfrak{m})} : \mathscr{O}^{\mathrm{gp}}_{T,s} \to \mathscr{O}^{\mathrm{gp}}_{T',v} \times \mathbb{Z} \quad : \quad z \mapsto ((\log g)^{\mathrm{gp}}_v(z), \varphi^{\mathrm{gp}}(z)).$$

Let $(p_1, \dots, p_d)$ be the basis of $\mathscr{O}^\vee_{T,u}$, ordered in such a way that $p_i = \pi_i \circ \sigma$ for every $i = 1, \dots, e$. By a little abuse of notation, we may then denote $(p_j \mid j \neq i)$ the basis of $\mathscr{O}^\vee_{T',v}$, so that the dual group $(\mathscr{O}^{\mathrm{gp}}_{T',v} \times \mathbb{Z})^\vee$ admits the basis $\{p_j^{\mathrm{gp}} \mid j \neq i\} \cup \{q\}$, where q is the natural projection onto $\mathbb{Z}$. Set as well $p'_i := p_i \circ (\log g)^{\mathrm{gp}}_u$ for every $i = 1, \dots, d$. With this notation, the above transpose is the group homomorphism given by the rule :

$$p_j \mapsto p'_i \quad \text{for } j \neq i \qquad \text{and} \qquad q \mapsto b_1 p'_1 + \cdots + b_e p'_e$$

from which we deduce easily that $\delta_{(v,\mathfrak{m})} = b_i \cdot \delta_u$, whence (3.6.34). $\square$

Proposition 3.6.35. *Let $(\Gamma, +, 0)$ be a fine monoid, M a fine Γ-graded monoid. Then :*

(i) *There exists a finite set of generators $C := \{\gamma_1, \dots, \gamma_k\}$ of Γ, with the following property. For every $\gamma \in \Gamma$, we may find $a_1, \dots, a_k \in \mathbb{N}$ such that:*

$$\gamma = a_1\gamma_1 + \cdots + a_k\gamma_k \quad \textit{and} \quad M_\gamma = M_{\gamma_1}^{a_1} \cdots M_{\gamma_k}^{a_k}.$$

(ii) *There exists a subgroup $H \subset \Gamma^{\mathrm{gp}}$ of finite index, such that :*
$$M_{a\gamma} = M_\gamma^a \qquad \text{for every } \gamma \in H \cap \Gamma \text{ and every integer } a > 0.$$
(In (i) *and* (ii) *we use the multiplication law of $\mathscr{P}(M)$, as in* (3.1.1)*).*

Proof. Obviously we may assume that M maps surjectively onto Γ, in which case $G := \Gamma^{\mathrm{gp}}$ is finitely generated, and its image G' into $G_{\mathbb{R}} := G \otimes_{\mathbb{Z}} \mathbb{R}$ is a free abelian group of finite rank. The same holds as well for the image L of $\log M^{\mathrm{gp}}$ in $M^{\mathrm{gp}}_{\mathbb{R}} := \log M^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{R}$. Let $p : G \to G'$ be the natural projection. According to proposition 3.3.22(iii), we have :

$$M_{\mathbb{Q}} = M_{\mathbb{R}} \cap M^{\mathrm{gp}}_{\mathbb{Q}} \tag{3.6.36}$$

(notation of (3.3.20)). Let $f^{\mathrm{gp}}_{\mathbb{R}} : M^{\mathrm{gp}}_{\mathbb{R}} \to G_{\mathbb{R}}$ be the induced $\mathbb{R}$-linear map, and denote by $f_{\mathbb{R}} : M_{\mathbb{R}} \to G_{\mathbb{R}}$ the restriction of $f^{\mathrm{gp}}_{\mathbb{R}}$. By proposition 3.3.28(ii), we may find a G'-rational subdivision Δ of $f_{\mathbb{R}}(M_{\mathbb{R}})$ such that :

$$f^{-1}_{\mathbb{R}}(a+b) = f^{-1}_{\mathbb{R}}(a) + f^{-1}_{\mathbb{R}}(b) \tag{3.6.37}$$

for every $\sigma \in \Delta$ and every $a, b \in \sigma$. After choosing a refinement, we may assume that Δ is a simplicial fan (theorem 3.6.31). Let $\tau \in \Delta$ be any cone; by proposition 3.3.22(i), the monoid $N := \tau \cap G'$ is finitely generated, and then the same holds for $M \times_{G'} N$, by corollary 3.4.2. However, the latter is just $M' := \bigoplus_{\gamma \in p^{-1}N} M_\gamma$ (lemma 2.3.29(iii)). Set $\Gamma' := \Gamma \cap p^{-1}N$.

Claim 3.6.38. $p(\Gamma')$ generates τ.

Proof of the claim. Clearly the Γ-grading of M induces a surjection $M_{\mathbb{Q}} \to \Gamma_{\mathbb{Q}}$ (notation of (3.3.20)). By proposition 3.3.22(iii), we have $\Gamma_{\mathbb{Q}} = f_{\mathbb{R}}(M_{\mathbb{R}}) \cap G_{\mathbb{Q}}$, hence $N \subset \tau \cap G_{\mathbb{Q}} \subset \Gamma_{\mathbb{Q}}$. Thus, for every $n \in N$ we may find $a \in \mathbb{N}$ such that $a \cdot n \in p(\Gamma)$, hence $a \cdot n \in p(\Gamma')$. On the other hand, N generates τ, since the latter is G'-rational. The claim follows. ◊

In view of claim 3.6.38, we may replace M by M', and Γ by Γ', which allows to assume that $f_{\mathbb{R}}(M_{\mathbb{R}})$ is a simplicial cone, and (3.6.37) holds for every $a, b \in f_{\mathbb{R}}(M_{\mathbb{R}})$. Next, let $S := \{e_1, \dots, e_n\} \subset p(\Gamma)$ be a set of generators of the cone $f_{\mathbb{R}}(M_{\mathbb{R}})$; from the discussion in (3.3.15) we see that, up to replacing S by a subset, the rays $\mathbb{R}_+ \cdot e_i$ (with $i = 1, \dots, n$) are precisely the extremal rays of $f_{\mathbb{R}}(M_{\mathbb{R}})$, especially, the vectors $e_1, \dots, e_n$ are $\mathbb{R}$-linearly independent. Choose $g_1, \dots, g_n \in \Gamma$ such that $p(g_i) = e_i$ for every $i = 1, \dots, n$. According to corollary 3.4.3, there exist finite subsets $\Sigma_1 \dots, \Sigma_n \subset M$, such that :
$$M_{g_i} = M_0 \cdot \Sigma_i \qquad \text{for every } i = 1, \dots, n.$$

Claim 3.6.39. $(M_0)_{\mathbb{Q}} = f^{-1}_{\mathbb{R}}(0) \cap M^{\mathrm{gp}}_{\mathbb{Q}}$.

Proof of the claim. To begin with, we may write $f^{-1}_{\mathbb{R}}(0) = (f^{\mathrm{gp}}_{\mathbb{R}})^{-1}(0) \cap M_{\mathbb{R}}$, hence $f^{-1}_{\mathbb{R}}(0) \cap M^{\mathrm{gp}}_{\mathbb{Q}} = (f^{\mathrm{gp}}_{\mathbb{Q}})^{-1}(0) \cap M_{\mathbb{Q}}$, by (3.6.36). Now, suppose $x \in (f^{\mathrm{gp}}_{\mathbb{Q}})^{-1}(0) \cap M_{\mathbb{Q}}$; then we may find an integer $a > 0$ such that $ax = m \otimes 1$ for some $m \in \log M$. Say that $m \in \log M_\gamma$; then $f^{\mathrm{gp}}_{\mathbb{Q}}(\gamma) = 0$, therefore γ is a torsion element of G', and consequently $bm \in \log M_0$ for an integer $b > 0$ large enough. We conclude that $x = (bm) \otimes (ba)^{-1} \in (M_0)_{\mathbb{Q}}$, as claimed. ◊

On the other hand, since $\mathbb{R}_+ e_i$ is a G'-rational polyhedral cone, $f^{-1}_{\mathbb{R}}(\mathbb{R}_+ e_i)$ is an L-rational polyhedral cone (proposition 3.3.21(ii,iii)), hence it admits a finite set of generators $S_i \subset L$. Up to replacing the elements of S_i by some positive rational multiples, we may assume that $f_{\mathbb{R}}(s)$ is either 0 or e_i, for every $s \in S_i$. In this case, it follows easily that

$$f^{-1}_{\mathbb{R}}(e_i) = f^{-1}_{\mathbb{R}}(0) + T_i \tag{3.6.40}$$

where :
$$T_i := \left\{ \sum_{s \in S_i} t_s \cdot s \mid t_s \in \mathbb{R}_+, \sum_{s \in S_i} t_s = 1 \right\}$$

is the convex hull of S_i (and as usual, the addition of sets in (3.6.40) refers to the addition law of $\mathscr{P}(M^{\mathrm{gp}}_{\mathbb{R}})$, see (3.1.1)). Next, notice that $S_i \subset M_{\mathbb{Q}}$, by (3.6.36); thus, we may find an integer $a > 0$ such that $a \cdot s$ lies in the image of $\log M$, for every $s \in S_i$. After replacing e_i by $a \cdot e_i$ and S_i by $\{a \cdot s \mid s \in S_i\}$, we may then achieve that (3.6.40) holds, and furthermore S_i lies in the image of $\log M$, therefore in the image of $\log M_{g_i}$. It follows easily that (3.6.40) still holds with S_i replaced by the set $\Sigma_i \otimes 1 := \{m \otimes 1 \mid m \in \Sigma_i\}$. Let $\Sigma_0 \subset \log M$ be a finite set of generators for the monoid M_0; claim 3.6.39 implies that Σ_0 is also a set of generators for the L-rational polyhedral cone $f_{\mathbb{R}}^{-1}(0)$. Let $P \subset M$ be the submonoid generated by $\Sigma := \Sigma_0 \cup \Sigma_1 \cup \cdots \cup \Sigma_n$, and $\Delta \subset \Gamma$ the submonoid generated by $g_1, \dots, g_n$; clearly the Γ-grading of M restricts to a Δ-grading on P. Notice that $g_1 \otimes 1, \dots, g_n \otimes 1$ are linearly independent in $G_{\mathbb{Q}}$, since the same holds for $e_1, \dots, e_n$; especially, $\Delta \simeq \mathbb{N}^{\oplus n}$.

Claim 3.6.41. (i) The set $\Sigma \otimes 1 := \{s \otimes 1 \mid s \in \Sigma\}$ generates the cone $M_{\mathbb{R}}$.

(ii) $P_{a+b} = P_a \cdot P_b$ for every $a, b \in \Delta$.

(iii) There exists a finite set $A \subset M$ such that $M = A \cdot P$.

Proof of the claim. By (3.6.40) and the foregoing discussion, we know that $(\Sigma_0 \cup \Sigma_i) \otimes 1$ generate $f_{\mathbb{R}}^{-1}(\mathbb{R}_+ e_i)$, for every $i = 1, \dots, n$. Since the additivity property (3.6.37) holds for every $a, b \in f_{\mathbb{R}}(M_{\mathbb{R}})$, assertion (i) follows. (ii) is a straightforward consequence of the definitions. Next, from (i) and proposition 3.3.22(iii), we deduce that $M_{\mathbb{Q}} = P_{\mathbb{Q}}$. Thus, let $m_1, \dots, m_r$ be a system of generators for the monoid M; it follows that there are integers $k_1, \dots, k_r > 0$, such that $m_1^{k_1}, \dots, m_r^{k_r} \in P$, and therefore the subset $A := \{\prod_{i=1}^r m_i^{t_i} \mid 0 \leq t_i < k_i \text{ for every } i \leq r\}$ fulfills the condition of (iii). ♢

We introduce a partial ordering on G, by declaring that $a \leq b$ for two elements $a, b \in G$, if and only if $b - a \in \Delta$. Now, let $a \in G$ be any element; we set

$$G(a) := \{g \in G \mid g \leq a\}.$$

The subset $G(a)$ inherits a partial ordering from G, and a is the maximum of the elements of $G(a)$; moreover, notice that every finite subset $S \subset G(a)$ admits a supremum $\sup S \in G(a)$. Indeed, it suffices to show the assertion for a set of two elements $S = \{b_1, b_2\}$; we may then write $a - b_i = \sum_{j=1}^n k_{ij} g_j$ for certain $k_{ij} \in \mathbb{N}$, and then $\sup(b_1, b_2) = a - \sum_{j=1}^n \min(k_{1j}, k_{2j}) \cdot g_j$.

Let A be as in claim 3.6.41(iii), and denote by $B \subset \Gamma$ the image of A; then

$$M = M_0 \cdot M_B \qquad \text{where } M_B := \bigoplus_{b \in B} \log M_b.$$

For every $a \in G$, let also $B(a) := B \cap G(a)$; invoking several times claim 3.6.41(ii), we get :

$$\begin{aligned} M_a &= \bigcup_{b \in B(a)} M_b \cdot P_{a-b} \\ &= \bigcup_{b \in B(a)} M_b \cdot P_{\sup B(a) - b} \cdot P_{a - \sup B(a)} \\ &\subset M_{\sup B(a)} \cdot P_{a - \sup B(a)}. \end{aligned}$$

Finally, say that $a - \sup B(a) = \sum_{i=1}^n t_i g_i$ for certain $t_1, \dots, t_n \in \mathbb{N}$; applying once more claim 3.6.41(ii), we conclude that :

$$M_a \subset M_{\sup B(a)} \cdot \prod_{i=1}^n M_{g_i}^{t_i}.$$

The converse inclusion is clear, and therefore the set $C := \{g_1, \dots, g_n\} \cup \{\sup B(a) \mid a \in G\}$ fulfills condition (i) of the proposition.

(ii): For $h \in \Delta^{\mathrm{gp}}$, say $h = \sum_{i=1}^n a_i g_i$, with integers $a_1, \dots, a_n$, we let $|h| := \sum_{i=1}^n |a_i| g_i \in \Delta$. Choose any positive integer α such that :

$$|b| \leq \alpha \cdot \sum_{i=1}^n g_i \qquad \text{for every } b \in B \cap \Delta^{\mathrm{gp}} \tag{3.6.42}$$

and let $H \subset \Delta^{\mathrm{gp}}$ be the subgroup generated by $\alpha g_1, \dots, \alpha g_n$.

Claim 3.6.43. $B(h) = B(kh)$ for every $h \in H$ and every integer $k > 0$.

Proof of the claim. Let $h := \sum_{i=1}^n \alpha_i g_i \in H$, and suppose that $b \in B(kh)$ for some $k > 0$; therefore $kh - b \in \Delta$, hence $b \in \Delta^{\mathrm{gp}}$, and we can write $b = \sum_{i=1}^n \beta_i g_i$ for integers $\beta_1, \dots, \beta_n$, such that $k\alpha_i - \beta_i \geq 0$ for every $i = 1, \dots, n$. In this case, (3.6.42) implies that $\alpha_i \geq 0$ for every $i \leq n$, and $\alpha_i \geq \alpha \geq \beta_i$ whenever $\beta_i > 0$. It follows easily that $k'h - b \in \Delta$ for every integer $k' > 0$, whence the claim. $\diamondsuit$

Using claims 3.6.41(ii) and 3.6.43, and arguing as in the foregoing, we may compute :

$$\begin{aligned} M_{ah} &= M_{\sup B(ah)} \cdot P_{ah - \sup B(ah)} \\ &= M_{\sup B(h)} \cdot P_{ah - \sup B(h)} \\ &= M_{\sup B(h)} \cdot P_{h - \sup B(h)} \cdot P_h^{a-1} \\ &\subset M_h^a. \end{aligned}$$

The converse inclusion is clear, so (ii) holds. $\square$

3.6.44. Let M be an integral monoid, and $w \in \log M^{\mathrm{gp}}$ any element. For $\varepsilon \in \{1, -1\}$ we have a natural inclusion

$$j_\varepsilon : \log M \to M(\varepsilon) := \log M + \varepsilon \mathbb{N} w$$

(*i.e.* $M(\varepsilon)$ is the submonoid of M^{gp} generated by M and w^ε). Let us write $w := b^{-1}a$ for some $a, b \in M$; then the induced morphisms of affine schemes $\iota_\varepsilon := \mathrm{Spec}\, \mathbb{Z}[j_\varepsilon]$ have a natural geometric interpretation. Namely, let $f : X \to \mathrm{Spec}\, \mathbb{Z}[M]$ be the blow up of the ideal $I \subset \mathbb{Z}[M]$ generated by a and b; we have $X = U_1 \cup U_{-1}$, where U_ε, for $\varepsilon = \pm 1$, is the largest open subscheme of X such that $w^\varepsilon \in \mathscr{O}_X(U_\varepsilon)$. Then ι_ε is naturally identified with the restriction $U_\varepsilon \to \mathrm{Spec}\, \mathbb{Z}[M]$ of the blow up f. More generally, by adding to M any finite number of elements of M^{gp}, we may construct in a combinatorial fashion, the standard affine charts of a blow up of an ideal of $\mathbb{Z}[M]$ generated by finitely many elements of M. These considerations explain the significance of the following *flattening theorem* :

Theorem 3.6.45. *Let $j : M \to N$ be an inclusion of fine monoids. Then there exists a finite set $\Sigma \subset \log M^{\mathrm{gp}}$, and an integer $k > 0$ such that the following holds :*

(i) *For every mapping $\varepsilon : \Sigma \to \{\pm 1\}$, the induced inclusion :*

$$M(\varepsilon) := \log M + \sum_{\sigma \in \Sigma} \varepsilon(\sigma) \mathbb{N}\sigma \to N(\varepsilon) := \log N + \sum_{\sigma \in \Sigma} \varepsilon(\sigma) \mathbb{N}\sigma$$

is a flat morphism of fine monoids.

(ii) *Suppose that j is a flat morphism, and let $\iota : M \to M^{\mathrm{sat}}$ be the natural inclusion. Denote by Q the push-out of the diagram $N \xleftarrow{j} M \xrightarrow{\iota \circ \boldsymbol{k}_M} M^{\mathrm{sat}}$ (where $\boldsymbol{k}_M$ is the k-Frobenius). Then the natural map $M^{\mathrm{sat}} \to Q^{\mathrm{sat}}$ is flat and saturated.*

Proof. (i): (Notice that $\log M$, $\log N$ and $P(\varepsilon) := \sum_{\sigma \in \Sigma} \varepsilon(\sigma) \mathbb{N}\sigma$ may be regarded as submonoids of $\log N^{\mathrm{gp}}$, and then the above sum is taken in the monoid $(\mathscr{P}(\log N^{\mathrm{gp}}), +)$ defined as in (3.1.1).) Set $G := N^{\mathrm{gp}}/M^{\mathrm{gp}}$, and let $N^{\mathrm{gp}} = \bigoplus_{\gamma \in G} N^{\mathrm{gp}}_\gamma$ be the j-grading of N^{gp} (remark 3.2.5(iii)); notice that this grading restricts to j-gradings for N and $N(\varepsilon)$. Let also $\Gamma := \{\gamma \in G \mid N_\gamma \neq \varnothing\}$, and choose a finite generating set $\{\gamma_1, \dots, \gamma_r\}$ of Γ with the properties of proposition 3.6.35(i). According to corollary 3.4.3, for every $i \leq r$ there exists a finite subset $\Sigma_i := \{t_{i1}, \dots, t_{in_i}\} \subset M$ such that $N_{\gamma_i} = N_0 \cdot \Sigma_i$. Moreover, N_0 is a finitely generated monoid, by corollary 3.4.2. Let Σ_0 be a finite system of generators for N_0, and for every $i \leq r$ define $\Sigma'_i := \{t_{ij} - t_{il} \mid 1 \leq j < l \leq n_i\}$. We claim that the subset $\Sigma := \Sigma_0 \cup \Sigma'_1 \cup \dots \cup \Sigma'_r$ will do. Indeed, first of all notice that $\Sigma \subset \log M^{\mathrm{gp}}$. We shall apply the flatness criterion of remark 3.2.5(iv). Thus, we have to show that, for every $g \in \Gamma$, the $M(\varepsilon)$-module $N(\varepsilon)_g$ is a filtered union of cosets $\{m\} + M(\varepsilon)$ (for certain $m \in N_g$). Hence, let $x_1, x_2 \in N(\varepsilon)_g$; we may write $x_i = m_i + p_i$ for some $m_i \in \log N$ and $p_i \in P(\varepsilon)$ $(i = 1, 2)$; since $\{x_i\} + M(\varepsilon) \subset \{m_i\} + M(\varepsilon)$, we may then assume that $p_1 = p_2 = 0$, hence $x_1, x_2 \in N$. Thus, it suffices to show that N_g is contained in a filtered union of cosets as above. However, by assumption there exist $a_1, \dots, a_r \in \mathbb{N}$ such that $g = \sum_{i=1}^r a_i\gamma_i$ and $N_g = N_{\gamma_1}^{a_1} \cdots N_{\gamma_r}^{a_r}$; we are then easily reduced to the case where $g = \gamma_i$ for some $i \leq r$. Therefore, $x_j = \sigma_j + b_j$, where $\sigma_j \in \Sigma_i$ and $b_j \in N_0$, for $j = 1, 2$. Notice that $P(\varepsilon)$ contains either $\sigma_1 - \sigma_2$, or $\sigma_2 - \sigma_1$ (or both); in the first occurrence, set $\sigma := \sigma_2$, and otherwise, let $\sigma := \sigma_1$. Likewise, say that $\Sigma_0 = \{y_1, \dots, y_n\}$, so that $b_j = \sum_{s=1}^n a_{js}y_s$ for certain $a_{js} \in \mathbb{N}$ $(j = 1, 2)$; for every $s \leq n$, we set $a^*_s := \min(a_{1s}, a_{2s})$ if $y_s \in P(\varepsilon)$, and otherwise we set $a^*_s := \max(a_{1s}, a_{2s})$. One sees easily that $x_1, x_2 \in \{\sigma + \sum_{s=1}^n a^*_s y_s\} + M(\varepsilon)$, whence the contention.

(ii): By proposition 3.6.35(ii), there exists a subgroup $H \subset G$ of finite index, such that :

$$\pi^{-1}(h^n) = \pi^{-1}(h)^n$$

for every integer $n > 0$ and every $h \in H$. Let $k := (G : H)$, and define N' as the fibre product in the cartesian diagram :

(3.6.46)
$$\begin{tikzcd} N' \arrow[r, "\pi'"] \arrow[d, "\mu"'] & G \arrow[d, "\boldsymbol{k}_G"] \\ N \arrow[r, "\pi"] & G \end{tikzcd}$$

The trivial morphism $\mathbf{0}_M : M \to G$ (*i.e.* the unique one that factors through $\{1\}$) and the inclusion j satisfy the identity : $\boldsymbol{k}_G \circ \mathbf{0}_M = \pi \circ j$, hence they determine a well-defined map $\varphi : M \to N'$.

Claim 3.6.47. φ is flat and saturated.

Proof of the claim. For the flatness, we shall apply the criterion of remark 3.2.5(iv). First, φ is injective, since $\mu \circ \varphi = j$. Next, notice that the sequence of abelian groups :

$$0 \to M^{\mathrm{gp}} \xrightarrow{\varphi^{\mathrm{gp}}} (N')^{\mathrm{gp}} \xrightarrow{(\pi')^{\mathrm{gp}}} G \to 0$$

is short exact; indeed, this is none else than the pullback $\mathscr{E} * \boldsymbol{k}_G^{\mathrm{gp}}$ along the morphism $\boldsymbol{k}_G$, of the short exact sequence

(3.6.48)
$$\mathscr{E} := (0 \to M^{\mathrm{gp}} \xrightarrow{j^{\mathrm{gp}}} N^{\mathrm{gp}} \xrightarrow{\pi^{\mathrm{gp}}} G \to 0).$$

It follows that φ is flat if and only if, for every $x \in \pi'(N')$, the preimage $(\pi')^{-1}(x)$ is a filtered union of cosets of the form $\{n\} \cdot \varphi(M')$. However, the induced map $(\pi')^{-1}(g) \to \pi^{-1}(g^k)$ is a bijection for every $g \in G$, and the flatness of j implies that $\pi^{-1}(x^k)$ is a filtered union of cosets, whence the contention. Notice also that $\mathrm{Im}\, \boldsymbol{k}_G \subset H$; hence, by the same token, we derive that $(\pi')^{-1}(g^n) = (\pi')^{-1}(g)^n$ for every $g \in G$, therefore φ is quasi-saturated, by proposition 3.2.31. Since we know already that j is integral (theorem 3.2.3), the claim follows. $\diamondsuit$

Next, we wish to consider the commutative diagram of monoids :

$$
\begin{tikzcd}
M \arrow[r, "j"] \arrow[d, "\boldsymbol{k}_M"'] & N \arrow[d, "\psi"] \\
M \arrow[r, "\varphi"] & N'
\end{tikzcd}
\tag{3.6.49}
$$

such that ψ is the map determined by the pair of morphisms $(f, \boldsymbol{k}_N)$. Let P be the push-out of the maps j and $\boldsymbol{k}_M$; the maps φ and ψ determine a morphism $\tau : P \to N'$.

Claim 3.6.50. (i) The diagram $(3.6.46)^{\mathrm{gp}}$ of associated abelian groups, is cartesian.

(ii) The diagram of abelian groups $(3.6.49)^{\mathrm{gp}}$ is cocartesian (*i.e.* τ^{gp} is an isomorphism).

(iii) There exists a morphism $\lambda : N' \to P$ such that $\lambda \circ \tau = \boldsymbol{k}_P$ and $\tau \circ \lambda = \boldsymbol{k}_{N'}$.

Proof of the claim. (i): Suppose that $x \in (N')^{\mathrm{gp}}$ is any element such that $(\pi')^{\mathrm{gp}}(x) = 1$ and $\mu^{\mathrm{gp}}(x) = 1$; we may write $x = b^{-1}a$ for some $a, b \in N'$, and it follows that $\pi'(a) = \pi'(b)$ and $\mu(a) = \mu(b)$, hence $a = b$ in N', since the forgetful functor $\mathbf{Mnd} \to \mathbf{Set}$ commutes with fibre products. Thus $x = 1$. On the other hand, suppose that $\pi^{\mathrm{gp}}(b^{-1}a) = x^k$ for some $a, b \in N$ and $x \in G$; therefore, $\pi(b^{k-1}a) = (bx)^k$, so there exists $c \in N'$ such that $\mu(c) = b^{k-1}a$ and $\pi'(c) = bx$. Likewise, there exists $d \in N'$ with $\mu(d) = b^k$ and $\pi'(d) = b$. Consequently, $(\pi')^{\mathrm{gp}}(d^{-1}c) = x$ and $\mu(d^{-1}c) = b^{-1}a$. The assertion follows.

(ii): Quite generally, let $E := (0 \to A \to B \to C \to 0)$ be a short exact sequence of objects in an abelian category $\mathscr{C}$, and for every object X of $\mathscr{C}$, let $\boldsymbol{k}_X := k \cdot \mathbf{1}_X : X \to X$. Then one has a natural map of complexes $\alpha_E : E * \boldsymbol{k}_C \to E$ (resp. $\beta_E : E \to \boldsymbol{k}_A * E$) from E to the pull-back of E along $\boldsymbol{k}_C$ (resp. from the push-out of E along $\boldsymbol{k}_A$ to E), and a natural isomorphism $\omega_E : \boldsymbol{k}_A * E \xrightarrow{\sim} E * \boldsymbol{k}_C$ in the category $\mathbf{Ext}_{\mathscr{C}}(C, A)$ of extensions of C by A (this is the category whose objects are all short exact sequences in $\mathscr{C}$ of the form $0 \to A \to X \to C \to 0$, and whose morphisms are the maps of complexes which are the identity on A and C). These maps are related by the identities :

$$
\beta_E \circ \alpha_E \circ \omega_E = k \cdot \mathbf{1}_{\boldsymbol{k}_A * E} \qquad \omega_E \circ \beta_E \circ \alpha_E = k \cdot \mathbf{1}_{E * \boldsymbol{k}_C}. \tag{3.6.51}
$$

We leave to the reader the construction of ω_E. In the case at hand, we obtain a natural isomorphism $\omega_{\mathscr{E}} : \boldsymbol{k}_M^{\mathrm{gp}} * \mathscr{E} \xrightarrow{\sim} \mathscr{E} * \boldsymbol{k}_G$, where $\mathscr{E}$ is the short exact sequence of (3.6.48). An inspection of the construction shows that the map τ^{gp} is precisely the isomorphism defined by $\omega_{\mathscr{E}}$.

(iii): Let $\mu' : N \to P$ be the natural map, and set $\lambda := \mu' \circ \mu$. By inspecting the constructions, one checks easily that λ^{gp} is the map defined by $\beta_{\mathscr{E}} \circ \alpha_{\mathscr{E}}$. Then the assertion follows from (3.6.51). ◊

Let P' be the push-out of the diagram $N' \xleftarrow{\varphi} M \xrightarrow{\iota} M^{\mathrm{sat}}$; from claim 3.6.47, lemma 3.2.2(i) and corollary 3.2.25(iii), we deduce that the natural map $M^{\mathrm{sat}} \to P'$ is flat and saturated, hence P' is saturated. On the other hand, directly from the definitions we get a cocartesian diagram

$$
\begin{tikzcd}
P \arrow[r] \arrow[d, "\tau"'] & Q \arrow[d, "\tau'"] \\
N' \arrow[r] & P'.
\end{tikzcd}
\tag{3.6.52}
$$

The induced diagram $(3.6.52)^{\mathrm{sat}}$ of saturated monoids is still cocartesian (remark 2.3.41(v)); however, claim 3.6.50(iii) implies easily that τ^{sat} is an isomorphism, therefore the same holds for $(\tau')^{\mathrm{sat}}$, and assertion (ii) follows. □

4. Complements of commutative and homological algebra

This chapter is a miscellanea of results of commutative algebra that shall be needed in the rest of the treatise.

4.1. **Complexes in an abelian category.** Let $\mathscr{A}$ be any abelian category. We denote by $\mathsf{C}(\mathscr{A})$ the category of (cochain) complexes of objects of $\mathscr{A}$, and by $\mathsf{D}(\mathscr{A})$ the derived category of $\mathscr{A}$. Hence, an object of $\mathsf{C}(\mathscr{A})$ or $\mathsf{D}(\mathscr{A})$ is a pair

$$K^\bullet := (K^\bullet, d_K^\bullet)$$

consisting of a system of objects $(K^n \mid n \in \mathbb{Z})$ and morphisms $(d_K^n : K^n \to K^{n+1} \mid n \in \mathbb{Z})$ of $\mathscr{A}$, called the *differentials* of the complex $K^\bullet$, such that

$$d_K^{n+1} \circ d_K^n = 0 \qquad \text{for every } n \in \mathbb{Z}.$$

The morphisms $\varphi^\bullet : K^\bullet \to L^\bullet$ in $\mathsf{C}(\mathscr{A})$ are the systems of morphisms $(\varphi^n : K^n \to L^n \mid n \in \mathbb{Z})$ of $\mathscr{A}$ such that

$$\varphi^{n+1} \circ d_K^n = d_L^{n+1} \circ \varphi^n \qquad \text{for every } n \in \mathbb{Z}.$$

We will usually omit the subscript when referring to the differentials of a complex, unless there is a danger of confusion. Let $i \in \mathbb{Z}$ be any integer; the *cohomology of $K^\bullet$ in degree i* is the object of $\mathscr{A}$:

$$H^i K^\bullet := \operatorname{Ker} d^i / \operatorname{Im} d^{i-1}.$$

Clearly, for every $i \in \mathbb{Z}$, the rule $K^\bullet \mapsto H^i K^\bullet$ extends to a functor

$$H^i : \mathsf{C}(\mathscr{A}) \to \mathscr{A}.$$

Also, let $I \subset \mathbb{Z}$ be any (bounded or unbounded) interval, *i.e.* I is either of the form $\mathbb{Z} \cap [a, +\infty[$, or $\mathbb{Z} \cap]-\infty, b]$ (for some $a, b \in \mathbb{N}$), or the intersection of any of these two. We shall denote by

$$\mathsf{C}^I(\mathscr{C}) \qquad (\text{ resp. } \mathsf{D}^I(\mathscr{C}) \text{ })$$

the full subcategory of $\mathsf{C}(\mathscr{C})$ (resp. of $\mathsf{D}(\mathscr{C})$) whose objects are the complexes $K^\bullet$ such that $K^i = 0$ whenever $i \notin I$ (resp. such that $H^i K^\bullet = 0$ whenever $i \in\notin \mathbb{Z}$). For instance, if $I = \mathbb{Z} \cap [a, +\infty[$, then $\mathsf{C}^I(\mathscr{C})$ (resp. $\mathsf{D}^I(\mathscr{C})$) is also denoted $\mathsf{C}^{\geq a}(\mathscr{C})$ (resp. $\mathsf{D}^{\geq a}(\mathsf{C})$), and likewise for the case of an upper bounded interval. Moreover, we set

$$\mathsf{C}^-(\mathscr{A}) := \bigcup_{n \in \mathbb{Z}} \mathsf{C}^{\leq n}(\mathscr{A}) \qquad \mathsf{C}^+(\mathscr{A}) := \bigcup_{n \in \mathbb{Z}} \mathsf{C}^{\geq n}(\mathscr{A})$$

so $\mathsf{C}^-(\mathscr{A})$ (resp. $\mathsf{C}^+(\mathscr{A})$) is the full subcategory of $\mathsf{C}(\mathscr{A})$ whose objects are the *bounded above* (resp. *bounded below*) complexes of $\mathscr{A}$. Likewise we define the full subcategories $\mathsf{D}^-(\mathscr{A})$ and $\mathsf{D}^+(\mathscr{A})$ of $\mathsf{D}(\mathscr{A})$. Recall that, for any interval I, the category $\mathsf{D}^I(\mathscr{C})$ is also naturally equivalent to the localization of $\mathsf{C}^I(\mathscr{C})$ by the multiplicative set of all morphisms in $\mathsf{C}^I(\mathscr{C})$ that are quasi-isomorphisms, and likewise for $\mathsf{D}^-(\mathscr{A})$ and $\mathsf{D}^+(\mathscr{A})$ (verification left to the reader).

For every $a \in \mathbb{Z}$, the inclusion functor

$$\mathsf{C}^{\geq a}(\mathscr{A}) \to \mathsf{C}(\mathscr{A}) \qquad (\text{resp. } \mathsf{C}^{\leq a}(\mathscr{A}) \to \mathsf{C}(\mathscr{A}) \text{ })$$

admits a right (resp. left) adjoint

$$t^{\geq a} : \mathsf{C}(\mathscr{A}) \to \mathsf{C}^{\geq a}(\mathscr{A}) \qquad (\text{resp. } t^{\leq a} : \mathsf{C}(\mathscr{A}) \to \mathsf{C}^{\leq a}(\mathscr{A}) \text{ })$$

called the *brutal truncation functor*; namely, for any complex $K^\bullet$, we let $t^{\geq a}(K^\bullet)$ be the unique object of $\mathsf{C}^{\geq a}(\mathscr{A})$ that agrees with $K^\bullet$ in all degrees $\geq a$, and with the same differentials as $K^\bullet$, in this range of degrees (and likewise for $t^{\leq a}(K^\bullet)$).

4.1.1. There is an obvious functor

$$\mathscr{A} \to \mathsf{C}(\mathscr{A}) \quad : \quad A \mapsto A[0]$$

that sends any object A of $\mathscr{A}$ to the complex with A *placed in degree zero*, *i.e.* such that $A[0]^i$ equals A if $i = 0$, and equals 0 otherwise (clearly, there is a unique such complex). On the other hand, the *shift operator* is the functor

$$\mathsf{C}(\mathscr{A}) \to \mathsf{C}(\mathscr{A}) \quad : \quad K^\bullet \to K^\bullet[1]$$

given by the rule :

$$K^\bullet[1]^n := K^{n+1} \qquad d^n_{K[1]} := -d^{n+1}_K \qquad \text{for every } n \in \mathbb{Z}.$$

Clearly the shift operator is an automorphism of $\mathsf{C}(\mathscr{A})$, and one defines the operator $K^\bullet \mapsto K^\bullet[n]$, for every $n \in \mathbb{Z}$, as the n-th power of the shift operator (in the automorphism group of $\mathsf{C}(\mathscr{A})$). Then, we can combine the two previous operators, to define the complex

$$A[n] := (A[0])[n] \qquad \text{for every } n \in \mathbb{Z} \text{ and every } A \in \mathrm{Ob}(\mathscr{A}).$$

Definition 4.1.2. Let $\mathscr{A}$ be an abelian category, $\varphi^\bullet, \psi^\bullet : K^\bullet \to L^\bullet$ two morphisms in $\mathsf{C}(\mathscr{A})$.

(i) A *(chain) homotopy* from $\varphi^\bullet$ to $\psi^\bullet$ is the datum $(s^n : K^n \to L^{n-1} \mid n \in \mathbb{Z})$ of a system of morphisms in $\mathscr{A}$ such that

$$\varphi^n - \psi^n = s^{n+1} \circ d^n_K + d^{n+1}_L \circ s^n \qquad \text{for every } n \in \mathbb{Z}.$$

(ii) We say that $\varphi^\bullet$ and $\psi^\bullet$ as in (i) are *chain homotopic* if there is a chain homotopy between them. It is easily seen that this defines an equivalence relation $\sim$ on the set $\mathrm{Hom}_{\mathsf{C}(\mathscr{A})}(K^\bullet, L^\bullet)$, which is preserved by composition of morphisms : if $\varphi^\bullet \sim \psi^\bullet$ and $\alpha^\bullet : K'^\bullet \to K^\bullet$, $\beta^\bullet : L^\bullet \to L'^\bullet$ are any two morphisms, then $\varphi^\bullet \circ \alpha^\bullet \sim \psi^\bullet \circ \alpha^\bullet$ and $\beta^\bullet \circ \varphi^\bullet \sim \beta^\bullet \circ \psi^\bullet$. It follows that there exists a well defined *homotopy category*

$$\mathsf{Hot}(\mathscr{A})$$

whose objects are the same as those of $\mathsf{C}(\mathscr{A})$, and whose morphisms are the homotopy classes of morphisms of complexes, and a natural functor

$$\mathsf{C}(\mathscr{A}) \to \mathsf{Hot}(\mathscr{A})$$

which is the identity on objects, and the quotient map on Hom-sets.

(iii) We also say that a morphism $\varphi : K^\bullet \to L^\bullet$ is a *homotopic equivalence*, if the class of $\varphi^\bullet$ is an isomorphism in $\mathsf{Hot}(\mathscr{A})$, *i.e.* if there exists a morphism $\psi^\bullet : L^\bullet \to K^\bullet$ such that $\psi^\bullet \circ \varphi^\bullet \sim \mathbf{1}_{K^\bullet}$ and $\varphi \circ \psi \sim \mathbf{1}_{L^\bullet}$. We say that a complex $K^\bullet$ is *homotopically trivial*, if the zero endomorphism $0 \cdot \mathbf{1}_{K^\bullet}$ is a homotopy equivalence.

Remark 4.1.3. (i) If $F : \mathscr{A} \to \mathscr{A}'$ is any additive functors of abelian categories, we get induced functors

$$\mathsf{C}(F) : \mathsf{C}(\mathscr{A}) \to \mathsf{C}(\mathscr{A}') \qquad \mathsf{Hot}(F) : \mathsf{Hot}(\mathscr{A}) \to \mathsf{Hot}(\mathscr{A}')$$

by the rule :

$$F(K^\bullet)^n := F(K^n) \quad \text{and} \quad d^n_{F(K)} := F(d^n_K) \qquad \text{for every } n \in \mathbb{Z} \text{ and every } K^\bullet \in \mathrm{Ob}(\mathsf{C}(\mathscr{A})).$$

(ii) Furthermore, if $\varphi^\bullet, \psi^\bullet : K^\bullet \to L^\bullet$ are chain homotopic morphisms in $\mathsf{C}(\mathscr{A})$, then it is easily seen that, for every $i \in \mathbb{Z}$, the induced morphisms in cohomology

$$H^i\varphi^\bullet, H^i\psi^\bullet : H^iK^\bullet \to H^iL^\bullet$$

coincide. Hence, the cohomology functor H^i factors (uniquely) through a functor

$$H^i : \mathsf{Hot}(\mathscr{A}) \to \mathscr{A} \qquad \text{for every } i \in \mathbb{Z}.$$

Remark 4.1.4. The indexing notation that makes use of superscripts to denote the degrees in a complex, is known traditionally as *cohomological degree notation*. Sometimes it is more natural to switch to the *homological degree notation*, that makes use of subscript indexing; namely, one associates to any cochain complex $K^\bullet$, the *chain complex* $K_\bullet$ given by the rule :

$$K_n := K^{-n} \qquad \text{and} \qquad d_n := d^{-n} : K_n \to K_{n-1} \qquad \text{for every } n \in \mathbb{Z}.$$

Likewise, one sets $H_n K_\bullet := H^{-n} K_n$ for every $n \in \mathbb{Z}$, and calls this object of $\mathscr{A}$ the *homology of* $K_\bullet$ in degree n.

4.1.5. A *double complex* of $\mathscr{A}$ is an object of $\mathsf{C}(\mathsf{C}(\mathscr{A}))$ (and likewise for a morphism of double complexes). In other words, a double complex is a triple

$$K^{\bullet\bullet} := (K^{\bullet\bullet}, d_h^{\bullet\bullet}, d_v^{\bullet\bullet})$$

consisting of a system $(K^{pq} \mid p, q \in \mathbb{Z})$ of objects of $\mathscr{A}$, and morphisms d_h^{pq}, d_v^{pq} called respectively the *horizontal* and *vertical* differentials, fitting into a commutative diagram

$$\begin{tikzcd} K^{pq} \arrow[r, "d_h^{pq}"] \arrow[d, "d_v^{pq}"'] & K^{p+1,q} \arrow[d, "d_v^{p+1,q}"] \\ K^{p,q+} \arrow[r, "d_h^{pq}"] & K^{p+1,q+1} \end{tikzcd} \qquad \text{for every } p, q \in \mathbb{Z}$$

and such that

$$d_h^{p+1,q} \circ d_h^{pq} = 0 \qquad d_v^{p,q+1} \circ d_v^{pq} = 0 \qquad \text{for every } p, q \in \mathbb{Z}.$$

4.1.6. There are natural functors

$$\mathsf{C}(\mathsf{C}(\mathscr{A})) \to \mathsf{C}(\mathsf{C}(\mathscr{A})) \ : \ K^{\bullet\bullet} \mapsto \mathsf{fl}(K^{\bullet\bullet}) \qquad\qquad \mathsf{C}(\mathsf{C}(\mathscr{A})) \to \mathsf{C}(\mathscr{A}) \ : \ K^{\bullet\bullet} \mapsto (K^{\bullet\bullet})^\Delta$$

where :

- The *flip* $\mathsf{fl}(K^{\bullet\bullet})$ of $K^{\bullet\bullet}$ is the double complex $F^{\bullet\bullet}$ such that $F^{pq} := K^{qp}$ for every $p, q \in \mathbb{Z}$, with differentials deduced from those of $K^{\bullet\bullet}$, in the obvious way
- The *diagonal* $(K^{\bullet\bullet})^\Delta$ is the complex $D^\bullet$ such that $D^p := K^{pp}$ for every $p \in \mathbb{Z}$ and with differentials

$$d_v^{p+1,q} \circ d_h^{pq} : D^p \to D^{p+1} \qquad \text{for every } p \in \mathbb{Z}.$$

Suppose that all coproducts (resp. all products) are representable in $\mathscr{A}$. Then there are two other natural functors

$$\mathrm{Tot}^\oplus : \mathsf{C}(\mathsf{C}(\mathscr{A})) \to \mathsf{C}(\mathscr{A}) \qquad (\text{ resp. } \mathrm{Tot}^\Pi : \mathsf{C}(\mathsf{C}(\mathscr{A})) \to \mathsf{C}(\mathscr{A}) \ )$$

defined as follows. The *total complex* $\mathrm{Tot}^\oplus(K^{\bullet\bullet})$ (resp. $\mathrm{Tot}^\Pi(K^{\bullet\bullet})$) is the complex $T^\bullet$ such that

$$T^n := \bigoplus_{p+q=n} K^{pq} \qquad (\text{ resp. } T^n := \prod_{p+q=n} K^{pq} \ )$$

and with differentials $T^n \to T^{n+1}$ given by the sum (resp. the product) of the morphisms

$$d_h^{pq} + (-1)^p \cdot d_v^{pq} : K^{pq} \to K^{p,q+1} \oplus K^{p+1,q} \qquad \text{for all } p, q \in \mathbb{Z} \text{ such that } p + q = n.$$

We often omit the superscript $\oplus$ when dealing with the total complex functor; to avoid confusion, we stipulate that *the notation* Tot *shall always refer to the functor* $\mathrm{Tot}^\oplus$, so if we need to use the other total complex functor, we shall denote it explicitly by Tot^Π. Notice that we have natural isomorphisms

$$\mathrm{Tot}(K^{\bullet\bullet}) \xrightarrow{\sim} \mathrm{Tot}(\mathsf{fl}(K^{\bullet\bullet})) \qquad (\text{resp. } \mathrm{Tot}^\Pi(K^{\bullet\bullet}) \xrightarrow{\sim} \mathrm{Tot}^\Pi(\mathsf{fl}(K^{\bullet\bullet})) \ )$$

given, in each degree $n \in \mathbb{Z}$, by the direct sum (resp. the direct product) of the morphisms $(-1)^{pq} \cdot \mathbf{1}_{K^{pq}}$, for every $p, q \in \mathbb{Z}$ such that $p + q = n$.

Example 4.1.7. (i) To any two objects $K^\bullet$ and $L^\bullet$ of $\mathsf{C}(\mathscr{A})$, we may attach the double complex of abelian groups $\mathrm{Hom}^{\bullet\bullet}_{\mathscr{A}}(K^\bullet, L^\bullet)$, given by the rule :

$$\mathrm{Hom}^{p,q}_{\mathscr{A}}(K^\bullet, L^\bullet) := \mathrm{Hom}_{\mathscr{A}}(K^{-p}, L^q) \qquad \text{for every } p, q \in \mathbb{Z}$$

with differentials

$$d_h^{p,q} := \mathrm{Hom}_{\mathscr{A}}(d_K^{-p}, \mathbf{1}_{L^q}) \qquad d_v^{p,q} := (-1)^{q+1} \cdot \mathrm{Hom}_{\mathscr{A}}(\mathbf{1}_{K^{-p}}, d_L^q) \qquad \text{for every } p, q \in \mathbb{Z}.$$

Then set

$$(\mathrm{Hom}^\bullet_{\mathscr{A}}(K^\bullet, L^\bullet), d^\bullet_{K,L}) := \mathrm{Tot}^\Pi \mathrm{Hom}^{\bullet\bullet}_{\mathscr{A}}(K^\bullet, L^\bullet).$$

Let $n \in \mathbb{Z}$ be any integer; with this notation, a simple inspection shows that :

- $\mathrm{Hom}_{\mathsf{C}(\mathscr{A})}(K^\bullet, L^\bullet[n]) = \mathrm{Ker}\, d^n_{K,L}$.
- Let $\varphi^\bullet, \psi^\bullet : K^\bullet \to L^\bullet[n]$ be any two morphisms; then the set of homotopies from $\varphi^\bullet$ to $\psi^\bullet$ is naturally identified with the subset
$$\{s^\bullet \in \mathrm{Hom}^{-1-n}_{\mathscr{A}}(K^\bullet, L^\bullet) \mid d^{-1-n}_{K,L}(s^\bullet) = \psi^\bullet - \varphi^\bullet\}.$$
- Consequently, we have a natural identification :
$$\mathrm{Hom}_{\mathsf{Hot}(\mathscr{A})}(K^\bullet, L^\bullet[n]) \xrightarrow{\sim} H^n \mathrm{Hom}^\bullet_{\mathscr{A}}(K^\bullet, L^\bullet).$$

(ii) The constructions of (i) can be used to endow $\mathsf{C}(\mathscr{A})$ with a natural 2-category structure, whose 2-cells are given by homotopies of complexes. Indeed, let us write

$$s^\bullet : \varphi^\bullet \Rightarrow \psi^\bullet$$

if $s^\bullet := (s^n \mid n \in \mathbb{N})$ is a homotopy from $\varphi^\bullet$ to $\psi^\bullet$. Then, if $\lambda^\bullet : K^\bullet \to L^\bullet$ is another morphism, and $t^\bullet : \psi^\bullet \Rightarrow \lambda^\bullet$ another homotopy, we define a composition law by setting

$$s^\bullet \odot t^\bullet := (s^n + t^n \mid n \in \mathbb{N})$$

and it is immediate that $s^\bullet \odot t^\bullet$ is a homotopy $\varphi^\bullet \Rightarrow \lambda^\bullet$. Moreover, if $\beta^\bullet, \gamma^\bullet : L^\bullet \to P^\bullet$ are two other morphisms in $\mathsf{C}(\mathscr{A})$, and $u : \beta^\bullet \Rightarrow \gamma^\bullet$ another homotopy, we have a Godement composition law by the rule

$$u^\bullet * s^\bullet := (\beta^{n+1} \circ s^n + u^n \circ \psi^n \mid n \in \mathbb{N}) : \beta^\bullet \circ \varphi^\bullet \Rightarrow \gamma^\bullet \circ \psi^\bullet.$$

The associativity of the laws $*$ and $\oplus$ thus defined are easily checked by direct computation.

Now, suppose that $\delta^\bullet : L^\bullet \to P^\bullet$ is yet another morphism, and $v : \gamma \Rightarrow \delta$ another homotopy. A direct calcultation yields the identity

$$(u^\bullet * s^\bullet) \odot (v^\bullet * t^\bullet) = (u^\bullet \odot v^\bullet) * (s^\bullet \odot t^\bullet) + c^\bullet$$

where $c^n := d_P^{n-2} \circ u^{n-1} \circ t^n - u^n \circ t^{n+1} \circ d^n_K$ for every $n \in \mathbb{Z}$, which can be rewritten as

$$c^\bullet = d^{-2}_{K,L}((u \circ t)^\bullet) \qquad \text{where } (u \circ t)^n := u^{n-1} \circ t^n \text{ for every } n \in \mathbb{Z}.$$

Summing up, we conclude that $\mathsf{C}(\mathscr{A})$ carries a 2-category structure, whose 2-cells $\varphi \Rightarrow \psi$ (for any two 1-cells $\varphi, \psi : K^\bullet \to L^\bullet$) are the classes

$$\overline{s} \in \mathrm{Hom}^{-1}_{\mathscr{A}}(K^\bullet, L^\bullet)/\mathrm{Im}(d^{-2}_{K,L}) \qquad \text{such that } d^{-1}_{K,L}(\overline{s}) = \psi^\bullet - \varphi^\bullet.$$

(iii) Moreover, if F is any additive functor as in remark 4.1.3(i), the induced functor $\mathsf{C}(F)$ extends to a pseudo-functor for the 2-category structures given by (ii); indeed, if $s^\bullet : K^\bullet \Rightarrow L^\bullet$ is a homotopy, obviously the system $(Fs^n \mid n \in \mathbb{N})$ is a homotopy $Fs^\bullet : F(K^\bullet) \Rightarrow F(L^\bullet)$.

Example 4.1.8. (i) Suppose that $(\mathscr{A}, \otimes, \Phi, \Psi)$ is a tensor abelian category. Then there exists a natural functor

$$- \boxtimes - : \mathsf{C}(\mathscr{A}) \times \mathsf{C}(\mathscr{A}) \to \mathsf{C}(\mathsf{C}(\mathscr{A}))$$

defined as follows. Given two complexes $K^\bullet$ and $L^\bullet$, we let

$$(K^\bullet \boxtimes_A L^\bullet)^{p,q} := K^p \otimes L^q \qquad \text{for every } p, q \in \mathbb{Z}$$

with differentials

$$d_h^{p,q} := d_K^p \otimes \mathbf{1}_{L^q} \qquad d_v^{p,q} := \mathbf{1}_{K^p} \otimes d_L^q \qquad \text{for every } p,q \in \mathbb{Z}.$$

If all coproducts are representables in $\mathscr{A}$, we may set as well :

$$K^\bullet \otimes L^\bullet := \mathrm{Tot}(K^\bullet \boxtimes L^\bullet).$$

The commutativity constraints for $(\mathscr{A}, \otimes)$ yield natural isomorphisms $K^\bullet \boxtimes L^\bullet \xrightarrow{\sim} \mathsf{fl}(L^\bullet \boxtimes K^\bullet)$, as well as

$$\Psi^\bullet_{K,L} : K^\bullet \otimes L^\bullet \xrightarrow{\sim} L^\bullet \otimes K^\bullet. \tag{4.1.9}$$

Namely, one takes the direct sum of the maps $(-1)^{pq} \cdot \Psi_{K^p,L^q}$, for every $p,q \in \mathbb{Z}$.

(ii) Likewise, if $P^\bullet$ is another complex of $\mathscr{A}$, the double complexes $(K^\bullet \otimes L^\bullet) \boxtimes P^\bullet$ and $K^\bullet \boxtimes (L^\bullet \otimes P^\bullet)$ are not isomorphic, but the associativity constraints of $\mathscr{A}$ induce natural isomorphisms in $\mathsf{C}(\mathscr{A})$:

$$\Phi^\bullet_{K,L,P} : K^\bullet \otimes (L^\bullet \otimes P^\bullet) \xrightarrow{\sim} (K^\bullet \otimes L^\bullet) \otimes P^\bullet.$$

Namely, one takes the direct sum of the morphisms Φ_{K^i,L^j,P^k} (for every $i,j,k \in \mathbb{Z}$). With these natural isomorphisms, $\mathsf{C}(\mathscr{A})$ is then naturally a tensor abelian category as well.

(iii) In the situation of (i), notice that the natural morphism $K^i \otimes L^j \to (K^\bullet \otimes L^\bullet)^{i+j}$ induces morphisms

$$\begin{aligned} &\mathrm{Ker}\,(d_K^i) \otimes \mathrm{Ker}\,(d_L^j) \to \mathrm{Ker}\,(d_{K\otimes L}^{i+j}) \\ &(\mathrm{Ker}\,(d_K^i) \otimes \mathrm{Im}\,(d_L^{j-1})) \oplus (\mathrm{Im}\,(d_K^{i-1}) \otimes \mathrm{Ker}\,(d_L^j)) \to \mathrm{Im}\,(d_{K\otimes L}^{i+j}) \end{aligned}$$

so, the induced map $\mathrm{Ker}\,(d_K^i) \otimes \mathrm{Ker}\,(d_L^j) \to H^{i+j}(K^\bullet \otimes L^\bullet)$ factors through a natural pairing :

$$H^i(K^\bullet) \otimes H^j(L^\bullet) \to H^{i+j}(K^\bullet \otimes L^\bullet) \qquad \text{for every } i,j \in \mathbb{Z}.$$

(iv) Moreover, if $P^\bullet$ is a third complex, in view of (ii) we get a commutative diagram

$$\begin{tikzcd} H^iK^\bullet \otimes (H^jL^\bullet \otimes H^kP^\bullet) \arrow[r, "\alpha"] \arrow[d, "\gamma"'] & (H^iK^\bullet \otimes H^jL^\bullet) \otimes H^kP^\bullet \arrow[r, "\beta"] & H^{i+j}(K^\bullet \otimes L^\bullet) \otimes H^kP^\bullet \arrow[d, "\delta"] \\ H^iK^\bullet \otimes H^{j+k}(L^\bullet \otimes P^\bullet) \arrow[r, "\lambda"] & H^{i+j+k}(K^\bullet \otimes (L^\bullet \otimes P^\bullet)) \arrow[r, "\sigma"] & H^{i+j+k}((K^\bullet \otimes L^\bullet) \otimes P^\bullet) \end{tikzcd}$$

where α is the associativity constraint, β, γ, δ and λ are given by the above pairing, and σ is deduced from $\Phi^\bullet_{K,L,P}$.

(v) If $\mathscr{A}$ also admits an internal Hom functor, we may define as well a functor

$$\mathscr{H}\!om^{\bullet\bullet} : \mathsf{C}(\mathscr{A})^o \times \mathsf{C}(\mathscr{A}) \to \mathsf{C}(\mathsf{C}(\mathscr{A}))$$

following the trace of example 4.1.7(i); namely, we set

$$\mathscr{H}\!om^{p,q}(K^\bullet, L^\bullet) := \mathscr{H}\!om(K^{-p}, L^q) \qquad \text{for every } p,q \in \mathbb{Z}$$

with differentials

$$d_h^{p,q} := \mathscr{H}\!om(d_K^{-p}, \mathbf{1}_{L^q}) \qquad d_v^{p,q} := (-1)^{q+1} \cdot \mathscr{H}\!om(\mathbf{1}_{K^{-p}}, d_L^q) \qquad \text{for every } p,q \in \mathbb{Z}.$$

If all products are representable in $\mathscr{A}$, we may then define

$$\mathscr{H}\!om^\bullet(K^\bullet, L^\bullet) := \mathrm{Tot}^\Pi \mathscr{H}\!om^{\bullet\bullet}(K^\bullet, L^\bullet).$$

(vi) Suppose that $\mathscr{A}$ is both complete and cocomplete, and $P^\bullet$ is any other complex; to ease notation, set $H^\bullet := \mathscr{H}\!om^\bullet(K^\bullet, L^\bullet)$ and $N^\bullet := P^\bullet \otimes K^\bullet$; we have natural isomorphisms

$$\mathrm{Hom}_{\mathsf{C}(\mathscr{A})}(P^\bullet, H^\bullet) \xrightarrow{\sim} \mathrm{Equal}(\mathrm{Hom}^0_{\mathscr{A}}(P^\bullet, H^\bullet) \underset{d_h}{\overset{d_v}{\rightrightarrows}} \mathrm{Hom}^1_{\mathscr{A}}(P^\bullet, H^\bullet))$$

where

$$d_h := \prod_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathscr{A}}(P^n, d_H^n) \qquad \text{and} \qquad d_v := \prod_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathscr{A}}(d_P^n, H^{n+1})$$

(see example 4.1.7(i)). However, notice that

$$\begin{aligned}\mathrm{Hom}^a_{\mathscr{A}}(P^\bullet, H^\bullet) &= \prod_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathscr{A}}(P^n, \prod_{p+q=n+a} \mathscr{H}om(K^{-p}, L^q)) \\ &= \prod_{n\in\mathbb{Z}} \prod_{p+q=n+a} \mathrm{Hom}_{\mathscr{A}}(P^n, \mathscr{H}om(K^{-p}, L^q)) \\ &= \prod_{n\in\mathbb{Z}} \prod_{p+q=n+a} \mathrm{Hom}_{\mathscr{A}}(P^n \otimes K^{-p}, L^q)) \\ &= \prod_{q\in\mathbb{Z}} \mathrm{Hom}_{\mathscr{A}}(\bigoplus_{n-p=q-a} P^n \otimes K^{-p}, L^q))\end{aligned}$$

for every $n, a \in \mathbb{Z}$, whence

$$\mathrm{Hom}_{\mathsf{C}(\mathscr{A})}(P^\bullet, H^\bullet) \xrightarrow{\sim} \mathrm{Equal}(\mathrm{Hom}^0_{\mathscr{A}}(N^\bullet, L^\bullet) \underset{d'_h}{\overset{d'_v}{\rightrightarrows}} \mathrm{Hom}^1_{\mathscr{A}}(N^\bullet, L^\bullet))$$

where

$$d'_h := \prod_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathscr{A}}(d_N^q, L^q) \qquad \text{and} \qquad d'_v := \prod_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathscr{A}}(N^q, d_L^q)$$

so finally :

$$\mathrm{Hom}_{\mathsf{C}(\mathscr{A})}(P^\bullet, H^\bullet) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{C}(\mathscr{A})}(N^\bullet, L^\bullet)$$

which says that $\mathscr{H}om^\bullet$ is an internal Hom functor for $\mathsf{C}(\mathscr{A})$.

(vii) In the situation of (vi), set

$$\mathscr{H}om_{\mathsf{C}(\mathscr{A})}(K^\bullet, L^\bullet) := \mathrm{Ker}\,(d^0 : H^0 \to H^1)$$

and take $P^\bullet := Z[0]$, where Z is any object of $\mathscr{A}$; it is easily seen that the natural map

$$\mathrm{Hom}_{\mathscr{A}}(Z, \mathscr{H}om_{\mathsf{C}(\mathscr{A})}(K^\bullet, L^\bullet)) \to \mathrm{Hom}_{\mathsf{C}(\mathscr{A})}(P^\bullet, H^\bullet) \to \mathrm{Hom}_{\mathsf{C}(\mathscr{A})}(Z[0] \otimes K^\bullet, L^\bullet)$$

is an isomorphism : details left to the reader.

4.1.10. Suppose that $\mathscr{A}$ is a small abelian category, and let $(G^\bullet, d_G^\bullet)$ be any complex of finitely generated abelian groups; with the notation of (1.2.43), we obtain an object $(G^\bullet_{\mathscr{A}}, d^\bullet_{\mathscr{A}})$ of $\mathsf{C}(\mathscr{A}^\dagger)$; on the other hand, if $(K^\bullet, d_K^\bullet)$ is any object of $\mathsf{C}(\mathscr{A})$, we may also consider the object $h_K^\dagger := (h^\dagger_{K^n}, h^\dagger_{d^n} \mid n \in \mathbb{N})$ of $\mathsf{C}(\mathscr{A}^\dagger)$, and since $\mathscr{A}^\dagger$ is an abelian tensor category, we may form the tensor product

$$G^\bullet \boxtimes_{\mathbb{Z}} K^\bullet := G^\bullet_{\mathscr{A}} \boxtimes h_K^\dagger$$

according to example 4.1.8(i). Arguing as in (1.2.43), we see that this object of $\mathsf{C}(\mathsf{C}(\mathscr{A}^\dagger))$ is isomorphic to an object of $\mathsf{C}(\mathsf{C}(\mathscr{A}))$, and after choosing representing objects, we get a functor

$$\mathsf{C}(\mathbb{Z}\text{-}\mathbf{Mod}_{\mathrm{fg}}) \times \mathsf{C}(\mathscr{A}) \to \mathsf{C}(\mathsf{C}(\mathscr{A})) \qquad (G^\bullet, K^\bullet) \mapsto G^\bullet \boxtimes_{\mathbb{Z}} K^\bullet$$

which is additive in both arguments. Arguing as in remark 1.2.45(ii,iii), we may also define more generally this functor in case $\mathscr{A}$ is an arbitrary abelian category, and if $\mathscr{A}$ is cocomplete, we can extend the functor to the whole of $\mathsf{C}(\mathbb{Z}\text{-}\mathbf{Mod})$. It is then natural to define

$$G^\bullet \otimes_{\mathbb{Z}} K^\bullet := \mathrm{Tot}\, G^\bullet \boxtimes_{\mathbb{Z}} K^\bullet \qquad \text{for every } G^\bullet \text{ and } K^\bullet \text{ as above.}$$

Likewise, we set $K^\bullet \boxtimes_{\mathbb{Z}} G^\bullet := \mathsf{fl}(G^\bullet \boxtimes_{\mathbb{Z}} K^\bullet)$ and $K^\bullet \otimes_{\mathbb{Z}} G^\bullet := \mathrm{Tot}(K^\bullet \boxtimes_{\mathbb{Z}} G^\bullet)$.

Remark 4.1.11. (i) With the notation of (4.1.10), notice the natural isomorphism

$$K^\bullet[1] \xrightarrow{\sim} \mathbb{Z}[1] \otimes_{\mathbb{Z}} K^\bullet \qquad \text{for every } K^\bullet \in \mathrm{Ob}(\mathsf{C}(\mathscr{A}))$$

which explains the sign convention in the definition of the shift operator in (4.1.1).

(ii) Moreover, denote by $K\langle 1\rangle^\bullet \in \mathrm{Ob}(\mathsf{C}^{[-1,0]}(\mathbb{Z}\text{-}\mathbf{Mod}))$ the object such that $K\langle 1\rangle^{-1} := \mathbb{Z}$, $K\langle 1\rangle^0 := \mathbb{Z} \oplus \mathbb{Z}$, and with differential d^{-1} given by the rule : $n \mapsto (n, -n)$ for every $n \in \mathbb{Z}$. Let $e_0 := (0,1)$ and $e_1 := (1,0)$ be the canonical basis of $K\langle 1\rangle^0$; we have two morphisms

$$\iota_i : \mathbb{Z}[0] \to K\langle 1\rangle^\bullet \qquad \text{for } i = 0, 1$$

given, in degree zero, by the rule : $n \mapsto n \cdot e_i$ for every $n \in \mathbb{N}$. For any pair of morphisms $\varphi^\bullet, \psi^\bullet : L^\bullet \to M^\bullet$ in $\mathsf{C}(\mathscr{A})$, and any homotopy $s^\bullet := (s^n \mid n \in \mathbb{Z})$ from $\varphi^\bullet$ to $\psi^\bullet$, we obtain a morphism of complexes

$$\sigma^\bullet : K\langle 1\rangle^\bullet \otimes_{\mathbb{Z}} L^\bullet \to M^\bullet$$

as follows. For every $n \in \mathbb{Z}$, the morphism $\sigma^n : L^n \oplus L^n \oplus L^{n+1} \to M^n$ restricts to φ^n (resp. ψ^n, resp. s^n) on the first (resp. second, resp. third) summand. Conversely, the datum of a morphism $\sigma^\bullet : K\langle 1\rangle^\bullet \otimes_{\mathbb{Z}} L^\bullet \to M^\bullet$ yields a homotopy from $\sigma^\bullet \circ (\iota_1^\bullet \otimes_{\mathbb{Z}} L^\bullet)$ to $\sigma^\bullet \circ (\iota_0^\bullet \otimes_{\mathbb{Z}} L^\bullet)$.

4.1.12. In the situation of example 4.1.8, suppose now that $\mathscr{A}$ is a category with enough projective objects. Then, for every bounded above complex $K^\bullet$ we may find a quasi-isomorphism $\rho_K^\bullet : P_K^\bullet \to K^\bullet$ with $P_K^\bullet$ a bounded above complex of projective objects, and if $K^\bullet \to L^\bullet$ is any morphism of complexes, there exists a commutative diagram in $\mathsf{C}^-(\mathscr{A})$

$$\begin{tikzcd} P_K^\bullet \arrow[r, "P_\varphi^\bullet"] \arrow[d, "\rho_K^\bullet"'] & P_L^\bullet \arrow[d, "\rho_L^\bullet"] \\ K^\bullet \arrow[r, "\varphi^\bullet"] & L^\bullet \end{tikzcd}$$

where $P_\varphi^\bullet$ is unique up to homotopy, so the rules $K^\bullet \mapsto P_K^\bullet$ and $\varphi^\bullet \mapsto P_\varphi^\bullet$ yield a well defined functor

$$\mathsf{C}^-(\mathscr{A}) \to \mathsf{Hot}(\mathscr{A})$$

and moreover notice that, if $\varphi^\bullet$ as above is a quasi-isomorphism, then $P_\varphi^\bullet$ is a homotopic equivalence. Hence, let $K^\bullet$ and $L^\bullet$ be any two objects of $\mathsf{C}^-(\mathscr{A})$; we may define two functors

$$K^\bullet \overset{\mathbf{L}}{\otimes} - \quad (\text{ resp. } - \overset{\mathbf{L}}{\otimes} L^\bullet) \quad : \quad \mathsf{C}(\mathscr{A}) \to \mathsf{Hot}(\mathscr{A})$$

by the rules :

$$M^\bullet \mapsto K^\bullet \otimes P_M^\bullet \qquad (\text{ resp. } M \mapsto P_M^\bullet \otimes L^\bullet\ )$$

and in light of example 4.1.7(iii), we see that both functors transform quasi-isomorphisms into homotopic equivalences, so they induce well defined functors on the derived categories, and moreover it follows that the notation $K^\bullet \overset{\mathbf{L}}{\otimes} L^\bullet$ is unambiguous : we may compute this object by applying the functor $K^\bullet \overset{\mathbf{L}}{\otimes} -$ to $L^\bullet$, or by applying the functor $- \overset{\mathbf{L}}{\otimes} L^\bullet$ to $K^\bullet$, and the two resulting complexes are naturally isomorphic in $\mathsf{D}(\mathscr{A})$ to the same complex $P_K^\bullet \otimes P_L^\bullet$. The latter assertion also implies that $K^\bullet \overset{\mathbf{L}}{\otimes} L^\bullet$ is independent, up to unique isomorphism in $\mathsf{D}(\mathscr{A})$, of the choices of $P_M^\bullet$ and $P_L^\bullet$. Hence we have obtained a natural functor

$$- \overset{\mathbf{L}}{\otimes} - : \mathsf{D}^-(\mathscr{A}) \times \mathsf{D}^-(\mathscr{A}) \to \mathsf{D}^-(\mathscr{A})$$

called the *derived tensor product*. We also let

$$\mathrm{Tor}_i^{\mathscr{A}}(K^\bullet, L^\bullet) := H_i(K^\bullet \overset{\mathbf{L}}{\otimes} L^\bullet) \qquad \text{for every } i \in \mathbb{Z}.$$

In case $\mathscr{A} = A\text{-}\mathbf{Mod}$ for some ring A, it is customary to denote this functor by $- \overset{\mathbf{L}}{\otimes}_A -$, and then one also writes Tor_i^A instead of $\mathrm{Tor}_i^{A\text{-}\mathbf{Mod}}$.

Remark 4.1.13. (i) Using the commutativity and associativity constraints for the tensor product in $\mathscr{A}$, we deduce – in light of example 4.1.8(i,ii) – natural *associativity isomorphisms*

$$K^\bullet \overset{\mathbf{L}}{\otimes} (L^\bullet \overset{\mathbf{L}}{\otimes} Q^\bullet) \xrightarrow{\sim} (K^\bullet \overset{\mathbf{L}}{\otimes} L^\bullet) \overset{\mathbf{L}}{\otimes} Q^\bullet \qquad \text{in } \mathsf{D}^-(\mathscr{A})$$

as well as *commutativity isomorphisms*

$$K^\bullet \overset{\mathbf{L}}{\otimes} L^\bullet \xrightarrow{\sim} L^\bullet \overset{\mathbf{L}}{\otimes} K^\bullet \qquad \text{in } \mathsf{D}^-(\mathscr{A})$$

for any bounded above complexes $K^\bullet$, $L^\bullet$, and $Q^\bullet$.

(ii) Also, it is easily seen that, if $K^\bullet \in \mathrm{Ob}(\mathsf{D}^{\leq a}(\mathscr{A}))$ and $L^\bullet \in \mathrm{Ob}(\mathsf{D}^{\leq b}(\mathscr{A}))$, then $K^\bullet \overset{\mathbf{L}}{\otimes} L^\bullet \in \mathrm{Ob}(\mathsf{D}^{\leq a+b}(\mathscr{A}))$, for every $a, b \in \mathbb{Z}$.

(iii) In the situation of (4.1.12), take $\mathscr{A} = A\text{-}\mathbf{Mod}$ for some ring A, and suppose furthermore that $\varphi : A \to B$ is a ring homomorphism, $K^\bullet$ a bounded above complex of A-modules, and $L^\bullet$ a complex of B-modules. Notice that $P^\bullet \otimes_A L^\bullet$ is naturally a complex of B-modules; also, if $L^\bullet \to Q^\bullet$ is any morphism of complexes of B-modules, then the induced map $P^\bullet \otimes_A L^\bullet \to P^\bullet \otimes_A Q^\bullet$ is B-linear. It follows easily that the derived tensor product yields a functor

$$\mathsf{D}^-(A\text{-}\mathbf{Mod}) \times \mathsf{D}^-(B\text{-}\mathbf{Mod}) \to \mathsf{D}^-(B\text{-}\mathbf{Mod}) \qquad (K^\bullet, L^\bullet) \mapsto K^\bullet \overset{\mathbf{L}}{\otimes}_A L^\bullet$$

such that, denoting $\varphi^* : \mathsf{D}^-(B\text{-}\mathbf{Mod}) \to \mathsf{D}^-(A\text{-}\mathbf{Mod})$ the "forgetful" functor, we have a natural isomorphism

$$K^\bullet \overset{\mathbf{L}}{\otimes}_A \varphi^* L^\bullet \xrightarrow{\sim} \varphi^*(K^\bullet \overset{\mathbf{L}}{\otimes}_A L^\bullet) \qquad \text{in } \mathsf{D}^-(A\text{-}\mathbf{Mod}).$$

4.1.14. Let now $M_1^\bullet$, $M_2^\bullet$, $N_1^\bullet$, $N_2^\bullet$ be any four objects of $\mathsf{C}^-(\mathscr{A})$, and to ease notation, set

$$M_{12}^\bullet := M_1^\bullet \otimes M_2^\bullet \qquad N_{12}^\bullet := N_1^\bullet \otimes N_2^\bullet \qquad P_{12}^\bullet := P_{M_{12}}^\bullet \qquad \rho_{12}^\bullet := \rho_{M_{12}}^\bullet$$

as well as $P_i^\bullet := P_{M_i}^\bullet$ and $\rho_i^\bullet := \rho_{M_i}^\bullet$ for $i = 1, 2$. There is a commutative diagram in $\mathsf{C}^-(\mathscr{A})$

$$\begin{tikzcd} P_1^\bullet \otimes P_2^\bullet \arrow[rr, "\varphi_{12}^\bullet"] \arrow[dr, "\rho_1^\bullet \otimes \rho_2^\bullet"'] & & P_{12}^\bullet \arrow[dl, "\rho_{12}^\bullet"] \\ & M_{12}^\bullet & \end{tikzcd}$$

where $\varphi_{12}^\bullet$ is uniquely determined up to homotopy, whence a map

$$(M_1^\bullet \overset{\mathbf{L}}{\otimes} N_1^\bullet) \otimes (M_2^\bullet \overset{\mathbf{L}}{\otimes} N_2^\bullet) \xrightarrow{\sim} (P_1^\bullet \otimes P_2^\bullet) \otimes N_{12}^\bullet \xrightarrow{\varphi_{12}^\bullet \otimes N_{12}^\bullet} P_{12}^\bullet \otimes N_{12}^\bullet \xrightarrow{\sim} M_{12}^\bullet \overset{\mathbf{L}}{\otimes} N_{12}^\bullet.$$

Taking into account example 4.1.8(iii), we deduce a bilinear pairing

$$\mathrm{Tor}_i^{\mathscr{A}}(M_1^\bullet, N_1^\bullet) \otimes \mathrm{Tor}_j^{\mathscr{A}}(M_2^\bullet, N_2^\bullet) \to \mathrm{Tor}_{i+j}^{\mathscr{A}}(M_{12}^\bullet, N_{12}^\bullet) \qquad \text{for every } i, j \in \mathbb{Z}.$$

Moreover, suppose that $M_3^\bullet$ and $N_3^\bullet$ are two other bounded above complexes of $\mathscr{A}$; by inspecting the constructions, we find a commutative diagram

$$\begin{tikzcd}
P_1^\bullet \otimes (P_2^\bullet \otimes P_3^\bullet) \arrow[r, "P_1^\bullet\otimes\varphi_{23}^\bullet"] \arrow[dr, "\rho_1^\bullet\otimes(\rho_2^\bullet\otimes\rho_3^\bullet)"'] \arrow[ddd, "\Phi_P^\bullet"'] & P_1^\bullet \otimes P_{23}^\bullet \arrow[r, "\varphi_{1,23}^\bullet"] \arrow[d, "\rho_1^\bullet\otimes\rho_{23}^\bullet"] & P_{1,23}^\bullet \arrow[dl, "\rho_{1,23}^\bullet"] \arrow[ddd, "P^\bullet_{\Phi_M}"] \\
& M_{1,23}^\bullet \arrow[d, "\Phi_M^\bullet"] & \\
& M_{12,3}^\bullet & \\
(P_1^\bullet \otimes P_2^\bullet) \otimes P_3^\bullet \arrow[ur, "(\rho_1^\bullet\otimes\rho_2^\bullet)\otimes\rho_3^\bullet"] \arrow[r, "\varphi_{12}^\bullet\otimes P_3^\bullet"] & P_{12}^\bullet \otimes P_3^\bullet \arrow[u, "\rho_{12}^\bullet\otimes\rho_3^\bullet"] \arrow[r, "\varphi_{12,3}^\bullet"] & P_{12,3}^\bullet \arrow[ul, "\rho_{12,3}^\bullet"']
\end{tikzcd}$$

where $M_{1,23}^\bullet := M_{12}^\bullet \otimes M_3^\bullet$, $M_{23}^\bullet := M_2^\bullet \otimes M_3^\bullet$, $M_{1,23}^\bullet := M_1^\bullet \otimes M_{23}^\bullet$, and likewise for $P_{1,23}^\bullet$, $P_{23}^\bullet$, and $P_{12,3}^\bullet$ and the morphism $\varphi_{23}^\bullet$, $\varphi_{1,23}^\bullet$, $\varphi_{12,3}^\bullet$, $\rho_{23}^\bullet$, $\rho_{1,23}^\bullet$, $\rho_{12,3}^\bullet$. Here $\Phi_M^\bullet$ and $\Phi_P^\bullet$ are the associativity constraints.

Therefore, set $T_i^j := \mathrm{Tor}_i^{\mathscr{A}}(M_j^\bullet, N_j^\bullet)$ for every $i \in \mathbb{Z}$ and $j = 1, 2, 3$, and also

$$T_i^{jk} := \mathrm{Tor}_i^{\mathscr{A}}(M_{jk}^\bullet, N_{jk}^\bullet) \qquad T_i^{1,23} := \mathrm{Tor}_i^{\mathscr{A}}(M_{1,23}^\bullet, N_{1,23}^\bullet) \qquad T_i^{12,3} := \mathrm{Tor}_i^{\mathscr{A}}(M_{12,3}^\bullet, N_{12,3}^\bullet)$$

for every $i \in \mathbb{Z}$, with $j = 1, 2$ and $k = j + 1$; in light of example 4.1.8(iv), we deduce a commutative diagram in $\mathscr{A}$:

$$\begin{tikzcd}
T_i^1 \otimes (T_j^2 \otimes T_k^3) \arrow[r] \arrow[d] & T_i^1 \otimes T_{j+k}^{23} \arrow[r] & T_{i+j+k}^{1,23} \arrow[d] \\
(T_i^1 \otimes T_j^2) \otimes T_k^3 \arrow[r] & T_{i+j}^{12} \otimes T_k^3 \arrow[r] & T_{i+j+k}^{12,3}
\end{tikzcd} \tag{4.1.15}$$

whose horizontal arrows are given by the above bilinear pairing, and whose left (resp. right) vertical arrow is the associativity constraint (resp. is induced by the associativity constraint $\Phi_M^\bullet$).

4.1.16. *Koszul complex and regular sequences.* Let $\mathbf{f} := (f_i \mid i = 1, \dots, r)$ be a finite system of elements of a ring A, and $(\mathbf{f}) \subset A$ the ideal generated by this sequence; we recall the definition of the *Koszul complex* $\mathbf{K}_\bullet(\mathbf{f})$ (see [28, Ch.III, §1.1]). First, suppose that $r = 1$, so $\mathbf{f} = (f)$ for a single element $f \in A$; in this case :

$$\mathbf{K}_\bullet(f) := (0 \to A \xrightarrow{f} A \to 0)$$

concentrated in cohomological degrees 0 and -1. In the general case one lets :

$$\mathbf{K}_\bullet(\mathbf{f}) := \mathbf{K}_\bullet(f_1) \otimes_A \cdots \otimes_A \mathbf{K}_\bullet(f_r).$$

For every complex of A-modules $M^\bullet$ one sets :

$$\mathbf{K}_\bullet(\mathbf{f}, M^\bullet) := M^\bullet \otimes_A \mathbf{K}_\bullet(\mathbf{f}) \qquad \mathbf{K}^\bullet(\mathbf{f}, M^\bullet) := \mathrm{Tot}^\bullet(\mathrm{Hom}_A^\bullet(\mathbf{K}_\bullet(\mathbf{f}), M^\bullet))$$

and denotes by $H_\bullet(\mathbf{f}, M^\bullet)$ (resp. $H^\bullet(\mathbf{f}, M^\bullet)$) the homology of $\mathbf{K}_\bullet(\mathbf{f}, M^\bullet)$ (resp. the cohomology of $\mathbf{K}^\bullet(\mathbf{f}, M^\bullet)$). Especially, if M is any A-module :

$$H_0(\mathbf{f}, M) = M/(\mathbf{f})M \qquad H^0(\mathbf{f}, M) = \mathrm{Hom}_A(A/(\mathbf{f}), M)$$

(where, as usual, we regard M as a complex placed in degree 0).

4.1.17. Let $r > 0$ and $\mathbf{f}$ be as in (4.1.16), and set $\mathbf{f}' := (f_1, \dots, f_{r-1})$. We have a short exact sequence of complexes : $0 \to A[0] \to \mathbf{K}_\bullet(f_r) \to A[1] \to 0$, and after tensoring with $\mathbf{K}_\bullet(\mathbf{f}')$ we derive a distinguished triangle:

$$\mathbf{K}_\bullet(\mathbf{f}') \to \mathbf{K}_\bullet(\mathbf{f}) \to \mathbf{K}_\bullet(\mathbf{f}')[1] \xrightarrow{\partial} \mathbf{K}_\bullet(\mathbf{f}')[1].$$

By inspecting the definitions one checks easily that the boundary map ∂ is induced by multiplication by f_r. There follow exact sequences :

(4.1.18) $$0 \to H_0(f_r, H_p(\mathbf{f}', M)) \to H_p(\mathbf{f}, M) \to H^0(f_r, H_{p-1}(\mathbf{f}', M)) \to 0$$

for every A-module M and for every $p \in \mathbb{N}$, whence the following :

Lemma 4.1.19. *With the notation of* (4.1.17)*, the following conditions are equivalent :*

(a) $H_i(\mathbf{f}, M) = 0$ *for every* $i > 0$.
(b) *The scalar multiplication by* f_r *is a bijection on* $H_i(\mathbf{f}', M)$ *for every* $i > 0$*, and is an injection on* $M/(\mathbf{f}')M$. ☐

Definition 4.1.20. The sequence $\mathbf{f}$ is said to be *completely secant* on the A-module M, if we have $H_i(\mathbf{f}, M) = 0$ for every $i > 0$.

The interest of definition 4.1.20 is due to its relation to the notion of *regular sequence* of elements of A (see *e.g.* [15, Ch.X, §9, n.6]). Namely, we have the following criterion :

Proposition 4.1.21. *With the notation of* (4.1.16)*, the following conditions are equivalent :*

(a) *The sequence* $\mathbf{f}$ *is* M*-regular.*
(b) *For every* $j \leq r$*, the sequence* $(f_1, \dots, f_j)$ *is completely secant on* M.

Proof. Lemma 4.1.19 shows that (b) implies (a). Conversely, suppose that (a) holds; we show that (b) holds, by induction on r. If $r = 0$, there is nothing to prove. Assume that the assertion is already known for all $j < r$. Since $\mathbf{f}$ is M-regular by assumption, the same holds for the subsequence $\mathbf{f}' := (f_1, \dots, f_{r-1})$, and f_r is regular on $M/(\mathbf{f}')M$. Hence $H_p(\mathbf{f}', M) = 0$ for every $p > 0$, by inductive assumption. Then lemma 4.1.19 shows that $H_p(\mathbf{f}, M) = 0$ for every $p > 0$, as claimed. ☐

Notice that any permutation of a completely secant sequence is again completely secant, whereas a permutation of a regular sequence is not always regular. As an application of the foregoing, we point out the following :

Corollary 4.1.22. *If a sequence* (f, g) *of elements of* A *is* M*-regular, and* M *is* f*-adically separated, then* (g, f) *is* M*-regular.*

Proof. According to proposition 4.1.21, we only need to show that the sequence (g) is completely secant, *i.e.* that g is regular on M. Hence, suppose that $gm = 0$ for some $m \in M$; it suffices to show that $m \in f^n M$ for every $n \in \mathbb{N}$. We argue by induction on n. By assumption g is regular on M/fM, hence $m \in fM$, which shows the claim for $n = 1$. Let $n > 1$, and suppose we already know that $m = f^{n-1}m'$ for some $m' \in M$. Hence $0 = gf^{n-1}m'$, so $gm' = 0$ and the foregoing case shows that $m' = fm''$ for some $m'' \in M$, thus $m = f^n m''$, as required. ☐

4.1.23. In the situation of (4.1.16), suppose that $\mathbf{g} := (g_i \mid i = 1, \dots, r)$ is another sequence of elements of A; we set $\mathbf{fg} := (f_i g_i \mid i = 1, \dots, r)$ and define a map of complexes

$$\boldsymbol{\varphi}_{\mathbf{g}} : \mathbf{K}_\bullet(\mathbf{fg}) \to \mathbf{K}_\bullet(\mathbf{f})$$

as follows. First, suppose that $r = 1$; then $\mathbf{f} = (f)$, $\mathbf{g} = (g)$ and the sought map $\boldsymbol{\varphi}_g$ is the commutative diagram :

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \xrightarrow{fg} & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle g} & & \| & & \\ 0 & \longrightarrow & A & \xrightarrow{f} & A & \longrightarrow & 0. \end{array}$$

For the general case we let :

$$\boldsymbol{\varphi}_{\mathbf{g}} := \boldsymbol{\varphi}_{g_1} \otimes_A \cdots \otimes_A \boldsymbol{\varphi}_{g_r}.$$

Especially, for every $m, n \geq 0$ we have maps $\boldsymbol{\varphi}_{\mathbf{f}^n} : \mathbf{K}_\bullet(\mathbf{f}^{n+m}) \to \mathbf{K}_\bullet(\mathbf{f}^m)$, whence maps

$$\boldsymbol{\varphi}^\bullet_{\mathbf{f}^n} : \mathbf{K}^\bullet(\mathbf{f}^m, M) \to \mathbf{K}^\bullet(\mathbf{f}^{m+n}, M)$$

and clearly $\boldsymbol{\varphi}^\bullet_{\mathbf{f}^{p+q}} = \boldsymbol{\varphi}^\bullet_{\mathbf{f}^p} \circ \boldsymbol{\varphi}^\bullet_{\mathbf{f}^q}$ for every $m, p, q \geq 0$.

4.1.24. Let A be any ring, $I \subset A$ a finitely generated ideal, $\mathbf{f} := (f_i \mid i = 1, \dots, r)$ a finite system of generators for I. Moreover, for every $n > 0$ let $I^{(n)} \subset A$ denote the ideal generated by $\mathbf{f}^n := (f_i^n \mid i = 1, \dots, r)$. For all $m \geq n > 0$ we deduce natural commutative diagrams of complexes :

$$\begin{array}{ccc} \mathbf{K}_\bullet(\mathbf{f}^m) & \longrightarrow & A/I^{(m)}[0] \\ {\scriptstyle \boldsymbol{\varphi}_{\mathbf{f}^{m-n}}}\downarrow & & \downarrow{\scriptstyle \pi_{mn}} \\ \mathbf{K}_\bullet(\mathbf{f}^n) & \longrightarrow & A/I^{(n)}[0] \end{array}$$

(notation of (4.1.23)) where π_{mn} is the natural surjection, whence a compatible system of maps:

$$\mathrm{Hom}_{\mathsf{D}(A\text{-}\mathbf{Mod})}(A/I^{(n)}[0], C^\bullet) \to \mathrm{Hom}_{\mathsf{D}(A\text{-}\mathbf{Mod})}(\mathbf{K}_\bullet(\mathbf{f}^n), C^\bullet). \tag{4.1.25}$$

for every $n \geq 0$ and every complex $C^\bullet$ in $\mathsf{D}^+(A\text{-}\mathbf{Mod})$. Especially, let us take $C^\bullet$ of the form $M[-i]$, for some A-module M and integer $i \in \mathbb{N}$; since $\mathbf{K}_\bullet(\mathbf{f}^n)$ is a complex of free A-modules, (4.1.25) translates as a direct system of maps :

$$\mathrm{Ext}^i_A(A/I^{(n)}, M) \to H^i(\mathbf{f}^n, M) \qquad \text{for all } n \in \mathbb{N} \text{ and every } i \in \mathbb{N}. \tag{4.1.26}$$

Notice that $I^{nr-r+1} \subset I^{(n)} \subset I^n$ for every $n > 0$, hence the colimit of the system (4.1.26) is equivalent to a natural map :

$$\operatorname*{colim}_{n \in \mathbb{N}} \mathrm{Ext}^i_A(A/I^n, M) \to \operatorname*{colim}_{n \in \mathbb{N}} H^i(\mathbf{f}^n, M) \qquad \text{for every } i \in \mathbb{N}. \tag{4.1.27}$$

Lemma 4.1.28. *With the notation of* (4.1.24), *the following conditions are equivalent :*

(a) *The map* (4.1.27) *is an isomorphism for every A-module M and every $i \in \mathbb{N}$.*
(b) $\operatorname*{colim}_{n \in \mathbb{N}} H^i(\mathbf{f}^n, J) = 0$ *for every $i > 0$ and every injective A-module J.*
(c) *The inverse system $(H_i\mathbf{K}_\bullet(\mathbf{f}^n) \mid n \in \mathbb{N})$ is essentially zero whenever $i > 0$,* i.e. *for every $p \in \mathbb{N}$ there exists $q \geq p$ such that $H_i\mathbf{K}_\bullet(\mathbf{f}^q) \to H_i\mathbf{K}_\bullet(\mathbf{f}^p)$ is the zero map.*

Proof. (a) ⇒ (b) is obvious. Next, if J is an injective A-module, we have natural isomorphisms

$$H^i(\mathbf{f}^n, J) \simeq \mathrm{Hom}_A(H_i\mathbf{K}_\bullet(\mathbf{f}^n), J) \qquad \text{for all } n \in \mathbb{N}. \tag{4.1.29}$$

which easily implies that (c) ⇒ (b).

(b) ⇒ (c) : Indeed, for any $p \in \mathbb{N}$ let us choose an injection $\varphi : H_i\mathbf{K}_\bullet(\mathbf{f}^n) \to J$ into an injective A-module J. By (4.1.29) we can regard φ as an element of $H^i(\mathbf{f}^n, J)$; by (b) the image of φ in $\mathrm{Hom}_A(H_i\mathbf{K}_\bullet(\mathbf{f}^q), J)$ must vanish if $q > p$ is large enough. This can happen only if $H_i\mathbf{K}_\bullet(\mathbf{f}^q) \to H_i\mathbf{K}_\bullet(\mathbf{f}^p)$ is the zero map.

(b) ⇒ (a) : Let $M \to J^\bullet$ be an injective resolution of the A-module M. The double complex $\operatorname*{colim}_{n\in\mathbb{N}} \mathrm{Hom}^\bullet_A(\mathbf{K}_\bullet(\mathbf{f}^n), J^\bullet)$ determines two spectral sequences :

$$E_1^{pq} := \operatorname*{colim}_{n\in\mathbb{N}} \mathrm{Hom}_A(\mathbf{K}_p(\mathbf{f}^n), H^q J^\bullet) \Rightarrow \operatorname*{colim}_{n\in\mathbb{N}} \mathrm{Ext}^{p+q}_A(\mathbf{K}_\bullet(\mathbf{f}^n), M)$$
$$F_1^{pq} := \operatorname*{colim}_{n\in\mathbb{N}} H^p(\mathbf{f}^n, J^q) \simeq \operatorname*{colim}_{n\in\mathbb{N}} \mathrm{Hom}_A(H_p\mathbf{K}_\bullet(\mathbf{f}^n), J^q) \Rightarrow \operatorname*{colim}_{n\in\mathbb{N}} \mathrm{Ext}^{p+q}_A(\mathbf{K}_\bullet(\mathbf{f}^n), M).$$

Clearly $E_1^{pq} = 0$ whenever $q > 0$, and (b) says that $F_1^{pq} = 0$ for $p > 0$. Hence these two spectral sequences degenerate and we deduce natural isomorphisms :

$$\operatorname*{colim}_{n\in\mathbb{N}} \mathrm{Ext}^q_A(A/I^{(n)}, M) \simeq F_2^{0q} \xrightarrow{\sim} E_2^{q0} \simeq \operatorname*{colim}_{n\in\mathbb{N}} H^q(\mathbf{f}^n, M).$$

By inspection, one sees easily that these isomorphisms are the same as the maps (4.1.27). □

Lemma 4.1.30. *In the situation of* (4.1.24), *suppose that the following holds. For every finitely presented quotient B of A, and every $b \in B$, there exists $p \in \mathbb{N}$ such that*

$$\mathrm{Ann}_B(b^q) = \mathrm{Ann}_B(b^p) \qquad \textit{for every } q \geq p.$$

Then the inverse system $(H_i\mathbf{K}_\bullet(\mathbf{f}^n) \mid n \in \mathbb{N})$ is essentially zero for every $i > 0$.

Proof. We shall argue by induction on r. If $r = 1$, then $\mathbf{f} = (f)$ for a single element $f \in A$. In this case, our assumption ensures that there exists $p \in \mathbb{N}$ such that $\mathrm{Ann}_A(f^q) = \mathrm{Ann}_A(f^p)$ for every $q \geq p$. It follows easily that $H_1(\boldsymbol{\varphi}_{f^p}) : H_1\mathbf{K}_\bullet(f^{p+k}) \to H_1\mathbf{K}_\bullet(f^k)$ is the zero map for every $k \geq 0$ (notation of (4.1.23)), whence the claim.

Next, suppose that $r > 1$ and that the claim is known for all sequences of less than r elements. Set $\mathbf{g} := (f_1, \dots, f_{r-1})$ and $f := f_r$. Specializing (4.1.18) to our current situation, we derive short exact sequences :

$$0 \to H_0(f^n, H_p\mathbf{K}_\bullet(\mathbf{g}^n)) \to H_p\mathbf{K}_\bullet(\mathbf{f}^n) \to H^0(f^n, H_{p-1}\mathbf{K}_\bullet(\mathbf{g}^n)) \to 0$$

for every $p > 0$ and $n \geq 0$; for a fixed p, this is an inverse system of exact sequences, where the transition maps on the rightmost term are given by f^{m-n}. By induction, the inverse system $(H_i\mathbf{K}_\bullet(\mathbf{g}^n) \mid n \in \mathbb{N})$ is essentially zero for $i > 0$, so we deduce already that the inverse system $(H_i\mathbf{K}_\bullet(\mathbf{f}^n) \mid n \in \mathbb{N})$ is essentially zero for all $i > 1$. To conclude, we are thus reduced to showing that the inverse system $(T_n := H^0(f^n, H_0\mathbf{K}_\bullet(\mathbf{g}^n)) \mid n \in \mathbb{N})$ is essentially zero. However $A_n := H_0\mathbf{K}_\bullet(\mathbf{g}^n) = A/(g_1^n, \dots, g_{r-1}^n)$ is a finitely presented quotient of A for any fixed $n \in \mathbb{N}$, hence the foregoing case $r = 1$ shows that the inverse system $(T_{mn} := \mathrm{Ann}_{A_n}(f^m) \mid m \in \mathbb{N})$ is essentially zero. Let $m \geq n$ be chosen so that $T_{mn} \to T_{nn}$ is the zero map; then the composition $T_m = T_{mm} \to T_{mn} \to T_{nn} = T_n$ is zero as well. □

Remark 4.1.31. Notice that the condition of lemma 4.1.30 is verified when A is noetherian.

4.1.32. *Minimal resolutions.* Let A be a local ring, k its residue field, M an A-module of finite type, and :

$$\cdots \xrightarrow{d_3} L_2 \xrightarrow{d_2} L_1 \xrightarrow{d_1} L_0 \xrightarrow{\varepsilon} M$$

a resolution of M by free A-modules. We say that $(L_\bullet, d_\bullet, \varepsilon)$ is a *finite-free resolution* if each L_i has finite rank. We say that $(L_\bullet, d_\bullet, \varepsilon)$ is a *minimal free resolution* of M if it is a finite-free resolution, and moreover the induced maps $k \otimes_A L_i \to k \otimes_A \mathrm{Im}\, d_i$ are isomorphisms for all $i \in \mathbb{N}$ (where we let $d_0 := \varepsilon$). One verifies easily that if A is a coherent ring, then every finitely presented A-module admits a minimal resolution.

4.1.33. Let $\underline{L} := (L_\bullet, d_\bullet, \varepsilon)$ and $\underline{L}' := (L'_\bullet, d'_\bullet, \varepsilon')$ be two free resolutions of M. A *morphism of resolutions* $\underline{L} \to \underline{L}'$ is a map of complexes $\varphi_\bullet : (L_\bullet, d_\bullet) \to (L'_\bullet, d'_\bullet)$ that extends to a commutative diagram

$$\begin{tikzcd} L_\bullet \arrow[r, "\varepsilon"] \arrow[d, "\varphi_\bullet"'] & M \arrow[d, equal] \\ L'_\bullet \arrow[r, "\varepsilon'"] & M \end{tikzcd}$$

Lemma 4.1.34. *Let $\underline{L} := (L_\bullet, d_\bullet, \varepsilon)$ be a minimal free resolution of an A-module M of finite type, $\underline{L}' := (L'_\bullet, d'_\bullet, \varepsilon')$ any other finite-free resolution, $\varphi_\bullet : \underline{L}' \to \underline{L}$ a morphism of resolutions. Then $\varphi_\bullet$ is an epimorphism (in the category of complexes of A-modules), $\mathrm{Ker}\,\varphi_\bullet$ is a null homotopic complex of free A-modules, and there is an isomorphism of complexes :*

$$L'_\bullet \xrightarrow{\sim} L_\bullet \oplus \mathrm{Ker}\,\varphi_\bullet.$$

Proof. Suppose first that $\underline{L} = \underline{L}'$. We set $d_0 := \varepsilon$, $L_{-1} := M$, $\varphi_{-1} := \mathbf{1}_M$ and we show by induction on n that φ_n is an isomorphism. Indeed, this holds for $n = -1$ by definition. Suppose that $n \geq 0$ and that the assertion is known for all $j < n$; by a little diagram chasing (or the five lemma) we deduce that φ_{n-1} induces an automorphism $\mathrm{Im}\, d_n \xrightarrow{\sim} \mathrm{Im}\, d_n$, therefore $\varphi_n \otimes_A \mathbf{1}_k : k \otimes_A L_n \to k \otimes_A L_n$ is an automorphism (by minimality of $\underline{L}$), so the same holds for φ_n (*e.g.* by looking at the determinant of φ_n).

For the general case, by standard arguments we construct a morphism of resolutions: $\psi_\bullet : \underline{L} \to \underline{L}'$. By the foregoing case, $\varphi_\bullet \circ \psi_\bullet$ is an automorphism of $\underline{L}$, so $\varphi_\bullet$ is necessarily an epimorphism, and $\underline{L}'$ decomposes as claimed. Finally, it is also clear that $\mathrm{Ker}\,\varphi_\bullet$ is an acyclic bounded above complex of free A-modules, hence it is null homotopic. ☐

Remark 4.1.35. (i) Suppose that A is a coherent local ring, and let $\underline{L} := (L_\bullet, d_\bullet, \varepsilon)$ and $\underline{L}' := (L'_\bullet, d'_\bullet, \varepsilon')$ be two minimal resolutions of the finitely presented A-module M. It follows easily from lemma 4.1.34 that $\underline{L}$ and $\underline{L}'$ are isomorphic as resolutions of M.

(ii) Moreover, any two isomorphisms $\underline{L} \to \underline{L}'$ are homotopic, hence the rule: $M \mapsto L_\bullet$ extends to a functor

$$A\text{-}\mathbf{Mod}_{\mathrm{coh}} \to \mathsf{Hot}(A\text{-}\mathbf{Mod})$$

from the category of finitely presented A-modules to the homotopy category of complexes of A-modules.

(iii) The sequence of A-modules $(\mathrm{Syz}^i_A M := \mathrm{Im}\, d_i \mid i > 0)$ is determined uniquely by M (up to non-unique isomorphism). The graded module $\mathrm{Syz}^\bullet_A M$ is sometimes called the *syzygy* of the module M. Moreover, if $\underline{L}''$ is any other finite free resolution of M, then we can choose a morphism of resolutions $\underline{L}'' \to \underline{L}$, which will be a split epimorphism by lemma 4.1.34, and the submodule $d_\bullet(L''_\bullet) \subset L_\bullet$ decomposes as a direct sum of $\mathrm{Syz}^\bullet_A M$ and a free A-module of finite rank.

Lemma 4.1.36. *Let $A \to B$ a faithfully flat homomorphism of coherent local rings, M a finitely presented A-module. Then there exists an isomorphism of graded B-modules :*

$$B \otimes_A \mathrm{Syz}^\bullet_A M \to \mathrm{Syz}^\bullet_B(B \otimes_A M).$$

Proof. Left to the reader. ☐

Proposition 4.1.37. *Let $A \to B$ be a flat and essentially finitely presented local ring homomorphism of local rings, M an A-flat finitely presented B-module. Then the B-module M admits a minimal free resolution*

$$\Sigma_\bullet \quad : \quad \cdots \xrightarrow{d_3} L_2 \xrightarrow{d_2} L_1 \xrightarrow{d_1} L_0 \xrightarrow{d_{-1}} L_{-1} := M.$$

Moreover, Σ *is* universally A-exact, i.e. *for every* A*-module* N*, the complex* $\Sigma_\bullet \otimes_A N$ *is still exact.*

Proof. To start out, let us notice :

Claim 4.1.38. In order to prove the proposition, it suffices to show that, for every A-flat finitely presented B-module N, and every B-linear surjective map $d : L \to N$, from a free B-module L of finite rank, $\mathrm{Ker}\, d$ is also an A-flat finitely presented B-module.

Proof of the claim. Indeed, in that case, we can build inductively a minimal resolution $\Sigma_\bullet$ of M, such that $N_i := \mathrm{Ker}(d_i : L_i \to L_{i-1})$ is an A-flat finitely presented B-module for every $i \in \mathbb{N}$. Namely, suppose that a complex $\Sigma_\bullet^{(i)}$ with these properties has already been constructed, up to degree i, and let κ_B be the residue field of B; by Nakayama's lemma, we may find a surjection $d_{i+1} : L_{i+1} \to N_i$, where L_{i+1} is a free B-module of rank $\dim_{\kappa_B}(N_i \otimes_B \kappa_B)$. Under the assumption of the claim, the resulting complex $\Sigma_\bullet^{(i+1)} : (L_{i+1} \to L_i \to L_{i-1} \to \cdots \to M)$ fulfills the sought conditions, up to degree $i + 1$.

It is easily seen that the complex $\Sigma_\bullet$ thus obtained shall be universally A-exact. $\diamondsuit$

Let us write A as the union of the filtered family $(A_\lambda \mid \lambda \in \Lambda)$ of its noetherian subalgebras. Say that $B = C_\mathfrak{p}$, for some finitely presented A-algebra C, and a prime ideal $\mathfrak{p} \subset C$, and $M = N_\mathfrak{p}$ for some finitely presented C-module N. We may find $\lambda \in \Lambda$, a finitely generated A_λ-algebra C_λ and a C_λ-module N_λ such that $C = C_\lambda \otimes_{A_\lambda} A$, and $N = N_\lambda \otimes_{A_\lambda} A$; for every $\mu \geq \lambda$, let $C_\mu := A_\mu \otimes_{A_\lambda} C_\lambda$ and $N_\mu := A_\mu \otimes_{A_\lambda} N_\lambda$; also, denote by $\mathfrak{p}_\mu$ the preimage of $\mathfrak{p}$ in C_μ, and set $B_\mu := (C_\mu)_{\mathfrak{p}_\mu}$, $M_\mu := (N_\mu)_{\mathfrak{p}_\mu}$. According to [32, Ch.IV, Cor.11.2.6.1(ii)], we may assume that M_μ is a flat A_μ-module, for every $\mu \geq \lambda$. Moreover, suppose that $d : L \to M$ is a B-linear surjection from a free B-module L of rank r; then we may find $\mu \in \Lambda$ such that d descends to a B_μ-linear surjection $d_\mu : L_\mu \to M_\mu$ from a free B_μ-module L_μ of rank r. It follows easily that $K_\mu := \mathrm{Ker}\, d_\mu$ is a flat A_μ-module, for every $\mu \geq \lambda$, and the induced map $K_\mu \otimes_{B_\mu} B \to K := \mathrm{Ker}\, d$ is a surjection, whose kernel is a quotient of $\mathrm{Tor}_1^{A_\mu}(A, M_\mu)_\mathfrak{p}$; the latter vanishes, since M_μ is A_μ-flat. Hence, K is A-flat; furthermore, K_μ is clearly a finitely generated B_μ-module, hence K is a finitely presented B-module. Then the proposition follows from claim 4.1.38. $\square$

4.2. **Simplicial objects.**

In this section, we introduce the simplicial formalism, which provides the language for the homotopical algebra of section 4.5.

Definition 4.2.1. Let $\mathscr{C}$ be any category, and $k \in \mathbb{N}$ any integer.

(i) We denote by Δ the *simplicial category*, whose objects are the finite ordered sets :
$$[n] := \{0 < 1 < \cdots < n\} \qquad \text{for every } n \in \mathbb{N}$$
and whose morphisms are the non-decreasing functions.

(ii) Δ is a full subcategory of the *augmented simplicial category* $\Delta^\wedge$, whose set of objects is $\mathrm{Ob}(\Delta) \cup \{\varnothing\}$, with $\mathrm{Hom}_{\Delta^\wedge}(\varnothing, [n])$ consisting of the unique mapping of sets $\varnothing \to [n]$, for every $n \in \mathbb{N}$. It is convenient to set $[-1] := \varnothing$.

(iii) The *augmented* k*-truncated simplicial category* $\Delta_k^\wedge$, is the full subcategory of $\Delta^\wedge$ whose objects are the elements of $\mathrm{Ob}(\Delta^\wedge)$ of cardinality $\leq k+1$. The k*-truncated simplicial category* is the full subcategory Δ_k of $\Delta_k^\wedge$ whose set of objects is $\mathrm{Ob}(\Delta_k^\wedge) \setminus \{\varnothing\}$.

(iv) A *simplicial object* (resp. an *augmented simplicial object*, resp. a k*-truncated simplicial object*, resp. a k*-truncated augmented simplicial object*) of $\mathscr{C}$ is a functor $\Delta^o \to \mathscr{C}$ (resp. $(\Delta^\wedge)^o \to \mathscr{C}$, resp. $\Delta_k^o \to \mathscr{C}$, resp. $(\Delta_k^\wedge)^o$). The morphisms of simplicial objects of $\mathscr{C}$ are just the natural transformations (and likewise for the truncated or augmented variants). Clearly, these objects form a category, and we use the notation

$$s.\mathscr{C} := \mathbf{Fun}(\Delta^o, \mathscr{C}) \qquad \widehat{s}.\mathscr{C} := \mathbf{Fun}((\Delta^\wedge)^o, \mathscr{C})$$
$$s_k.\mathscr{C} := \mathbf{Fun}(\Delta_k^o, \mathscr{C}) \qquad \widehat{s}_k.\mathscr{C} := \mathbf{Fun}((\Delta_k^\wedge)^o, \mathscr{C}).$$

(v) Dually, a *cosimplicial object* $F^\bullet$ of $\mathscr{C}$ is a functor $F : \Delta \to \mathscr{C}$, or – which is the same – a simplicial object in $\mathscr{C}^o$. Likewise one defines the truncated or augmented cosimplicial variants, and we set

$$c.\mathscr{C} := \mathbf{Fun}(\Delta, \mathscr{C}) \qquad \widehat{c}.\mathscr{C} := \mathbf{Fun}(\Delta^\wedge, \mathscr{C})$$
$$c_k.\mathscr{C} := \mathbf{Fun}(\Delta_k, \mathscr{C}) \qquad \widehat{c}_k.\mathscr{C} := \mathbf{Fun}(\Delta_k^\wedge, \mathscr{C}).$$

4.2.2. Notice the *front-to-back* involution :

$$\Delta \to \Delta \quad : \quad (\alpha : [n] \to [m]) \mapsto (\alpha^\vee : [n] \to [m]) \quad \text{for every } n, m \in \mathbb{N} \tag{4.2.3}$$

defined as the endofunctor which induces the identity on $\mathrm{Ob}(\Delta)$, and such that :

$$\alpha^\vee(i) := m - \alpha(n-i) \qquad \text{for every } \alpha \in \mathrm{Hom}_\Delta([n],[m]) \text{ and every } i \in [n].$$

Another construction of interest is the endofunctor

$$\gamma : \Delta \to \Delta$$

given by the rule : $[n] \mapsto [n+1]$ for every $n \in \mathbb{N}$, and which takes any morphism $\alpha : [n] \to [m]$ of Δ, to the morphism $\gamma(\alpha) : [n+1] \to [m+1]$ which is the unique extension of α such that $\gamma(\alpha)(n+1) := m+1$. Notice that γ restricts to functors $\gamma_k : \Delta_{k+1} \to \Delta_k$ for every $k \in \mathbb{N}$.

• Given a simplicial object F of $\mathscr{C}$, one gets a cosimplicial object F^o of $\mathscr{C}$, by the (obvious) rule : $(F^o)[n] := F[n]$ for every $n \in \mathbb{N}$, and $F^o(\alpha) := F(\alpha)^o$ for every morphism α in Δ.

• Moreover, by composing a simplicial (resp. cosimplicial) object F (resp. G) with the involution (4.2.3), one obtains a simplicial (resp. cosimplicial) object $F^\vee$ (resp. $G^\vee$). Likewise, given a morphism $\alpha : F_1 \to F_2$, the Godement product $\alpha^\vee := \alpha * (4.2.3)$ is a morphism $F_1^\vee \to F_2^\vee$.

• For F and G as above, we may also consider the simplicial (resp. cosimplicial) object $\gamma F := F \circ \gamma^o$ (resp. $\gamma G := G \circ \gamma$), and this definition extends again to morphisms, by taking Godement products. The object γF (resp. γG) is called the *path space* of F (resp. of G). If F is a $(k+1)$-truncated simplicial object, then we can consider $\gamma_k F := F \circ \gamma_k^o$, which is a k-truncated simplicial object (and likewise for truncated cosimplicial objects).

4.2.4. There is an obvious fully faithful functor :

$$\mathscr{C} \to s.\mathscr{C} \quad : \quad A \mapsto s.A \qquad (\text{ resp. } \quad \mathscr{C} \to s_k.\mathscr{C} \quad : \quad A \mapsto s_k.A \ )$$

that assigns to each object A of $\mathscr{C}$ the *constant simplicial object* $s.A$ (resp. *constant truncated simplicial object* $s_k.A$) such that $s.A[n] := A$ for every $n \in \mathbb{N}$ (resp. for every $n \leq k$), and $s.A(\alpha) := \mathbf{1}_A$ for every morphism α of Δ (resp. of Δ_k). Of course, we have as well augmented variants $\widehat{s}.A$ and $\widehat{s}_k.A$, and cosimplicial versions $c.A$, $c_k.A$, $\widehat{c}.A$, $\widehat{c}_k.A$.

Moreover, we have, for every integer $k \in \mathbb{N}$, the *k-truncation functor*

$$s.\mathrm{trunc}_k : s.\mathscr{C} \to s_k.\mathscr{C} \qquad (\text{ resp. } \quad \widehat{s}.\mathrm{trunc}_k : \widehat{s}.\mathscr{C} \to \widehat{s}_k.\mathscr{C} \ )$$

that assigns to any simplicial (resp. augmented simplicial) object $F : \Delta^o \to \mathscr{C}$ (resp. $F : (\Delta^\wedge)^o \to \mathscr{C}$) its composition with the inclusion functor $\Delta_k^o \to \Delta^o$ (resp. $(\Delta_k^\wedge)^o \to (\Delta^\wedge)^o$). Again, we have as well the corresponding cosimplicial versions $c.\mathrm{trunc}_k$ and $\widehat{c}.\mathrm{trunc}_k$. Also, for every $n \in \mathbb{N}$, we have the functor

$$\bullet[n] : s.\mathscr{A} \to \mathscr{A} \qquad A \mapsto A[n].$$

Lastly, any functor $\varphi : \mathscr{B} \to \mathscr{C}$ induces functors

$$s.\varphi : s.\mathscr{B} \to s.\mathscr{C} \qquad s_k.\varphi : s_k.\mathscr{B} \to s_k.\mathscr{C} \quad : \quad F \mapsto \varphi \circ F$$

and there are of course augmented variants $\widehat{s}.\varphi$ and $\widehat{s}_k.\varphi$, as well as the corresponding cosimplicial versions.

4.2.5. For given $n \in \mathbb{N}$, and every $i = 0, \dots, n$, let

$$\varepsilon_i : [n-1] \to [n] \qquad \text{(resp. } \eta_i : [n+1] \to [n]\text{)}$$

be the unique injective map in $\Delta^\wedge$ whose image misses i (resp. the unique surjective map in Δ with two elements mapping to i). The morphisms ε_i (resp. η_i) are called *face maps* (resp. *degeneracy maps*). By direct inspection, one checks that they fulfill the identities :

$$\varepsilon_j \circ \varepsilon_i = \varepsilon_i \circ \varepsilon_{j-1} \qquad \text{if } i < j$$
$$\eta_j \circ \eta_i = \eta_i \circ \eta_{j+1} \qquad \text{if } i \leq j$$
$$\eta_j \circ \varepsilon_i = \begin{cases} \varepsilon_i \circ \eta_{j-1} & \text{if } i < j \\ \mathbf{1} & \text{if } i = j \text{ or } i = j+1 \\ \varepsilon_{i-1} \circ \eta_j & \text{if } i > j+1. \end{cases}$$

Example 4.2.6. (i) For instance, notice the identities :

$$\varepsilon_i^\vee = \varepsilon_{n-i} \qquad \eta_i^\vee = \eta_{n-i} \qquad \text{for every } n \in \mathbb{N} \text{ and every } i = 0, \dots, n.$$

(ii) For every $r, s \in \mathbb{N}$, we set

$$\varepsilon_{r,0}^s := \varepsilon_0 \circ \cdots \circ \varepsilon_0 : [r] \to [r+s].$$

This is the injective mapping whose image is $\{s, \dots, r+s\}$; therefore, its front-to-back dual

$$\varepsilon_{r,0}^{s\vee} := \varepsilon_{r+s} \circ \cdots \circ \varepsilon_{r+1} : [r] \to [r+s]$$

is just the natural inclusion map. We shall also use the notation

$$\varepsilon_{-1,0}^s : [-1] \to [s] \qquad \text{for every } s \in \mathbb{N}$$

for the unique morphism $\varnothing \to [s]$ in $\Delta^\wedge$; of course, we have $\varepsilon_{-1,0}^{s\vee} = \varepsilon_{-1,0}^s$ for every $s \in \mathbb{N}$.

4.2.7. It is easily seen that every morphism $\alpha : [n] \to [m]$ in Δ admits a unique factorization $\alpha = \varepsilon \circ \eta$, where the monomorphism ε is uniquely a composition of faces :

$$\varepsilon = \varepsilon_{i_1} \circ \cdots \circ \varepsilon_{i_s} \qquad \text{with } 0 \leq i_s \leq \cdots \leq i_1 \leq m$$

and the epimorphism η is uniquely a composition of degeneracy maps :

$$\eta = \eta_{j_1} \circ \cdots \circ \eta_{j_t} \qquad \text{with } 0 \leq j_1 < \cdots < j_t \leq m$$

(see [75, Lemma 8.1.2]). It follows that, to give a simplicial object $A[\bullet]$ of a category $\mathscr{C}$, it suffices to give a sequence of objects $(A[n] \mid n \in \mathbb{N})$ of $\mathscr{C}$, together with *face operators*

$$\partial_i := A[\varepsilon_i] : A[n] \to A[n-1] \qquad i = 0, \dots, n$$

for every integer $n > 0$ and *degeneracy operators*

$$\sigma_i := A[\eta_i] : A[n] \to A[n+1] \qquad i = 0, \dots, n$$

for every $n \in \mathbb{N}$, satisfying the following *simplicial identities* :

$$\partial_i \circ \partial_j = \partial_{j-1} \circ \partial_i \qquad \text{if } i < j$$
$$\sigma_i \circ \sigma_j = \sigma_{j+1} \circ \sigma_i \qquad \text{if } i \leq j$$
$$\partial_i \circ \sigma_j = \begin{cases} \sigma_{j-1} \circ \partial_i & \text{if } i < j \\ \mathbf{1} & \text{if } i = j \text{ or } i = j+1 \\ \sigma_j \circ \partial_{i-1} & \text{if } i > j+1. \end{cases} \tag{4.2.8}$$

Under this correspondence we have $\partial_i = A(\varepsilon_i)$ and $\sigma_i = A(\eta_i)$ ([75, Prop.8.1.3]). Likewise, a k-truncated simplicial object of $\mathscr{C}$ is the same as the datum of a sequence $(A[n] \mid n = 0, \dots, k)$

of objects of $\mathscr{C}$, and of a system of face and degeneracy operators restricted to this sequence of objects, and fulfilling the same identities (4.2.8).

Dually, a cosimplicial object $A[\bullet]$ of $\mathscr{C}$ is the same as the datum of a sequence $(A[n] \mid n \in \mathbb{N})$ of objects of $\mathscr{C}$, together with *coface operators*

$$\partial^i : A[n-1] \to A[n] \qquad i = 0, \dots, n$$

and *codegeneracy operators*

$$\sigma^i : A[n+1] \to A[n] \qquad i = 0, \dots, n$$

which satisfy the *cosimplicial identities* :

$$\begin{aligned} &\partial^j \circ \partial^i = \partial^i \circ \partial^{j-1} && \text{if } i < j \\ &\sigma^j \circ \sigma^i = \sigma^i \circ \sigma^{j+1} && \text{if } i \leq j \\ &\sigma^j \circ \partial^i = \begin{cases} \partial^i \circ \sigma^{j-1} & \text{if } i < j \\ \mathbf{1} & \text{if } i = j \text{ or } i = j+1 \\ \partial^{i-1} \circ \sigma^j & \text{if } i > j+1 \end{cases} \end{aligned} \tag{4.2.9}$$

and likewise for k-truncated cosimplicial objects.

4.2.10. An augmented simplicial object of a category $\mathscr{C}$ can be viewed as the datum of a simplicial object $A[\bullet]$ of $\mathscr{C}$, together with an object $A[-1] \in \mathrm{Ob}(\mathscr{C})$, and a morphism $\varepsilon : A[0] \to A[-1]$, which is an *augmentation, i.e.* such that :

$$\varepsilon \circ \partial_0 = \varepsilon \circ \partial_1.$$

Dually, an augmented cosimplicial object of $\mathscr{C}$ can be viewed as a cosimplicial object $A[\bullet]$, together with a morphism $\eta : A[-1] \to A[0]$ in $\mathscr{C}$, such that η^o is an augmentation for $A^o[\bullet]$. We say that η is an *augmentation* for $A[\bullet]$.

Remark 4.2.11. Let A be a simplicial object of the category $\mathscr{C}$.

(i) For every $n \in \mathbb{N}$ we have $\gamma A[n] := A[n+1]$ (notation of (4.2.2)), and the face operators $\gamma A[\varepsilon_i] : \gamma A[n+1] \to \gamma A[n]$ for $i \leq n+1$ (resp. degeneracy operators $\gamma A[\eta_i] : \gamma A[n] \to \gamma A[n+1]$ for $i \leq n$) of γA are $\partial_i : A[n+2] \to A[n+1]$ (resp. $\sigma_i : A[n+1] \to A[n+2]$); *i.e.* we drop ∂_{n+2} and σ_{n+1}. Likewise for the truncated variants.

(ii) The discarded faces ∂_{n+2} and degeneracies σ_{n+1} can be used to produce natural morphisms

$$s.A[0] \xrightarrow{f_A} \gamma A \xrightarrow{g_A} A.$$

Namely, we set

$$f_A[n] := \sigma_n \circ \cdots \circ \sigma_1 \qquad g_A[n] := \partial_{n+1} \qquad \text{for every } n \in \mathbb{N}.$$

For every $k \in \mathbb{N}$, the same operation on an object $A \in \mathrm{Ob}(s_{k+1}.\mathscr{C})$ yields natural morphisms

$$s.\mathrm{trunc}_k A \xrightarrow{f_A} \gamma_k A \xrightarrow{g_A} s.\mathrm{trunc}_k A.$$

(iii) Since there is a unique morphism $\sigma_{n,0} : [n] \to [0]$ in Δ for every $n \in \mathbb{N}$, it is easily seen that the system $(A[\sigma_{n,0}] : A[0] \to A[n])$ defines a natural morphism

$$s.A[0] \to A \qquad \text{in } s.\mathscr{C}.$$

(iv) Likewise, suppose $\varepsilon : A[0] \to A[-1]$ is an augmentation for A; since there is exactly one morphism $\varepsilon_{-1,0}^{n+1} : \varnothing \to [n]$ in $\Delta^\wedge$ for every $n \in \mathbb{N}$, we see that the system $(A[\varepsilon_{-1,0}^{n+1}] \mid n \in \mathbb{N})$ defines a natural morphism

$$A \to s.A[-1] \qquad \text{in } s.\mathscr{C}.$$

Definition 4.2.12. Let $\mathscr{C}$ be any category. Denote by

$$e_i : \Delta^o \to [1]/\Delta^o \qquad i = 0, 1$$

the functor that assigns to each $[n] \in \mathrm{Ob}(\Delta)$ the unique morphism $[n] \to [1]$ of Δ whose image is $\{i\}$. Let also $\mathtt{t} : [1]/\Delta^o \to \Delta^o$ be the target functor (see (1.1.12)). Let A and B be two simplicial objects of $\mathscr{C}$, and $f, g : A \to B$ two morphisms.

(i) A *homotopy* from f to g is the datum of a natural transformation

$$u : A \circ \mathtt{t} \Rightarrow B \circ \mathtt{t}$$

such that $u * e_0 = f$ and $u * e_1 = g$.

(ii) If $A[\bullet]$ is an augmented simplicial object of $\mathscr{C}$, with augmentation given by a morphism $\varphi : A \to s.A[-1]$ in $s.\mathscr{C}$, then we say that A is *homotopically trivial*, if there exists a morphism $\psi : s.A[-1] \to A$ and homotopies u and v, respectively from $\mathbf{1}_{s.A[-1]}$ to $\varphi \circ \psi$, and from $\mathbf{1}_A$ to $\psi \circ \varphi$.

(iii) Dually, if C and D are cosimplicial objects of $\mathscr{C}$, and $p, q : C \to D$ any two morphisms in $c.\mathscr{C}$, then a *homotopy* from p to q is a homotopy from p^o to q^o in $s.\mathscr{C}^o$. Likewise we define homotopically trivial cosimplicial objects.

Remark 4.2.13. (i) In the situation of definition 4.2.12, suppose that $p : A' \to A$ and $q : B \to B'$ are any two morphisms in $s.\mathscr{C}$; then the natural transformation

$$(q * \mathtt{t}) \circ u \circ (p * \mathtt{t}) : A' \circ \mathtt{t} \Rightarrow B' \circ \mathtt{t}$$

is a homotopy from $f \circ p$ to $q \circ g$. Moreover, if $F : \mathscr{C} \to \mathscr{D}$ is any functor, then

$$F * u : FA * \mathtt{t} \Rightarrow FB * \mathtt{t}$$

is a homotopy from Ff to Fg.

(ii) However, unlike the case for chain homotopies, simplicial homotopies cannot be composed in this generality; hence, the simplicial category $s.\mathscr{C}$ cannot be made into a 2-category, by taking the homotopies as 2-cells.

(iii) In the same vein, for a general category $\mathscr{C}$, the relation "there exists a homotopy from f to g" on morphisms of $s.\mathscr{C}$ is neither symmetric nor transitive (though it will follow from theorem 4.2.58 that this is an equivalence relation, in case $\mathscr{C}$ is abelian).

4.2.14. Notice that there are exactly $n+1$ morphisms $\varphi : [n] \to [1]$ for every $[n] \in \mathrm{Ob}(\Delta)$, and they can be labeled by the cardinality of $\varphi^{-1}(0)$: for every $n \in \mathbb{N}$ and every $k \leq n+1$, we shall write $\varphi_{n,k} : [n] \to [1]$ for the unique morphism such that $\varphi_{n,k}^{-1}(0)$ has cardinality k. With this notation, notice that

$$\varphi_{n,k} \circ \varepsilon_i = \begin{cases} \varphi_{n-1,k} & \text{if } i \geq k \\ \varphi_{n-1,k-1} & \text{if } i < k \end{cases} \qquad \text{and} \qquad \varphi_{n,k} \circ \eta_i = \begin{cases} \varphi_{n+1,k} & \text{if } i \geq k \\ \varphi_{n+1,k+1} & \text{if } i < k. \end{cases}$$

Hence, a homotopy u from f to g as in definition 4.2.12, is the same as a system of morphisms

$$u_{n,k} : A[n] \to B[n] \qquad \text{for every } n \in \mathbb{N} \text{ and every } k \leq n+1$$

such that $u_{n,n+1} = f[n]$ and $u_{n,0} = g[n]$ for every $n \in \mathbb{N}$, and the diagrams

$$\begin{tikzcd} A[n] \arrow[r, "u_{n,k}"] \arrow[d, "\partial_i"'] & B[n] \arrow[d, "\partial_i"] \\ A[n-1] \arrow[r, "u_{n-1,k-a}"] & B[n-1] \end{tikzcd} \qquad \begin{tikzcd} A[n] \arrow[r, "u_{n,k}"] \arrow[d, "\sigma_i"'] & B[n] \arrow[d, "\sigma_i"] \\ A[n+1] \arrow[r, "u_{n+1,k+a}"] & B[n+1] \end{tikzcd}$$

commute for every $n \in \mathbb{N}$ and every $k \leq n+1$, where $a := 0$ if $i \geq k$, and $a := 1$ if $i < k$.

4.2.15. A *bisimplicial object* in a category $\mathscr{C}$ is an object of the category $s.(s.\mathscr{C})$. The latter can also be regarded as the category of all functors $\Delta^o \times \Delta^o \to \mathscr{C}$; it follows that a bisimplicial object of $\mathscr{C}$ is the same as a system $(A[p,q] \mid (p,q) \in \mathbb{N} \times \mathbb{N})$ of objects of $\mathscr{C}$, together with morphisms

$$A[\alpha,\beta] : A[p,q] \to A[p',q'] \qquad \text{for all morphisms } \alpha : [p'] \to [p], \beta : [q'] \to [q] \text{ of } \Delta$$

compatible with compositions of morphisms in Δ, in the obvious way. More generally, we may define inductively the category of n-simplicial objects $s^n.\mathscr{A}$, for every $n \in \mathbb{N}$, by letting $s^0.\mathscr{A} := \mathscr{A}$, and $s^n.\mathscr{A} := s.(s^{n-1}.\mathscr{A})$, for every $n > 0$. The *diagonal functor*

$$\Delta \to \Delta \times \Delta \qquad [n] \mapsto ([n],[n]) \qquad \alpha \mapsto (\alpha,\alpha) \qquad \text{for all } n \in \mathbb{N} \text{ and all morphisms } \alpha \text{ of } \Delta$$

induces a functor

$$\Delta_{\mathscr{C}} : s^2.\mathscr{C} \to s.\mathscr{C} \qquad A \mapsto A^\Delta.$$

Especially, we have $A^\Delta[n] := A[n,n]$ for every $n \in \mathbb{N}$, and the face operators ∂_i on $A^\Delta[n]$ are of the form $A[\varepsilon_i,\varepsilon_i]$, for every $i = 0,\dots,n$ (and likewise for the degeneracies). Also, the *flip functor*

$$\Delta \times \Delta \to \Delta \times \Delta \qquad ([m],[n]) \mapsto ([n],[m])$$

induces an endofunctor

$$\mathsf{fl} : s^2.\mathscr{A} \to s^2.\mathscr{A}$$

in the obvious way. Furthermore, the endofunctors

$$\Delta \times \Delta \xrightarrow{\gamma \times \mathbf{1}_\Delta} \Delta \times \Delta \xleftarrow{\mathbf{1}_\Delta \times \gamma} \Delta \times \Delta$$

induce functors

$$s^2.\mathscr{A} \xleftarrow{\gamma_1} s^2.\mathscr{A} \xrightarrow{\gamma_2} s^2.\mathscr{A}$$

that admit descriptions as in remark 4.2.11(i). Correspondingly, we get natural morphisms

$$g_A^{(i)} : \gamma_i A \to A \qquad \text{for } i = 1,2 \text{ and every } A \in \mathrm{Ob}(s^2.\mathscr{A})$$

as in remark 4.2.11(ii).

Remark 4.2.16. (i) Let $\mathscr{C}$ be a category whose finite coproducts are representable. Let also $\mathbf{f}.\mathbf{Set}$ be the category of finite sets. To every object S of $s.\mathbf{f}.\mathbf{Set}$ and every $X \in \mathrm{Ob}(s.\mathscr{C})$, we attach a bisimplicial object $S \boxtimes X$ of $\mathscr{C}$ as follows. For every $n,m \in \mathbb{N}$, we let $S \boxtimes X[n,m]$ be the coproduct of finitely many copies of $X[m]$, indexed by the elements of $S[n]$; hence, for every $a \in S[n]$ we have a natural morphism $i_a : X[m] \to S \boxtimes X[n,m]$. If $\varphi : [n] \to [n']$ and $\psi : [m] \to [m']$ are any two morphisms in Δ^o, we let $S \boxtimes X[\varphi,\psi] : S \boxtimes X[n,m] \to S \boxtimes X[n',m']$ be the unique morphism such that $S \boxtimes X[\varphi,\psi] \circ i_a = i_{S[\varphi](a)} \circ X[\psi]$ for every $a \in S[n]$. Clearly, this rules extends to a well defined functor

$$s.\mathbf{f}.\mathbf{Set} \times s.\mathscr{C} \to s^2.\mathscr{C} \qquad (S,X) \mapsto S \boxtimes X.$$

Likewise, we define $X \boxtimes S := \mathsf{fl}(S \boxtimes X)$ (notation of (4.2.15)). If all coproducts of $\mathscr{C}$ are representable, we may even extend the above construction to arbitrary simplicial sets.

(ii) In the same vein, let $\mathscr{A}$ be any abelian category, and M any object of $s.\mathbb{Z}\text{-}\mathbf{Mod}_{\mathrm{fg}}$ (notation of (1.2.43)). For any $A \in \mathrm{Ob}(s.\mathscr{A})$, we may define a bisimplicial object $M \boxtimes_\mathbb{Z} A$ of $\mathscr{A}$, by the rule : $[n,m] \mapsto M[n] \otimes_\mathbb{Z} A[n]$ for every $n,m \in \mathbb{N}$ and $[\varphi,\psi] \mapsto M[\varphi] \otimes_\mathbb{Z} A[\psi]$ for all morphisms φ,ψ of Δ (where these mixed tensor products are as defined in (1.2.43)). Clearly these rules yield a well defined functor

$$s.\mathbb{Z}\text{-}\mathbf{Mod}_{\mathrm{fg}} \times s.\mathscr{A} \to s^2.\mathscr{A} \qquad (M,A) \mapsto M \boxtimes_\mathbb{Z} A.$$

Likewise, we set $A \boxtimes_\mathbb{Z} M := \mathsf{fl}(M \boxtimes_\mathbb{Z} A)$.

(iii) Furthermore, if $(\mathscr{C}, \otimes)$ is any tensor category, and X, Y any two simplicial objects of $\mathscr{C}$, we may define a bisimplicial object $X \boxtimes Y$, by the same rule as in (ii). This yields a functor

$$s.\mathscr{C} \times s.\mathscr{C} \to s^2.\mathscr{C} \qquad (X, Y) \mapsto X \boxtimes Y.$$

In this situation (resp. in the situation of (i), resp. of (ii)), we shall let also

$$X \otimes Y := (X \boxtimes Y)^\Delta \qquad \text{(resp. } S \otimes X := (S \boxtimes X)^\Delta, \text{ resp. } M \otimes_{\mathbb{Z}} A := (M \boxtimes_{\mathbb{Z}} A)^\Delta \text{)}$$

which we shall call the *tensor product* of X and Y (resp. of S and X, resp. of M and A). Notice the natural identifications

$$S \boxtimes A \xrightarrow{\sim} (S \otimes s.\mathbb{Z}) \boxtimes_{\mathbb{Z}} A \qquad \text{for every } S \in \mathrm{Ob}(s.\mathbf{f}.\mathbf{Set}) \text{ and every } A \in \mathrm{Ob}(s.\mathscr{A}).$$

Likewise, if U is any unit object for $(\mathscr{C}, \otimes)$, we get natural identifications

$$S \boxtimes X \xrightarrow{\sim} (S \otimes s.U) \boxtimes X \qquad \text{for every } S \in \mathrm{Ob}(s.\mathbf{f}.\mathbf{Set}) \text{ and every } X \in \mathrm{Ob}(s.\mathscr{C})$$

via the isomorphisms $u_X : X \xrightarrow{\sim} U \otimes X$ provided by proposition 1.2.6.

(iv) Notice also that, if $f, g : S \to T$ are any two morphisms of simplicial finite sets, then – in light of remark 4.2.13(i) – any homotopy u from f to g induces a homotopy $u \otimes X$ from $f \otimes X$ to $g \otimes X$. Moreover, for every $k \in \mathbb{N}$, let $\mathbf{\Delta}_k$ be the simplicial finite set given by the rule

$$\mathbf{\Delta}_k[n] := \mathrm{Hom}_\Delta([n], [k]) \quad \text{and} \quad \mathbf{\Delta}_k[\varphi] := \mathrm{Hom}_\Delta(\varphi, [k])$$

for every $n \in \mathbb{N}$ and every morphism φ in Δ. Especially, $\mathbf{\Delta}_0$ is the constant simplicial set associated to the set with one element, and therefore $\mathbf{\Delta}_0 \otimes X = X$ for every $X \in \mathrm{Ob}(\mathscr{C})$. Also, the rules : $[n] \mapsto (e_i[n] : [n] \to [1])$ (for $i = 0, 1$; here e_i is the functor introduced in definition 4.2.12) define morphisms

$$e_i^* : \mathbf{\Delta}_0 \to \mathbf{\Delta}_1 \qquad i = 0, 1$$

and we notice that the datum of a homotopy u from f to g as in definition 4.2.12(i), is the same as that of a morphism

$$\widetilde{u} : \mathbf{\Delta}_1 \otimes A \to B \qquad \text{such that } \widetilde{u} \circ (e_0^* \otimes A) = f \text{ and } \widetilde{u} \circ (e_1^* \otimes A) = g.$$

Indeed, given u, we construct $\widetilde{u}$ as follows. For every $n \in \mathbb{N}$ and every $\varphi \in \mathbf{\Delta}_1[n]$, let $\widetilde{u}[n]$ be the unique morphism such that $\widetilde{u}[n] \circ i_\varphi = u_\varphi$. The naturality of u easily implies that this rule amounts to a morphism $\widetilde{u}$ as sought. Conversely, given $\widetilde{u}$, we can construct a natural transformation u, by reversing the foregoing rule.

Remark 4.2.17. (i) Let $(\mathscr{C}, \otimes)$ be a tensor category with internal Hom functor $\mathscr{H}\!om$, and unit object U. To every two objects X, Y of $s.\mathscr{C}$, we may attach an object

$$\mathscr{H}\!om(X, Y) \qquad \text{of } c.s.\mathscr{C}$$

as follows. For every $n, m \in \mathbb{N}$, we let $\mathscr{H}\!om(X, Y)[n, m] := \mathscr{H}\!om(X[n], Y[m])$, and for every two morphisms φ, ψ of Δ, we define $\mathscr{H}\!om(X, Y)[\varphi, \psi] := \mathscr{H}\!om(X[\varphi], Y[\psi])$. Since this is a mixed simplicial-cosimplicial object, we cannot extract a diagonal object from it; however, if $\mathscr{C}$ is complete, we can at least define

$$\mathscr{H}\!om_{s.\mathscr{C}}(X, Y) := \mathrm{Equal}(\textstyle\prod_{n \in \mathbb{N}} \mathscr{H}\!om(X, Y)[n, n] \overset{d_1}{\underset{d_0}{\rightrightarrows}} \textstyle\prod_{\varphi:[n] \to [m]} \mathscr{H}\!om(X, Y)[m, n])$$

where the second product ranges over the morphisms φ of Δ, and where

$$d_0 := \prod_{\varphi:[n]\to[m]} \mathscr{H}\!om(X[\varphi], Y[n]) \qquad \text{and} \qquad d_1 := \prod_{\varphi:[n]\to[m]} \mathscr{H}\!om(X[m], Y[\varphi]).$$

Arguing as in example 4.1.8(vi), it is easily seen that there are natural isomorphisms

$$\mathrm{Hom}_{\mathscr{C}}(Z, \mathscr{H}\!om_{s.\mathscr{C}}(X, Y)) \xrightarrow{\sim} \mathrm{Hom}_{s.\mathscr{C}}(s.Z \otimes X, Y)$$

for every $Z \in \mathrm{Ob}(\mathscr{C})$ and every $X, Y \in \mathrm{Ob}(s.\mathscr{C})$.

(ii) Suppose additionally, that all finite coproducts of $\mathscr{C}$ are representable. For every $Z \in \mathrm{Ob}(\mathscr{C})$, and every $X \in \mathrm{Ob}(s.\mathscr{C})$, consider the simplicial set

$$\mathrm{Hom}_{\mathscr{C}}(Z, X)$$

such that $\mathrm{Hom}_{\mathscr{C}}(Z, X)[n] := \mathrm{Hom}_{\mathscr{C}}(Z, X[n])$ for every $n \in \mathbb{N}$, with face and degeneracies deduced from those of X, in the obvious way. For any $k \in \mathbb{N}$, let also $\mathbf{\Delta}_k$ be the simplicial set defined in remark 4.2.16(iv); we have natural isomorphisms

$$\begin{aligned}\mathrm{Hom}_{\mathscr{C}}(Z, \mathscr{H}om_{s.\mathscr{C}}(\mathbf{\Delta}_k \otimes s.U, X)) &\xrightarrow{\sim} \mathrm{Hom}_{s.\mathscr{C}}(\mathbf{\Delta}_k \otimes s.Z, X) \\ &\xrightarrow{\sim} \mathrm{Hom}_{s.\mathbf{Set}}(\mathbf{\Delta}_k, \mathrm{Hom}_{\mathscr{C}}(Z, X)) \\ &\xrightarrow{\sim} \mathrm{Hom}_{\mathscr{C}}(Z, X)[k]\end{aligned}$$

where the last isomorphism follows from Yoneda's lemma (proposition 1.1.20(ii) : details left to the reader). Applying again Yoneda's lemma, we deduce a natural isomorphism

$$\mathscr{H}om_{s.\mathscr{C}}(\mathbf{\Delta}_k \otimes s.U, X) \xrightarrow{\sim} X[k] \qquad \text{for every } k \in \mathbb{N}.$$

(iii) Notice that every morphism $\varphi : [k] \to [k']$ in Δ induces a morphism

$$\mathbf{\Delta}_\varphi := \mathrm{Hom}_\Delta(-, \varphi) : \mathbf{\Delta}_k \to \mathbf{\Delta}_{k'}$$

and clearly $\mathbf{\Delta}_\psi \circ \mathbf{\Delta}_\varphi = \mathbf{\Delta}_{\psi\circ\varphi}$, if $\psi : [k'] \to [k'']$ is any other morphism of Δ. Hence, the system $(\mathbf{\Delta}_i \mid i \in \mathbb{N})$ amounts to an object of $c.s.\mathbf{Set}$. Moreover, since the Yoneda isomorphisms are natural in both X and $\mathbf{\Delta}_k$, we get a commutative diagram

$$\begin{tikzcd} \mathscr{H}om_{s.\mathscr{C}}(\mathbf{\Delta}_{k'} \otimes s.U, X) \arrow[r, "\sim"] \arrow[d, "\mathscr{H}om_{s.\mathscr{C}}(\mathbf{\Delta}_\varphi \otimes s.U, X)"'] & X[k'] \arrow[d, "X[\varphi]"] \\ \mathscr{H}om_{s.\mathscr{C}}(\mathbf{\Delta}_k \otimes s.U, X) \arrow[r, "\sim"] & X[k] \end{tikzcd}$$

for every morphism φ as above.

(iv) Notice as well that the considerations of (ii) and (iii) can be repeated, *mutatis mutandi*, for truncated simplicial objects : if X is an object of $s_n.\mathscr{C}$, then $\mathrm{Hom}_{\mathscr{C}}(Z, X)$ shall be an object of $s_k.\mathbf{Set}$, and we shall have natural isomorphisms

$$\mathscr{H}om_{s_n.\mathscr{C}}(s.\mathrm{trunc}_n(\mathbf{\Delta}_k \otimes s.U), X) \xrightarrow{\sim} X[k] \qquad \text{for every } k \leq n$$

and similarly for the commutative diagrams of (iii) (details left to the reader).

4.2.18. Let $\mathscr{C}$ be a category with small Hom-sets, and suppose that all finite colimits of $\mathscr{C}$ are representable. Then proposition 1.1.34 and remark 1.1.38(v) say that, for every integer $k \in \mathbb{N}$, the k-truncation functor on $s.\mathscr{C}$ admits a left adjoint

$$\mathrm{sk}_k : s_k.\mathscr{C} \to s.\mathscr{C}$$

which is called the *k-th skeleton functor*. By inspecting the proof of *loc.cit.* we see that, for every k-truncated simplicial object F, this adjoint is calculated by the rule :

$$\mathrm{sk}_k A[n] := \operatorname*{colim}_{\varphi:[i]\to[n]} F[i] \tag{4.2.19}$$

where i ranges over all the integers $\leq k$, and φ over all the morphisms $[i] \to [n]$ in Δ^o, and the transition maps $F[i] \to F[j]$ in the colimit are the morphisms $F[\psi]$ given by all commutative triangles

$$\begin{tikzcd} {[i]} \arrow[rr, "\psi"] \arrow[dr, "\varphi"'] & & {[j]} \arrow[dl, "\varphi"] \\ & {[n].} & \end{tikzcd} \tag{4.2.20}$$

A morphism $\alpha : [n] \to [m]$ in Δ^o induces a morphism $\mathrm{sk}_k F[\alpha] : \mathrm{sk}_k A[n] \to \mathrm{sk}_k A[m]$; namely, for every $\varphi : [i] \to [m]$ one has a natural morphism $j_\varphi : F[i] \to \mathrm{cosk}_k F[m]$, and $\mathrm{sk}_k F[\alpha]$ is the colimit of the system of morphisms

$$j_{\alpha\circ\varphi} : F[i] \to \mathrm{sk}_k A[m] \qquad \text{for all } \varphi : [i] \to [n].$$

It is clear that, for every $n, m \leq k$ and every $\alpha : [n] \to [m]$, the colimit (4.2.19) is realized by $F[n]$, and under this identification, $\mathrm{sk}_k A[\alpha]$ agrees with $F[\alpha]$, so the unit of adjunction

$$F \to s.\mathrm{trunc}_k \circ \mathrm{sk}_k F$$

is an isomorphism. Dually, if all finite limits are representable in $\mathscr{C}$, the truncation functor admits a right adjoint

$$\mathrm{cosk}_k : s_k.\mathscr{C} \to s.\mathscr{C}$$

called the k-*th coskeleton functor*, and a simple inspection of the proof of *loc.cit.* yields the rule:

$$\mathrm{cosk}_k F[n] := \lim_{\varphi:[n]\to[i]} F[i]$$

where i ranges over the integers $\leq k$, and $\varphi : [n] \to [i]$ over the morphisms in Δ^o, and the transition maps are as in the foregoing (except that the downwards arrows in the commutative triangles (4.2.20) are reversed). Especially, we easily deduce that the counit of adjunction

$$s.\mathrm{trunc}_k \circ \mathrm{cosk}_k F \to F$$

is an isomorphism. Moreover, if $\mathscr{B}$ is another category with small Hom-sets, whose finite colimits (resp. finite limits) are all representable, and $\varphi : \mathscr{B} \to \mathscr{C}$ is any functor, then for any $F \in \mathrm{Ob}(s_k.\mathscr{B})$ there is a natural transformation

$$\mathrm{sk}_k(s_k.\varphi F) \to s.\varphi(\mathrm{sk}_k F) \qquad (\text{ resp. } s.\varphi(\mathrm{cosk}_k F) \to \mathrm{cosk}_k(s_k.\varphi F)\ )$$

which is an isomorphism, if φ is right exact (resp. if φ is left exact).

4.2.21. Let $\mathscr{A}$ be an abelian category, and A any object of $s.\mathscr{A}$. For every $n > 0$, set

$$d_n := \sum_{i=0}^{n} (-1)^i \cdot \partial_i \qquad A[n] \to A[n-1].$$

Directly from the simplicial identities (4.2.8) we may compute

$$\begin{aligned} d_n \circ d_{n+1} &= \sum_{i=0}^{n} \sum_{j=0}^{n+1} (-1)^{i+j} \cdot \partial_i \circ \partial_j \\ &= \sum_{i=0}^{n} \sum_{j>i}^{n+1} (-1)^{i+j} \cdot \partial_{j-1} \circ \partial_i + \sum_{i=0}^{n} \partial_i \circ \partial_i + \sum_{i=0}^{n} \sum_{j<i}^{n} (-1)^{i+j} \cdot \partial_i \circ \partial_j \\ &= \sum_{i=0}^{n} \sum_{j-1>i}^{n+1} (-1)^{i+j} \cdot \partial_{j-1} \circ \partial_i + \sum_{i=0}^{n} \sum_{j<i}^{n} (-1)^{i+j} \cdot \partial_i \circ \partial_j \\ &= 0 \end{aligned}$$

for every $n \in \mathbb{N}$, so we are led to the following :

Definition 4.2.22. Let $\mathscr{A}$ be an abelian category, and $k \in \mathbb{N}$ any integer.

(i) If $A[\bullet]$ is any simplicial object of $\mathscr{A}$, with face operators ∂_i, the *unnormalized complex associated to* $A[\bullet]$ is the complex $(A_\bullet, d_\bullet) \in \mathrm{Ob}(\mathsf{C}^{\leq 0}(\mathscr{A}))$ such that $A_n := A[n]$ for every $n \in \mathbb{N}$, and d_n is defined as in (4.2.21), for every $n > 0$.

(ii) If $A[\bullet]$ is a k-truncated simplicial object of $\mathscr{A}$, the *unnormalized complex associated to* $A[\bullet]$ as the complex $(A_\bullet, d_\bullet)$ where A_n and d_n are defined as in (i) for every $n \leq k$, and $A_n := 0$, $d_n := 0$ for every $n > k$.

(iii) If $A[\bullet]$ is as in (i) (resp. as in (ii)), the *normalized complex associated to* $A[\bullet]$ is the subcomplex $N_\bullet A$ of $A_\bullet$ such that

$$N_0 A := A[0] \quad \text{and} \quad N_n A := \bigcap_{i=1}^{n} \mathrm{Ker}\, \partial_i \qquad \text{for every } n > 0 \qquad (\text{resp. for } 0 < n \leq k).$$

So, the differential $N_n A \to N_{n-1} A$ equals ∂_0, for every $n \in \mathbb{N}$ (resp. for every $n \leq k$).

(iv) If $A[\bullet]$ is as in (i) or (ii), the *homology of* A in degree n is

$$H_n A := H_n A_\bullet \qquad \text{for every } n \in \mathbb{N}.$$

(v) Let $\varepsilon : A \to A_{-1}$ be an augmentation for $A[\bullet]$. One says that the augmented simplicial object (A, ε) is *aspherical*, if $H_n A = 0$ for every $n > 0$, and ε induces an isomorphism $H_0 A \xrightarrow{\sim} A_{-1}$.

4.2.23. Let $\mathscr{A}$ be an abelian category, and recall that $s^n.\mathscr{A}$ and $s_k.\mathscr{A}$ are both abelian categories as well, for every $n, k \in \mathbb{N}$ (remark 1.2.36(ii)); also, clearly the rule of definition 4.2.22(iii) yields natural additive functors

$$\mathsf{N}_{\mathscr{A}} : s.\mathscr{A} \to \mathsf{C}^{\leq 0}(\mathscr{A}) \qquad \mathsf{N}_{\mathscr{A},k} : s_k.\mathscr{A} \to \mathsf{C}^{[-k,0]}(\mathscr{A}) \qquad A[\bullet] \mapsto N_\bullet A \qquad \text{for every } k \in \mathbb{N}$$

and the rules of definition 4.2.22(i,ii) yield additive functors

$$\mathsf{U}_{\mathscr{A}} : s.\mathscr{A} \to \mathsf{C}^{\leq 0}(\mathscr{A}) \qquad \mathsf{U}_{\mathscr{A},k} : s_k.\mathscr{A} \to \mathsf{C}^{[-k,0]}(\mathscr{A}) \qquad A[\bullet] \mapsto A_\bullet \qquad \text{for every } k \in \mathbb{N}.$$

Remark 4.2.24. Let $\mathscr{A}$ and A be as in (4.2.21); directly from the simplicial identities (4.2.8) we may compute

$$d_n \circ \sigma_k = \sum_{i=0}^{k-1} (-1)^i \cdot \sigma_{k-1} \circ \partial_i + \sum_{i=k+2}^{n} (-1)^i \cdot \sigma_k \circ \partial_{i-1}.$$

Especially, if we let

$$D_0 := 0 \qquad \text{and} \qquad D_n A := \sum_{i=0}^{n-1} \mathrm{Im}\,(\sigma_i : A[n-1] \to A[n]) \qquad \text{for every } n > 0$$

we see that d_n restricts to a morphism $d'_n : D_n A \to D_{n-1} A$ for every $n > 0$, hence $(D_\bullet A, d'_\bullet)$ is a subcomplex of $A_\bullet$, called the *degenerate subcomplex*, and clearly we obtain an additive functor

$$\mathsf{D}_{\mathscr{A}} : s.\mathscr{A} \to \mathsf{C}(\mathscr{A}) \qquad A \mapsto D_\bullet A.$$

Proposition 4.2.25. *The natural injections induce a decomposition*

$$A_\bullet = N_\bullet A \oplus D_\bullet A \qquad \textit{in } \mathsf{C}^{\leq 0}(\mathscr{A}).$$

Proof. First, we notice that $N_n A \cap D_n A = 0$ for every $n \in \mathbb{N}$. Indeed, it suffices to check that $N_n A \cap \mathrm{Im}(\sigma_i) = 0$ for every $i = 0, \dots, n-1$, but the latter follows easily from the identity $\partial_{i+1} \circ \sigma_i = \mathbf{1}_{A[n-1]}$. To conclude the proof, it suffices to show that $N_n A + D_n A = A_n$ for every $n \in \mathbb{N}$. Indeed, set $K_0 := A_n$ and define inductively $K_i := K_{i-1} \cap \mathrm{Ker}\, \partial_i$ for every $i = 1, \dots, n$; to prove the latter assertion, it suffices to check that

$$(\mathbf{1}_{A_n} - \sigma_i \cdot \partial_i)(K_i) \subset K_{i+1} \qquad \text{for every } i = 0, \dots, n. \tag{4.2.26}$$

However, we have :

$$\partial_j \circ (\mathbf{1}_{A_n} - \sigma_i \cdot \partial_i) = \partial_j - \sigma_{i-1} \circ \partial_{i-1} \circ \partial_j \qquad \text{whenever } j < i$$

and $\partial_i \circ (\mathbf{1}_{A_n} - \sigma_i \cdot \partial_i) = \partial_i - \partial_i = 0$, whence (4.2.26). □

Example 4.2.27. (i) Let S be any simplicial set, and set $\mathbb{Z}^S := S \otimes s.\mathbb{Z}$ (notation of remark 4.2.16(iii)); we wish to give an explicit description of the complex $N_\bullet \mathbb{Z}^S$. To begin with, notice that $\mathbb{Z}^S[n]$ is the free abelian group with basis indexed by $S[n]$, for every $n \in \mathbb{N}$, and for every $x \in S[n]$ denote by $e_x \in \mathbb{Z}^S[n]$ the corresponding basis element. Next, set

$$D_n S := \bigcup_{i=0}^{n-1} \mathrm{Im}\,(\sigma_i : S[n-1] \to S[n]) \quad \text{and} \quad N_n S := S[n] \setminus D_n S \qquad \text{for every } n \in \mathbb{N}.$$

From proposition 4.2.25, we see that $N_n \mathbb{Z}^S$ can be naturally identified with the direct summand of $\mathbb{Z}^S[n]$ generated by the system $(e_x \mid x \in N_n S)$, for every $n \in \mathbb{N}$. In order to describe the differential d_n, let us define $\overline{e}_x := e_x$ if $x \in N_n S$, and $\overline{e}_x := 0$ if $x \in D_n S$. Then we may write

$$d_n(\overline{e}_x) = \sum_{i=0}^{n} (-1)^i \cdot \overline{e}_{\partial_i x} \qquad \text{for every } n \in \mathbb{N} \text{ and every } x \in N_n S.$$

The verifications are straightforward, and shall be left to the reader.

(ii) For instance, for any $i \in \mathbb{N}$ consider the simplicial set $\mathbf{\Delta}_i$ as in remark 4.2.16(iv), and define

$$K\langle i \rangle_\bullet := N_\bullet \mathbb{Z}^{\mathbf{\Delta}_i}$$

where $\mathbb{Z}^{\mathbf{\Delta}_i}$ is defined as in (i). Notice that this notation agrees with that of remark 4.1.11(ii). It is easily seen that a morphism $[n] \to [i]$ of Δ lies in $D_n \mathbf{\Delta}_i$ if and only if it is not injective; hence $N_n \mathbf{\Delta}_i$ is the set of all injective maps $\varphi : [n] \to [i]$ of Δ. We deduce a natural isomorphism

$$K\langle i \rangle_n \xrightarrow{\sim} \Lambda^{n+1}_{\mathbb{Z}} \mathbb{Z}^{\oplus i+1} \qquad \text{for every } i, n \in \mathbb{N}.$$

Namely, to any map φ as above, we assign the exterior product $e_{\varphi(0)} \wedge \cdots \wedge e_{\varphi(n)}$, where $e_0, \dots, e_i$ denotes the canonical basis of $\mathbb{Z}^{\oplus i+1}$. Under this isomorphism, the differential of $K\langle i \rangle_\bullet$ gets identified with the differential of the Koszul complex attached to the sequence $\mathbf{1}_{i+1} := (1, \dots, 1) \in \mathbb{Z}^{\oplus i+1}$ (see example 4.4.48). Summing up, we obtain natural short exact sequences

$$0 \to K\langle i \rangle_\bullet[1] \to \mathbf{K}_\bullet(\mathbf{1}_{i+1}) \to \mathbb{Z}[0] \to 0 \qquad \text{for every } i \in \mathbb{N}$$

where $[1]$ denotes the shift operator, and $\mathbb{Z}[0]$ is the complex with $\mathbb{Z}$ placed in degree zero : see (4.1.1).

(iii) Let $\mathscr{A}$ be any abelian category, A any object of $s.\mathscr{A}$ and Z any object of $\mathscr{A}$. Notice that the simplicial set $\mathrm{Hom}_{\mathscr{A}}(Z, A)$ defined in remark 4.2.17(ii) is actually a simplicial abelian group, and a direct inspection of the arguments of *loc.cit.* yields a natural isomorphism of abelian groups

$$\mathrm{Hom}_{s.\mathbb{Z}\text{-}\mathbf{Mod}}(\mathbb{Z}^{\mathbf{\Delta}_i}, \mathrm{Hom}_{\mathscr{A}}(Z, A)) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{A}}(Z, A[i]) \qquad \text{for every } i \in \mathbb{N}.$$

4.2.28. Keep the notation of remark 4.2.24, and let

$$\mathsf{j}^A_\bullet : N_\bullet A \to A_\bullet \qquad \mathsf{q}^A_\bullet : A_\bullet \to N_\bullet A$$

be respectively the injection and the projection with kernel $D_\bullet A$. Then the rules : $A \mapsto \mathsf{j}^A_\bullet$ and $A \mapsto \mathsf{q}^A_\bullet$ define natural transformations

$$\mathsf{j}_\bullet : \mathsf{N}_{\mathscr{A}} \Rightarrow \mathsf{U}_{\mathscr{A}} \qquad \mathsf{q}_\bullet : \mathsf{U}_{\mathscr{A}} \Rightarrow \mathsf{N}_{\mathscr{A}}.$$

With this notation, we may state :

Theorem 4.2.29. *With the notation of remark* (4.2.28)*, we have :*

(i) *The injection* $\mathsf{j}^A_\bullet$ *is a homotopy equivalence, and* $D_\bullet A$ *is homotopically trivial.*
(ii) *More precisely, there exist natural modifications*

$$\mathsf{j}_\bullet \circ \mathsf{q}_\bullet \rightsquigarrow \mathbf{1}_{\mathsf{U}_{\mathscr{A}}} \qquad \mathsf{q}_\bullet \circ \mathsf{j}_\bullet \rightsquigarrow \mathbf{1}_{\mathsf{N}_{\mathscr{A}}}$$

on the 2*-category* $\mathsf{C}(\mathscr{A})$ *as in example* 4.1.7(ii) *(see definition* 1.3.7*).*
(iii) *Especially, the natural map*

$$H_i N_\bullet A \to H_i A$$

is an isomorphism, for every $i \in \mathbb{N}$*.*

Proof. Clearly (ii)⇒(i), and (i)⇒(iii), by virtue of remark 4.1.3(ii). To show (ii), set $\varphi^A_0 := \mathbf{1}_{A_0}$ and $\varphi^A_n := \mathbf{1}_{A[n]} - \sigma_{n-1} \circ \partial_n : A_n \to A_n$ for every $n > 0$. We notice :

Claim 4.2.30. The system $(\varphi^A_n \mid n \in \mathbb{N})$ defines an endomorphism of $A_\bullet$, which is homotopically equivalent to $\mathbf{1}_{A_\bullet}$.

Proof of the claim. Indeed, let $s_n := (-1)^n \cdot \sigma_n : A_n \to A_{n+1}$ for every $n \in \mathbb{N}$, and define $s_n := 0$ for every $n < 0$. Using the simplicial identities (4.2.8) we compute :

$$\begin{aligned} s_{n-1} \circ d_n + d_{n+1} \circ s_n &= (-1)^{n-1} \cdot \sum_{j=0}^{n} (-1)^j \cdot (\sigma_{n-1} \circ \partial_j - \partial_j \circ \sigma_n) - \mathbf{1}_{A[n]} \\ &= \varphi^A_n + (-1)^{n-1} \cdot (\sum_{j=0}^{n-1} (-1)^j \cdot (\sigma_{n-1} \circ \partial_j - \sigma_{n-1} \circ \partial_j) - \mathbf{1}_{A[n]} \\ &= \varphi^A_n - \mathbf{1}_{[n]} \end{aligned}$$

for every $n > 0$, and for $n = 0$ we have as well $s_{-1} \circ d_0 + d_1 \circ s_0 = 0$, whence the claim. ◊

Notice that the homotopy exhibited in the proof of claim 4.2.30 is natural in A, so it already yields the first sought modification. Next, for every simplicial object B of $\mathscr{A}$, let γB and $g_B : \gamma B \to B$ be as in remark 4.2.11(i,ii), and define inductively

$$\beta_0 A := A \qquad \text{and} \qquad \beta^{n+1} A := \operatorname{Ker} g_{\beta^n A} \qquad \text{for every } n \in \mathbb{N}.$$

A simple inspection shows that

$$N_{n+i} A \subset \beta^n A[i] \qquad \text{and} \qquad \beta^n A[0] = N_n A \qquad \text{for every } i, n \in \mathbb{N}.$$

Hence, we may define a subcomplex $B^{(n)}_\bullet$ of $A_\bullet$ for every $n \in \mathbb{N}$, as follows. For $i = 0, \dots, n$ we let $B^{(n)}_i := N_i A$, and for $i > n$ we let $B^{(n)}_i := (\beta^n A)[n + i]$ (notation of (4.1.1)). The differential of $B^{(n)}_\bullet$ is of course just the restriction of that of $A_\bullet$. By construction, $N_\bullet A$ is a subcomplex of $B^{(n)}_\bullet$, for all $n \in \mathbb{N}$. Next, we define an endomorphism $f^{(n)}_\bullet$ of $B^{(n)}_\bullet$, by setting

$$f^{(n)}_i := \mathbf{1}_{B^{(n)}_i} \qquad \text{for every } i \leq n, \text{ and} \qquad f^{(n)}_i := \varphi^{\beta^n A}_i \qquad \text{for } i > n.$$

Notice that the restriction of $f^{(n)}_\bullet$ to the subcomplex $N_\bullet A$ is just the inclusion map $N_\bullet \to B^{(n)}_\bullet$. Clearly $B^{(n+1)} \subset B^{(n)}$ for every $n \in \mathbb{N}$, and we remark that $f^{(n)}_\bullet$ factors through the inclusion map $B^{(n+1)} \subset B^{(n)}$. Indeed, since $B^{(n)}_i = B^{(n+1)}_i$ for every $i \leq n$, the assertion is obvious for this range of degrees; so we have only to check that $\varphi^{\beta^n A}_\bullet$ factors through $\beta^{n+1} A_\bullet$ for every $n \in \mathbb{N}$, and an easy induction reduces to checking that $\varphi^A_\bullet$ factors through $B^{(1)}_\bullet$. But the latter assertion comes down to the identity $\partial_i \circ \varphi^A_i = 0$ for every $i > 0$, which follows easily from the simplicial identities (4.2.8).

If $(s_i \mid i \in \mathbb{N})$ is the homotopy between $\mathbf{1}_{\beta^n A_\bullet}$ and $\varphi^{\beta^n A}_\bullet$ supplied by claim 4.2.30, then we obtain a homotopy $(t^{(n)}_i \mid i \in \mathbb{N})$ between $\mathbf{1}_{B^{(n)}_\bullet}$ and $f^{(n)}_\bullet$, by setting

$$t^{(n)}_i := 0 \qquad \text{for every } i < n, \text{ and} \qquad t^{(n)}_i := s_{i-n} \qquad \text{for } i \geq n.$$

Notice that $\mathrm{Im}\, t_i^{(n)} \subset D_{i+1}A$ for every $i, n \in \mathbb{N}$. Thus, for every $n \in \mathbb{N}$, let $h_\bullet^{(n)} : B_\bullet^{(n)} \to B_\bullet^{(n+1)}$ be the morphism of complexes deduced from $f_\bullet^{(n)}$ and $j_\bullet^{(n)} : B^{(n)} \to A_\bullet$ the inclusion map, and set $q_\bullet^{(n)} := h_\bullet^{(n)} \circ \cdots \circ h_\bullet^{(0)}$; it follows easily that the composition

$$p_\bullet^{(n)} := j_\bullet^{(n+1)} \circ q_\bullet^{(n)} : A_\bullet \to A_\bullet$$

is homotopically equivalent to $\mathbf{1}_{A_\bullet}$, for every $n \in \mathbb{N}$. More precisely, a direct inspection shows that the system of morphisms

$$\tau_i^{(n)} := \sum_{k=0}^{n-1} j_\bullet^{(k+1)} \circ t_\bullet^{(k+1)} \circ q_\bullet^{(k)} \qquad \text{for every } i \in \mathbb{N}$$

provides a homotopy between $\mathbf{1}_{A_\bullet}$ and $p_\bullet^{(n)}$. Furthermore, it is clear that

$$p_i^{(n)} = p_i^{(m)} \qquad \text{and} \qquad \tau_i^{(n)} = \tau_i^{(m)} \qquad \text{for every } m \geq n \text{ and every } i \leq n$$

so, we finally get an endomorphism $p_\bullet : A_\bullet \to A_\bullet$ by setting $p_i := p_i^{(i)}$ for every $i \in \mathbb{N}$, and a homotopy $\tau_\bullet$ between $p_\bullet$ and $\mathbf{1}_{A_\bullet}$, with $\tau_i := \tau_i^{(i)}$ for every $i \in \mathbb{N}$. By construction, $p_\bullet$ factors through $\mathsf{j}_\bullet^A$, and moreover, the restriction of $p_\bullet$ to the subcomplex $N_\bullet A$ is just $\mathsf{j}_\bullet^A$, so $\mathsf{j}_\bullet^A$ is a homotopy equivalence, via natural homotopies, as stated.

Lastly, notice that $\mathrm{Im}\, \tau_i \subset D_{i+1}A$ for every $i \in \mathbb{N}$, from which it follows easily that $p_i(D_iA) \subset D_iA$ for every $i \in \mathbb{N}$, and then the foregoing implies that $\mathrm{Ker}\, p_\bullet = D_\bullet A$. We conclude that $D_\bullet A$ is homotopically trivial, as stated. □

4.2.31. By iterating U, we get a functor from bisimplicial objects to double complexes

$$\mathsf{U}^2_{\mathscr{A}} : s^2.\mathscr{A} \xrightarrow{s.\mathsf{U}_{\mathscr{A}}} s.\mathsf{C}(\mathscr{A}) \xrightarrow{\mathsf{U}_{\mathsf{C}(\mathscr{A})}} \mathsf{C}(\mathsf{C}(\mathscr{A})) \qquad A[\bullet, \bullet] \mapsto A_{\bullet\bullet}$$

and notice that this functor is naturally isomorphic to the functor

$$s^2.\mathscr{A} \xrightarrow{\mathsf{U}_{s.\mathscr{A}}} \mathsf{C}(s.\mathscr{A}) \xrightarrow{\mathsf{C}(\mathsf{U}_{\mathscr{A}})} \mathsf{C}(\mathsf{C}(\mathscr{A})).$$

In the same vein, theorem 4.2.29 yields natural decompositions of additive functors :

$$\mathsf{U}^2_{\mathscr{A}} = (\mathsf{C}(\mathsf{N}_{\mathscr{A}}) \oplus \mathsf{C}(\mathsf{D}_{\mathscr{A}})) \circ (\mathsf{N}_{s.\mathscr{A}} \oplus \mathsf{D}_{s.\mathscr{A}}) \tag{4.2.32}$$

and if we let

$$\mathsf{N}^2_{\mathscr{A}} := \mathsf{C}(\mathsf{N}_{\mathscr{A}}) \circ \mathsf{N}_{s.\mathscr{A}} : s^2.\mathscr{A} \to \mathsf{C}(\mathsf{C}(\mathscr{A})) \qquad A \mapsto N_{\bullet\bullet}A$$

the natural morphism

$$\mathsf{i}_\bullet^A : \mathrm{Tot}\, N_{\bullet\bullet}A \to \mathrm{Tot}\, A_{\bullet\bullet}$$

is a homotopy equivalence.

4.2.33. We wish next to exhibit two natural transformations

$$A_\bullet^\Delta \xrightarrow{\mathsf{AW}_\bullet^A} \mathrm{Tot} A_{\bullet\bullet} \xrightarrow{\mathsf{Sh}_\bullet^A} A_\bullet^\Delta \qquad \text{for every } A \in \mathrm{Ob}(s^2.\mathscr{A}).$$

Namely, for every $n \in \mathbb{N}$, define :

- Sh_n^A as the sum, for all $p, q \in \mathbb{N}$ such that $p + q = n$, of the *shuffle maps*

$$\mathsf{Sh}_{p,q}^A := \sum_{\mu\nu} \varepsilon_{\mu\nu} \cdot A[\eta_{\nu_1} \circ \cdots \circ \eta_{\nu_q}, \eta_{\mu_1} \circ \cdots \circ \eta_{\mu_p}] : A_{p,q} \to A_{n,n}$$

  where the sum ranges over the *shuffle permutations* (μ, ν) of type (p, q) of the set $\{0, \dots, p + q - 1\}$ (these are the permutations described in [36, §4.3.15]), and $\varepsilon_{\mu\nu}$ is the sign of the permutation (μ, ν)
- AW_n^A as the sum, for all $p, q \in \mathbb{N}$ such that $p + q = n$, of the *Alexander-Whitney maps*

$$\mathsf{AW}_{p,q}^A := A[\varepsilon_{p,0}^{q\vee}, \varepsilon_{q,0}^p] : A_{n,n} \to A_{p,q}$$

(notation of example 4.2.6(ii)). Clearly, for every $p,q\in\mathbb{N}$ the rule $A\mapsto A_{p,q}$ defines a functor

$$\bullet[p,q] : s^2.\mathscr{A}\to\mathscr{A}$$

and the maps $\mathsf{Sh}^A_{p,q}$ and $\mathsf{AW}^A_{p,q}$ yield natural transformations

$$\mathsf{Sh}_{p,q} : \bullet[p,q]\Rightarrow\bullet[p+q,p+q] \qquad \mathsf{AW}_{p,q} : \bullet[p+q,p+q]\Rightarrow\bullet[p,q].$$

Proposition 4.2.34. *With the notation of* (4.2.23)*, the sequence* $(\mathsf{Sh}^A_n\mid n\in\mathbb{N})$ *defines a morphism of chain complexes*

$$\mathsf{Sh}^A_\bullet : \mathrm{Tot}A_{\bullet\bullet}\to A^\Delta_\bullet \qquad \textit{for every } A\in\mathrm{Ob}(s^2.\mathscr{A}).$$

Proof. For any bisimplicial object A of $\mathscr{A}$, set $A_{-1,q}=A_{p,-1}:=0$ for every $p,q\in\mathbb{Z}$, and let

$$A[\varepsilon_0,\mathbf{1}_{[q]}] : A_{0,q}\to A_{-1,q} \qquad A[\mathbf{1}_{[p]},\varepsilon_0] : A_{p,0}\to A_{p,-1} \qquad A[\varepsilon_0,\varepsilon_0] : A_{0,0}\to A_{-1,-1}$$

be the zero maps; likewise, let $\mathsf{Sh}^A_{-1,q} : A_{-1,q}\to A_{q-1,q-1}$ and $\mathsf{Sh}^A_{p,-1} : A_{p,-1}\to A_{p-1,p-1}$ be the zero maps, and notice that also $\mathsf{Sh}^A_{0,0}$ is the zero map. We define

$$d^{A,h}_{p,q} := \sum_{i=0}^{p}(-1)^i\cdot A[\varepsilon_i,\mathbf{1}_{[q]}] : A_{p,q}\to A_{p-1,q} \qquad d^{A,v}_{p,q} := \sum_{i=0}^{q}(-1)^j\cdot A[\mathbf{1}_{[p]},\varepsilon_i] : A_{p,q}\to A_{p,q-1}$$

so, for every $n\in\mathbb{N}$ the differential d_n in degree n of $\mathrm{Tot}A_{\bullet\bullet}$ is the sum of the maps

$$d^{A,h}_{p,q}+(-1)^p\cdot d^{A,v}_{p,q} : A_{p,q}\to A_{p-1,q}\oplus A_{p,q-1} \qquad \text{for all } p,q\in\mathbb{N} \text{ such that } p+q=n$$

whereas the differential of $A^\Delta_\bullet$ is the morphism

$$d^A_n := \sum_{i=1}^{n} A[\varepsilon_i,\varepsilon_i] : A_{n,n}\to A_{n-1,n-1}$$

and we have to check the identity

$$d^A_{p+q}\circ\mathsf{Sh}^A_{p,q}=\mathsf{Sh}^A_{p-1,q}\circ d^{A,h}_{p,q}+(-1)^p\cdot\mathsf{Sh}^A_{p,q-1}\circ d^{A,v}_{p,q} \qquad \text{for all } p,q\in\mathbb{N}. \tag{4.2.35}$$

Now, we set $B:=\gamma_1 A$, $C:=\gamma_2 A$, $D:=\gamma_2 B$ (notation of (4.2.15)), but for the purpose of this proof, we shall modify the differentials of the double complexes $B_{\bullet\bullet}$ and $C_{\bullet\bullet}$ in certain low degrees : namely, we define

$$d^{B,h}_{0,q}:=A[\varepsilon_0,\mathbf{1}_{[q]}] \qquad d^{C,v}_{p,0}:=A[\mathbf{1}_{[p]},\varepsilon_0] \qquad \text{for every } p,q\in\mathbb{N}.$$

Claim 4.2.36. With the foregoing notation, the following holds :

(i) $\mathsf{Sh}^A_{p,q}=(-1)^q\cdot\mathsf{Sh}^D_{p-1,q}\circ A[\mathbf{1}_{[p]},\eta_q]+\mathsf{Sh}^D_{p,q-1}\circ A[\eta_p,\mathbf{1}_{[q]}]$ for every $p,q\in\mathbb{N}$.

(ii) $A[\varepsilon_{p+q-1},\varepsilon_{p+q-1}]\circ\mathsf{Sh}^D_{p-1,q-1}=\mathsf{Sh}^A_{p-1,q-1}\circ A[\varepsilon_p,\varepsilon_q]$ for every $p,q>0$.

(iii) $A[\varepsilon_{p+q},\varepsilon_{p+q}]\circ\mathsf{Sh}^A_{p,q}=(-1)^q\cdot\mathsf{Sh}^A_{p-1,q}\circ A[\varepsilon_p,\mathbf{1}_{[q]}]+\mathsf{Sh}^A_{p,q-1}\circ A[\mathbf{1}_{[p]},\varepsilon_q]$ for every $p,q\in\mathbb{N}$.

Proof of the claim. (i): First, we notice the identities :

$$\begin{aligned} A[\mathbf{1}_{[p+q]},\eta_{p+q}]\circ\mathsf{Sh}^A_{p,q} &= \mathsf{Sh}^C_{p,q}\circ A[\mathbf{1}_{[p]},\eta_q] \\ A[\eta_{p+q},\mathbf{1}_{[p+q]}]\circ\mathsf{Sh}^A_{p,q} &= \mathsf{Sh}^B_{p,q}\circ A[\eta_p,\mathbf{1}_{[q]}]. \end{aligned} \qquad \text{for every } p,q\in\mathbb{N} \tag{4.2.37}$$

that are deduced from the identities

$$\begin{aligned} \eta_{\mu_1}\circ\cdots\circ\eta_{\mu_p}\circ\eta_{p+q} &= \eta_q\circ\eta_{\mu_1}\circ\cdots\circ\eta_{\mu_p} \\ \eta_{\nu_1}\circ\cdots\circ\eta_{\nu_q}\circ\eta_{p+q} &= \eta_p\circ\eta_{\nu_1}\circ\cdots\circ\eta_{\nu_q} \end{aligned}$$

which in turn follow from the simplicial identities for degeneracy maps. By applying the first (resp. second) identity (4.2.37) with A replaced by B (resp. by C) and p replaced by $p-1$ (resp. with q replaced by $q-1$) we get :

(4.2.38)
$$\begin{aligned} A[\mathbf{1}_{[p+q]},\eta_{p+q-1}]\circ\mathsf{Sh}^B_{p-1,q} &= \mathsf{Sh}^D_{p-1,q}\circ A[\mathbf{1}_{[p]},\eta_q] \\ A[\eta_{p+q-1},\mathbf{1}_{[p+q]}]\circ\mathsf{Sh}^C_{p,q-1} &= \mathsf{Sh}^D_{p,q-1}\circ A[\eta_p,\mathbf{1}_{[q]}]. \end{aligned} \qquad \text{for every } p,q\in\mathbb{N}.$$

Now, suppose first that $p,q>0$, and let (μ,ν) be any (p,q)-shuffle of $\{0,\dots,p+q-1\}$; then either $\mu_p=p+q-1$ or $\nu_q=p+q-1$. In the first (resp. second) case, after removing μ_p (resp. ν_q) we get a $(p-1,q)$-shuffle (resp. a $(p,q-1)$-shuffle) $(\overline{\mu},\overline{\nu})$ with

$$\varepsilon_{\overline{\mu\nu}}=(-1)^q\cdot\varepsilon_{\mu\nu} \qquad (\text{resp. } \varepsilon_{\overline{\mu\nu}}=\varepsilon_{\mu\nu}).$$

However, the first (resp. second) left-hand side of (4.2.38) contains precisely all the terms of the first (resp. second) type occurring in the definition of $\mathsf{Sh}^A_{p,q}$, so we get (i) in this case. The cases where either $p=0$ or $q=0$ can be dealt with by a similar, but simpler, argument.

(ii) follows by naturality of $\mathsf{Sh}_{p-1,q-1}$, applied to the morphism $g^{(1)}_A\circ g^{(2)}_B:D\to A$ from (4.2.15).

(iii): The case where $p=q=0$ is obvious, and the other cases follow by composing both sides of (i) with $A[\varepsilon_{p+q},\varepsilon_{p+q}]$, applying (ii), and recalling that $\eta_q\circ\varepsilon_q=\mathbf{1}_{[q]}$: details left to the reader. ♢

Now, a simple inspection shows that (4.2.35) follows from claim 4.2.36(iii) and the following

Claim 4.2.39. $d^D_{p+q-1}\circ\mathsf{Sh}^A_{p,q}=\mathsf{Sh}^A_{p-1,q}\circ d^{B,h}_{p-1,q}+(-1)^p\cdot\mathsf{Sh}^A_{p,q-1}\circ d^{C,v}_{p,q-1}$ for every $p,q\in\mathbb{N}$.

Proof of the claim. Consider first the case where $p=0$. If $q\leq 1$, it is easily seen that both sides of the stated identity vanish; if $q\geq 2$, the left-hand side is

$$d^D_{q-1}\circ A[\eta_0\circ\cdots\circ\eta_{q-1},\mathbf{1}_{[q]}]=\sum_{i=0}^{q-1}(-1)^i\cdot A[\eta_0\circ\cdots\circ\eta_{q-1}\circ\varepsilon_i,\varepsilon_i]$$

and the right-hand side is

$$A[\eta_0\circ\cdots\circ\eta_{q-2},\mathbf{1}_{[q]}]\circ d^{C,v}_{0,q-1}=\sum_{i=0}^{q-1}(-1)^i\cdot A[\eta_0\circ\cdots\circ\eta_{q-2},\varepsilon_i]$$

so in this case the assertion comes down to the obvious identity

$$\eta_0\circ\cdots\circ\eta_{q-1}\circ\varepsilon_i=\eta_0\circ\cdots\circ\eta_{q-2}:[q-2]\to[0] \qquad \text{for every } i=0,\dots,q-1.$$

Likewise we deal with the case where $q=0$. It follows already that (4.2.35) holds for these values of (p,q), and for every bisimplicial object A of $\mathscr{A}$; especially, we get

(4.2.40)
$$d^D_q\circ\mathsf{Sh}^D_{0,q}=\mathsf{Sh}^D_{0,q-1}\circ d^{D,v}_{0,q} \qquad \text{for every } q\in\mathbb{N}.$$

Next, for $p=q=1$, a direct computation shows that both sides equal

$$A[\varepsilon_0\circ\eta_0,\mathbf{1}_{[1]}]-A[\mathbf{1}_{[1]},\varepsilon_0\circ\eta_0]:A_{1,1}\to A_{1,1}.$$

We prove now, by induction on q, that the assertion holds for $p=1$. This is already known for $q\leq 1$, so suppose that $r>1$, and that the assertion holds for $p=1$ and every $q<r$. The latter implies that also (4.2.35) holds for these values of (p,q), and for every bisimplicial object A of $\mathscr{A}$; especially, we get

(4.2.41)
$$d^D_r\circ\mathsf{Sh}^D_{1,r-1}=\mathsf{Sh}^D_{0,r-1}\circ d^{D,h}_{1,r-1}-\mathsf{Sh}^D_{1,r-2}\circ d^{D,v}_{1,r-1}.$$

On the other hand, claim 4.2.36(i) says that

$$d^D_r\circ\mathsf{Sh}^A_{1,r}=(-1)^r\cdot d^D_r\circ\mathsf{Sh}^D_{0,r}\circ A[\mathbf{1}_{[1]},\eta_r]+d^D_r\circ\mathsf{Sh}^D_{1,r-1}\circ A[\eta_1,\mathbf{1}_{[r]}]$$

Combining with (4.2.40) and (4.2.41) we obtain

$$d^D_r \circ \mathsf{Sh}^A_{1,r} = (-1)^r \cdot \mathsf{Sh}^D_{0,r-1} \circ d^{D,v}_{0,r} \circ A[\mathbf{1}_{[1]}, \eta_r] + (\mathsf{Sh}^D_{0,r-1} \circ d^{D,h}_{1,r-1} - \mathsf{Sh}^D_{1,r-2} \circ d^{D,v}_{1,r-1}) \circ A[\eta_1, \mathbf{1}_{[r]}].$$

However, we have

$$d^{D,h}_{1,r-1} \circ A[\eta_1, \mathbf{1}_{[r]}] = A[\eta_1 \circ \varepsilon_0, \mathbf{1}_{[r]}] - A[\mathbf{1}_{[1]}, \mathbf{1}_{[r]}] = A[\varepsilon_0 \circ \eta_0, \mathbf{1}_{[r]}] - \mathbf{1}_{A[1,r]}$$

so we can rewrite $d^D_r \circ \mathsf{Sh}^A_{1,r} = \varphi_1 + \varphi_2 - \varphi_3$, where

$$\begin{aligned}
\varphi_1 &:= \mathsf{Sh}^D_{0,r-1} \circ ((-1)^r \cdot d^{D,v}_{0,r} \circ A[\mathbf{1}_{[1]}, \eta_r] - \mathbf{1}_{A[1,r]}) = (-1)^r \cdot \mathsf{Sh}^D_{0,r-1} \circ A[\mathbf{1}_{[1]}, \eta_{r-1}] \circ d^{C,v}_{1,r-1} \\
\varphi_2 &:= \mathsf{Sh}^D_{0,r-1} \circ A[\varepsilon_0 \circ \eta_0, \mathbf{1}_{[r]}] \\
\varphi_3 &:= \mathsf{Sh}^D_{1,r-2} \circ d^{D,v}_{1,r-1} \circ A[\eta_1, \mathbf{1}_{[r]}].
\end{aligned}$$

On the other hand, using claim 4.2.36(i), we can compute

$$\begin{aligned}
\mathsf{Sh}^A_{0,r} \circ d^{B,h}_{0,r-1} &= \mathsf{Sh}^D_{0,r-1} \circ A[\eta_0, \mathbf{1}_{[r]}] \circ d^{B,h}_{0,r-1} = \varphi_2 \\
\mathsf{Sh}^A_{1,r-1} \circ d^{C,v}_{1,r-1} &= ((-1)^{r-1} \cdot \mathsf{Sh}^D_{0,r-1} \circ A[\mathbf{1}_{[1]}, \eta_{r-1}] + \mathsf{Sh}^D_{1,r-2} \circ A[\eta_1, \mathbf{1}_{[r-1]}]) \circ d^{C,v}_{1,r-1} \\
&= \varphi_3 - \varphi_1
\end{aligned}$$

which shows that the claim holds for $p = 1$ and $q = r$, and concludes the induction.

Lastly, we prove the claim for every $p, q \in \mathbb{N}$, by induction on $p+q$; notice that the foregoing already shows that the assertion holds whenever $p + q \leq 2$. Thus, let $r > 2$, and suppose that the claim is already known for every pair (p, q) such that $p+q < r$; then also (4.2.35) holds for such values of p and q, and especially we get

$$\text{(4.2.42)} \qquad d^D_{p+q} \circ \mathsf{Sh}^D_{p,q} = \mathsf{Sh}^D_{p-1,q} \circ d^{D,h}_{p,q} + (-1)^p \cdot \mathsf{Sh}^D_{p,q-1} \circ d^{D,v}_{p,q} \qquad \text{whenever } p+q < r.$$

Let (p', q') be a pair such that $p' + q' = r$; combining (4.2.42) with claim 4.2.36(i), we get

$$\begin{aligned}
d^D_{p'+q'-1} \circ \mathsf{Sh}^A_{p',q'} = {} & (-1)^{q'} \cdot (\mathsf{Sh}^D_{p'-2,q'} \circ d^{D,h}_{p'-1,q'} - (-1)^{p'} \cdot \mathsf{Sh}^D_{p'-1,q'-1} \circ d^{D,v}_{p'-1,q'}) \circ A[\mathbf{1}_{[p']}, \eta_{q'}] \\
& + (\mathsf{Sh}^D_{p'-1,q'-1} \circ d^{D,h}_{p',q'-1} + (-1)^{p'} \cdot \mathsf{Sh}^D_{p',q'-2} \circ d^{D,v}_{p',q'-1}) \circ A[\eta_{p'}, \mathbf{1}_{[q']}].
\end{aligned}$$

On the other hand, after noticing that

$$\begin{aligned}
d^{D,h}_{p',q'-1} \circ A[\eta_{p'}, \mathbf{1}_{[q']}] - (-1)^{p'} \cdot \mathbf{1}_{A[p',q']} &= A[\eta_{p'-1}, \mathbf{1}_{[q']}] \circ d^{B,h}_{p'-1,q'} \\
d^{D,v}_{p'-1,q'} \circ A[\mathbf{1}_{[p']}, \eta_{q'}] - (-1)^{q'} \cdot \mathbf{1}_{A[p',q']} &= A[\mathbf{1}_{[p']}, \eta_{q'-1}] \circ d^{C,v}_{p',q'-1}
\end{aligned}$$

we may apply claim 4.2.36(i), to compute

$$\begin{aligned}
\mathsf{Sh}^A_{p'-1,q'} \circ d^{B,h}_{p'-1,q'} &= ((-1)^{q'} \cdot \mathsf{Sh}^D_{p'-2,q'} \circ A[\mathbf{1}_{[p'-1]}, \eta_{q'}] + \mathsf{Sh}^D_{p'-1,q'-1} \circ A[\eta_{p'-1}, \mathbf{1}_{[q']}]) \circ d^{B,h}_{p'-1,q'} \\
&= (-1)^{q'} \cdot \mathsf{Sh}^D_{p'-2,q'} \circ d^{D,h}_{p'-1,q'} \circ A[\mathbf{1}_{[p']}, \eta_{q'}] - (-1)^{p'} \cdot \mathsf{Sh}^D_{p'-1,q'-1} \\
&\quad + \mathsf{Sh}^D_{p'-1,q'-1} \circ d^{D,h}_{p',q'-1} \circ A[\eta_{p'}, \mathbf{1}_{[q']}]
\end{aligned}$$

and

$$\begin{aligned}
\mathsf{Sh}^A_{p',q'-1} \circ d^{C,v}_{p',q'-1} &= ((-1)^{q'-1} \cdot \mathsf{Sh}^D_{p'-1,q'-1} \circ A[\mathbf{1}_{[p']}, \eta_{q'-1}] \\
&\qquad + \mathsf{Sh}^D_{p',q'-2} \circ A[\eta_{p'}, \mathbf{1}_{[q'-1]}]) \circ d^{C,v}_{p',q'-1} \\
&= (-1)^{q'-1} \cdot \mathsf{Sh}^D_{p'-1,q'-1} \circ d^{D,v}_{p'-1,q'} \circ A[\mathbf{1}_{[p']}, \eta_{q'}] + \mathsf{Sh}^D_{p'-1,q'-1} \\
&\qquad + \mathsf{Sh}^D_{p',q'-2} \circ d^{D,v}_{p',q'-1} \circ A[\eta_{p'}, \mathbf{1}_{[q']}].
\end{aligned}$$

Finally, the sought identity for the pair (p', q') follows by comparing the last two identities with the previous one for $d^D_{p'+q'-1} \circ \mathsf{Sh}^A_{p',q'}$, and this concludes the proof of the inductive step. $\square$

Proposition 4.2.43. *With the notation of* (4.2.23)*, the sequence* $(\mathsf{AW}^A_n \mid n \in \mathbb{N})$ *defines a morphism of chain complexes*

$$\mathsf{AW}^A_\bullet : A^\Delta_\bullet \to \mathrm{Tot} A_{\bullet\bullet} \qquad \textit{for every } A \in \mathrm{Ob}(s^2.\mathscr{A}).$$

Proof. Denote by $d^T_\bullet$ the differential of $\mathrm{Tot}A_{\bullet\bullet}$, and keep the notation of the proof of proposition 4.2.34; we have to check the identity

$$\mathsf{AW}_{n-1} \circ d^A_n = d^T_n \circ \mathsf{AW}_n \qquad \text{for every } n \in \mathbb{N}. \tag{4.2.44}$$

However, say that $p, q \in \mathbb{N}$ and $p + q = n$; the projection of the right-hand side of (4.2.44) on the direct summand $A_{p-1,q}$ of $(\mathrm{Tot}A_{\bullet\bullet})_{n-1}$ equals

$$d^{A,h}_{p,q} \circ \mathsf{AW}_{p,q} + (-1)^{p-1} \cdot d^{A,v}_{p-1,q+1} \circ \mathsf{AW}_{p-1,q+1}. \tag{4.2.45}$$

Unwinding the definition, (4.2.45) is found to be

$$\sum_{i=0}^{p} (-1)^i \cdot A[\varepsilon^{q\vee}_{p,0} \circ \varepsilon_i, \varepsilon^p_{q,0}] - (-1)^p \cdot \sum_{i=0}^{q+1} (-1)^i \cdot A[\varepsilon^{q+1\vee}_{p-1,0}, \varepsilon^{p-1}_{q+1,0} \circ \varepsilon_i]$$

which we may rewrite as

$$\sum_{i=0}^{p-1} (-1)^i \cdot A[\varepsilon^{q\vee}_{p,0} \circ \varepsilon_i, \varepsilon^p_{q,0}] + \sum_{i=p}^{p+q} (-1)^i \cdot A[\varepsilon^{q+1\vee}_{p-1,0}, \varepsilon^{p-1}_{q+1,0} \circ \varepsilon_{i-p+1}]. \tag{4.2.46}$$

On the other hand, the projection of the left-hand side of (4.2.44) onto $A_{p-1,q}$ equals

$$\sum_{i=0}^{n} (-1)^i \cdot A[\varepsilon_i \circ \varepsilon^{q\vee}_{p-1,0}, \varepsilon_i \circ \varepsilon^{p-1}_{q,r}].$$

To compare the latter with (4.2.46), it suffices to remark that

$$\varepsilon_i \circ \varepsilon^{q\vee}_{p-1,0} = \begin{cases} \varepsilon^{q\vee}_{p,0} \circ \varepsilon_i & \text{if } i < p \\ \varepsilon^{q+1\vee}_{p-1,0} & \text{if } i \geq p \end{cases} \qquad \text{and} \qquad \varepsilon_i \circ \varepsilon^{p-1}_{q,r} = \begin{cases} \varepsilon^{p}_{q,0} & \text{if } i < p \\ \varepsilon^{p-1}_{q+1,0} \circ \varepsilon_i & \text{if } i \geq p \end{cases}$$

which are all deduced from the simplicial identities for the ε_i. The proposition follows. $\square$

4.2.47. Propositions 4.2.34 and 4.2.43 yield natural transformations of functors

$$\mathrm{Tot}\, \mathsf{U}^2_{\mathscr{A}} \underset{\mathsf{AW}_\bullet}{\overset{\mathsf{Sh}_\bullet}{\rightleftarrows}} \mathsf{U}_{\mathscr{A}} \circ \Delta_{\mathscr{A}}.$$

Now, denote by $\mathsf{h}_{\mathscr{A}} : \mathsf{C}(\mathscr{A}) \to \mathsf{Hot}(\mathscr{A})$ the natural functor (this induces the identity on the objects, and the projection on the group of morphisms); there follow natural transformations

$$\mathsf{h}_{\mathscr{A}} \circ \mathrm{Tot}\, \mathsf{U}^2_{\mathscr{A}} \underset{\mathsf{h}_{\mathscr{A}} * \mathsf{AW}_\bullet}{\overset{\mathsf{h}_{\mathscr{A}} * \mathsf{Sh}_\bullet}{\rightleftarrows}} \mathsf{h}_{\mathscr{A}} \circ \mathsf{U}_{\mathscr{A}} \circ \Delta_{\mathscr{A}}. \qquad \text{between functors } s^2.\mathscr{A} \to \mathsf{Hot}(\mathscr{A}).$$

Theorem 4.2.48 (Eilenberg-Zilber-Cartier)**.** *With the notation of* (4.2.47)*, we have :*

(i) $\mathsf{h}_{\mathscr{A}} * \mathsf{AW}_\bullet$ *and* $\mathsf{h}_{\mathscr{A}} * \mathsf{Sh}_\bullet$ *are mutually inverse isomorphisms of functors.*
(ii) *More precisely, there exist natural modifications (see definition* 1.3.7*)*

$$\mathsf{AW}_\bullet \circ \mathsf{Sh}_\bullet \rightsquigarrow \mathbf{1}_{\mathrm{Tot}\, \mathsf{U}^2_{\mathscr{A}}} \qquad \mathsf{Sh}_\bullet \circ \mathsf{AW}_\bullet \rightsquigarrow \mathbf{1}_{\mathsf{U}_{\mathscr{A}} \circ \Delta_{\mathscr{A}}}.$$

Proof. Let $\mathsf{p}^A_\bullet : \mathrm{Tot}\, A_{\bullet\bullet} \to \mathrm{Tot}\, N_{\bullet\bullet}A$ be the projection whose kernel is the sum of the remaining three direct summands in the decomposition $\mathrm{Tot}\,(4.2.32)$; explicitly, in each degree $n \in \mathbb{N}$, this kernel is the sum of the subobjects $\mathrm{Im}\, A[\eta_i, \mathbf{1}_{[q]}]$ and $\mathrm{Im}\, A[\mathbf{1}_{[p]}, \eta_j]$ of $A_{p,q}$, for every $i = 0, \dots, p-1$, $j = 0, \dots, q-1$, and every $p, q \in \mathbb{N}$ such that $p + q = n$. Likewise, let $\mathsf{j}^A_\bullet : N_\bullet A^\Delta \to A^\Delta_\bullet$ and $\mathsf{q}^A_\bullet : A^\Delta_\bullet \to N_\bullet A^\Delta$ be as in (4.2.28); theorem 4.2.29 and the discussion of (4.2.31) show that the pairs $(\mathsf{p}^A_\bullet, \mathsf{i}^A_\bullet)$ and $(\mathsf{j}^A_\bullet, \mathsf{q}^A_\bullet)$ induce mutually inverse isomorphisms in $\mathsf{Hot}(\mathscr{A})$, so it suffices to check that the same holds for the compositions

$$\overline{\mathsf{Sh}}^A_\bullet := \mathsf{q}^A_\bullet \circ \mathsf{Sh}^A_\bullet \circ \mathsf{i}^A_\bullet : \mathrm{Tot}\, N_{\bullet\bullet}A \to N_\bullet A^\Delta \qquad \overline{\mathsf{AW}}^A_\bullet := \mathsf{p}^A_\bullet \circ \mathsf{AW}^A_\bullet \circ \mathsf{j}^A_\bullet : N_\bullet A^\Delta \to \mathrm{Tot}\, N_{\bullet\bullet}A$$

for every bisimplicial object A of $\mathscr{A}$. However, we have

Claim 4.2.49. $\mathsf{p}^A_\bullet \circ \mathsf{AW}^A_\bullet \circ \mathsf{Sh}^A_\bullet \circ \mathsf{i}^A_\bullet = \mathbf{1}_{\mathrm{Tot}\, N_{\bullet\bullet}A}$.

Proof of the claim. More precisely, say that $p + q = n$, consider the composition

$$f_{p,q} : A_{p,q} \xrightarrow{\mathsf{Sh}^A_{p,q}} A_{p+q,p+q} \xrightarrow{\mathsf{AW}^A_n} (\mathrm{Tot}\, A_{\bullet\bullet})_n$$

and let $g_{p,q} : A_{p,q} \to (\mathrm{Tot}\, A_{\bullet\bullet})_n$ be the inclusion map; we shall show that $f_{p,q} - g_{p,q} \subset \mathrm{Ker}\, \mathsf{p}^A_n$. Indeed, we have

$$f_{p,q} = \sum_{i=0}^{n} \sum_{(\mu,\nu)} \varepsilon_{\mu\nu} \cdot A[\eta_{\nu_1} \circ \cdots \circ \eta_{\nu_q} \circ \varepsilon^{n-i\vee}_{i,0}, \eta_{\mu_1} \circ \cdots \circ \eta_{\mu_p} \circ \varepsilon^i_{n-i,0}]$$

(notation of (4.2.33), and $\varepsilon^i_{n-i,0}$ is defined as in the proof of proposition 4.2.43). However, if $i < p$, then $\eta_{\mu_1} \circ \cdots \circ \eta_{\mu_p} \circ \varepsilon^i_{n-i,0}$ is of the form $\tau \circ \eta_{\mu_p}$ for some $\tau : [p+q-i-1] \to [q]$, so the corresponding term does lie in $\mathrm{Ker}\, \mathsf{p}^A_n$. If $i > p$, then $\eta_{\nu_1} \circ \cdots \circ \eta_{\nu_q} \circ \varepsilon^{n-i\vee}_{i,0}$ is of the form $\tau \circ \eta_k$ for some $k \leq \nu_q$ and some $\tau : [i-1] \to [p]$, so this term likewise lies in $\mathrm{Ker}\, \mathsf{p}^A_n$. It remains to consider the terms with $i = p$; however, a simple inspection shows that $\eta_{\nu_1} \circ \cdots \circ \eta_{\nu_q} \circ \varepsilon^{n-p\vee}_{p,0}$ is of the same form as above, unless ν_1 is either $p-1$ or p (details left to the reader); furthermore, if $\nu_1 = p - 1$, then $\mu_p \geq p$, in which case $\eta_{\mu_1} \circ \cdots \circ \eta_{\mu_p} \circ \varepsilon^i_{n-i,0}$ is of the form described above. In all these cases, the corresponding term again lies in $\mathrm{Ker}\, \mathsf{p}^A_n$. So, it remains only to consider the single case where (μ, ν) is the identity permutation, whose sign equals 1; in this case, the corresponding term is none else than $A[\mathbf{1}_{[p]}, \mathbf{1}_{[q]}]$, whence the claim. $\Diamond$

Claim 4.2.49 and theorem 4.2.29(ii) already yield the existence of the sought modification $\mathsf{AW}_\bullet \circ \mathsf{Sh}_\bullet \rightsquigarrow \mathbf{1}_{\mathrm{Tot}\, \mathsf{U}^2_{\mathscr{A}}}$. Next we define, for every $n \in \mathbb{N}$, a natural transformation

$$s_n : \bullet[n] \circ \Delta_{\mathscr{A}} \Rightarrow \bullet[n+1] \circ \Delta_{\mathscr{A}}$$

(notation of (4.2.4); so s^A_n is a morphism $A[n,n] \to A[n+1,n+1]$ for every object A of $s^2.\mathscr{A}$). The construction is by induction on n : for $n = 0$ we let $s^A_0 : A[0,0] \to A[1,1]$ be the zero morphism; for $n > 0$ we set

$$s^A_n := \mathsf{Sh}^{A'}_n \circ \mathsf{AW}^{A'}_n \circ A[\eta_0, \eta_0] - s^{A'}_{n-1} \qquad \text{where } A' := (\gamma_2 \circ \gamma_1(A^\vee))^\vee$$

(notation of (4.2.15) and (4.2.2) : explicitly, we have $A'[p,q] := A[p+1, q+1]$ for every $p, q \in \mathbb{N}$, and $A'[\varepsilon_i, \varepsilon_j] := A[\varepsilon_{i+1}, \varepsilon_{j+1}]$, and likewise for $A'[\varepsilon_i, \eta_j]$, $A'[\eta_i, \varepsilon_j]$ and $A'[\eta_i, \eta_j]$, for every face and degeneracy map of Δ). We have

Claim 4.2.50. The system $(\mathsf{q}^A_{n+1} \circ s^A_n \circ \mathsf{j}^A_n \mid n \in \mathbb{N})$ is a homotopy $\mathsf{q}^A_\bullet \circ \mathsf{Sh}^A_\bullet \circ \mathsf{AW}^A_\bullet \circ \mathsf{j}^A_\bullet \Rightarrow \mathbf{1}_{N_\bullet A^\Delta}$.

Proof of the claim. Denote by $d^A_\bullet$ the differential of $A^\Delta_\bullet$, and let $d^A_0 : A^\Delta_0 \to 0$ and $s^A_{-1} : 0 \to A^\Delta_0$ be the zero morphisms. We check, more precisely, that

$$\mathsf{q}^A_n \circ (d^A_{n+1} \circ s^A_n + s^A_{n-1} \circ d^A_n) = \mathsf{q}^A_n \circ \mathsf{Sh}^A_n \circ \mathsf{AW}^A_n - \mathsf{q}^A_n \qquad \text{for every } n \in \mathbb{N}.$$

We argue by induction on n, and the assertion is clear for $n = 0$. For $n = 1$, notice that

$$\mathsf{Sh}^A_1 \circ \mathsf{AW}^A_1 = A[\varepsilon_1 \circ \eta_0, \mathbf{1}_{[1]}] + A[\mathbf{1}_{[1]}, \varepsilon_0 \circ \eta_0] : A_{1,1} \to A_{1,1}.$$

It follows that

$$s^A_1 = \mathsf{Sh}^{A'}_1 \circ \mathsf{AW}^{A'}_1 \circ A[\eta_0, \eta_0] = (A[\varepsilon_1 \circ \eta_0 \circ \eta_0, \eta_0] + A[\eta_0, \eta_1])$$

whence

$$d_2^A \circ s_1^A = \mathsf{Sh}_1^A \circ \mathsf{AW}_1^A - \mathbf{1}_{A[1,1]} + A[\varepsilon_1 \circ \varepsilon_1 \circ \eta_0, \varepsilon_1 \circ \eta_0]$$

(details left to the reader). Since $\mathrm{Im}\, A[\eta_0, \eta_0] \subset \mathrm{Ker}\, \mathsf{q}_2^A$, the assertion follows in this case. Next, suppose that $r > 1$, and that the sought identity is already known for every $n < r$, and every bisimplicial object A; especially, it holds for A', so if we let $d_\bullet^{A'}$ be the differential of $A'_\bullet$, we get

$$\mathsf{q}_{r-1}^{A'} \circ (d_r^{A'} \circ s_{r-1}^{A'} + s_{r-2}^{A'} \circ d_{r-1}^{A'}) = \mathsf{q}_{r-1}^{A'} \circ \mathsf{Sh}_{r-1}^{A'} \circ \mathsf{AW}_{r-1}^{A'} - \mathsf{q}_{r-1}^{A'}. \tag{4.2.51}$$

On the other hand, after noticing that

$$d_r^{A'} \circ A[\eta_0, \eta_0] = \mathbf{1}_{A[r,r]} - A[\eta_0, \eta_0] \circ d_{r-1}^{A'}$$

we may compute

$$\begin{aligned} d_r^{A'} \circ s_r^A &= d_r^{A'} \circ \mathsf{Sh}_r^{A'} \circ \mathsf{AW}_r^{A'} \circ A[\eta_0, \eta_0] - d_r^{A'} \circ s_{r-1}^{A'} \\ &= \mathsf{Sh}_r^{A'} \circ \mathsf{AW}_r^{A'} \circ d_r^{A'} \circ A[\eta_0, \eta_0] - d_r^{A'} \circ s_{r-1}^{A'} \\ &= \mathsf{Sh}_r^{A'} \circ \mathsf{AW}_r^{A'} - \mathsf{Sh}_r^{A'} \circ \mathsf{AW}_r^{A'} \circ A[\eta_0, \eta_0] \circ d_{r-1}^{A'} - d_r^{A'} \circ s_{r-1}^{A'} \end{aligned}$$

which, combined with (4.2.51), implies

$$\mathsf{q}_{r-1}^{A'} \circ (d_r^{A'} \circ s_r^A + s_{r-1}^A \circ d_{r-1}^{A'}) = -\mathsf{q}_{r-1}^{A'}. \tag{4.2.52}$$

Furthermore, notice the natural morphism in $s^2.\mathscr{A}$

$$(g_{A^\vee}^{(1)} \circ g_{\gamma_1 A^\vee}^{(2)})^\vee : A' \to A$$

supplied by (4.2.15); explicitly, for every $p, q \in \mathbb{N}$, this morphism is given by the discarded face operator $A[\varepsilon_0, \varepsilon_0] : A[p+1, q+1] \to A[p, q]$. Then, the naturality of $s_\bullet$, $\mathsf{Sh}_\bullet$ and $\mathsf{AW}_\bullet$ implies

$$\begin{aligned} A[\varepsilon_0, \varepsilon_0] \circ s_r^A &= A[\varepsilon_0, \varepsilon_0] \circ \mathsf{Sh}_r^{A'} \circ \mathsf{AW}_r^{A'} \circ A[\eta_0, \eta_0] - A[\varepsilon_0, \varepsilon_0] \circ s_{r-1}^{A'} \\ &= \mathsf{Sh}_r^A \circ \mathsf{AW}_r^A \circ A[\eta_0 \circ \varepsilon_0, \eta_0 \circ \varepsilon_0] - s_{r-1}^A \circ A[\varepsilon_0, \varepsilon_0] \\ &= \mathsf{Sh}_r^A \circ \mathsf{AW}_r^A - s_{r-1}^A \circ A[\varepsilon_0, \varepsilon_0]. \end{aligned}$$

Lastly, recalling that $d_r^A = A[\varepsilon_0, \varepsilon_0] - d_{r-1}^{A'}$, the latter identity can be added to (4.2.52), to deduce

$$\mathsf{q}_{r-1}^{A'} \circ (d_{r+1}^A \circ s_r^A + s_{r-1}^A \circ d_r^A) = \mathsf{q}_{r-1}^{A'} \circ \mathsf{Sh}_r^A \circ \mathsf{AW}_r^A - \mathsf{q}_{r-1}^{A'}.$$

Now, to prove the assertion in degree r, it suffices to observe that q_r^A factors through $\mathsf{q}_{r-1}^{A'}$. $\diamondsuit$

Claim 4.2.50 and theorem 4.2.29(ii) supply the second sought modification, and conclude the proof of the theorem. $\square$

4.2.53. Let $(\mathscr{A}, \otimes)$ be an abelian tensor category, $A[\bullet]$ and $B[\bullet]$ two objects of $s.\mathscr{A}$, and define the bisimplicial objects $A \boxtimes B$ and $B \boxtimes A$, as well as the simplicial objects $A \otimes B$ and $B \otimes A$ of $\mathscr{A}$ as in remark 4.2.16(iii). Notice that the system of commutativity constraints $(\Psi_{A[n],B[n]} \mid n \in \mathbb{N})$ amounts to an isomorphism

$$\Psi_{A\otimes B} : A \otimes B \xrightarrow{\sim} B \otimes A \qquad \text{in } s.\mathscr{A}$$

whence an isomorphism $\Psi_{(A\otimes B)_\bullet} : (A \otimes B)_\bullet \xrightarrow{\sim} (B \otimes A)_\bullet$ on the respective unnormalized complexes.

Proposition 4.2.54. *With the notation of* (4.2.53)*, the diagram of chain complexes*

$$\begin{tikzcd} \mathrm{Tot}(A \boxtimes B)_{\bullet\bullet} \arrow[r, "\Psi^\bullet_{A_\bullet, B_\bullet}"] \arrow[d, "\mathsf{Sh}_\bullet^{A\boxtimes B}"'] & \mathrm{Tot}(B \boxtimes A)_{\bullet\bullet} \arrow[d, "\mathsf{Sh}_\bullet^{B\boxtimes A}"] \\ (A \otimes B)_\bullet \arrow[r, "\Psi_{(A\otimes B)_\bullet}"] & (B \otimes A)_\bullet \end{tikzcd}$$

commutes, where $\Psi^\bullet_{A_\bullet,B_\bullet}$ *is the commutativity constraint for the unnormalized chain complexes* $A_\bullet$ *and* $B_\bullet$*, as in* (4.1.9)*.*

Proof. The assertion boils down to the identity

$$(-1)^{pq} \cdot \mathsf{Sh}^{B\boxtimes A}_{q,p} \circ \Psi_{A[p],B[q]} = \Psi_{A[n],B[n]} \circ \mathsf{Sh}^{A\boxtimes B}_{p,q} \tag{4.2.55}$$

for every $p, q \in \mathbb{N}$ with $p + q = n$. For the latter, suppose first that $p = 0$, in which case

$$\mathsf{Sh}^{A\boxtimes B}_{p,q} = A[\eta_0 \circ \cdots \circ \eta_{q-1}] \otimes \mathbf{1}_{B[q]} \qquad \text{and} \qquad \mathsf{Sh}^{B\boxtimes A}_{q,p} = \mathbf{1}_{B[q]} \otimes A[\eta_0 \circ \cdots \circ \eta_{q-1}]$$

from which we derive (4.2.55), using the naturality of Ψ (details left to the reader). Likewise we argue for the case where $q = 0$. For the general case, we proceed by induction on n. The cases $n = 0, 1$ have already been dealt with, so suppose $r \geq 2$, and that the sought identity is already known for every pair of integers whose sum is $< n$, and every objects A, B of $s.\mathscr{A}$. By the foregoing, we may also assume that both $p, q > 0$, and then claim 4.2.36(i) implies that

$$\mathsf{Sh}^{A\boxtimes B}_{p,q} = (-1)^q \cdot \mathsf{Sh}^{\gamma A\boxtimes\gamma B}_{p-1,q} \circ (\mathbf{1}_{A[p]} \otimes B[\eta_q]) + \mathsf{Sh}^{\gamma A\boxtimes\gamma B}_{p,q-1} \circ (A[\eta_p] \otimes \mathbf{1}_{B[q]}).$$

However, it follows easily from remark 1.2.34(iii) that Ψ is an additive functor in both of its arguments; combining with the inductive assumption, we deduce that

$$\begin{aligned}
\Psi_{A[n],B[n]} \circ \mathsf{Sh}^{A\boxtimes B}_{p,q} &= (-1)^{pq} \cdot \mathsf{Sh}^{\gamma B\boxtimes\gamma A}_{q,p-1} \circ \Psi_{A[p],B[q+1]} \circ (\mathbf{1}_{A[p]} \otimes B[\eta_q]) \\
&\quad + (-1)^{p(q-1)} \cdot \mathsf{Sh}^{\gamma B\boxtimes\gamma A}_{q-1,p} \circ \Psi_{A[p+1],B[q]} \circ (A[\eta_p] \otimes \mathbf{1}_{B[q]}) \\
&= (-1)^{pq} \cdot \mathsf{Sh}^{\gamma B\boxtimes\gamma A}_{q,p-1} \circ (B[\eta_q] \otimes \mathbf{1}_{A[p]}) \circ \Psi_{A[p],B[q]} \\
&\quad + (-1)^{p(q-1)} \cdot \mathsf{Sh}^{\gamma B\boxtimes\gamma A}_{q-1,p} \circ (\mathbf{1}_{B[q]} \otimes A[\eta_p]) \circ \Psi_{A[p],B[q]} \\
&= (-1)^{pq} \cdot \mathsf{Sh}^{B\boxtimes A}_{q,p} \circ \Psi_{A[p],B[q]}
\end{aligned}$$

where the last equality follows again from claim 4.2.36(i) and the additivity of Ψ. ☐

4.2.56. The shuffle map is also *associative*, in the following sense. Let

$$A := (A[p, q, r] \mid p, q, r \in \mathbb{N})$$

be any triple simplicial object of the abelian category $\mathscr{A}$, and denote by $A^{(1,2)}$ (resp. $A^{(2,3)}$) the diagonal bisimplicial object of $\mathscr{A}$ extracted from A by the rule

$$A^{(1,2)}[p, q] := A[p, p, q] \qquad (\text{ resp. } A^{(2,3)}[p, q] := A[p, q, q]\ ) \qquad \text{for every } p, q \in \mathbb{N}.$$

Let also $A_{\bullet\bullet\bullet}$ be the triple chain complex associated to A, and $A^\Delta_{\bullet\bullet\bullet}$ the diagonal chain complex extracted from $A_{\bullet\bullet\bullet}$. Moreover, denote by A' (resp. A'') the bisimplicial object of $s.\mathscr{A}$ given by the rule :

$$[p, q] \mapsto A[p, q, \bullet] \quad (\text{ resp. } [p, q] \mapsto A[\bullet, p, q]\ ) \qquad \text{for every } p, q \in \mathbb{N}.$$

Proposition 4.2.57. *With the notation of* (4.2.56)*, we have a commutative diagram in* $\mathsf{C}(\mathscr{A})$

$$\begin{tikzcd}
\mathrm{Tot}(A_{\bullet\bullet\bullet}) \arrow[r] \arrow[d] & \mathrm{Tot}(A^{(1,2)}_{\bullet\bullet}) \arrow[d, "\mathsf{Sh}^{A^{(1,2)}}_\bullet"] \\
\mathrm{Tot}(A^{(2,3)}_{\bullet\bullet}) \arrow[r, "\mathsf{Sh}^{A^{(2,3)}}_\bullet"] & A^\Delta_{\bullet\bullet\bullet}
\end{tikzcd}$$

whose top horizontal (resp. left vertical) arrow is obtained as the composition of $\mathsf{Sh}^{A'}_\bullet$ *(resp.* $\mathsf{Sh}^{A''}_\bullet$*) with the functor* $\mathrm{Tot} \circ \mathsf{C}(\mathsf{U}_{\mathscr{A}}) : \mathsf{C}(s.\mathscr{A}) \to \mathsf{C}(\mathsf{C}(\mathscr{A})) \to \mathsf{C}(\mathscr{A})$*.*

Proof. For every $p, q, r \in \mathbb{N}$, denote by

$$\mathsf{Sh}^{A'}_{p,q}[r] : A[p,q,r] \to A[p+q,p+q,r] = A^{(1,2)}[p+q,r]$$
$$\mathsf{Sh}^{A''}_{q,r}[p] : A[p,q,r] \to A^{(2,3)}[p,q+r]$$

respectively the $[r]$-component of $\mathsf{Sh}^{A'}_{p,q}$ and the $[p]$-component of $\mathsf{Sh}^{A''}_{q,r}$. The assertion boils down to the identity :

$$\mathsf{Sh}^{A^{(1,2)}}_{p+q,r} \circ \mathsf{Sh}^{A'}_{p,q}[r] = \mathsf{Sh}^{A^{(2,3)}}_{p,q+r} \circ \mathsf{Sh}^{A''}_{q,r}[p] \qquad \text{for every } p,q,r \in \mathbb{N}.$$

To check the latter, set

$$D' := \gamma_2 \circ \gamma_1 A' \qquad D^{(1,2)} := \gamma_2 \circ \gamma_1(A^{(1,2)})$$

and define likewise D'' and $D^{(2,3)}$ (notation of (4.2.15)). Also, let $A[p,q,r] := 0$ whenever one of the indices p, q, r is < 0, and define $\mathsf{Sh}_{p,q}$ to be the zero map, when either p or q is strictly negative (cp. the proof of proposition 4.2.34). We argue by induction on $n := p+q+r$; the case $n = 0$ is trivial, so suppose that $n > 0$, and that the assertion is already known for all indices whose sum is $< n$, and all triple simplicial objects of $\mathscr{A}$. Notice as well that the assertion trivially holds as well if any of the indices p, q, r is strictly negative, since in this case both sides are the zero map. Hence, we may assume that $p, q, r \in \mathbb{N}$; in this case, by applying claim 4.2.36(i), first to $A^{(1,2)}$ and then to A', we compute

$$\begin{aligned}
\mathsf{Sh}^{A^{(1,2)}}_{p+q,r} \circ \mathsf{Sh}^{A'}_{p,q}[r] &= (-1)^r \cdot \mathsf{Sh}^{D^{(1,2)}}_{p+q-1,r} \circ A[\mathbf{1}_{[p+q]}, \mathbf{1}_{[p+q]}, \eta_r] \circ \mathsf{Sh}^{A'}_{p,q}[r] \\
&\quad + \mathsf{Sh}^{D^{(1,2)}}_{p+q,r-1} \circ A[\eta_{p+q}, \eta_{p+q}, \mathbf{1}_{[r]}] \circ \mathsf{Sh}^{A'}_{p,q}[r] \\
&= (-1)^r \cdot \mathsf{Sh}^{D^{(1,2)}}_{p+q-1,r} \circ \mathsf{Sh}^{A'}_{p,q}[r+1] \circ A[\mathbf{1}_{[p]}, \mathbf{1}_{[q]}, \eta_r] \\
&\quad + \mathsf{Sh}^{D^{(1,2)}}_{p+q,r-1} \circ \mathsf{Sh}^{D'}_{p,q}[r] \circ A[\eta_p, \eta_q, \mathbf{1}_{[r]}] && \text{(by (4.2.37))} \\
&= (-1)^{q+r} \cdot \mathsf{Sh}^{D^{(1,2)}}_{p+q-1,r} \circ \mathsf{Sh}^{D'}_{p-1,q}[r+1] \circ A[\mathbf{1}_{[p]}, \eta_q, \eta_r] \\
&\quad + (-1)^r \cdot \mathsf{Sh}^{D^{(1,2)}}_{p+q-1,r} \circ \mathsf{Sh}^{D'}_{p,q-1}[r+1] \circ A[\eta_p, \mathbf{1}_{[q]}, \eta_r] \\
&\quad + \mathsf{Sh}^{D^{(1,2)}}_{p+q,r-1} \circ \mathsf{Sh}^{D'}_{p,q}[r] \circ A[\eta_p, \eta_q, \mathbf{1}_{[r]}] \\
&= (-1)^{q+r} \cdot \mathsf{Sh}^{D^{(2,3)}}_{p-1,q+r} \circ \mathsf{Sh}^{D''}_{q,r}[p] \circ A[\mathbf{1}_{[p]}, \eta_q, \eta_r] \\
&\quad + (-1)^r \cdot \mathsf{Sh}^{D^{(2,3)}}_{p,q+r-1} \circ \mathsf{Sh}^{D''}_{q-1,r}[p+1] \circ A[\eta_p, \mathbf{1}_{[q]}, \eta_r] \\
&\quad + \mathsf{Sh}^{D^{(2,3)}}_{p,q+r-1} \circ \mathsf{Sh}^{D''}_{q,r-1}[p+1] \circ A[\eta_p, \eta_q, \mathbf{1}_{[r]}]
\end{aligned}$$

where the last identity holds by inductive assumption. On the other hand, by applying claim 4.2.36(i) to $A^{(2,3)}$ we get

$$\begin{aligned}
\mathsf{Sh}^{A^{(2,3)}}_{p,q+r} \circ \mathsf{Sh}^{A''}_{q,r}[p] &= (-1)^{q+r} \cdot \mathsf{Sh}^{D^{(2,3)}}_{p-1,q+r} \circ A[\mathbf{1}_{[p]}, \eta_{q+r}, \eta_{q+r}] \circ \mathsf{Sh}^{A''}_{q,r}[p] \\
&\quad + \mathsf{Sh}^{D^{(2,3)}}_{p,q+r-1} \circ A[\eta_p, \mathbf{1}_{[q+r]}, \mathbf{1}_{[q+r]}] \circ \mathsf{Sh}^{A''}_{q,r}[p] \\
&= (-1)^{q+r} \cdot \mathsf{Sh}^{D^{(2,3)}}_{p-1,q+r} \circ \mathsf{Sh}^{D''}_{q,r}[p] \circ A[\mathbf{1}_{[p]}, \eta_q, \eta_r] \\
&\quad + \mathsf{Sh}^{D^{(2,3)}}_{p,q+r-1} \circ \mathsf{Sh}^{A''}_{q,r}[p+1] \circ A[\eta_p, \mathbf{1}_{[q]}, \mathbf{1}_{[r]}] && \text{(by (4.2.37))}
\end{aligned}$$

and after applying again claim 4.2.36(i) to A'' and comparing with the foregoing expression for $\mathsf{Sh}^{A^{(1,2)}}_{p+q,r} \circ \mathsf{Sh}^{A'}_{p,q}[r]$, we obtain the sought identity. $\square$

Theorem 4.2.58 (Dold-Puppe-Kan)**.** *For any abelian category $\mathscr{A}$, and any $k \in \mathbb{N}$, we have :*

(i) *The functors $\mathsf{N}_{\mathscr{A}}$ and $\mathsf{N}_{\mathscr{A},k}$ are equivalences.*

(ii) *If f, g are any two morphisms in $s.\mathscr{A}$, then there exists a simplicial homotopy from f to g if and only if there exists a chain homotopy from $N_\bullet f$ to $N_\bullet g$.*

Proof. We easily reduce to the case where $\mathscr{A}$ is small, and then there exists a fully faithful imbedding $\mathscr{A} \to \mathscr{B}$, where $\mathscr{B}$ is a complete and cocomplete abelian tensor category, with internal Hom functor (lemma 1.2.42).

(i): We first construct an explicit quasi-inverse for the functors $\mathsf{N}_{\mathscr{B}}$ and $\mathsf{N}_{\mathscr{B},k}$, as follows. For every $i \in \mathbb{N}$, consider the cochain complex $K\langle i\rangle_\bullet$ defined in example 4.2.27(ii); notice that every morphism $\varphi : [i] \to [i']$ in Δ induces a morphism

$$K\langle\varphi\rangle_\bullet := N_\bullet \mathbb{Z}^{\mathbf{\Delta}_\varphi} : K\langle i\rangle_\bullet \to K\langle i'\rangle_\bullet$$

(notation of remark 4.2.17(iii)). Hence, the system $(K\langle i\rangle_\bullet \mid i \in \mathbb{N})$ amounts to a cosimplicial object of $\mathsf{C}^{\leq 0}(\mathbb{Z}\text{-}\mathbf{Mod})$. Now, let U be a unit of the tensor category $\mathscr{B}$; for any object $C_\bullet$ of $\mathsf{C}^{\leq 0}(\mathscr{B})$ (resp. of $\mathsf{C}^{[-k,0]}(\mathscr{B})$), we set

$$K_C[i] := \mathscr{H}om_{\mathsf{C}(\mathscr{B})}(K\langle i\rangle_\bullet \otimes_\mathbb{Z} U[0], C_\bullet) \qquad \text{for every } i \in \mathbb{N} \text{ (resp. for every } i \leq k)$$

where $\mathscr{H}om_{\mathsf{C}(\mathscr{B})}$ is the functor constructed in example 4.1.8(vii), and the mixed tensor product is defined as in (4.1.10). By the foregoing, it is clear that the system $(K_C[i] \mid i \in \mathbb{N})$ amounts to an object of $s.\mathscr{B}$ (resp. of $s_k.\mathscr{B}$). In order to compute $N_\bullet K_C$, let us set

$$\mathbb{Z}_+^{\mathbf{\Delta}_i} := \sum_{n=1}^{i} \mathrm{Im}\,(\mathbb{Z}^{\mathbf{\Delta}_{\varepsilon_n}} : \mathbb{Z}^{\mathbf{\Delta}_{i-1}} \to \mathbb{Z}^{\mathbf{\Delta}_i}) \qquad \mathbb{Z}_0^{\mathbf{\Delta}_i} := \mathbb{Z}^{\mathbf{\Delta}_i}/\mathbb{Z}_+^{\mathbf{\Delta}_i} \qquad \overline{K}\langle i\rangle_\bullet := N_\bullet \mathbb{Z}_0^{\mathbf{\Delta}_i}$$

for every $i > 0$, as well as $\mathbb{Z}_0^{\mathbf{\Delta}_0} := \mathbb{Z}^{\mathbf{\Delta}_0}$ and $\overline{K}\langle 0\rangle_\bullet := K\langle 0\rangle_\bullet$; thus, $\overline{K}\langle i\rangle_\bullet$ is also the quotient of $K\langle i\rangle_\bullet$ by the sum of the images of the morphisms $K\langle\varepsilon_n\rangle$, for $n = 1, \dots, i$. With this notation, a simple inspection of the definition shows that

$$N_i K_C = \mathscr{H}om_{\mathsf{C}(\mathscr{B})}(\overline{K}\langle i\rangle_\bullet \otimes_\mathbb{Z} U[0], C_\bullet) \qquad \text{for every } i \in \mathbb{N} \text{ (resp. for every } i \leq k).$$

On the other hand, using the explicit description of example 4.2.27(ii), it is easily seen that

$$\overline{K}\langle i\rangle_n \xrightarrow{\sim} \begin{cases} \mathbb{Z} & \text{if } n = i \text{ or } n = i-1 \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

More precisely, $\overline{K}\langle i\rangle_i$ (resp. $\overline{K}\langle i\rangle_{i-1}$) is generated by the basis element $e_\mathbf{1}$ of $K\langle i\rangle_i$ (resp. e_{ε_0} of $K\langle i\rangle_{i-1}$) corresponding to the identity map $\mathbf{1}_{[i]}$ (resp. corresponding to $\varepsilon_0 : [i-1] \to [i]$). Furthermore, example 4.2.27(i) shows that the differential $K\langle i\rangle_i \to K\langle i\rangle_{i-1}$ maps $e_\mathbf{1}$ to e_{ε_0}, so it corresponds to the identity map $\mathbb{Z} \to \mathbb{Z}$, under the foregoing identification. Taking into account remark 1.2.12(iv), we deduce that $N_i K_C$ is the kernel of the morphism

$$C_i \oplus C_{i-1} \to C_{i-1} \oplus C_{i-2} \tag{4.2.59}$$

given by the matrix

$$\begin{pmatrix} (-1)^i \cdot d_i^C & \mathbf{1}_{C_{i-1}} \\ 0 & (-1)^{i+1} \cdot d_{i-1}^C \end{pmatrix}$$

and since $d_{i-1}^C \circ d_i^C = 0$, the latter is just C_i; more precisely, C_i is identified with this kernel, via the monomorphism

$$(\mathbf{1}_{C_i}, (-1)^{i+1} \cdot d_i^C) : C_i \to C_i \oplus C_{i-1}. \tag{4.2.60}$$

This identification $N_i K_C \xrightarrow{\sim} C_i$ can also be described as follows. For every $Z \in \mathrm{Ob}(\mathscr{B})$, let

$$\mathrm{Hom}_\mathscr{B}(Z, C_\bullet)$$

be the cochain complex such that $\mathrm{Hom}_\mathscr{B}(Z, C_\bullet)_n := \mathrm{Hom}_\mathscr{B}(Z, C_n)$ for every $n \in \mathbb{Z}$, with differentials induced by those of $C_\bullet$, in the obvious way; then we have natural identifications

$$\mathrm{Hom}_{\mathsf{C}(\mathbb{Z}\text{-}\mathbf{Mod})}(\overline{K}\langle i\rangle_\bullet, \mathrm{Hom}_\mathscr{B}(Z, C_\bullet)) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{C}(\mathscr{B})}(\overline{K}\langle i\rangle_\bullet \otimes_\mathbb{Z} Z[0], C_\bullet) \xrightarrow{\sim} \mathrm{Hom}_\mathscr{B}(Z, N_i K_C)$$

(see example 4.1.8(vii)) whose composition with the induced isomorphism

$$\mathrm{Hom}_\mathscr{B}(Z, N_i K_C) \xrightarrow{\sim} \mathrm{Hom}_\mathscr{B}(Z, C_i)$$

is given by the rule :

$$(4.2.61)\qquad (\varphi_\bullet : \overline{K}\langle i\rangle_\bullet \to \mathrm{Hom}_{\mathscr{B}}(Z, C_\bullet)) \mapsto (\varphi_i(\varepsilon_{\mathbf{1}}) : Z \to C_i).$$

It remains to determine the differential $d^N_{i+1} : N_{i+1}K_C \to N_iK_C$; by definition, the latter is induced by the morphism $\overline{K}\langle\varepsilon_0\rangle_\bullet : \overline{K}\langle i\rangle_\bullet \to \overline{K}\langle i+1\rangle_\bullet$. In turn, the foregoing description says that $\overline{K}\langle\varepsilon_0\rangle_\bullet$ is naturally identified with the morphism of $\mathsf{C}(\mathbb{Z}\text{-}\mathbf{Mod})$

$$\begin{tikzcd} \overline{K}\langle i\rangle_\bullet \arrow[d] & 0 \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d, "\mathbf{1}_{\mathbb{Z}}"'] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \\ \overline{K}\langle i+1\rangle_\bullet & 0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r] & 0 \arrow[r] & 0 \end{tikzcd}$$

(where the two copies of $\mathbb{Z}$ on the top horizontal row are placed in homological degrees i and $i-1$), so d^N_{i+1} is deduced from the morphism in $\mathsf{C}(\mathscr{B})$

$$\begin{tikzcd} 0 \arrow[r] & C_{i+1}\oplus C_i \arrow[r] \arrow[d] & C_i\oplus C_{i-1} \arrow[r] \arrow[d, "\mathbf{1}"'] & 0 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & 0 \arrow[r] & C_i\oplus C_{i-1} \arrow[r] & C_{i-1}\oplus C_{i-2} \arrow[r] & 0 \end{tikzcd}$$

whose rows are given by the morphisms (4.2.59). Therefore, via the identification (4.2.60), the morphism d^N_{i+1} becomes none else than $(-1)^{i+1}\cdot d^C_{i+1}$, *i.e.* we have obtained a natural isomorphism

$$\omega^C_\bullet : N_\bullet K_C \xrightarrow{\sim} C_\bullet$$

for every complex in $\mathsf{C}^{\leq 0}(\mathscr{B})$ (resp. in $\mathsf{C}^{[-k,0]}(\mathscr{B})$).

Conversely, let $B[\bullet]$ be any object of $s.\mathscr{B}$, and Z any object of $\mathscr{B}$; clearly

$$N_\bullet(\mathbf{\Delta}_k \otimes s.Z) = K\langle i\rangle_\bullet \otimes_{\mathbb{Z}} Z[0] \qquad \text{for every } i\in\mathbb{N}.$$

In view of example 4.1.8(vii) and remark 4.2.17(ii), we deduce a natural transformation

$$(4.2.62)\qquad \begin{tikzcd} \mathrm{Hom}_{\mathscr{B}}(Z, B[i]) \arrow[r, "a"] & \mathrm{Hom}_{s.\mathscr{B}}(\mathbf{\Delta}_i\otimes s.Z, B) \arrow[d, "b"] \\ \mathrm{Hom}_{\mathscr{B}}(Z, K_{N_\bullet B}[i]) & \mathrm{Hom}_{\mathsf{C}(\mathscr{B})}(K\langle i\rangle_\bullet\otimes_{\mathbb{Z}} Z[0], N_\bullet B) \arrow[l, "c"'] \end{tikzcd}$$

which, by Yoneda's lemma, comes from a unique morphism in $\mathscr{B}$

$$\psi^B_i : B[i] \to K_{N_\bullet B}[i] \qquad \text{for every } i\in\mathbb{N}.$$

The same construction applies, in case B is an object of $s_k.\mathscr{B}$: we need only replace the group $\mathrm{Hom}_{s.\mathscr{B}}(\mathbf{\Delta}_i\otimes s.Z, B)$ by $\mathrm{Hom}_{s_k.\mathscr{B}}(s.\mathrm{trunc}_k(\mathbf{\Delta}_i\otimes s.Z), B)$ in (4.2.62) : see remark 4.2.17(iv). Moreover, remark 4.2.17(iii) implies that the system $\psi^B := (\psi^B_i \mid i\in\mathbb{N})$ amounts to a morphism $B\to K_{N_\bullet B}$ in $s.\mathscr{B}$ (resp. in $s_k.\mathscr{B}$). By the same token, the $\mathbb{Z}$-linear isomorphism a maps the abelian subgroup $\mathrm{Hom}_{\mathscr{B}}(Z, N_iB)$ isomorphically onto the subgroup

$$\mathrm{Hom}_{s.\mathscr{B}}(\mathbb{Z}_0^{\mathbf{\Delta}_i}\otimes_{\mathbb{Z}} s.Z, B) \xrightarrow{\sim} \mathrm{Hom}_{s.\mathbb{Z}\text{-}\mathbf{Mod}}(\mathbb{Z}_0^{\mathbf{\Delta}_i}, \mathrm{Hom}_{\mathscr{B}}(Z,B))$$

for every $Z\in\mathrm{Ob}(\mathscr{B})$ and every $i\in\mathbb{N}$, where $\mathrm{Hom}_{\mathscr{B}}(Z,B)$ is the simplicial abelian group as in example 4.2.27(iii). Explicitly, if $\beta : Z\to N_iB$ is any morphism, then $a(\beta)$ is the unique morphism $\mathbb{Z}_0^{\mathbf{\Delta}_i}\to\mathrm{Hom}_{\mathscr{B}}(Z,B)$ of simplicial abelian groups such that $a(\beta)(e_{\mathbf{1}}) = \beta$, where $e_{\mathbf{1}}\in\overline{K}\langle i\rangle_i\subset\mathbb{Z}_0^{\mathbf{\Delta}_i}[i]$ is the basis element described in the foregoing. Likewise, we have natural identifications

$$\mathrm{Hom}_{\mathsf{C}(\mathbb{Z}\text{-}\mathbf{Mod})}(K\langle i\rangle_\bullet, \mathrm{Hom}_{\mathscr{B}}(Z, N_\bullet B)) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{C}(\mathscr{B})}(K\langle i\rangle_\bullet\otimes_{\mathbb{Z}} Z[0], N_\bullet B)$$

and b restricts to a map

$$\mathrm{Hom}_{s.\mathbb{Z}\text{-}\mathbf{Mod}}(\mathbb{Z}_0^{\mathbf{\Delta}_i}, \mathrm{Hom}_{\mathscr{B}}(Z,B)) \to \mathrm{Hom}_{\mathsf{C}(\mathbb{Z}\text{-}\mathbf{Mod})}(\overline{K}\langle i\rangle_\bullet, \mathrm{Hom}_{\mathscr{B}}(Z, N_\bullet B)) \qquad \varphi \mapsto N_\bullet\varphi.$$

Again, the same applies to an object of $s_k.\mathscr{B}$, by taking suitable truncated variants of the above constructions. Taking into account (4.2.61), we conclude that ψ^B induces an isomorphism

$$N_\bullet\psi^B : N_\bullet B \xrightarrow{\sim} N_\bullet K_{N_\bullet B}$$

which is inverse to $\omega^{N_\bullet B}$. To finish the proof of assertion (i) for the category $\mathscr{B}$, it then suffices to remark :

Claim 4.2.63. The functors $\mathsf{N}_{\mathscr{B}}$ and $\mathsf{N}_{\mathscr{B},k}$ are conservative.

Proof of the claim. We show, by induction on n, that if a morphism h in $s.\mathscr{B}$ or $s_k.\mathscr{B}$ (for any $k \in \mathbb{N}$) induces an isomorphism $N_\bullet h$, then $h[n]$ is an isomorphism for every $n \in \mathbb{N}$ (resp. for every $n \leq k$). The assertion is obvious for $n = 0$, hence suppose that $n > 0$, and that $h[n-1]$ is known to be an isomorphism whenever h is a morphism in $s.\mathscr{B}$ or in $s_k.\mathscr{B}$ (for arbitrary $k \in \mathbb{N}$) such that $N_\bullet h$ is an isomorphism. Let $h : A \to B$ be any such morphism in $s.\mathscr{B}$ or in $s_k.\mathscr{B}$. If h is a morphism in $s_0.\mathscr{B}$, we are done, hence we may suppose that either h is a morphism in $s.\mathscr{B}$ or $k > 0$. Set

$$A' := \mathrm{Ker}(g_A : \gamma A \to A) \qquad \text{(resp. } A' := \mathrm{Ker}(g_A : \gamma_{k-1}A \to s.\mathrm{trunc}_{k-1}A) \text{)}$$

as well as $B' := \mathrm{Ker}(g_B)$ (notation of remark 4.2.11(ii)). Notice that g_A is an epimorphism, since ∂_{n+2} admits the section σ_{n+1}, for every $n \in \mathbb{N}$ (resp. for every $n \leq k$). Therefore, we have a commutative diagram in $s.\mathscr{B}$ with exact rows

$$\begin{tikzcd} 0 \arrow[r] & A' \arrow[r] \arrow[d, "h'"'] & \gamma A \arrow[r] \arrow[d, "\gamma h"'] & A \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & A' \arrow[r] & \gamma A \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

(resp. a corresponding diagram in $s_{k-1}.\mathscr{B}$). By inspecting the definitions, it is easily seen that $N_\bullet A' = (N_\bullet A)[-1]$, $N_\bullet A' = (N_\bullet A)[-1]$, and $N_\bullet h' = (N_\bullet h)[-1]$; especially, $N_\bullet h'$ is an isomorphism, so $h'[n-1]$ is an isomorphism, by inductive assumption. The same holds also for $h[n-1]$, and we conclude that $\gamma h[n]$ is an isomorphism. But $\gamma h[n] = h[n+1]$, so we are done.

Lastly, in order to prove assertion (i) for the original category $\mathscr{A}$, it suffices to notice :

Claim 4.2.64. Let $C_\bullet$ be any object of $\mathsf{C}(\mathscr{A})$ (resp. of $\mathsf{C}^{[-k,0]}(\mathscr{A})$), and regard $C_\bullet$ as an object of $\mathsf{C}(\mathscr{B})$ (resp. of $\mathsf{C}^{[-k,0]}(\mathscr{B})$), via the fully faithful imbedding $\mathscr{A} \to \mathscr{B}$. Then K_C is isomorphic to an object of $s.\mathscr{A}$ (resp. $s_k.\mathscr{A}$), regarded as a full subcategory of $s.\mathscr{B}$ (resp. of $s_k.\mathscr{B}$), via the same imbedding.

Proof of the claim. This follows easily, by remarking that $K\langle i\rangle_\bullet$ lies in $\mathsf{C}^{[-i,0]}(\mathbb{Z}\text{-}\mathbf{Mod})$ for every $i \in \mathbb{N}$, and $K\langle i\rangle_j$ is a finitely generated abelian group for every $i, j \in \mathbb{N}$: details left to the reader. $\Diamond$

(ii): First, let $f, g : A \to B$ be two morphisms in $s.\mathscr{A}$, and $u : \mathbf{\Delta}_1 \otimes A \to B$ a homotopy from f to g (see remark 4.2.16(iv)); especially,

$$u \circ (\mathbf{\Delta}_{\varepsilon_1} \otimes A) = f \qquad \text{and} \qquad u \circ (\mathbf{\Delta}_{\varepsilon_0} \otimes A) = g.$$

Notice that

$$\mathbb{Z}_\bullet^{\mathbf{\Delta}_1} \otimes_{\mathbb{Z}} A_\bullet = \mathrm{Tot}(\mathbf{\Delta}_1 \boxtimes A)_{\bullet\bullet}$$

(notation of remark 4.2.16(iii) and example 4.2.27(i)); there follows a morphism in $\mathsf{C}(\mathscr{A})$

$$\widetilde{u}_\bullet : \mathbb{Z}_\bullet^{\mathbf{\Delta}_1} \otimes_{\mathbb{Z}} A_\bullet \xrightarrow{\mathsf{Sh}_\bullet} (\mathbf{\Delta}_1 \otimes A)_\bullet \xrightarrow{\mathsf{q}_\bullet} N_\bullet(\mathbf{\Delta}_1 \otimes A) \xrightarrow{N_\bullet u} N_\bullet B$$

where $\mathsf{Sh}_\bullet$ denotes the shuffle map for the bisimplicial object $A \boxtimes \mathbf{\Delta}_1$, and $\mathsf{q}_\bullet$ is the projection defined in (4.2.28). Moreover, the maps $\mathbf{\Delta}_{\varepsilon_i}[0] : \mathbf{\Delta}_0[0] \to \mathbf{\Delta}_1[0]$ $(i = 0, 1)$ induce morphisms

$$\widetilde{e}_{i,n} := \mathbb{Z}^{\mathbf{\Delta}_{\varepsilon_i}[0]} \otimes_{\mathbb{Z}} A_n : A_n \to \mathbb{Z}^{\mathbf{\Delta}_1[0]} \otimes_{\mathbb{Z}} A_n \subset (\mathbb{Z}_\bullet^{\mathbf{\Delta}_1} \otimes_{\mathbb{Z}} A_\bullet)_n$$

that amount to morphisms of cochain complexes $\widetilde{e}_{i,\bullet} : A_\bullet \to \mathbb{Z}_\bullet^{\mathbf{\Delta}_1} \otimes_{\mathbb{Z}} A_\bullet$ $(i = 0, 1)$, and a simple inspection of the definitions shows that

$$\mathsf{Sh}_\bullet \circ \widetilde{e}_{i,\bullet} = (\Delta_{\varepsilon_i} \otimes A)_\bullet \qquad \text{for } i = 0, 1$$

whence

$$\widetilde{u}_\bullet \circ \widetilde{e}_{1,\bullet} = f_\bullet \qquad \text{and} \qquad \widetilde{u}_\bullet \circ \widetilde{e}_{0,\bullet} = g_\bullet.$$

The construction makes it clear that $\widetilde{e}_i$ restricts to a morphism $N_\bullet A \to K\langle 1\rangle_\bullet \otimes_{\mathbb{Z}} N_\bullet A$, and the latter is none else than the map $\iota_i \otimes_{\mathbb{Z}} N_\bullet A$, with the notation of remark 4.1.11(ii). We conclude that the morphism

$$\overline{u}_\bullet : K\langle 1\rangle_\bullet \otimes_{\mathbb{Z}} N_\bullet A \hookrightarrow \mathbb{Z}_\bullet^{\mathbf{\Delta}_1} \otimes_{\mathbb{Z}} A_\bullet \xrightarrow{\widetilde{u}_\bullet} N_\bullet B$$

is a homotopy from $N_\bullet f$ to $N_\bullet g$ (notation of (4.2.28)).

Conversely, suppose that $\overline{v}_\bullet : K\langle 1\rangle_\bullet \otimes_{\mathbb{Z}} N_\bullet A \to N_\bullet B$ is a homotopy from $N_\bullet f$ to $N_\bullet g$. Since $K\langle 1\rangle_\bullet$ is a direct summand of $\mathbb{Z}_\bullet^{\mathbf{\Delta}_1}$ and $N_\bullet A$ is a direct summand of $A_\bullet$, we may extend $\overline{v}_\bullet$ to a morphism

$$\widetilde{v}_\bullet : \mathbb{Z}_\bullet^{\mathbf{\Delta}_1} \otimes_{\mathbb{Z}} A_\bullet \to N_\bullet B$$

such that $\widetilde{v}_\bullet$ is the zero morphism on the direct summands other than $K\langle 1\rangle_\bullet \otimes_{\mathbb{Z}} N_\bullet A$. Next, consider the composition

$$v_\bullet : N_\bullet(\mathbf{\Delta}_1 \otimes A) \xrightarrow{\mathsf{j}_\bullet} (\mathbf{\Delta}_1 \otimes A)_\bullet \xrightarrow{\mathsf{AW}_\bullet} A_\bullet \otimes_{\mathbb{Z}} \mathbb{Z}_\bullet^{\mathbf{\Delta}_1} \xrightarrow{\Psi_\bullet} \mathbb{Z}_\bullet^{\mathbf{\Delta}_1} \otimes_{\mathbb{Z}} A_\bullet \xrightarrow{\widetilde{v}_\bullet} N_\bullet B$$

where $\mathsf{j}_\bullet$ is the natural injection (see (4.2.28)), $\mathsf{AW}_\bullet$ is the Alexander-Whitney map for the bisimplicial object $A \boxtimes \mathbf{\Delta}_1$ (notice that $\mathbf{\Delta}_1 \otimes A = (A \boxtimes \mathbf{\Delta}_1)^\Delta$), and $\Psi_\bullet$ is the commutativity constraint (see example 4.1.8(i)). By (i), the morphism $v_\bullet$ comes from a unique morphism

$$v : \mathbf{\Delta}_1 \otimes A \to B \qquad \text{in } s.\mathscr{A}.$$

On the other hand, since $N_{p+q}A$ is contained in the kernel of $A[\varepsilon_{p,0}^{q\vee}]$ for every $p, q \in \mathbb{N}$, it is easily seen that the diagram

$$\begin{tikzcd}
N_\bullet A \arrow[r, "\iota_i \otimes_{\mathbb{Z}} N_\bullet A"] \arrow[d, "N_\bullet(\mathbf{\Delta}_{\varepsilon_i} \otimes A)"'] & K\langle 1\rangle_\bullet \otimes_{\mathbb{Z}} N_\bullet A \arrow[d] \\
N_\bullet(\mathbf{\Delta}_1 \otimes A) \arrow[r, "\Psi_\bullet \circ \mathsf{AW}_\bullet \circ \mathsf{j}_\bullet"] & \mathbb{Z}_\bullet^{\mathbf{\Delta}_1} \otimes_{\mathbb{Z}} A_\bullet
\end{tikzcd}$$

commutes for $i = 0, 1$. We conclude that v is a homotopy from f to g. □

Corollary 4.2.65. *Let $\mathscr{A}$ be any abelian category. We have :*

(i) *If $k \in \mathbb{N}$ is any integer, and A any k-truncated simplicial object of $\mathscr{A}$, then*

$$H_i(\mathrm{cosk}_k A) = 0 \qquad \textit{for every } i \geq k.$$

(ii) *Every homotopically trivial augmented simplicial object of $\mathscr{A}$ is aspherical.*

Proof. (i): Denote by $t_{\leq k} : \mathsf{C}^{\leq 0}(\mathscr{A}) \to \mathsf{C}^{[-k,0]}(\mathscr{A})$ the brutal truncation functor (see (4.1)). In light of theorem 4.2.58, we see that $t_{\leq k}$ admits a right adjoint $v_k : \mathsf{C}^{[-k,0]}(\mathscr{A}) \to \mathsf{C}^{\leq 0}(\mathscr{A})$, and clearly there are natural isomorphisms

$$N_\bullet \mathrm{cosk}_k A \xrightarrow{\sim} v_k N_\bullet A \qquad \text{for every } A \in \mathrm{Ob}(s_k.\mathscr{A}).$$

Taking into account theorem 4.2.29(iii), we are then reduced to showing

Claim 4.2.66. $H_i(v_k K_\bullet) = 0$ for every $(K_\bullet, d_\bullet) \in \mathrm{Ob}(\mathsf{C}^{[-k,0]})$ and every $i \geq k$.

Proof of the claim. Indeed it is easily seen that :

$$(v_k K_\bullet)_i = \begin{cases} K_i & \text{for } \leq k \\ \operatorname{Ker} d_k & \text{for } i = k+1 \\ 0 & \text{for } i > k+1 \end{cases}$$

and the differential of $v_k K_\bullet$ in degree $\leq k$ agrees with that of $K_\bullet$, whereas in degree $k+1$ it is the natural inclusion map (details left to the reader). The claim follows immediately. ◊

(ii) follows directly from theorems 4.2.58(ii) and 4.2.29(iii), and remark 4.1.3(ii). □

4.3. **Injective modules, flat modules and indecomposable modules.**

4.3.1. *Indecomposable modules.* Recall that a unitary (not necessarily commutative) ring R is said to be *local* if $R \neq 0$ and, for every $x \in R$ either x or $1-x$ is invertible. If R is commutative, this definition is equivalent to the usual one.

4.3.2. Let $\mathscr{C}$ be any abelian category and M an object of $\mathscr{C}$. One says that M is *indecomposable* if it is non-zero and cannot be presented in the form $M = N_1 \oplus N_2$ with non-zero objects N_1 and N_2. If such a decomposition exists, then $\operatorname{End}_{\mathscr{C}}(N_1) \times \operatorname{End}_{\mathscr{C}}(N_2) \subset \operatorname{End}_{\mathscr{C}}(M)$, especially the unitary ring $\operatorname{End}_{\mathscr{C}}(M)$ contains an idempotent element $e \neq 1, 0$ and therefore it is not a local ring.

However, if M is indecomposable, it does not necessarily follow that $\operatorname{End}_{\mathscr{C}}(M)$ is a local ring. Nevertheless, one has the following:

Theorem 4.3.3 (Krull-Remak-Schmidt). *Let $(A_i)_{i\in I}$ and $(B_j)_{j\in J}$ be two finite families of objects of $\mathscr{C}$, such that:*

(a) $\oplus_{i\in I} A_i \simeq \oplus_{j\in J} B_j$.
(b) $\operatorname{End}_{\mathscr{C}}(A_i)$ *is a local ring for every* $i \in I$, *and* $B_j \neq 0$ *for every* $j \in J$.

Then we have :

(i) *There is a surjection $\varphi : I \to J$, and isomorphisms $B_j \xrightarrow{\sim} \oplus_{i\in\varphi^{-1}(j)} A_i$, for every $j \in J$.*
(ii) *Especially, if B_j is indecomposable for every $j \in J$, then I and J have the same cardinality, and φ is a bijection.*

Proof. Clearly, (i)⇒(ii). To show (i), let us begin with the following :

Claim 4.3.4. Let M_1, M_2 be two objects of $\mathscr{C}$, and set $M := M_1 \oplus M_2$. Denote by $e_i : M_i \to M$ (resp. $p_i : M \to M_i$) the natural injection (resp. projection) for $i = 1, 2$. Suppose that $\alpha : P \to M$ is a subobject of M, such that $p_1\alpha : P \to M_1$ is an isomorphism. Then the natural morphism $\beta : P \oplus M_2 \to M$ is an isomorphism.

Proof of the claim. Denote by $\pi_P : P \oplus M_2 \to P$ and $\pi_2 : P \oplus M_2 \to M_2$ the natural projections; then $\beta := \alpha\pi_P + e_2\pi_2$. Of course, the assertion follows easily by applying the 5-lemma (which holds in any abelian category) to the commutative ladder with exact rows :

$$\begin{tikzcd}
0 \arrow[r] & M_2 \arrow[r] \arrow[d, equal] & P \oplus M_2 \arrow[r, "\pi_P"] \arrow[d, "\beta"'] & P \arrow[r] \arrow[d, "p_1\alpha"] & 0 \\
0 \arrow[r] & M_2 \arrow[r, "e_2"] & M \arrow[r, "p_1"] & M_1 \arrow[r] & 0.
\end{tikzcd}$$

Equivalently, one can argue directly as follows. By definition, $\operatorname{Coker} \beta$ represents the functor

$$\mathscr{C} \to \mathbb{Z}\text{-}\mathbf{Mod} \quad : \quad X \mapsto \operatorname{Ker} \operatorname{Hom}_{\mathscr{C}}(\beta, X)$$

and likewise for $\operatorname{Coker} p_1\alpha$; however, it is easily seen that these functors are naturally isomorphic, hence the natural morphism $\operatorname{Coker} \beta \to \operatorname{Coker} p_1\alpha$ is an isomorphism, so β is an epimorphism. Next, let $t : \operatorname{Ker} \beta \to P \oplus M_2$ be the natural morphism; since $p_1\alpha$ is an isomorphism,

$\pi_P \circ t = 0$, so t factors through a morphism $\overline{t} : \mathrm{Ker}\,\beta \to M_2$. It follows that $e_2 \circ \overline{t} = \beta \circ t = 0$, therefore $\overline{t} = 0$, and finally $\mathrm{Ker}\,\beta = 0$, since t is a monomorphism. ◊

Set $A := \oplus_{i\in I} A_i$, and denote $e_i : A_i \to A$ (resp. $p_i : A \to A_i$) the natural injection (resp. projection), for every $i \in I$. Also, endow $I' := I \cup \{\infty\}$ with any total ordering $(I', <)$ for which ∞ is the maximal element.

Claim 4.3.5. Let $(a_j : A \to A)_{j\in J}$ be a finite system of endomorphisms such that $\sum_{j\in J} a_j = \mathbf{1}_A$. Then there exists a mapping $\varphi : I \to J$ such that the following holds :

(i) $a_{\varphi(i)} e_i : A_i \to A$ is a monomorphism, for every $i \in I$, and let $P_i := \mathrm{Im}(a_{\varphi(i)} e_i)$.
(ii) For every $l \in I'$, the natural morphism $\beta_l : \bigoplus_{i<l} P_i \oplus \bigoplus_{i\geq l} A_i \to A$ is an isomorphism.

Proof of the claim. We both define $\varphi(l)$ and prove (ii), by induction on l. If l is the smallest element of I, then $\beta_l = \mathbf{1}_A$, so (ii) is obvious. Hence, let $l \in I$ be any element, $l' \in I'$ the successor of l, and assume that β_l is an isomorphism. Set $a'_j := \beta_l^{-1} a_j \beta_l$ for every $j \in J$, and $M_l := \bigoplus_{i<l} P_i \oplus \bigoplus_{i\geq l} A_i$; also, let $e'_l : A_l \to M_l$ and $p'_l : M_l \to A_l$ be the natural morphisms. Clearly $\sum_{j\in J} a'_j = \mathbf{1}_{M_l}$, so $\sum_{j\in J} p'_l a'_j e'_l = \mathbf{1}_{A_l}$; since $\mathrm{End}_{\mathscr{C}}(A_l)$ is a local ring, it follows that there exists $j_l \in J$ such that $p'_l a'_{j_l} e'_l$ is an automorphism of A_l. However,

$$a'_{j_l} e'_l = \beta_l^{-1} a_{j_l} \beta_l e'_l = \beta_l^{-1} a_{j_l} e_l \tag{4.3.6}$$

so assertion (i) follows for the index l, by setting $\varphi(l) := j_l$.

Next, set $M_{l,1} := A_l$ and $M_{l,2} := \bigoplus_{i<l} P_i \oplus \bigoplus_{i>l} A_i$. The foregoing argument provides a subobject $\alpha : P'_l := \mathrm{Im}(a'_{\varphi(l)} e'_l) \to M_l = M_{l,1} \oplus M_{l,2}$ such that $p'_l \alpha : P'_l \to M_{l,1}$ is an isomorphism. By claim 4.3.4, it follows that the natural map $\beta'_{l'} : P'_l \oplus M_{l,2} \to M_l$ is an isomorphism. On the other hand, (4.3.6) implies that β_l restricts to an isomorphism $\gamma_l : P'_l \xrightarrow{\sim} P_l$. We deduce a commutative diagram

$$\begin{tikzcd} P'_l \oplus M_{l,2} \arrow[r, "\beta'_{l'}"] \arrow[d, "\gamma_l \oplus \mathbf{1}_{M_{l,2}}"'] & M_l \arrow[d, "\beta_l"] \\ M_{l'} \arrow[r, "\beta_{l'}"] & A \end{tikzcd}$$

whose vertical arrows and top arrow are isomorphisms. Thus, the bottom arrow is an isomorphism as well, as required. ◊

For every $j \in J$, let $e'_j : B_j \to A$ (resp. $p'_j : A \to B_j$) be the injection (resp. projection) deduced from a given isomorphism as in (a), and set $a_j := e'_j p'_j$ for every $j \in J$. Then $\sum_{j\in J} a_j = \mathbf{1}_A$, so claim 4.3.5 yields a mapping $\varphi : I \to J$ such that the following holds. Set $P_j := \oplus_{i\in\varphi^{-1}(j)} \mathrm{Im}(a_j e_i)$ for every $j \in J$; then the natural morphism $\beta : \oplus_{j\in J} P_j \xrightarrow{\sim} A$ is an isomorphism. However, clearly $\beta(P_j)$ is a subobject of $\mathrm{Im}(e'_j)$, for every $j \in J$. Hence, φ must be a surjection, and β restricts to isomorphisms $P_j \xrightarrow{\sim} \mathrm{Im}(e'_j)$ for every $j \in J$, whence isomorphisms $P_j \xrightarrow{\sim} B_j$, from which (ii) follows immediately. □

Remark 4.3.7. There are variants of theorem 4.3.3, that hold under different sets of assumptions. For instance, in [37, Ch.I, §6, Th.1] it is stated that the theorem still holds when I and J are no longer finite sets, provided that the category $\mathscr{C}$ admits generators and that all filtered colimits are representable and exact in $\mathscr{C}$.

4.3.8. Let us now specialize to the case of the category A-$\mathbf{Mod}$, where A is a (commutative) local ring, say with residue field k. Let M be any finitely generated A-module; arguing by induction on the dimension of $M \otimes_A k$, one shows easily that M admits a finite decomposition $M = \oplus_{i=1}^r M_i$, where M_i is indecomposable for every $i \leq r$.

Lemma 4.3.9. *Let A be a henselian local ring, and M an A-module of finite type. Then M is indecomposable if and only if $\mathrm{End}_A(M)$ is a local ring.*

Proof. In view of (4.3.2), we can assume that M is indecomposable, and we wish to prove that $\mathrm{End}_A(M)$ is local. Thus, let $\varphi \in \mathrm{End}_A(M)$.

Claim 4.3.10. The subalgebra $A[\varphi] \subset \mathrm{End}_A(M)$ is integral over A.

Proof of the claim. This is standard: choose a finite system of generators $(f_i)_{1\leq i\leq r}$ for M, then we can find a matrix $\mathbf{a} := (a_{ij})_{1\leq i,j\leq r}$ of elements of A, such that $\varphi(f_i) = \sum_{j=1}^r a_{ij} f_j$ for every $i \leq r$. The matrix $\mathbf{a}$ yields an endomorphism ψ of the free A-module $A^{\oplus r}$; on the other hand, let $e_1, \dots, e_r$ be the standard basis of $A^{\oplus r}$ and define an A-linear surjection $\pi : A^{\oplus r} \to M$ by the rule: $e_i \mapsto f_i$ for every $i \leq r$. Then $\varphi \circ \pi = \pi \circ \psi$; by Cayley-Hamilton, ψ is annihilated by the characteristic polynomial $\chi(T)$ of the matrix $\mathbf{a}$, whence $\chi(\varphi) = 0$. $\diamondsuit$

Since A is henselian, claim 4.3.10 implies that $A[\varphi]$ decomposes as a finite product of local rings. If there were more than one non-zero factor in this decomposition, the ring $A[\varphi]$ would contain an idempotent $e \neq 1, 0$, whence the decomposition $M = eM \oplus (1-e)M$, where both summands would be non-zero, which contradicts the assumption. Thus, $A[\varphi]$ is a local ring, so that either φ or $\mathbf{1}_M - \varphi$ is invertible. Since φ was chosen arbitrarily in $\mathrm{End}_A(M)$, the claim follows. $\square$

Corollary 4.3.11. *Let A be a henselian local ring. Then:*

(i) *If $(M_i)_{i\in I}$ and $(N_j)_{j\in J}$ are two finite families of indecomposable A-modules of finite type such that $\oplus_{i=1}^r M_i \simeq \oplus_{j=1}^s N_j$, then there is a bijection $\beta : I \xrightarrow{\sim} J$ such that $M_i \simeq N_{\beta(i)}$ for every $i \in I$.*

(ii) *If M and N are two finitely generated A-modules such that $M^{\oplus k} \simeq N^{\oplus k}$ for some integer $k > 0$, then $M \simeq N$.*

(iii) *If M, N and X are three finitely generated A-modules such that $X \oplus M \simeq X \oplus N$, then $M \simeq N$.*

Proof. It follows easily from theorem 4.3.3 and lemma 4.3.9; the details are left to the reader. $\square$

Proposition 4.3.12. *Let $A \to B$ be a faithfully flat map of local rings, M and N two finitely presented A-modules with a B-linear isomorphism $\omega : B \otimes_A M \xrightarrow{\sim} B \otimes_A N$. Then $M \simeq N$.*

Proof. Under the standing assumptions, the natural B-linear map

$$B \otimes_A \mathrm{Hom}_A(M, N) \to \mathrm{Hom}_B(B \otimes_A M, B \otimes_A N)$$

is an isomorphism ([36, Lemma 2.4.29(i.a)]). Hence we can write $\omega = \sum_{i=1}^r b_i \otimes \varphi_i$ for some $b_i \in B$ and $\varphi_i : M \to N$ $(1 \leq i \leq r)$. Denote by k and K the residue fields of A and B respectively; we set $\overline{\varphi}_i := \mathbf{1}_k \otimes_A \varphi_i : k \otimes_A M \to k \otimes_A N$. From the existence of ω we deduce easily that $n := \dim_k k \otimes_A M = \dim_k k \otimes_A N$. Hence, after choosing bases, we can view $\overline{\varphi}_1, \dots, \overline{\varphi}_r$ as endomorphisms of the k-vector space $k^{\oplus n}$. We consider the polynomial $p(T_1, \dots, T_r) := \det(\sum_{i=1}^r T_i \cdot \overline{\varphi}_i) \in k[T_1, \dots, T_r]$. Let $\overline{b}_1, \dots, \overline{b}_r$ be the images of $b_1, \dots, b_r$ in K; it follows that $p(\overline{b}_1, \dots, \overline{b}_r) \neq 0$, especially $p(T_1, \dots, T_r) \neq 0$.

Claim 4.3.13. The proposition holds if k is an infinite field or if $k = K$.

Proof of the claim. Indeed, in either of these cases we can find $\overline{a}_1, \dots, \overline{a}_r \in k$ such that $p(\overline{a}_1, \dots, \overline{a}_r) \neq 0$. For every $i \leq r$ choose an arbitrary representative $a_i \in A$ of $\overline{a}_i$, and set $\varphi := \sum_{i=1}^r a_i \varphi_i$. By construction, $\mathbf{1}_k \otimes_A \varphi : k \otimes_A M \to k \otimes_A N$ is an isomorphism. By Nakayama's lemma we deduce that φ is surjective. Exchanging the roles of M and N, the same argument yields an A-linear surjection $\psi : N \to M$. Finally, [61, Ch.1, Th.2.4] shows that both $\psi \circ \varphi$ and $\varphi \circ \psi$ are isomorphisms, whence the claim. $\diamondsuit$

Let A^{h} be the henselization of A, and A^{sh}, B^{sh} the strict henselizations of A and B respectively. Now, the induced map $A^{\mathrm{sh}} \to B^{\mathrm{sh}}$ is faithfully flat, the residue field of A^{sh} is infinite, and ω induces a B^{sh}-linear isomorphism $B^{\mathrm{sh}} \otimes_A M \xrightarrow{\sim} B^{\mathrm{sh}} \otimes_A N$. Hence, claim 4.3.13 yields an A^{sh}-linear isomorphism $\beta : A^{\mathrm{sh}} \otimes_A M \xrightarrow{\sim} A^{\mathrm{sh}} \otimes_A N$. However, A^{sh} is the colimit of a filtered family $(A_\lambda \mid \lambda \in \Lambda)$ of finite étale A^{h}-algebras, so β descends to an A_λ-linear isomorphism $\beta_\lambda : A_\lambda \otimes_A M \simeq A_\lambda \otimes_A N$ for some $\lambda \in \Lambda$. The A^{h}-module A_λ is free, say of finite rank n, hence β_λ can be regarded as an A^{h}-linear isomorphism $(A^{\mathrm{h}} \otimes_A M)^{\oplus n} \xrightarrow{\sim} (A^{\mathrm{h}} \otimes_A N)^{\oplus n}$. Then $A^{\mathrm{h}} \otimes_A M \simeq A^{\mathrm{h}} \otimes_A N$, by corollary 4.3.11(ii). Since the residue field of A^{h} is k, we conclude by another application of claim 4.3.13. $\square$

4.3.14. *Injective hulls.* The notion of injective hull plays a central role in the theory of local duality : for a noetherian local ring, one constructs a dualizing module as the injective hull of the residue field (see [44, Exp.IV, Th.4.7]), and in section 5.9, injective hulls of the residue fields of a monoid algebra will also enable us to perform a certain computation of local cohomology, which is a crucial step in the proof of Hochster's theorem. We present here the basic results on injective hulls, in the context of arbitrary abelian categories.

Definition 4.3.15. Let $\mathscr{A}$ be any abelian category, and $f : N \to M$ a monomorphism in $\mathscr{A}$.

(i) We say that M is an *essential extension of* N if the following holds. For any subobject $P \subset M$ we have either $P = 0$ or $P \cap \mathrm{Im}\, f \neq 0$ (here 0 denotes the zero object of $\mathscr{A}$: see remark 1.2.29(i)).

(ii) We say that M is a *proper essential extension of* N if it is an essential extension of N, and f is not an isomorphism.

(iii) We say that M is an *injective hull* of N, if M is both an essential extension of N, and an injective object of $\mathscr{A}$.

Lemma 4.3.16. *Let $\mathscr{A}$ be an abelian category, and I an object of $\mathscr{A}$. Suppose that :*

(a) *$\mathscr{A}$ is cocomplete.*

(b) *All colimits of $\mathscr{A}$ are universal (see example* 1.1.24(v)*).*

Then we have :

(i) *I is injective if and only if it does not admit any proper essential extensions.*

(ii) *If $N \to M$ is any monomorphism, the set of all essential extensions of N contained in M admits maximal elements.*

Proof. (i): Suppose that I is injective, and let $f : I \to M$ be any monomorphism which is not an isomorphism. Then f admits a left inverse, so I is a direct summand of M, hence M is not an essential extension of I. Conversely, suppose that I does not admit proper essential extensions; let $f : N \to M$ be a monomorphism in $\mathscr{A}$ and $g : N \to I$ any morphism in $\mathscr{A}$. We consider the cocartesian diagram in $\mathscr{A}$

$$\begin{tikzcd} N \arrow[r, "f"] \arrow[d, "g"'] & M \arrow[d, "g'"] \\ I \arrow[r, "f'"] & P \end{tikzcd}$$

and notice that f' is a monomorphism, since the same holds for f. By Zorn's lemma – and due to conditions (a) and (b) – we may find a maximal subobject $Q \subset P$ such that $Q \cap I = 0$, and clearly P/Q is an essential extension of I (via h); therefore $P/Q = I$, so $P = I \oplus Q$, and if we let $p : P \to I$ be the resulting projection, we see that $p \circ g' : M \to I$ is an extension of f. This shows that I is injective.

(ii): By Zorn's lemma, it suffices to check the following assertion. Suppose that $(P_j \mid j \in J)$ is a totally ordered family of essential extensions of N contained in N (for some totally ordered small indexing set J). Then $P := \bigcup_{j\in J} P_j$ is again an essential extension of N (notice that this union (*i.e.* colimit) is a subobject of M, due to condition (b)). However, say that $Q \subset P$ and $Q \cap N = 0$; due to condition (b), we have $Q = \bigcup_{j\in J}(Q \cap P_j)$, and clearly $(Q \cap P_j) \cap N = 0$, so $Q \cap P_j = 0$ for every $j \in J$, and finally $Q = 0$. □

Proposition 4.3.17. *Let $\mathscr{A}$ be an abelian category fulfilling condition* (a) *and* (b) *of lemma* 4.3.16*, and M any object of $\mathscr{A}$. We suppose moreover that :*

(c) *$\mathscr{A}$ has enough injective objects.*

Then we have :

(i) *M admits an injective hull. More precisely, if $M \to I$ is any monomorphism into an injective object, then a maximal essential extension of M in I is an injective hull of M.*
(ii) *Let $f : M \to E$ be an injective hull of M, and $g : M \to I$ any monomorphism into an injective object. Then g factors through f and a monomorphism $h : E \to I$.*
(iii) *If $f : M \to E$ and $g : M \to E'$ are two injective hulls of M, there exists an isomorphism $h : E \xrightarrow{\sim} E'$ such that $h \circ f = f'$.*

Proof. (i): Let $E \subset I$ be such a maximal essential extension of M (which exists by lemma 4.3.16(ii)); by virtue of lemma 4.3.16(i), it suffices to check that E does not admit proper essential extensions. However, suppose that $E \to E'$ is a proper essential extension; since I is injective, the inclusion morphism $E \to I$ extends to a morphism $f : E' \to I$. By maximality of E, we must then have $\mathrm{Ker}\, f \neq 0$; on the other hand, obviously $E \cap \mathrm{Ker}\, f = 0$, which is absurd, since E' is an essential extension of E.

(ii): Since I is injective, g extends to a morphism $h : E \to I$, and clearly $M \cap \mathrm{Ker}\, h = 0$. Since E is an essential extension of M, it follows that $\mathrm{Ker}\, h = 0$.

(iii): By (ii) there exists a monomorphism $h : E \to E'$ such that $h \circ f = f'$. Since E is injective, it follows that $\mathrm{Im}\, h$ is a direct summand of E'; but E' is an essential extension of M, so $\mathrm{Im}\, h = E'$, *i.e.* h is also an epimorphism, hence an isomorphism. □

4.3.18. We specialize now to the case where $\mathscr{A}$ is the category A-$\mathbf{Mod}$ of A-modules, with A an arbitrary noetherian ring. Recall that, if M is any A-module, then the set $\mathrm{Ass}\, M$ of all *associated primes* of M consists of the prime ideals $\mathfrak{p} \subset A$ such that there exists $m \in M$ with $\mathrm{Ann}_A(m) = \mathfrak{p}$. (This is the correct definition only for noetherian rings : we shall see in definition 5.5.1 a more general notion that is well behaved for arbitrary rings.) By proposition 4.3.17(i,iii) the injective hull of M exists and is well defined up to (in general, non-unique) isomorphism, and we shall denote by $E_A(M)$ a choice of such hull.

Lemma 4.3.19. *Let A be a noetherian ring. The following holds :*

(i) *If $(I_\lambda \mid \lambda \in \Lambda)$ is a (small) family of injective A-modules, then $I := \bigoplus_{\lambda\in\Lambda} I_\lambda$ is an injective A-module.*
(ii) *If $S \subset A$ is any multiplicative subset, and M any injective A-module, then M_S is an injective A_S-module.*

Proof. (i): Let us first recall :

Claim 4.3.20. Let R be any ring, M an R-module. The following conditions are equivalent :

(a) M is an injective R-module.
(b) For every ideal $J \subset R$, every R-linear map $J \to M$ extends to an R-linear map $R \to M$.
(c) $\mathrm{Ext}^1_R(R/J, M) = 0$ for every ideal $J \subset R$.

Proof of the claim. Of course, (a)⇒(b)⇒(c). To check that (c)⇒(a), let $N \subset P$ be an inclusion of R-modules, and $f : N \to M$ an A-linear map. We let S be the set of all pairs (N', f'), where $N' \subset P$ is an R-submodule containing N, and $f' : N' \to M$ an R-linear map extending f. The set S is partially ordered, by declaring that $(N', f') \geq (N'', f'')$ if $N'' \subset N'$ and $f'_{|N''} = f''$, for any two pairs $(N', f'), (N'', f'') \in S$. By Zorn's lemma, S admits a maximal element (Q, g). Suppose $Q \neq P$, and let $x \in P \setminus Q$. Set $Q' := Q + Rx$ and $J := \mathrm{Ann}_R(Q'/Q)$; there follows an exact sequence of R-modules

$$0 \to Q \to Q' \to R/J \to 0$$

and then (c) implies that the restriction map $\mathrm{Hom}_R(Q', M) \to \mathrm{Hom}_R(Q, M)$ is surjective; especially, g extetnds to an R-linear map $Q' \to M$, contradicting the maximality of Q. Hence $Q = P$, which shows (a). ♢

In view of claim 4.3.20, it suffices to show that every A-linear map $f : J \to I$ from any ideal $J \subset A$, extends to an A-linear map $A \to I$. However, since J is finitely generated, there exists a finite subset $\Lambda' \subset \Lambda$ such that the image of f is contained in $I' := \bigoplus_{\lambda \in \Lambda'} I_\lambda$. Clearly I' is an injective A-module, so f extends to an A-linear map $A \to I'$, and the assertion follows.

(ii): Let $J \subset A_S$ be any ideal, and write $J = I_S$ for some ideal $I \subset A$; since A is noetherian, we have

$$\mathrm{Ext}^1_{A_S}(A_S/J, M_S) = A_S \otimes_A \mathrm{Ext}^1_A(A, M)$$

so the assertion follows from claim 4.3.20. □

We may now state :

Proposition 4.3.21. *Let A be any noetherian ring, M any A-module. We have :*

(i) *For every $\mathfrak{p} \in \mathrm{Spec}\, A$, the A-module $E_A(A/\mathfrak{p})$ is indecomposable (see* (4.3.2)*).*

(ii) *Let I be any non-zero injective A-module, and $\mathfrak{p} \in \mathrm{Ass}\, I$ any associated prime. Then $E_A(A/\mathfrak{p})$ is a direct summand of I. Especially, if I is indecomposable, then I is isomorphic to $E_A(A/\mathfrak{p})$.*

(iii) $\mathrm{Ass}_A M = \mathrm{Ass}_A E_A(M)$.

(iv) *If $\mathfrak{p}, \mathfrak{q} \in \mathrm{Spec}\, A$ are any two prime ideals, then the A-modules $E_A(A/\mathfrak{p})$ and $E_A(A/\mathfrak{q})$ are isomorphic if and only if $\mathfrak{p} = \mathfrak{q}$.*

(v) *$E_{A_S}(M_S) \simeq A_S \otimes_A E_A(M)$ for any multiplicative subset $S \subset A$.*

Proof. (i): Set $B := A/\mathfrak{p}$, and suppose that $E_A(B)$ is decomposable; especially, there exist non-zero submodules $M_1, M_2 \subset E_A(B)$ with $M_1 \cap M_2 = 0$. Thus, $(M_1 \cap B) \cap (M_2 \cap B) = 0$, and since B is a domain, we deduce that $M_i \cap B = 0$ for $i = 1, 2$. But since $E_A(B)$ is an essential extension of B, this is absurd.

(ii): By assumption, there exists $x \in I$ such that $\mathrm{Ann}_A(x) = \mathfrak{p}$, so the submodule $Ax \subset I$ is isomorphic to $A/\mathfrak{p}$; then, by proposition 4.3.17(ii) there exists a monomorphism $f : E_A(A/\mathfrak{p}) \to I$, and since $E_A(A/\mathfrak{p})$ in injective, the image of f is a direct summand of I.

(iii): Clearly $\mathrm{Ass}_A(M) \subset \mathrm{Ass}_A E_A(M)$. Conversely, suppose $\mathfrak{p} \in \mathrm{Ass}_A E_A(M)$; then there exists an A-submodule $N \subset E_A(M)$ isomorphic to $A/\mathfrak{p}$. Since $E_A(M)$ is an essential extension of M, we have $N \cap M \neq 0$, so $\mathfrak{p} \in \mathrm{Ass}_A(M)$.

(iv) follows directly from (iii).

(v): We know already that $E' := A_S \otimes_A E_A(M)$ contains M_S and is injective (lemma 4.3.19(iii)), so it remains only to check that E' is an essential extension of M. Thus, let $x \in E' \setminus M_S$; we have to check that $N := A_S x \cap M_S \neq 0$, and clearly we may assume that $x \in E_A(M)$. Set $\mathscr{F} := \{\mathrm{Ann}_A(tx) \mid t \in S\}$; since A is noetherian, $\mathscr{F}$ admits maximal elements, and notice that $A_S x = A_S tx$ for any $t \in S$. Hence, we may replace x by tx for some $t \in S$, after which we may assume that $\mathrm{Ann}_A(x)$ is maximal in $\mathscr{F}$. We may write $Ax \cap M = Ix$ for some ideal $I \subset A$, so $N = I_S x$; let $a_1, \dots, a_k \in A$ be a system of generators for I, and notice that $Ix \neq 0$, since $E_A(M)$ is an essential extension of M. Suppose that $N = 0$; then there exists $t \in S$ such that the identity $ta_i x = 0$ holds in A for $i = 1, \dots, k$. However, $\mathrm{Ann}_A(x) = \mathrm{Ann}_A(tx)$ by construction, so $Ix = 0$, a contradiction. $\square$

Theorem 4.3.22. *Let A be a noetherian ring, I an injective A-module. We have :*

(i) *I decomposes as a direct sum of the form*

$$I \simeq \bigoplus_{\mathfrak{p} \in \mathrm{Spec}\, A} E_A(A/\mathfrak{p})^{(R_\mathfrak{p})} \qquad (4.3.23)$$

for a system of (small) sets $(R_\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Spec}\, A)$.

(ii) *Moreover, the cardinality of $R_\mathfrak{p}$ equals $\dim_{\kappa(\mathfrak{p})} \mathrm{Hom}_{A_\mathfrak{p}}(\kappa(\mathfrak{p}), I_\mathfrak{p})$, for every $\mathfrak{p} \in \mathrm{Spec}\, A$ (where $\kappa(\mathfrak{p}) := A_\mathfrak{p}/\mathfrak{p}_\mathfrak{p}$). Especially, this cardinality is independent of the decomposition* (4.3.23).

Proof. (i): Denote by $\mathscr{F}$ the set of all indecomposable injective submodules of I, and by $\mathscr{S}$ the set of all subsets $\mathscr{G} \subset \mathscr{F}$ such that the natural map $I_\mathscr{G} := \bigoplus_{G \in \mathscr{G}} G \to I$ is injective. The set $\mathscr{S}$ is partially ordered by inclusion, and by Zorn's lemma, $\mathscr{S}$ admits a maximal element $\mathscr{M}$; in light of proposition 4.3.21(ii), it suffices to check that $I_\mathscr{M} = I$. However, $I_\mathscr{M}$ is injective (lemma 4.3.19(i)), hence $I = I_\mathscr{M} \oplus J$ for some A-module J, and it is easily seen that J is injective as well. We are thus reduced to showing that $J = 0$. Now, if $J \neq 0$, let $\mathfrak{p} \in \mathrm{Ass}_A J$ be any associated prime ([61, Th.6.1(i)]); then $E_A(A/\mathfrak{p})$ is an indecomposable injective direct summand of J (proposition 4.3.21(i,ii)), and therefore $I_\mathscr{M} \oplus E_A(A/\mathfrak{p})$ is a submodule of I, contradicting the maximality of $\mathscr{M}$.

(ii): In light of (i) and proposition 4.3.21(iii,v) we have

$$\mathrm{Hom}_{A_\mathfrak{p}}(\kappa(\mathfrak{p}), I_\mathfrak{p}) = \mathrm{Hom}_{A_\mathfrak{p}}(\kappa(\mathfrak{p}), E_A(A/\mathfrak{p})_\mathfrak{p}^{(R_\mathfrak{p})}) = \kappa(\mathfrak{p})^{(R_\mathfrak{p})} \otimes_{\kappa(\mathfrak{p})} \mathrm{Hom}_{A_\mathfrak{p}}(\kappa(\mathfrak{p}), E_{A_\mathfrak{p}}(\kappa(\mathfrak{p})))$$

so we are reduced to showing the following :

Claim 4.3.24. Let $\mathfrak{m} \subset A$ be any maximal ideal, and set $\kappa := A/\mathfrak{m}$; then

$$d := \dim_\kappa \mathrm{Hom}_A(\kappa, E_A(\kappa)) = 1.$$

Proof of the claim. Since $\kappa \subset E_A(\kappa)$, obviously $d \geq 1$. On the other hand, we have a natural identification

$$\mathrm{Hom}_A(\kappa, E_A(\kappa)) = F := \{x \in E_A(\kappa) \mid \mathfrak{m}x = 0\}.$$

If $d > 1$, we may find $x \in F$ such that $Ax \cap \kappa = 0$; but this is absurd, since $E_A(\kappa)$ is an essential extension of κ. $\square$

4.3.25. When dealing with the more general coherent rings that appear in sections 5.7 and 5.8, the injective hull is no longer suitable for the study of local cohomology, but it turns out that one can use instead a "coh-injective hull", which works just as well.

Definition 4.3.26. Let A be a ring; we denote by $A\text{-}\mathbf{Mod}_{\mathrm{coh}}$ the full subcategory of $A\text{-}\mathbf{Mod}$ consisting of all coherent A-modules.

(i) An A-module J is said to be *coh-injective* if the functor

$$A\text{-}\mathbf{Mod}_{\mathrm{coh}} \to A\text{-}\mathbf{Mod}^o \quad : \quad M \mapsto \mathrm{Hom}_A(M, J)$$

is exact.

(ii) An A-module M is said to be *ω-coherent* if it is countably generated, and every finitely generated submodule of M is finitely presented.

Lemma 4.3.27. *Let A be a ring.*

(i) *Let M, J be two A-modules, $N \subset M$ a submodule. Suppose that J is coh-injective and both M and M/N are ω-coherent. Then the natural map:*

$$\mathrm{Hom}_A(M,J) \to \mathrm{Hom}_A(N,J)$$

is surjective.

(ii) *Let $(J_\lambda \mid \lambda \in \Lambda)$ be a filtered family of coh-injective A-modules. Then $\operatorname*{colim}_{\lambda\in\Lambda} J_\lambda$ is coh-injective.*

(iii) *Assume that A is coherent, let $M_\bullet$ be an object of $\mathsf{D}^-(A\text{-}\mathbf{Mod}_{\mathrm{coh}})$ and $I^\bullet$ a bounded below complex of coh-injective A-modules. Then the natural map*

$$\mathrm{Hom}^\bullet_A(M_\bullet, I^\bullet) \to R\mathrm{Hom}^\bullet_A(M_\bullet, I^\bullet)$$

is an isomorphism in $\mathsf{D}(A\text{-}\mathbf{Mod})$.

Proof. (i): Since M is ω-coherent, we may write it as an increasing union $\bigcup_{n\in\mathbb{N}} M_n$ of finitely generated, hence finitely presented submodules. For each $n \in \mathbb{N}$, the image of M_n in M/N is a finitely generated, hence finitely presented submodule; therefore $N_n := N \cap M_n$ is finitely generated, hence it is a coherent A-module, and clearly $N = \bigcup_{n\in\mathbb{N}} N_n$. Suppose $\varphi : N \to J$ is any A-linear map; for every $n \in \mathbb{N}$, set $P_{n+1} := M_n + N_{n+1} \subset M$, and denote by $\varphi_n : N_n \to J$ the restriction of φ. We wish to extend φ to a linear map $\varphi' : M \to J$, and to this aim we construct inductively a compatible system of extensions $\varphi'_n : P_n \to J$, for every $n > 0$. For $n = 1$, one can choose arbitrarily an extension φ'_1 of φ_1 to P_1. Next, suppose $n > 0$, and that φ'_n is already given; since $P_n \subset M_n$, we may extend φ'_n to a map $\varphi''_n : M_n \to J$. Since $M_n \cap N_{n+1} = N_n$, and since the restrictions of φ''_n and φ_{n+1} agree on N_n, there exists a unique A-linear map $\varphi'_{n+1} : P_{n+1} \to J$ that extends both φ''_n and φ_{n+1}.

(ii): Recall that a coherent A-module is finitely presented. However, it is well known that an A-module M is finitely presented if and only if the functor $Q \mapsto \mathrm{Hom}_A(M,Q)$ on A-modules, commutes with filtered colimits (see *e.g.* [36, Prop.2.3.16(ii)]). The assertion is an easy consequence.

(iii): Since A is coherent, one can find a resolution $\varphi : P_\bullet \to M_\bullet$ consisting of free A-modules of finite rank (cp. [36, §7.1.20]). One looks at the spectral sequences

$$E_1^{pq} : \mathrm{Hom}_A(P_p, I^q) \Rightarrow R^{p+q}\mathrm{Hom}^\bullet_A(M_\bullet, I^\bullet)$$
$$F_1^{pq} : \mathrm{Hom}_A(M_p, I^q) \Rightarrow H^{p+q}\mathrm{Hom}^\bullet_A(M_\bullet, I^\bullet).$$

The resolution φ induces a morphism of spectral sequences $F_1^{\bullet\bullet} \to E_1^{\bullet\bullet}$; on the other hand, since I^q is coh-injective, we have : $E_2^{pq} \simeq \mathrm{Hom}_A(H_p F_\bullet, I^q) \simeq \mathrm{Hom}_A(H_p M_\bullet, I^q) \simeq F_2^{pq}$, so the induced map $F_2^{pq} \to E_2^{pq}$ is an isomorphism for every $p, q \in \mathbb{N}$, whence the claim. $\square$

4.3.28. Let A be a coherent ring, $Z \subset \mathrm{Spec}\, A$ a constructible closed subset; we denote by

$$A\text{-}\mathbf{Mod}_{\mathrm{coh},Z}$$

the full subcategory of $A\text{-}\mathbf{Mod}_{\mathrm{coh}}$ whose objects are the (coherent) A-modules with support contained in Z. Let $I \subset A$ be any finitely generated ideal such that $Z = \mathrm{Spec}\, A/I$; it is easily seen that

$$\mathrm{Ob}(A\text{-}\mathbf{Mod}_{\mathrm{coh},Z}) = \bigcup_{n\in\mathbb{N}} \mathrm{Ob}(A/I^n\text{-}\mathbf{Mod}_{\mathrm{coh}}).$$

Now, consider a functor

$$T : A\text{-}\mathbf{Mod}^o_{\mathrm{coh},Z} \to \mathbb{Z}\text{-}\mathbf{Mod}.$$

Notice that TM is naturally an A-module, for every coherent A-module M : indeed, if $a \in A$ is any element, we may define the scalar multiplication by a on TM as the $\mathbb{Z}$-linear endomorphism $T(a \cdot \mathbf{1}_M)$. In other words, T factors through the forgetful functor $A\text{-}\mathbf{Mod} \to \mathbb{Z}\text{-}\mathbf{Mod}$;

especially, if we set

$$H_T := \operatorname*{colim}_{n\in\mathbb{N}} T(A/I^n)$$

we get a natural A-linear map

$$M \xrightarrow{\sim} \operatorname*{colim}_{n\in\mathbb{N}} \mathrm{Hom}_A(A/I^n, M) \to \operatorname*{colim}_{n\in\mathbb{N}} \mathrm{Hom}_A(TM, T(A/I^n)) \to \mathrm{Hom}_A(TM, H_T)$$

whence a bilinear pairing

$$M \times TM \to H_T$$

which in turns yields a natural transformation

$$\omega_M : TM \to \mathrm{Hom}_A(M, H_T) \qquad \text{for every } M \in \mathrm{Ob}(A\text{-}\mathbf{Mod}_{\mathrm{coh},Z}).$$

Lemma 4.3.29. *In the situation of* (4.3.28)*, the following conditions are equivalent :*

(a) ω *is an isomorphism of functors* $T \xrightarrow{\sim} \mathrm{Hom}_A(-, H_T)$.
(b) T *is left exact.*

Proof. Clearly (a)⇒(b). For the converse, suppose first that $Z = \mathrm{Spec}\, A$, in which case, notice that $H_T = TA$ and ω_A is the natural isomorphism $TA \xrightarrow{\sim} \mathrm{Hom}_A(A, TA)$; then, if M is any coherent A-module, pick a finite presentation

$$\Sigma \quad : \quad A^{\oplus n} \to A^{\oplus m} \to M \to 0$$

and apply the 5-lemma to the resulting ladder of A-modules ω_Σ with left exact rows, to deduce that ω_M is an isomorphism. Next, for a general ideal I and M an A-module with $\mathrm{Supp}\, M \subset Z$, we may find $n \in \mathbb{N}$ such that M is an A/I^n-module; since A/I^k is coherent, the foregoing case then shows that the induced map

$$TM \to \mathrm{Hom}_A(M, T(A/I^k))$$

is an isomorphism, for every $k \geq n$. To conclude, it suffices to remark that the natural map

$$\operatorname*{colim}_{k\in\mathbb{N}} \mathrm{Hom}_A(M, T(A/I^k)) \to \mathrm{Hom}_A(M, H_T)$$

is an isomorphism, since M is finitely presented ([36, Prop.2.3.16(ii)]). □

We wish next to present a criterion that allows to detect, among the functors T as in (4.3.28), those that are exact. A complete characterization shall be given only for a restricted class of coherent ring; namely, we make the following :

Definition 4.3.30. Let A be a coherent ring. We say that A is an *Artin-Rees* ring, if the following holds. For every finitely generated ideal $I \subset A$, every coherent A-module M, and every finitely generated A-submodule $N \subset M$, the I-adic topology of M induces the I-adic topology on N.

We can then state :

Proposition 4.3.31. *In the situation of* (4.3.28) *suppose furthermore that A is an Artin-Rees ring. Then the following conditions are equivalent :*

(a) H_T *is a coh-injective A-module.*
(b) T *is exact.*

Proof. Clearly (a)⇒(b). For the converse, let M be any coherent A-module, and $N \subset M$ any finitely generated A-submodule; it suffices to check that the induced map

$$\mathrm{Hom}_A(M, H_T) \to \mathrm{Hom}_A(N, H_T)$$

is surjective. However, let $f : N \to H_T$ be any A-linear map; since N is finitely presented, there exists $n \in \mathbb{N}$ such that f factors through an A-linear map $f_n : N \to T(A/I^n)$ ([36, Prop.2.3.16(ii)]), and clearly $I^n N \subset \mathrm{Ker}\, f_n$, so $I^n N \subset \mathrm{Ker}\, f$ as well. Since the I-adic topology of N agrees with the topology induced by the I-adic topology of M, there exists $k \in \mathbb{N}$ such that $I^kM \cap N \subset I^nN$. Set $\overline{N} := N/(I^kM \cap N)$; then $\overline{N}$ is a finitely generated A-submodule of M/I^kM, and especially it is a coherent A-module. By construction, f factors through an A-linear map $\overline{f} : \overline{N} \to H_T$; assumption (b) and lemma 4.3.29 imply that $\overline{f}$ extends to an A-linear map $M/I^kM \to H_T$, and the resulting map $M \to H_T$ extends f, whence (a). $\square$

Remark 4.3.32. (i) In the terminology of definition 4.3.30, the standard Artin-Rees lemma implies that every noetherian ring is an Artin-Rees ring. We shall see later that, if V is any valuation ring, then every essentially finitely presented V-algebra is an Artin-Rees ring (corollary 5.7.2 and theorem 5.7.29).

(ii) On the other hand, if A is noetherian, claim 4.3.20 easily implies that an A-module is coh-injective if and only if it is injective.

(iii) Combining (i) and (ii) with proposition 4.3.31, we recover [44, Exp.IV, Prop.2.1].

Example 4.3.33. Let κ_0 be a field, A a noetherian κ_0-algebra, $\mathfrak{m} \subset A$ a maximal ideal such that $\kappa := A/\mathfrak{m}$ is a finite extension of κ_0, and set $Z := \{\mathfrak{m}\} \subset \operatorname{Spec} A$. Then every object of $A\text{-}\mathbf{Mod}_{\mathrm{coh},Z}$ is a finite dimensional κ_0-vector space, and therefore the functor

$$T : A\text{-}\mathbf{Mod}_{\mathrm{coh},Z} \to \kappa_0\text{-}\mathbf{Mod} \qquad M \mapsto \operatorname{Hom}_{\kappa_0}(M, \kappa_0)$$

is exact. Proposition 4.3.31 and remark 4.3.32(ii) then say that

$$H_T := \operatorname*{colim}_{n\in\mathbb{N}} \operatorname{Hom}_{\kappa_0}(A/\mathfrak{m}^n, \kappa_0)$$

is an injective A-module. More precisely, notice that $\operatorname{Hom}_A(\kappa, H_T) = \operatorname{Ann}_{H_T}(\mathfrak{m}) = T(\kappa) \simeq \kappa$, therefore H_T is the injective hull of the residue field κ.

4.3.34. *Flatness criteria.* The following generalization of the local flatness criterion answers affirmatively a question raised in [32, Ch.IV, Rem.11.3.12].

Lemma 4.3.35. *Let A be a ring, $I \subset A$ an ideal, B a finitely presented A-algebra, $\mathfrak{p} \subset B$ a prime ideal containing IB, and M a finitely presented B-module. Then the following two conditions are equivalent:*

(a) *$M_\mathfrak{p}$ is a flat A-module.*
(b) *$M_\mathfrak{p}/IM_\mathfrak{p}$ is a flat A/I-module and* $\operatorname{Tor}_1^A(M_\mathfrak{p}, A/I) = 0$.

Proof. Clearly, it suffices to show that (b)⇒(a), hence we assume that (b) holds. We write A as the union of the filtered family $(A_\lambda \mid \lambda \in \Lambda)$ of its local noetherian subalgebras, and set $A' := A/I$, $I_\lambda := I \cap A_\lambda$, $A'_\lambda := A_\lambda/I_\lambda$ for every $\lambda \in \Lambda$. Then, for some $\lambda \in \Lambda$, the A-algebra B descends to an A_λ-algebra B_λ of finite type, and M descends to a finitely presented B_λ-module M_λ. For every $\mu \geq \lambda$ we set $B_\mu := A_\mu \otimes_{A_\lambda} B_\lambda$ and $M_\mu := B_\mu \otimes_{B_\lambda} M_\lambda$. Up to replacing Λ by a cofinal family, we can assume that B_μ and M_μ are defined for every $\mu \in \Lambda$. Then, for every $\lambda \in \Lambda$ let $g_\lambda : B_\lambda \to B$ be the natural map, and set $\mathfrak{p}_\lambda := g_\lambda^{-1}\mathfrak{p}$.

Claim 4.3.36. There exists $\lambda \in \Lambda$ such that $M_{\lambda,\mathfrak{p}_\lambda}/I_\lambda M_{\lambda,\mathfrak{p}_\lambda}$ is a flat A'_λ-module.

Proof of the claim. Set $B'_\lambda := B_\lambda \otimes_{A_\lambda} A'_\lambda$ and $M'_\lambda := M_\lambda \otimes_{A_\lambda} A'_\lambda$ for every $\lambda \in \Lambda$; clearly the natural maps

$$\operatorname*{colim}_{\lambda\in\Lambda} A'_\lambda \to A/IA \qquad \operatorname*{colim}_{\lambda\in\Lambda} B'_\lambda \to B/IB \qquad \operatorname*{colim}_{\lambda\in\Lambda} M'_\lambda \to M/IM$$

are isomorphisms. Then the claim follows from (b) and [32, Ch.IV, Cor.11.2.6.1(i)]. $\Diamond$

In view of claim 4.3.36, we can replace Λ by a cofinal subset, and thereby assume that $M_{\lambda,\mathfrak{p}_\lambda}/I_\lambda M_{\lambda,\mathfrak{p}_\lambda}$ is a flat A'_λ-module for every $\lambda \in \Lambda$.

Claim 4.3.37. (i) The natural map: $\operatorname*{colim}_{\lambda\in\Lambda} \operatorname{Tor}_1^{A_\lambda}(M_\lambda, A'_\lambda) \to \operatorname{Tor}_1^A(M, A')$ is an isomorphism.

(ii) For every $\lambda, \mu \in \Lambda$ with $\mu \geq \lambda$, the natural map:

$$f_{\lambda\mu} : A_\mu \otimes_{A_\lambda} \mathrm{Tor}_1^{A_\lambda}(M_\lambda, A'_\lambda) \to \mathrm{Tor}_1^{A_\mu}(M_\mu, A'_\mu)$$

is surjective.

Proof of the claim. For every $\lambda \in \Lambda$, let $L_\bullet(M_\lambda)$ denote the canonical free resolution of the A_λ-module M_λ ([17, Ch.X, §3, n.3]). Similarly, denote by $L_\bullet(M)$ the canonical free resolution of the A-module M. It follows from [18, Ch.II, §6, n.6, Cor.] and the exactness properties of filtered colimits, that the natural map: $\operatorname*{colim}_{\lambda\in\Lambda} L_\bullet(M_\lambda) \to L_\bullet(M)$ is an isomorphism. Hence:

$$\begin{aligned}\operatorname*{colim}_{\lambda\in\Lambda} H_\bullet(A'_\lambda \overset{\mathbf{L}}{\otimes}_{A_\lambda} M_\lambda) &\simeq \operatorname*{colim}_{\lambda\in\Lambda} H_\bullet(A'_\lambda \otimes_{A_\lambda} L_\bullet(M_\lambda)) \\ &\simeq H_\bullet(A' \otimes_A \operatorname*{colim}_{\lambda\in\Lambda} L_\bullet(M_\lambda)) \\ &\simeq H_\bullet(A' \otimes_A L_\bullet(M)) \\ &\simeq H_\bullet(A' \overset{\mathbf{L}}{\otimes}_A M)\end{aligned}$$

which proves (i). To show (ii) we use the base change spectral sequences for Tor ([75, Th.5.6.6])

$$\begin{aligned} E^2_{pq} &: \mathrm{Tor}_p^{\mathbb{A}_\mu}(\mathrm{Tor}_q^{A_\lambda}(M_\lambda, A_\mu), A'_\mu) \Rightarrow \mathrm{Tor}_{p+q}^{A_\lambda}(M_\lambda, A'_\mu) \\ F^2_{pq} &: \mathrm{Tor}_p^{A'_\lambda}(\mathrm{Tor}_q^{A_\lambda}(M_\lambda, A'_\lambda), A'_\mu) \Rightarrow \mathrm{Tor}_{p+q}^{A_\lambda}(M_\lambda, A'_\mu).\end{aligned}$$

Since $F^2_{10} = 0$, the natural map

$$F^2_{01} := A'_\mu \otimes_{A'_\lambda} \mathrm{Tor}_q^{A_\lambda}(M_\lambda, A'_\lambda) \to \mathrm{Tor}_1^{A_\lambda}(M_\lambda, A'_\mu)$$

is an isomorphism. On the other hand, we have a surjection:

$$\mathrm{Tor}_1^{A_\lambda}(M_\lambda, A'_\mu) \to E^2_{10} := \mathrm{Tor}_1^{A_\mu}(M_\mu, A'_\mu)$$

whence the claim. ◊

We deduce from claim 4.3.37(i) that the natural map

$$\operatorname*{colim}_{\lambda\in\Lambda} \mathrm{Tor}_1^{A_\lambda}(M_{\lambda,\mathfrak{p}_\lambda}, A'_\lambda) \to \mathrm{Tor}_1^{A}(M_{\mathfrak{p}}, A') = 0$$

is an isomorphism. However, $\mathrm{Tor}_1^{A_\lambda}(M_{\lambda,\mathfrak{p}_\lambda}, A'_\lambda)$ is a finitely generated $B_{\lambda,\mathfrak{p}_\lambda}$-module by [17, Ch.X, §6, n.4, Cor.]. We deduce that $f_{\lambda\mu,\mathfrak{p}_\lambda} = 0$ for some $\mu \geq \lambda$; therefore $\mathrm{Tor}_1^{A_\mu}(M_{\mu,\mathfrak{p}_\mu}, A'_\mu) = 0$, in view of claim 4.3.37(ii), and then the local flatness criterion of [28, Ch.0, §10.2.2] says that $M_{\mu,\mathfrak{p}_\mu}$ is a flat A_μ-module, so finally $M_\mathfrak{p}$ is a flat A-module, as stated. □

Lemma 4.3.38. *Let A be a ring, $I \subset A$ an ideal. Then :*

(i) *The following are equivalent :*
   (a) *The map $A \to A/I$ is flat.*
   (b) *The map $A \to A/I$ is a localization.*
   (c) *For every prime ideal $\mathfrak{p} \subset A$ containing I, we have $IA_\mathfrak{p} = 0$, especially, $V(I)$ is closed under generizations in* $\mathrm{Spec}\, A$.

(ii) *Suppose I fulfills the conditions* (a)-(c) *of* (i)*. Then the following are equivalent :*
   (a) *I is finitely generated.*
   (b) *I is generated by an idempotent.*
   (c) *$V(I) \subset \mathrm{Spec}\, A$ is open.*

Proof. (i): Clearly (b)⇒(a). Also (a)⇒(c), since every flat local homomorphism is faithfully flat. Suppose that (c) holds; we show that the natural map $B := (1 + I)^{-1}A \to A/I$ is an isomorphism. Indeed, notice that IB is contained in the Jacobson radical of B, hence it vanishes, since it vanishes locally at every maximal ideal of B.

(ii): Clearly (c)⇒(b)⇒(a). By condition (i.c), we see that every element of I vanishes on an open subset of $\mathrm{Spec}\, A$ containing $V(I)$, hence (a)⇒(c) as well. □

4.3.39. For every ring A, we let $\mathscr{I}(A)$ be the set of all ideals $I \subset A$ fulfilling the equivalent conditions (i.a)-(i.c) of lemma 4.3.38, and $\mathscr{Z}(A)$ the set of all closed subset $Z \subset \mathrm{Spec}\, A$ that are closed under generizations in $\mathrm{Spec}\, A$. In light of lemma 4.3.38(i.c), we have a natural mapping:

$$\mathscr{I}(A) \to \mathscr{Z}(A) \quad : \quad I \mapsto V(I). \tag{4.3.40}$$

Lemma 4.3.41. *The mapping* (4.3.40) *is a bijection, whose inverse assigns to any $Z \in \mathscr{Z}(A)$ the ideal $I[Z]$ consisting of all the elements $f \in A$ such that $fA_{\mathfrak{p}} = 0$ for every $\mathfrak{p} \in Z$.*

Proof. Notice first that $I[V(I)] = I$ for every $I \in \mathscr{I}(A)$; indeed, clearly $I \subset I[V(I)]$, and if $f \in I[V(I)]$, then $fA_{\mathfrak{p}} = 0$ for every prime ideal $\mathfrak{p}$ containing I, hence the image of f in A/I vanishes, so $f \in I$. To conclude the proof, it remains to show that, if Z is closed and closed under generizations, then $V(I[Z]) = Z$. However, say that $Z = V(J)$, and suppose that $f \in J$; then $\mathrm{Spec}\, A_{\mathfrak{p}} \subset Z$ for every $\mathfrak{p} \in Z$, hence f is nilpotent in $A_{\mathfrak{p}}$, so there exists $n \in \mathbb{N}$ and an open neighborhood $U \subset \mathrm{Spec}\, A$ of $\mathfrak{p}$ such that $f^n = 0$ in U. Since Z is quasi-compact, finitely many such U suffice to cover Z, hence we may find $n \in \mathbb{N}$ large enough such that $f^n \in I[Z]$, whence the contention. □

Proposition 4.3.42. *Let $f : X' \to X$ a quasi-compact and faithfully flat morphism of schemes. Then the topological space underlying X is the quotient of X' under the equivalence relation induced by f.*

Proof. The assertion means that a subset $Z \subset X$ is open (resp. closed) if and only if the same holds for the subset $f^{-1}Z$ of X'. For any subset Z of X (resp. of Y'), we denote by $\overline{Z}$ the topological closure of Z in X (resp. in Y'). The proposition will result from the following more general :

Claim 4.3.43. Let $g : Y' \to X$ be a flat morphism of schemes, $h : Y'' \to X$ a quasi-compact morphism of schemes, and set $Z := h(Y'')$. Then

$$g^{-1}\overline{Z} = \overline{g^{-1}Z}.$$

Proof of the claim. Quite generally, if T is a topological space, $U \subset T$ an open subset, $Z \subset T$ any subset, and $\overline{Z}$ the topological closure of Z in X, then $\overline{Z} \cap U$ is the topological closure of $Z \cap U$ in U (where the latter is endowed with the topology induced by X) : indeed, set $Z' := Z \cap U$, $W := X \setminus U$, and notice that

$$\overline{Z} \cup W = \overline{Z \cup W} = \overline{Z' \cup W} = \overline{Z'} \cup W$$

hence we may assume that $Z \subset U$, in which case the assertion is obvious.

Now, let $(U_i \mid i \in I)$ be any affine open covering of X, and set $U'_i := g^{-1}U_i$, $U'' := h^{-1}U_i$ for every $i \in I$; by the foregoing, we are reduced to showing the claim with g and h replaced by the morphisms $g_{|U'_i} : U'_i \to U_i$ and $h_{|U''_i} : U''_i \to U_i$, hence we may assume that X is affine. Likewise, by considering an affine open covering of Y' and arguing similarly, we reduce to the case where Y' is affine as well. In this situation, since h is quasi-compact, Y'' is a finite union of affine open subsets $U''_1, \dots, U''_n$, hence we may replace Y'' by the disjoint union of $U''_1, \dots, U''_n$, and assume that Y'' is also affine. Say that $X = \mathrm{Spec}\, A$, $Y' = \mathrm{Spec}\, B'$ and $Y'' = \mathrm{Spec}\, B''$, and denote by $I \subset A$ the kernel of the ring homomorphism $A \to B''$ corresponding to h; then $\overline{Z} = \mathrm{Spec}\, A/I$. Let $h' : Y' \times_X Y'' \to Y'$ be the morphism obtained from h by base change, and notice that $g^{-1}Z$ is the image of h'; therefore, if we let $I' \subset B'$ be the kernel of the induced map $B' \to B' \otimes_A B''$, we have $\overline{g^{-1}Z} = \mathrm{Spec}\, B'/I'$. However, since B' is a flat A-algebra, $I' = IB'$, whence the claim. ◊

Let now $Z \subset X$ be a subset, such that $Z' := f^{-1}Z$ is closed in X', and endow Z' with the reduced subscheme structure. The restriction $h : Z' \to X$ of f is then quasi-compact, and clearly $f^{-1}(hZ') = Z'$. Thus, claim 4.3.43 says that $f^{-1}\overline{Z} = Z'$, which means that $\overline{Z} = Z$, *i.e.* Z is closed, as stated. $\square$

Corollary 4.3.44. *Let A be a ring whose set* $\mathrm{Min}\, A$ *(resp.* $\mathrm{Max}\, A$*) of minimal prime (resp. maximal) ideals is finite. If $Z \subset \mathrm{Spec}\, A$ is a subset closed under specializations and generizations, then Z is open and closed.*

Proof. When $\mathrm{Min}\, A$ is finite, any such Z is a finite union of irreducible components, hence it is closed. But the same applies to the complement of Z, whence the contention. When A is semi-local, consider the faithfully flat map $A \to B$, where B is the product of the localizations of A at its maximal ideals. Clearly the assertion holds for B, hence for A, by proposition 4.3.42. $\square$

The following result is borrowed from [65, Cor.2.6].

Proposition 4.3.45. *Let R be a ring, R' any finitely generated R-algebra, and suppose that* $\mathrm{Spec}\, R$ *is a noetherian topological space. Then the same holds for* $\mathrm{Spec}\, R'$*.*

Proof. Let us first remark :

Claim 4.3.46. Let T be a quasi-compact and quasi-separated topological space. We have :

(i) T is noetherian if and only if every closed subset of T is constructible.
(ii) Denote by $\mathscr{C}$ the set of all non-constructible closed subsets of T, partially ordered by inclusion. If $\mathscr{C}$ is not empty, it admits minimal elements, and every minimal element of $\mathscr{C}$ is an irreducible subset of T.

Proof of the claim. (i) shall be left to the reader.

(ii): Let $\{Z_i \mid i \in I\}$ be a totally ordered subset of $\mathscr{C}$, and set $Z := \bigcap_{i\in I} Z_i$; we claim that $Z \in \mathscr{C}$. Indeed, if this fails, Z is constructible, hence its complement is a quasi-compact open subset of T, and therefore it equals $T \setminus Z_i$ for some $i \in I$; but then $Z = Z_i$, so Z_i is constructible, a contradiction. By Zorn's lemma, we deduce that $\mathscr{C}$ admits minimal elements. Let $Z \in \mathscr{C}$ be such a minimal element, and suppose that Z is not irreducible; then $Z = Z_1 \cup Z_2$ for some closed subsets Z_1, Z_2 strictly contained in Z. By minimality of Z, both Z_1 and Z_2 are constructible, and therefore the same must hold for Z, a contradiction. $\diamondsuit$

Let R' be a finitely generated R-algebra. In order to show that $\mathrm{Spec}\, R'$ is noetherian, we may assume that R' is a free polynomial R-algebra of finite type, and then an easy induction reduces to the case where $R' = R[X]$. Now, suppose $Y := \mathrm{Spec}\, R[X]$ is not noetherian, and let $f : Y \to X := \mathrm{Spec}\, R$; by claim 4.3.46 we may find an irreducible non-contructible closed subset $Z \subset Y$. By assumption, the topological closure W of $f(Z)$ in X is an irreducible constructible closed subset, and therefore $f^{-1}W$ is a constructible closed subset of Y (recall that every morphism of schemes is continuous for the constructible topology). It follows that every constructible subset of $f^{-1}W$ is also constructible as a subset of Y; especially, Z is not constructible in $f^{-1}W$. We may then replace X by W and Y by $f^{-1}W$, and assume from start that $f(Z)$ is a dense subset of X and R is a domain, in which case we set $K := \mathrm{Frac}\, R$. Let $\mathfrak{p} \in Y$ be the generic point of Z; so $\mathfrak{p}$ is a prime ideal of $R[X]$, and $\mathfrak{p}K[X]$ is a principal ideal generated by some polynomial $p(X) \in \mathfrak{p}$. Let $c \in R$ be the leading coefficient of $p(X)$, set $U := \mathrm{Spec}\, R[c^{-1}]$ and notice that $f(Z) \cap U \neq \varnothing$. Hence, $Z \setminus f^{-1}U$ is a closed subset of Y strictly contained in Z, so it is constructible. It follows that $Z \cap f^{-1}U$ is a non-constructible subset of Y; since $f^{-1}U$ is a constructible subset of Y, we conclude that $Z \cap f^{-1}U$ is a non-constructible subset of $f^{-1}U$. Hence, we may replace R by $R[c^{-1}]$ and assume from start that $\mathfrak{p}K[X]$ is generated by a monic polynomial $p(X) \in \mathfrak{p}$. However, we notice :

Claim 4.3.47. Let A be a domain, K the field of fractions of A, $\mathfrak{p} \subset A[X]$ any ideal, and $p(X) \in \mathfrak{p}$ a monic polynomial that generates $\mathfrak{p}K[X]$. Then $p(X)$ generates $\mathfrak{p}$.

Proof of the claim. This is elementary : consider any $g(X) \in \mathfrak{p}$, and let $g(X) = q(X) \cdot p(X) + r(X)$ be the euclidean division of g by p in $K[X]$. Since p generates $\mathfrak{p}K[X]$, we have $r = 0$, and since p is monic, it is easily seen that $q \in A[X]$, whence the claim. $\Diamond$

Claim 4.3.47 says that $\mathfrak{p}$ is a principal ideal of $R[X]$; but in that case, Z must be constructible, a contradiction. $\square$

4.4. **Graded rings, filtered rings and differential graded algebras.**

4.4.1. *Graded rings.* To motivate the results of this paragraph, let us review briefly the well known correspondance between Γ-gradings on a module M (for a given commutative group Γ), and actions of the diagonalizable group $D(\Gamma)$ on M.

Let S be a scheme; on the category $\mathbf{Sch}/S$ of S-schemes we have the presheaf of rings :

$$\mathscr{O}_{\mathbf{Sch}/S} : \mathbf{Sch}/S \to \mathbb{Z}\text{-}\mathbf{Alg} \qquad (X \to S) \mapsto \Gamma(X, \mathscr{O}_X).$$

Also, let M be an $\mathscr{O}_S$-module; following [24, Exp.I, Déf.4.6.1], we attach to M the $\mathscr{O}_{\mathbf{Sch}/S}$-module $\mathscr{W}_M$ given by the rule : $(f : X \to S) \mapsto \Gamma(X, f^*M)$. If M and N are two quasi-coherent $\mathscr{O}_S$-modules, and $f : S' \to S$ an affine morphism, it is easily seen that

$$\Gamma(S', \mathscr{H}om_{\mathscr{O}_{\mathbf{Sch}/S}}(\mathscr{W}_M, \mathscr{W}_N)) = \mathrm{Hom}_{\mathscr{O}_S}(M, f_*\mathscr{O}_{S'} \otimes_{\mathscr{O}_S} N) \tag{4.4.2}$$

(see [24, Exp.I, Prop.4.6.4] for the details).

Let $f : G \to S$ be a group S-scheme, *i.e.* a group object in the category $\mathbf{Sch}/S$. If f is affine, we say that G is an *affine group S-scheme*; in that case, the mutiplication law $G \times_S G \to G$ and the unit section $S \to G$ correspond respectively to morphisms of $\mathscr{O}_S$-algebras

$$\Delta_G : f_*\mathscr{O}_G \to f_*\mathscr{O}_G \otimes_{\mathscr{O}_S} f_*\mathscr{O}_G \qquad \varepsilon_G : \mathscr{O}_S \to f_*\mathscr{O}_G$$

which make commute the diagram :

$$\begin{tikzcd}
f_*\mathscr{O}_G \arrow[r, "\Delta_G"] \arrow[d, "\Delta_G"'] & f_*\mathscr{O}_G \otimes_{\mathscr{O}_S} f_*\mathscr{O}_G \arrow[d, "\mathbf{1}_{f_*\mathscr{O}_S}\otimes\Delta_G"] \\
f_*\mathscr{O}_G \otimes_{\mathscr{O}_S} f_*\mathscr{O}_G \arrow[r, "\Delta_G\otimes\mathbf{1}_{f_*\mathscr{O}_S}"] & f_*\mathscr{O}_G \otimes_{\mathscr{O}_S} f_*\mathscr{O}_G \otimes_{\mathscr{O}_S} f_*\mathscr{O}_G
\end{tikzcd}$$

as well as a similar diagram, which expresses the unit property of ε_G : see [24, Exp.I, §4.2].

Example 4.4.3. Let G be any commutative group. The presheaf of groups

$$D_S(G) : \mathbf{Sch}/S \to \mathbb{Z}\text{-}\mathbf{Mod} \qquad (X \to S) \mapsto \mathrm{Hom}_{\mathbb{Z}\text{-}\mathbf{Mod}}(G, \mathscr{O}_X^\times(X))$$

is representable by an affine group S-scheme $D_S(G)$, called the *diagonalizable group scheme* attached to G. Explicitly, if $S = \mathrm{Spec}\, R$ is an affine scheme, the underlying S-scheme of $D_S(G)$ is $\mathrm{Spec}\, R[G]$, and the group law is given by the map of R-algebras

$$\Delta_G : R[G] \to R[G] \otimes_R R[G] \xrightarrow{\sim} R[G \times G] \qquad g \mapsto (g, g) \quad \text{for every } g \in G$$

with unit $\varepsilon_G : R[G] \to R$ given by the standard augmentation (see [24, Exp.I, §4.4]). For a general scheme S, we have $D_S(G) = D_{\mathrm{Spec}\,\mathbb{Z}}(G) \times_{\mathrm{Spec}\,\mathbb{Z}} S$ (with the induced group law and unit section).

Definition 4.4.4. Let M be an $\mathscr{O}_S$-module, G a group S-scheme. A *G-module structure* on M is the datum of a morphism of presheaves of groups on $\mathbf{Sch}/S$:

$$h_G \to \mathscr{A}ut_{\mathscr{O}_{\mathbf{Sch}/S}}(\mathscr{W}_M)$$

(where h_G denotes the Yoneda imbedding : see (1.1.19)).

4.4.5. Suppose now that $f : G \to S$ is an affine group S-scheme, and M a quasi-coherent $\mathscr{O}_S$-module; in view of (4.4.2), a G-module structure on M is then the same as a map of $\mathscr{O}_S$-modules

$$\mu_M : M \to f_*\mathscr{O}_G \otimes_{\mathscr{O}_S} M$$

which makes commute the diagrams :

$$\begin{tikzcd} M \arrow[rr, "\mu_M"] \arrow[d, "\mu_M"'] & & f_*\mathscr{O}_G \otimes_{\mathscr{O}_S} M \arrow[d, "\Delta_G\otimes\mathbf{1}_M"] \\ f_*\mathscr{O}_G \otimes_{\mathscr{O}_S} M \arrow[rr, "\mathbf{1}_{f_*\mathscr{O}_G}\otimes\mu_M"] & & f_*\mathscr{O}_G \otimes_{\mathscr{O}_S} f_*\mathscr{O}_G \otimes_{\mathscr{O}_S} M \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "\mu_M"] \arrow[dr, "\mathbf{1}_M"'] & f_*\mathscr{O}_G \otimes_{\mathscr{O}_S} M \arrow[d, "\varepsilon_G\otimes\mathbf{1}_M"] \\ & M. \end{tikzcd}$$

Example 4.4.6. Let Γ be a commutative group; a $D_S(\Gamma)$-module structure on a quasi-coherent $\mathscr{O}_S$-module M is the datum of a morphism of $\mathscr{O}_S$-modules

$$\mu_M : M \to \mathscr{O}_S[\Gamma] \otimes_{\mathscr{O}_S} M = \mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}} M$$

which makes commute the diagrams of (4.4.5). If $S = \mathrm{Spec}\, R$ is affine, M is associated to an R-module which we denote also by M; in this case, μ_M is the same as a system $(\mu_M^{(\gamma)} \mid \gamma \in \Gamma)$ of $\mathscr{O}_S$-linear endomorphisms of M, such that :

- for every $x \in M$, the subset $\{\gamma \in \Gamma \mid \mu_M^{(\gamma)}(x) \neq 0\}$ is finite.
- $\mu_M^{(\gamma)} \circ \mu_M^{(\tau)} = \delta_{\gamma,\tau} \cdot \mathbf{1}_M$ and $\sum_{\gamma\in\Gamma} \mu_M = \mathbf{1}_M$.

In other words, the $\mu_M^{(\gamma)}$ form an orthogonal system of projectors of M, summing up to the identity $\mathbf{1}_M$. This is the same as the datum of a Γ-grading on M : namely, for a given $D_S(\Gamma)$-module structure μ_M, one lets

$$\mathrm{gr}_\gamma M := \mu_M^{(\gamma)}(M) \qquad \text{for every } \gamma \in \Gamma$$

and conversely, given a Γ-grading $\mathrm{gr}_\bullet M$ on M, one defines μ_M as the R-linear map given by the rule : $x \mapsto \gamma \otimes x$ for every $\gamma \in \Gamma$ and every $x \in \mathrm{gr}_\gamma M$.

4.4.7. Suppose now that $g : X \to S$ is an affine S-scheme, and $f : G \to S$ an affine group S-scheme. A *G-action* on X is a morphism of presheaves of groups :

$$h_G \to \mathscr{A}ut_{\mathbf{Sch}/S^\wedge}(h_X).$$

(notation of (1.1.19)); the latter is the same as a morphism of S-schemes

$$G \times_S X \to X \tag{4.4.8}$$

inducing a G-module structure on $g_*\mathscr{O}_X$:

$$g_*\mathscr{O}_X \to f_*\mathscr{O}_G \otimes_{\mathscr{O}_S} g_*\mathscr{O}_X$$

which is also a morphism of $\mathscr{O}_S$-algebras. For instance, if $S = \mathrm{Spec}\, R$ is affine, and $G = D_S(\Gamma)$ for an abelian group Γ, we may write $X = \mathrm{Spec}\, A$ for some R-algebra B, and in view of example 4.4.6, the G-action on X is the same as the datum of a Γ-graded R-algebra structure on B, in the sense of the following :

Definition 4.4.9. Let $(\Gamma, +)$ be a commutative monoid, R a ring.

(i) A *Γ-graded R-algebra* is a pair $\underline{B} := (B, \mathrm{gr}_\bullet B)$ consisting of an R-algebra B and a Γ-grading $B = \bigoplus_{\gamma\in\Gamma} \mathrm{gr}_\gamma B$ of the R-module B, such that

$$\mathrm{gr}_\gamma B \cdot \mathrm{gr}_{\gamma'} B \subset \mathrm{gr}_{\gamma+\gamma'} B \qquad \text{for every } \gamma, \gamma' \in \Gamma.$$

A morphism of Γ-graded R-algebras is a map of R-algebras which is compatible with the gradings, in the obvious way.

(ii) Let $\underline{B} := (B, \mathrm{gr}_\bullet B)$ be a Γ-graded R-algebra. A *Γ-graded $\underline{B}$-module* is a datum $\underline{M} := (M, \mathrm{gr}_\bullet M)$ consisting of a B-module M and a Γ-grading $M = \bigoplus_{\gamma\in\Gamma} \mathrm{gr}_\gamma M$ of the R-module underlying M, such that

$$\mathrm{gr}_\gamma B \cdot \mathrm{gr}_{\gamma'} M \subset \mathrm{gr}_{\gamma+\gamma'} M \qquad \text{for every } \gamma, \gamma' \in \Gamma.$$

A morphism of Γ-graded $\underline{B}$-module is a map $f : M \to N$ of B-modules, such that $f(\mathrm{gr}_\gamma M) \subset \mathrm{gr}_\gamma N$, for every $\gamma \in \Gamma$.

(iii) If $f : \Gamma' \to \Gamma$ is any morphism of commutative monoids, and M is any Γ-graded R-module, we define the Γ'-graded R-module $\Gamma' \times_\Gamma M$ by setting

$$\mathrm{gr}_\gamma(\Gamma' \times_\Gamma M) := \mathrm{gr}_{f(\gamma)} M \qquad \text{for every } \gamma \in \Gamma'.$$

Notice that if $\underline{B} := (B, \mathrm{gr}_\bullet B)$ is a Γ-graded R-algebra, then $\Gamma' \times_\Gamma B$ with its grading gives a Γ'-graded R-algebra $\Gamma' \times_\Gamma \underline{B}$, with the multiplication and addition laws deduced from those of B, in the obvious way. Likewise, if $(M, \mathrm{gr}_\bullet M)$ is a Γ-graded $\underline{B}$-module, then $\Gamma' \times_\Gamma M$ is naturally a Γ'-graded $\Gamma' \times_\Gamma \underline{B}$-module.

(iv) Furthermore, if N is any Γ'-graded R-module, we define the Γ-graded R-module $N_{/\Gamma}$ whose underlying R-module is the same as N, and whose grading is given by the rule

$$\mathrm{gr}_\gamma(N_{/\Gamma}) := \bigoplus_{\gamma' \in f^{-1}(\gamma)} \mathrm{gr}_\gamma N.$$

Just as in (iii), if $\underline{C} := (C, \mathrm{gr}_\bullet C)$ is a Γ'-graded R-algebra, then we get a Γ-graded R-algebra $\underline{C}_{/\Gamma}$, whose underlying R-algebra is the same as C. Lastly, if $(N, \mathrm{gr}_\bullet N)$ is a Γ'-graded $\underline{C}$-module, then $N_{/\Gamma}$ is a Γ-graded $\underline{C}_{/\Gamma}$-module.

Example 4.4.10. (i) For instance, the R-algebra $R[\Gamma]$ is naturally a Γ-graded R-algebra, when endowed with the Γ-grading such that $\mathrm{gr}_\gamma R[\Gamma] := \gamma R$ for every $\gamma \in \Gamma$.

(ii) Suppose that Γ is an integral monoid. Then, to a Γ-graded R-algebra B, the correspondance described in (4.4.7) associates a $D_S(\Gamma^{\mathrm{gp}})$-action on $\mathrm{Spec}\, B$ (where $S := \mathrm{Spec}\, R$), given by the map of R-algebras

$$\vartheta_B : B \to B[\Gamma] \subset B[\Gamma^{\mathrm{gp}}] \quad : \quad b \mapsto b \cdot \gamma \qquad \text{for every } \gamma \in \Gamma, \text{ and every } b \in \mathrm{gr}_\gamma B.$$

Remark 4.4.11. (i) Let R be a ring; consider a cartesian diagram of monoids

$$\begin{tikzcd} \Gamma_3 \arrow[r] \arrow[d] & \Gamma_1 \arrow[d] \\ \Gamma_2 \arrow[r] & \Gamma_0 \end{tikzcd}$$

and let B be any Γ_1-graded R-algebra. A simple inspection of the definitions yields an identity of Γ_2-graded R-algebras :

$$\Gamma_2 \times_{\Gamma_0} B_{/\Gamma_0} = (\Gamma_3 \times_{\Gamma_1} B)_{/\Gamma_2}.$$

(ii) Suppose that Γ is a finite abelian group, whose order is invertible in $\mathscr{O}_S$. Then $D_S(\Gamma)$ is an étale S-scheme. Indeed, in light of (2.3.52), the assertion is reduced to the case where $\Gamma = \mathbb{Z}/n\mathbb{Z}$ for some integer $n > 0$ which is invertible in $\mathscr{O}_S$. However, $R[\mathbb{Z}/n\mathbb{Z}] \simeq R[T]/(T^n - 1)$, which is an étale R-algebra, if $n \in R^\times$.

(iii) More generally, suppose that Γ is a finitely generated abelian group, such that the order of its torsion subgroup is invertible in $\mathscr{O}_S$. Then we may write $\Gamma = L \oplus \Gamma_{\mathrm{tor}}$, where L is a free abelian group of finite rank, and Γ_{tor} is a finite abelian group as in (ii). In view of (2.3.52) and (ii), we conclude that $D_S(\Gamma)$ is a smooth S-scheme in this case.

(iv) Let Γ be as in (iii), and suppose that X is an S-scheme with an action of $G := D_S(\Gamma)$. Then the corresponding morphism (4.4.8) and the projection $p_G : G \times_S X \to G$ induce an automorphism of the G-scheme $G \times_S X$, whose composition with the projection $p_X : G \times_S X \to X$

equals (4.4.8). We then deduce that both (4.4.8) and p_X are smooth morphisms. This observation, together with the above correspondance between Γ-graded algebras and $D_S(\Gamma)$-actions, is the basis for a general method, that allows to prove properties of graded rings, provided they can be translated as properties of the corresponding schemes *which are well behaved under smooth base changes*. We shall present hereafter a few applications of this method.

Proposition 4.4.12. *Let Γ be an integral monoid, B a Γ-graded (commutative, unitary) ring, and suppose that the order of any torsion element of Γ^{gp} is invertible in B. Then :*

(i) $\mathrm{nil}(B[\Gamma]) = \mathrm{nil}(B) \cdot B[\Gamma]$.

(ii) $\mathrm{nil}(B)$ *is a Γ-graded ideal of B.*

Proof. (i): Clearly $\mathrm{nil}(B) \cdot B[\Gamma] \subset \mathrm{nil}(B[\Gamma])$. To show the converse inclusion, it suffices to prove that $\mathrm{nil}(B[\Gamma]) \subset \mathfrak{p}B[\Gamma]$ for every prime ideal $\mathfrak{p} \subset B$, or equivalently that $B/\mathfrak{p}[\Gamma]$ is a reduced ring for every such $\mathfrak{p}$. Since the natural map $\Gamma \to \Gamma^{\mathrm{gp}}$ is injective, we may further replace Γ by Γ^{gp}, and assume that Γ is an abelian group. In this case, $B/\mathfrak{p}[\Gamma]$ is the filtered union of the rings $B/\mathfrak{p}[H]$, where H runs over the finitely generated subgroups of Γ; it suffices therefore to prove that each $B/\mathfrak{p}[H]$ is reduced, so we may assume that Γ is finitely generated, and the order of its torsion subgroup is invertible in B. In this case, $B/\mathfrak{p}[\Gamma]$ is a smooth $B/\mathfrak{p}$-algebra (remark 4.4.11(iii)), and the assertion follows from [33, Ch.IV, Prop.17.5.7].

(ii): The assertion to prove is that $\mathrm{nil}(B) = \bigoplus_{\gamma\in\Gamma} \mathrm{nil}(B) \cap \mathrm{gr}_\gamma B$. However, let $\vartheta_B : B \to B[\Gamma]$ be the map defined as in example 4.4.10(ii); now, (i) implies that ϑ_B restricts to a map $\mathrm{nil}(B) \to \mathrm{nil}(B) \cdot B[\Gamma]$, whence the contention. $\square$

For a ring homomorphism $A \to B$, let us denote by $(A, B)^\nu$ the integral closure of A in B. We have the following :

Proposition 4.4.13. *Let Γ be an integral monoid, $f : A \to B$ a morphism of Γ-graded rings, and suppose that the order of any torsion element of Γ^{gp} is invertible in A. Then :*

(i) $(A, B)^\nu[\Gamma^{\mathrm{gp}}] = (A[\Gamma^{\mathrm{gp}}], B[\Gamma^{\mathrm{gp}}])^\nu$.

(ii) *The grading of B restricts to a Γ-grading on the subring $(A, B)^\nu$.*

Proof. (i): We easily reduce to the case where Γ is finitely generated, in which case $A[\Gamma^{\mathrm{gp}}]$ is a smooth A-algebra (remark 4.4.11(iii)), and the assertion follows from [31, Ch.IV, Prop.6.14.4].

(ii): We consider the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\vartheta_A} & A[\Gamma] \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f[\Gamma]} \\ B & \xrightarrow{\vartheta_B} & B[\Gamma] \end{array}$$

where ϑ_A and ϑ_B are defined as in example 4.4.10(ii). Say that $b \in (A, B)^\nu$; then $\vartheta_B(b) \in (A[\Gamma], B[\Gamma])^\nu$. In light of (i), it follows that $\vartheta_B(b) \in (A, B)^\nu[\Gamma^{\mathrm{gp}}] \cap B[\Gamma] = (A, B)^\nu[\Gamma]$. The claim follows easily. $\square$

Corollary 4.4.14. *In the situation of proposition* 4.4.12, *suppose moreover, that B is a domain. Then we have :*

(i) *The integral closure B^ν of B in its field of fractions is Γ^{gp}-graded, and the inclusion map $B \to B^\nu$ is a morphism of Γ^{gp}-graded rings.*

(ii) *Suppose furthermore, that Γ is saturated. Then B^ν is a Γ-graded ring.*

Proof. (Notice that, since Γ is integral, the grading of B extends trivially to a Γ^{gp}-grading, by setting $\mathrm{gr}_\gamma B := 0$ for every $\gamma \in \Gamma^{\mathrm{gp}} \setminus \Gamma$.) Clearly B is the filtered union of its subalgebras $\Delta \times_\Gamma B$, where Δ ranges over the finitely generated submonoids of Γ. Hence, we are easily reduced to the case where Γ is finitely generated. Let S be the multiplicative system of all non-zero homogeneous elements of B. It is easily seen that $A := S^{-1}B$ is a Γ^{gp}-graded algebra, $K := \mathrm{gr}_0 A$ is a field, and $\dim_K \mathrm{gr}_\gamma A = 1$ for every $\gamma \in \Gamma^{\mathrm{gp}}$. Moreover, we have

$$\mathrm{gr}_\gamma A \cdot \mathrm{gr}_{\gamma'} A = \mathrm{gr}_{\gamma+\gamma'} A \qquad \text{for every } \gamma, \gamma' \in \Gamma^{\mathrm{gp}}.$$

Claim 4.4.15. A is a normal domain, and if Γ is saturated, $\Gamma \times_{\Gamma^{\mathrm{gp}}} A$ is normal as well.

Proof of the claim. Pick a decomposition $\Gamma^{\mathrm{gp}} = L \oplus T$, where L is a free abelian group, and T is the torsion subgroup of Γ^{gp}. It follows easily that the induced map of K-algebras

$$(L \times_{\Gamma^{\mathrm{gp}}} A) \otimes_K (T \times_{\Gamma^{\mathrm{gp}}} A) \to A$$

is an isomorphism. Moreover, $E := T \times_{\Gamma^{\mathrm{gp}}} A$ is a finite field extension of K, and $L \times_{\Gamma^{\mathrm{gp}}} A \simeq K[L]$. Summing up, A is isomorphic to $E[L]$, whence the first assertion.

Next, suppose that Γ is saturated; then $\Gamma \simeq \Gamma^\sharp \times \Gamma^\times$ (lemma 3.2.10), and we have a decomposition $\Gamma^\times = H \oplus T$, where H is a free abelian group. In this case, we may take $L := (\Gamma^\sharp)^{\mathrm{gp}} \oplus H$, and the foregoing isomorphism induces an identification

$$\Gamma \times_{\Gamma^{\mathrm{gp}}} A \simeq E[\Gamma^\sharp \oplus H]$$

so the second assertion follows, taking into account theorem 3.4.16(iii). ◊

The corollary now follows straightforwardly from claim 4.4.15 and proposition 4.4.13. □

Proposition 4.4.16. *Suppose that* $(\Gamma', +) \to (\Gamma, +)$ *is a morphism of fine monoids,* B *a finitely generated (resp. finitely presented)* Γ*-graded* R*-algebra, and* M *a finitely generated (resp. finitely presented)* Γ*-graded* B*-module. We have :*

(i) $\Gamma' \times_\Gamma B$ *is a finitely generated (resp. finitely presented)* R*-algebra.*
(ii) $\Gamma' \times_\Gamma M$ *is a finitely generated (resp. finitely presented)* $\Gamma' \times_\Gamma B$*-module.*
(iii) $\mathrm{gr}_\gamma M$ *is a finitely generated (resp. finitely presented)* $\mathrm{gr}_0 B$*-module, for every* $\gamma \in \Gamma$.

Proof. Let B_M denote the direct sum $B \oplus M$, endowed with the R-algebra structure given by the rule :

$$(b_1, m_1) \cdot (b_2, m_2) = (b_1 b_2, b_1 m_2 + b_2 m_1) \qquad \text{for every } b_1, b_2 \in B \text{ and } m_1, m_2 \in M.$$

Notice that B_M is characterized as the unique R-algebra structure for which M is an ideal with $M^2 = 0$, the natural projection $B_M \to B$ is a map of R-algebras, and the B-module structure on M induced via π agrees with the given B-module structure on M.

Claim 4.4.17. The following conditions are equivalent :

(a) The R-algebra B_M is finitely generated (resp. finitely presented).
(b) B is a finitely generated (resp. finitely presented) R-algebra and M is a finitely generated (resp. finitely presented) B-module.

Proof of the claim. (b)⇒(a): Suppose first that B is a finitely generated R-algebra, and M is a finitely generated B-module. Pick a system of generators $\Sigma_B := \{b_1, \dots, b_s\}$ for B and $\Sigma_M := \{m_1, \dots, m_k\}$ for M. Then it is easily seen that $\Sigma_B \cup \Sigma_M$ generates the R-algebra B_M.

For the finitely presented case, pick a surjection of R-algebras $\varphi : R[T_1, \dots, T_s] \to B$ and of B-module $\psi : B^{\oplus k} \to M$. Let Σ'_B be a finite system of generators of the ideal $\mathrm{Ker}\,\varphi$. Pick also a finite system $b_1, \dots, b_r$ of generators of the B-module $\mathrm{Ker}\,\psi$; we may write $b_i = \sum_{j=1}^k b_{ij} e_j$ for certain $b_{ij} \in B$ (where $e_1, \dots, e_k$ is the standard basis of $B^{\oplus k}$). For every $i \leq r$ and $j \leq k$, pick $P_{ij} \in \varphi^{-1}(b_{ij})$. It is easily seen that B_M is isomorphic to the R-algebra $R[T_1, \dots, T_{s+k}]/I$, where I is generated by $\Sigma'_B \cup \{\sum_{j=1}^k P_{ij} T_{j+s} \mid i = 1, \dots, r\} \cup \{T_{i+s} T_{j+s} \mid 0 \leq i, j \leq k\}$.

(a)⇒(b): Suppose that B_M is a finitely generated R-algebra, and let $c_1, \dots, c_n$ be a system of generators. For every $i = 1, \dots, n$, we may write $c_i = b_i + m_i$ for unique $b_i \in B$ and $m_i \in M$.

Since $M^2 = 0$, it is easily seen that $m_1, \dots, m_n$ is a system of generators for the B-module M, and clearly $b_1, \dots, b_n$ is a system of generators for the R-algebra B.

Next, suppose that B_M is finitely presented over R. We may find a system of generators of the type $b_1, \dots, b_s, m_1, \dots, m_k$ for certain $b_i \in B$ and $m_j \in M$. We deduce a surjection of R-algebras

$$\varphi : R[T_1, \dots, T_{s+k}]/(T_{s+i}T_{s+j} \mid 0 \leq i, j \leq k) \to B_M$$

such that $T_i \mapsto b_i$ for every $i \leq s$ and $T_{j+s} \mapsto m_j$ for every $j \leq k$. It is easily seen that $\mathrm{Ker}\,\varphi$ is generated by the classes of finitely many polynomials $P_1, \dots, P_r$, where

$$P_i = Q_i(T_1, \dots, T_s) + \sum_{j=1}^{k} T_{s+j} Q_{ij}(T_1, \dots, T_s) \qquad i = 1, \dots, r$$

for certain polynomials $Q_i, Q_{ij} \in R[T_1, \dots, T_s]$. It follows easily that $B = R[T_1, \dots, T_s]/I$, where I is the ideal generated by $Q_1, \dots, Q_r$, and M is isomorphic to the B-module $B^{\oplus k}/N$, where N is the submodule generated by the system $\{\sum_{j=1}^{k} Q_{ij}(b_1, \dots, b_s)e_j \mid i = 1, \dots, r\}$ $\Diamond$

Suppose now that B is a finitely generated R-algebra; then B is generated by finitely many homogeneous elements, say $b_1, \dots, b_s$ of degrees respectively $\gamma_1, \dots, \gamma_s$. Thus, we may define surjections of monoids

$$\mathbb{N}^{\oplus s} \to \Gamma \quad : \quad e_i \mapsto \gamma_i \qquad \text{for } i = 1, \dots, s \tag{4.4.18}$$

(where $e_1, \dots, e_s$ is the standard basis of $\mathbb{N}^{\oplus s}$) and of R-algebras $\varphi : C \to B$, where $C := R[\mathbb{N}^{\oplus s}]$ is a free polynomial R-algebra. Notice that C is a $\mathbb{N}^{\oplus s}$-graded R-algebra, and via (4.4.18) we may regard φ as a morphism of Γ-graded R-algebras $C_{/\Gamma} \to B$. Then $I := \ker\,\varphi$ is a Γ-graded ideal of C, and if we set $P := \mathbb{N}^{\oplus s} \times_\Gamma \Gamma'$ we deduce an isomorphism of Γ'-graded R-algebras

$$B' := \Gamma' \times_\Gamma B \xrightarrow{\sim} (P \times_{\mathbb{N}^{\oplus s}} C)_{/\Gamma'}/(\Gamma' \times_\Gamma I)$$

(see remark 4.4.11(i)).

Claim 4.4.19. $P \times_{\mathbb{N}^{\oplus s}} C$ is a finitely presented R-algebra.

Proof of the claim. Indeed, this R-algebra is none else than $R[P]$, hence the assertion follows from corollary 3.4.2 and lemma 3.1.7(i). $\Diamond$

From claim 4.4.19 it follows already that B' is a finitely generated R-algebra. Now, suppose that M is a finitely generated B-module, and set $M' := \Gamma' \times_\Gamma M$; notice that

$$\Gamma' \times_\Gamma (B_M) = B'_{M'}. \tag{4.4.20}$$

In view of claim 4.4.17, we deduce that M' is a finitely generated B'-module. Next, in case B is a finitely presented R-algebra, I is a finitely generated ideal of $C_{/\Gamma}$; as we have just seen, this implies that $\Gamma' \times_\Gamma I$ is a finitely generated $(\Gamma' \times_\Gamma C_{/\Gamma})$-module, and then claim 4.4.19 shows that B' is a finitely presented R-algebra. This concludes the proof of (i).

Lastly, if moreover M is a finitely presented B-module, assertion (i), together with (4.4.20) and claim 4.4.17 say that M' is a finitely presented B'-module; thus, also assertion (ii) is proven.

(iii): For any given $\gamma \in \Gamma$, let us consider the morphism $f : \mathbb{N} \to \Gamma$ such that $1 \mapsto \gamma$, and set $B' := \mathbb{N} \times_\Gamma B$. By (i), the R-algebra B' is finitely generated (resp. finitely presented), and the B'-module $M' := \mathbb{N} \times_\Gamma M$ is finitely generated (resp. finitely presented). After replacing B by B' and M by M', we may then assume from start that $\Gamma = \mathbb{N}$, and we are reduced to showing that $\mathrm{gr}_1 M$ is a finitely generated (resp. finitely presented) $\mathrm{gr}_0 B$-module.

Let $m_1, \dots, m_t$ be a system of generators of M consisting of homogeneous elements of degrees respectively $j_1, \dots, j_t$. We endow $B^{\oplus t}$ with the $\mathbb{N}$-grading such that

$$\mathrm{gr}_k B^{\oplus t} := \bigoplus_{i=1}^{t} \mathrm{gr}_{k-j_i} B e_i$$

(where $e_1, \dots, e_t$ is the standard basis of $B^{\oplus t}$); then the B-linear map $B^{\oplus t} \to M$ given by the rule $e_i \mapsto m_i$ for every $i = 1, \dots, t$ is a morphism of $\mathbb{N}$-graded B-modules, and if M is finitely presented, its kernel is generated by finitely many homogeneous elements $b_1, \dots, b_s$. In the latter case, endow again $B^{\oplus s}$ with the unique $\mathbb{N}$-grading such that the B-linear map $\varphi : B^{\oplus s} \to B^{\oplus t}$ given by the rule $e_i \mapsto b_i$ for every $i = 1, \dots, s$ is a morphism of $\mathbb{N}$-graded B-modules. Now, in order to check that $\mathrm{gr}_1 M$ is a finitely generated $\mathrm{gr}_0 B$-module, it suffices to show that the same holds for $\mathrm{gr}_1 B^{\oplus t}$. The latter is a direct sum of $\mathrm{gr}_0 B$-modules isomorphic to either $\mathrm{gr}_0 B$ or $\mathrm{gr}_1 B$. Likewise, if M is finitely presented, $\mathrm{gr}_1 M = \mathrm{Coker}\, \mathrm{gr}_1 \varphi$, and again, $\mathrm{gr}_1 B^{\oplus s}$ is a direct sum of $\mathrm{gr}_0 B$-modules isomorphic to either $\mathrm{gr}_0 B$ or $\mathrm{gr}_1 B$; hence in order to check that $\mathrm{gr}_1 M$ is a finitely presented $\mathrm{gr}_0 B$-module, it suffices to show that $\mathrm{gr}_1 B$ is a finitely presented $\mathrm{gr}_0 B$-module. In either event, we are reduced to the case where $\Gamma = \mathbb{N}$ and $M = B$.

However, from (ii) we deduce especially that $\mathrm{gr}_0 B = \{0\} \times_{\mathbb{N}} B$ is a finitely generated (resp. finitely presented) R-algebra, hence B is a finitely generated (resp. finitely presented) B_0-algebra as well; we may then assume that $R = \mathrm{gr}_0 B$. Let Σ be a system of homogeneous generators for the R-algebra B; we may then also assume that

$$\Sigma \cap \mathrm{gr}_0 B = \varnothing. \tag{4.4.21}$$

Then it is easily seen that the R-module $\mathrm{gr}_1 B$ is generated by $\Sigma \cap \mathrm{gr}_1 B$. Lastly, if B is a finitely presented R-algebra, we consider the natural surjection $\psi : R[\Sigma] \to B$ from the free polynomial R-algebra generated by the set Σ, and endow $R[\Sigma]$ with the unique grading for which ψ is a map of $\mathbb{N}$-graded R-algebras; then $I := \mathrm{Ker}\, \psi$ is a finitely generated $\mathbb{N}$-graded ideal with $\mathrm{gr}_0 I = 0$. As usual, we pick a finite system Σ' of homogeneous generators for I; clearly B_1 is isomorphic to $\mathrm{gr}_1 B_0[\Sigma]/\mathrm{gr}_1 I$. On the other hand, (4.4.21) easily implies that $\mathrm{gr}_1 R[\Sigma]$ is a free R-module of finite rank, and moreover $\mathrm{gr}_1 I$ is generated by $\Sigma' \cap \mathrm{gr}_1 I$; especially, $\mathrm{gr}_1 B$ is a finitely presented R-module in this case, and the proof is complete. $\square$

4.4.22. Let $(\Gamma, +)$ be a monoid, R a ring, $\underline{B} := (B, \mathrm{gr}_\bullet B)$ a Γ-graded R-algebra, and M a Γ-graded $\underline{B}$-module. We denote by $M[\gamma]$ the Γ-graded $\underline{B}$-module whose underlying B-module is M, and whose grading is given by the rule :

$$\mathrm{gr}_\beta M[\gamma] := \mathrm{gr}_{\beta+\gamma} M \qquad \text{for every } \gamma \in \Gamma.$$

Remark 4.4.23. (i) In the situation of (4.4.22), pick any system $\mathbf{x} := (x_i \mid i \in I)$ of homogeneous generators of M, and say that $x_i \in \mathrm{gr}_{\gamma_i} M$ for every $i \in I$. Then we may define a surjective map of Γ-graded $\underline{B}$-modules

$$L := \bigoplus_{i \in I} B[-\gamma_i] \to M \quad : \quad e_i \mapsto x_i \qquad \text{for every } i \in I$$

where $(e_i \mid i \in I)$ denotes the canonical basis of the free B-module L (notice that $e_i \in \mathrm{gr}_{\gamma_i} L$ for every $i \in I$).

(ii) In case M is a finitely generated B-module, we may pick a finite system $\mathbf{x}$ as above, and then L shall be a free B-module of finite rank.

(iii) Especially, suppose that B is a coherent ring and M is finitely presented as a B-module; then, in the situation of (ii), the kernel of the surjection $L \to M$ shall be again a finitely presented Γ-graded $\underline{B}$-module, so we can repeat the above construction, and find inductively a resolution

$$\Sigma \quad : \quad \cdots \to L_n \xrightarrow{d_n} L_{n-1} \to \cdots \to L_0 \xrightarrow{d_0} M$$

such that L_n is a free B-module of finite rank, and the map d_n is a morphism of Γ-graded $\underline{B}$-modules, for every $n \in \mathbb{N}$.

(iv) In the situation of (iii), suppose furthermore that B is a flat R-algebra, in which case $\mathrm{gr}_\gamma B$ is a flat R-module, for every $\gamma \in \Gamma$. Then it is clear that the resolution Σ yields, in each degree $\gamma \in \Gamma$ a flat resolution Σ_γ of the R-module $\mathrm{gr}_\gamma M$.

4.4.24. *Filtered rings and Rees algebras.* Some of the following material is borrowed from [9, Appendix III], where much more can be found.

Definition 4.4.25. Let R be a ring, A an R-algebra.

(i) An *R-algebra filtration on* A is an increasing exhaustive filtration $\mathrm{Fil}_\bullet A$ indexed by $\mathbb{Z}$ and consisting of R-submodules of A, such that :

$$1 \in \mathrm{Fil}_0 A \qquad \text{and} \qquad \mathrm{Fil}_i A \cdot \mathrm{Fil}_j A \subset \mathrm{Fil}_{i+j} A \qquad \text{for every } i, j \in \mathbb{Z}.$$

The pair $\underline{A} := (A, \mathrm{Fil}_\bullet A)$ is called a *filtered R-algebra*.

(ii) Let M be an A-module. An *$\underline{A}$-filtration* on M is an increasing exhaustive filtration $\mathrm{Fil}_\bullet M$ consisting of R-submodules, and such that :

$$\mathrm{Fil}_i A \cdot \mathrm{Fil}_j M \subset \mathrm{Fil}_{i+j} M \qquad \text{for every } i, j \in \mathbb{Z}.$$

The pair $\underline{M} := (M, \mathrm{Fil}_\bullet M)$ is called a *filtered $\underline{A}$-module*.

(iii) Let U be an indeterminate. The *Rees algebra* of $\underline{A}$ is the $\mathbb{Z}$-graded subring of $A[U, U^{-1}]$

$$\mathsf{R}(\underline{A})_\bullet := \bigoplus_{i \in \mathbb{Z}} U^i \cdot \mathrm{Fil}_i A.$$

(iv) Let $\underline{M} := (M, \mathrm{Fil}_\bullet M)$ be a filtered $\underline{A}$-module. The *Rees module* of $\underline{M}$ is the graded $\mathsf{R}(\underline{A})_\bullet$-module :

$$\mathsf{R}(\underline{M})_\bullet := \bigoplus_{i \in \mathbb{Z}} U^i \cdot \mathrm{Fil}_i M.$$

Lemma 4.4.26. *Let R be a ring, $\underline{A} := (A, \mathrm{Fil}_\bullet A)$ a filtered R-algebra, $\mathrm{gr}_\bullet \underline{A}$ the associated graded R-algebra, $\mathsf{R}(\underline{A})_\bullet \subset A[U, U^{-1}]$ the Rees algebra of $\underline{A}$. Then there are natural isomorphisms of graded R-algebras :*

$$\mathsf{R}(\underline{A})_\bullet / U\mathsf{R}(\underline{A})_\bullet \simeq \mathrm{gr}_\bullet \underline{A} \qquad \mathsf{R}(\underline{A})_\bullet[U^{-1}] \simeq A[U, U^{-1}]$$

and of R-algebras :

$$\mathsf{R}(\underline{A})_\bullet / (1-U)\mathsf{R}(\underline{A})_\bullet \simeq A.$$

Proof. The isomorphisms with $\mathrm{gr}_\bullet \underline{A}$ and with $A[U, U^{-1}]$ follow directly from the definitions. For the third isomorphism, it suffices to remark that $A[U, U^{-1}]/(1-U) \simeq A$. $\square$

Definition 4.4.27. Let R be a ring, $\underline{A} := (A, \mathrm{Fil}_\bullet A)$ a filtered R-algebra.

(i) Suppose that A is of finite type over R, let $\mathbf{x} := (x_1, \dots, x_n)$ be a finite set of generators for A as an R-algebra, and $\mathbf{k} := (k_1, \dots, k_n)$ a sequence of n integers; the *good filtration* $\mathrm{Fil}_\bullet A$ *attached to the pair* $(\mathbf{x}, \mathbf{k})$ is the R-algebra filtration such that $\mathrm{Fil}_i A$ is the R-submodule generated by all the elements of the form

$$\prod_{j=1}^n x_j^{a_j} \qquad \text{where :} \qquad \sum_{j=1}^n a_j k_j \leq i \quad \text{and} \quad a_1, \dots, a_n \geq 0$$

for every $i \in \mathbb{Z}$. A filtration $\mathrm{Fil}_\bullet A$ on A is said to be *good* if it is the good filtration attached to some system of generators $\mathbf{x}$ and some sequence of integers $\mathbf{k}$.

(ii) The filtration $\mathrm{Fil}_\bullet A$ is said to be *positive* if it is the good filtration associated to a pair $(\mathbf{x}, \mathbf{k})$ as in (i), such that moreover $k_i > 0$ for every $i = 1, \dots, n$.

(iii) Let M be a finitely generated A-module. An $\underline{A}$-filtration $\mathrm{Fil}_\bullet M$ is called a *good filtration* if $\mathsf{R}(M, \mathrm{Fil}_\bullet M)_\bullet$ is a finitely generated $\mathsf{R}(\underline{A})_\bullet$-module.

Example 4.4.28. Let $A := R[t_1, \dots, t_n]$ be the free R-algebra in n indeterminates. Choose any sequence $\mathbf{k} := (k_1, \dots, k_n)$ of integers, and denote by $\mathrm{Fil}_\bullet A$ the good filtration associated to $\mathbf{t} := (t_1, \dots, t_n)$ and $\mathbf{k}$. Then $\mathsf{R}(A, \mathrm{Fil}_\bullet A)_\bullet$ is isomorphic, as a graded R-algebra, to the free polynomial algebra $A[U] = R[U, t_1, \dots, t_n]$, endowed with the grading such that $U \in \mathrm{gr}_1 A[U]$ and $t_j \in \mathrm{gr}_{k_j} A[U]$ for every $j \leq n$. Indeed, a graded isomorphism can be defined by the rule : $U \mapsto U$ and $t_j \mapsto U^{k_j} \cdot t_j$ for every $j \leq n$. The easy verification shall be left to the reader.

4.4.29. Let $\underline{A}$ and M be as in definition 4.4.27(iii); suppose that $\mathbf{m} := (m_1, \dots, m_n)$ is a finite system of generators of M, and let $\mathbf{k} := (k_1, \dots, k_n)$ be an arbitrary sequence of n integers. To the pair $(\mathbf{m}, \mathbf{k})$ we associate a filtered $\underline{A}$-module $\underline{M} := (M, \mathrm{Fil}_\bullet M)$, by declaring that :

$$\mathrm{Fil}_i M := m_1 \cdot \mathrm{Fil}_{i-k_1} A + \cdots + m_n \cdot \mathrm{Fil}_{i-k_n} A \qquad \text{for every } i \in \mathbb{Z}. \tag{4.4.30}$$

Notice that the homogeneous elements $m_1 \cdot U^{k_1}, \dots, m_n \cdot U^{k_n}$ generate the graded Rees module $\mathsf{R}(\underline{M})_\bullet$, hence $\mathrm{Fil}_\bullet M$ is a good $\underline{A}$-filtration. Conversely :

Lemma 4.4.31. *For every good filtration $\mathrm{Fil}_\bullet M$ on M, there exists a sequence $\mathbf{m}$ of generators of M and a sequence of integers $\mathbf{k}$, such that $\mathrm{Fil}_\bullet M$ is of the form* (4.4.30).

Proof. Suppose that $\mathrm{Fil}_\bullet M$ is a good filtration; then $\mathsf{R}(\underline{M})_\bullet$ is generated by finitely many homogeneous elements $m_1 \cdot U^{k_1}, \dots, m_n \cdot U^{k_n}$. Thus,

$$\mathsf{R}(\underline{M})_i := U^i \cdot \mathrm{Fil}_i M = m_1 \cdot U^{k_1} \cdot A_{i-k_1} + \cdots + m_n \cdot U^{k_n} \cdot A_{i-k_n} \qquad \text{for every } i \in \mathbb{Z}$$

which means that the sequences $\mathbf{m} := (m_1, \dots, m_n)$ and $\mathbf{k} := (k_1, \dots, k_n)$ will do. ◻

4.4.32. Let $A \to B$ be a map of noetherian rings, $I \subset A$ an ideal, M a finitely generated A-module, and N a finitely generated B-module. It is easily seen that

$$T_i := \mathrm{Tor}_i^A(M, N)$$

is a finitely generated B-module, for every $i \in \mathbb{N}$. We endow A (resp. B) with its I-adic (resp. IB-adic) filtration, extended to negative integers, by the rule $I^k := A$ for every $k \leq 0$, and denote by $\underline{A}$ (resp. $\underline{B}$) the resulting filtered ring. Also, we define a $\underline{B}$-filtration on T_i, by the rule :

$$\mathrm{Fil}_n T_i := \mathrm{Im}\,(\mathrm{Tor}_i^A(I^n M, N) \to T_i) \qquad \text{for every } i \in \mathbb{N} \text{ and } n \in \mathbb{Z}.$$

Proposition 4.4.33. *The $\underline{B}$-filtration $\mathrm{Fil}_\bullet T_i$ is good, for every $i \in \mathbb{N}$.*

Proof. Pick a finite set of generators $f_1, \dots, f_r$ for I, and consider the surjective map of graded A-algebras

$$A[U, t_1, \dots, t_r] \to \mathsf{R}(\underline{A})_\bullet \quad : \quad U \mapsto 1 \in \mathsf{R}(\underline{A})_{-1} \quad t_i \mapsto f_i \qquad \text{for } i = 1, \dots, r \tag{4.4.34}$$

(for the grading of $A[U, t_1, \dots, t_r]$ that places the indeterminates $t_1, \dots, t_r$ in degree 1, and U in degree -1). The B-module

$$P_{i,\bullet} := \bigoplus_{n \in \mathbb{Z}} \mathrm{Tor}_i^A(I^n M, N) \tag{4.4.35}$$

carries a natural structure of graded $B \otimes_A \mathsf{R}(\underline{A})_\bullet$-module, whence a graded $B[U, t_1, \dots, t_r]$-module structure as well, via (4.4.34), and it suffices to show :

Claim 4.4.36. $P_{i,\bullet}$ is a finitely generated $B[U, t_1, \dots, t_r]$-module, for every $i \in \mathbb{N}$.

Proof of the claim. Let $\mathrm{Fil}_\bullet M$ be the I-adic filtration on M, and notice that $\mathsf{R}(M, \mathrm{Fil}_\bullet M)_\bullet$ is a finitely generated $\mathsf{R}(\underline{M})_\bullet$-module; *a fortiori*, it is a finitely generated graded $A[U, t_1, \dots, t_r]$-module. By remark 4.4.23(iv), we may then find a resolution

$$\cdots \to L_n \xrightarrow{d_n} L_{n-1} \to \cdots \to L_0 \xrightarrow{d_0} \mathsf{R}(M, \mathrm{Fil}_\bullet M)_\bullet$$

where each L_n is a free $A[U, t_1, \dots, t_r]$-module of finite rank, each d_n is a morphism of graded $A[U, t_1, \dots, t_r]$-modules, and the restriction of the resolution to the summands of degree i is a flat resolution of $I^i M$, for every $i \in \mathbb{N}$. There follows a natural isomorphism of graded B-modules

$$P_{i,\bullet} \xrightarrow{\sim} H_i(L_\bullet \otimes_A B) \qquad \text{for every } i \in \mathbb{N} \tag{4.4.37}$$

and a simple inspection shows that the $B[U, t_1, \dots, t_r]$-module structure on $P_{i,\bullet}$ deduced via (4.4.37) agrees with the foregoing one, so the assertion follows. □

4.4.38. In the situation of (4.4.32), set $A_n := A/I^{n+1}$ for every $n \in \mathbb{N}$. We deduce, for every $i \in \mathbb{N}$, a morphism of projective systems of B-modules

$$X^i_\bullet := (\mathrm{Tor}^A_i(M, N) \otimes_A A_n \mid n \in \mathbb{N}) \xrightarrow{\varphi^i_\bullet} Y^i_\bullet := (\mathrm{Tor}^A_i(M \otimes_A A_n, N) \mid n \in \mathbb{N})$$

where the transition maps of $X^i_\bullet$ and $Y^i_\bullet$ are induced by the projections $A_{n+1} \to A_n$, for every $n \in \mathbb{N}$, and the morphisms φ^i_n are induced by the projections $M \to M \otimes_A A_n$.

Corollary 4.4.39. *With the notation of* (4.4.38)*, we have :*

(i) *The morphism $\varphi^i_\bullet$ is an isomorphism of pro-B-modules, for every $i \in \mathbb{N}$.*

(ii) *The natural map*

$$\lim_{n \in \mathbb{N}} \mathrm{Tor}^A_i(M, N) \otimes_A A_n \to \lim_{n \in \mathbb{N}} \mathrm{Tor}^A_i(M/I^n M, N)$$

is an isomorphism, for every $i \in \mathbb{N}$.

Proof. (i): The assertion means that the systems $(\mathrm{Ker}\, \varphi^i_n \mid n \in \mathbb{N})$ and $(\mathrm{Coker}\, \varphi^i_n \mid n \in \mathbb{N})$ are essentially zero, for every $i \in \mathbb{N}$. However, set $U := 1 \in \mathsf{R}(\underline{A})_{-1}$, and notice that the long exact $\mathrm{Tor}^A_\bullet(-, N)$-sequence arising from the short exact sequence

$$0 \to I^n M \to M \to M/I^n M \to 0$$

yields a natural identification

$$\mathrm{Coker}\, \varphi_n = \mathrm{Ker}\, (U^n : P_{i,n} \to P_{i,0})$$

where $P_{i,\bullet}$ is defined as in (4.4.35), and U^n denotes the scalar multiplication by the same element, for the natural $\mathsf{R}(\underline{A})_\bullet$-module structure of $P_{i,\bullet}$. Moreover, under this identification, the system $(\mathrm{Coker}\, \varphi^i_\bullet \mid n \in \mathbb{N})$ becomes a direct summand of the system of B-modules

$$(\mathrm{Ker}\, U^n : P_{i,\bullet} \to P_{i,\bullet} \mid n \in \mathbb{N}) \tag{4.4.40}$$

whose transition maps are given by multiplication by U. By the same token, the projective system $(\mathrm{Ker}\, \varphi^i_n \mid n \in \mathbb{N})$ is a quotient of the projective system $Z^i_\bullet := (\mathrm{Fil}_n T_i \otimes_A A_n \mid n \in \mathbb{N})$, for every $i \in \mathbb{N}$. Now, it follows easily from proposition 4.4.33 and lemma 4.4.31, that, for every $i \in \mathbb{N}$, there exists $c \in \mathbb{N}$ such that

$$\mathrm{Fil}_{n+k+c} T_i \subset I^{n+k} T_i \subset I^k \mathrm{Fil}_n T_i \qquad \text{for every } k, n \in \mathbb{N}.$$

Taking $k = n$, we conclude that the system $Z^i_\bullet$ is essentially zero, so the same holds for $(\mathrm{Ker}\, \varphi^i_n \mid n \in \mathbb{N})$, for every $i \in \mathbb{N}$. Lastly, since $B[U, t_1, \dots, t_r]$ is noetherian, claim 4.4.36 implies that there exists $N \in \mathbb{N}$ such that $\mathrm{Ker}\, U^{N+k} = \mathrm{Ker}\, U^N$ for every $k \in \mathbb{N}$. It follows easily that the system (4.4.40) is essentially zero, and then the same holds for $(\mathrm{Coker}\, \varphi^i_n \mid n \in \mathbb{N})$, for every $i \in \mathbb{N}$.

(ii) is a standard consequence of (i) : see [75, Prop.3.5.7]. □

4.4.41. *Differential graded algebras.* The material of this paragraph shall be applied in section 4.5, in order to study certain strictly anti-commutative graded algebras constructed via homotopical algebra. Especially, the graded algebras appearing in this paragraph are usually *not* commutative, unless it is explicitly said otherwise.

Definition 4.4.42. Let A be any ring.

(i) A *differential graded* A*-algebra* is the datum of

- a complex $(B^\bullet, d_B^\bullet)$ of A-modules
- an A-linear map $\mu^{pq} : B^p \otimes_A B^q \to B^{p+q}$ for every $p, q \in \mathbb{Z}$

such that the following holds :

(a) Set $B := \bigoplus_{p\in\mathbb{Z}} B^p$; then the system of maps $\mu^{\bullet\bullet}$ adds up to a map $\mu : B \otimes_A B \to B$, and one requires that the resulting pair (B, μ) is an associative unitary $\mathbb{Z}$-graded A-algebra. Then, one sets $a \cdot b := \mu(a \otimes b)$, for every $a, b \in B$.

(b) We have the identities

$$d_B^{p+q}(a \cdot b) = d_B^p(a) \cdot b + (-1)^p \cdot a \cdot d_B^q(b) \qquad \text{for every } p, q \in \mathbb{Z} \text{ and every } a \in B^p,\, b \in B^q.$$

We call B the *associated graded* A*-algebra* of $B^\bullet$. A morphism $B^\bullet \to C^\bullet$ of differential graded A-algebras is a map of complexes of A-modules such that the induced map of associated graded A-modules is a map of A-algebras. We denote by

$$A\text{-}\mathbf{dga}$$

the resulting category of differential graded A-algebras.

(ii) We say that the differential graded A-algebra $B^\bullet$ is *strictly anti-commutative*, if we have

(i) $a \cdot b = (-1)^{pq} \cdot b \cdot a$ for every $p, q \in \mathbb{Z}$ and every $a \in B^p$, $b \in B^q$

(ii) $a \cdot a = 0$ for every $p \in \mathbb{Z}$ and every $a \in B^{2p+1}$.

If only condition (a) holds, we say that $B^\bullet$ is *anti-commutative*.

(iii) Let $(B^\bullet, d_B^\bullet)$ be a differential graded A-algebra, B its associated graded A-algebra, and $(M^\bullet, d_M^\bullet)$ a complex of A-modules.

(a) We say that $M^\bullet$ is a *left* $B^\bullet$*-module* if the A-module $M := \bigoplus_{p\in\mathbb{Z}} M^p$ is a graded left B-module (for the natural $\mathbb{Z}$-grading on M), and we have

$$d_M^{p+q}(b \cdot m) = (d_B^p b) \cdot m + (-1)^p \cdot b \cdot d_M^q(m)$$

for every $p, q \in \mathbb{Z}$, every $b \in B^p$, and every $m \in M^q$.

(b) We say that $M^\bullet$ is a *right* $B^\bullet$*-module* if M is a graded right B-module, and we have

$$d_M^{p+q}(m \cdot a) = d_M^q(m) \cdot a + (-1)^q \cdot m \cdot d_M^p(a)$$

for every $p, q \in \mathbb{Z}$, every $b \in B^p$, and every $m \in M^q$.

(c) We say that $M^\bullet$ is a B*-bimodule* if it is both a left and right $B^\bullet$-module, and with these $B^\bullet$-modules structures, the A-module M becomes a B-bimodule (*i.e.* the left multiplication commutes with the right multiplication).

We call M the *associated graded* B*-module* of M. A morphism $M^\bullet \to N^\bullet$ of left (resp. right, resp. bi-) $B^\bullet$-modules is a map of complexes of A-modules, such that the induced map of associated graded A-modules is a map of left (resp. right, resp. bi-) B-modules.

(iv) Let $C^\bullet$ be any $\mathbb{Z}$-graded A-algebra, and $M^\bullet$ any $\mathbb{Z}$-graded $C^\bullet$-bimodule. An A*-linear graded derivation* from $C^\bullet$ to $M^\bullet$ is a morphism of graded A-modules $\partial : C^\bullet \to M^\bullet$ such that

$$\partial(xy) = \partial(x) \cdot y + x \cdot \partial(y) \qquad \text{for every } x, y \in C.$$

Remark 4.4.43. Let $B^\bullet$ be any differential graded A-algebra.

(i) Notice that condition (b) of definition 4.4.42(i) is the same as saying that $\mu^{\bullet\bullet}$ induces a map of complexes

$$\mu^\bullet : B^\bullet \otimes_A B^\bullet \to B^\bullet.$$

Moreover, in light of example 4.1.8(i), we see that $B^\bullet$ is anti-commutative if and only if the diagram

$$\begin{tikzcd} B^\bullet \otimes_A B^\bullet \arrow[rr, "\sim"] \arrow[dr, "\mu^\bullet"'] & & B^\bullet \otimes_A B^\bullet \arrow[dl, "\mu^\bullet"] \\ & B^\bullet & \end{tikzcd}$$

commutes, where the horizontal arrow is the isomorphism (4.1.9) that swaps the two tensor factors. Hence, in some sense this is actually a commutativity condition.

(ii) Likewise, if $M^\bullet$ is a complex of A-modules with a graded left (right) B-module structure on the associated graded A-module M, then $M^\bullet$ is a left (resp. right) $B^\bullet$-module if and only if the scalar multiplication of the B-module M induces a map of complexes

$$B^\bullet \otimes_A M^\bullet \to M^\bullet \qquad (\text{ resp. } M^\bullet \otimes_A B^\bullet \to M^\bullet\ ).$$

(iii) It is easily seen that the multiplication maps μ^{pq} induce A-linear maps

$$(H^pB^\bullet) \otimes_A (H^qB^\bullet) \to H^{p+q}B^\bullet \qquad \text{for every } p, q \in \mathbb{Z}$$

anf if we let $H^\bullet B^\bullet := \bigoplus_{p\in\mathbb{Z}} H^pB^\bullet$, then the resulting map

$$(H^\bullet B^\bullet) \otimes_A (H^\bullet B^\bullet) \to H^\bullet B^\bullet$$

endows $H^\bullet B^\bullet$ with a structure of $\mathbb{Z}$-graded associative unitary A-algebra, which shall be strictly anti-commutative whenever the same holds for $B^\bullet$. Likewise, if $M^\bullet$ is any left (resp. right, resp. bi) $B^\bullet$-module, then $H^\bullet M^\bullet$ is naturally a $\mathbb{Z}$-graded left (resp. right, resp. bi) $H^\bullet B^\bullet$-module.

(iv) Let $M^\bullet$ be a left $B^\bullet$-module, and denote by $\mu_M^{pq} : B^p \otimes_A M^q \to M^{p+q}$ the (p, q)-graded component of the scalar multiplication of $M^\bullet$, for every $p, q \in \mathbb{Z}$. Then $M^\bullet[1]$ is also naturally a left $B^\bullet$-module, with scalar multiplication $\mu^{\bullet\bullet}_{M[1]}$ given by the rule :

$$\mu^{pq}_{M[1]} := (-1)^p \cdot \mu_M^{p,q+1} \qquad \text{for every } p, q \in \mathbb{Z}.$$

Likewise, if $N^\bullet$ is a right $B^\bullet$-module, with scalar multiplication $\mu_N^{\bullet\bullet}$, then $N^\bullet[1]$ is naturally a right $B^\bullet$-module, with multiplication $\mu^{\bullet\bullet}_{N[1]}$ given by the rule :

$$\mu^{pq}_{N[1]} := \mu_N^{p,q+1} \qquad \text{for every } p, q \in \mathbb{Z}.$$

Lastly, if $P^\bullet$ is a $B^\bullet$-bimodule, then the left and right $B^\bullet$-module structure defined above on $P^\bullet[1]$, amount to a natural B-bimodule structure on $P^\bullet[1]$.

(v) Let $C^\bullet$ be any $\mathbb{Z}$-graded A-algebra, $N^\bullet$ any $\mathbb{Z}$-graded left (resp. right, resp. bi-) B-module. We let $N[1]^\bullet$ be the graded A-module given by the rule $N[1]^p := N^{p+1}$ for every $p \in \mathbb{Z}$. We shall always view $N[1]^\bullet$ as a left (resp. right, resp. bi-) B-module, via the scalar multiplications obtained from those of $N^\bullet$, following the rules spelled out in (iv). This ensures that the functor from $B^\bullet$-modules to $H^\bullet B^\bullet$-modules that assigns to any $B^\bullet$-module $M^\bullet$ its homology $H^\bullet M^\bullet$, is compatible with shift operators.

4.4.44. Let A be ring, $(B^\bullet, d_B^\bullet)$ any differential graded A-algebra, and $I^\bullet \subset B^\bullet$ a (graded) two-sided ideal of $B^\bullet$ (*i.e.* a bi-submodule of the $B^\bullet$-bimodule $B^\bullet$). Let

$$\partial : H^\bullet(B^\bullet/I^\bullet) \to H^\bullet I^\bullet[1]$$

denote the natural map induced by the short exact sequence of complexes

$$0 \to I^\bullet \to B^\bullet \to B^\bullet/I^\bullet \to 0.$$

We have :

Lemma 4.4.45. *In the situation of* (4.4.44)*, the map ∂ is an A-linear graded derivation of the graded A-algebra $H^\bullet(B^\bullet/I^\bullet)$.*

Proof. Indeed, let a and b be any two cycles of the complex $B^\bullet/I^\bullet$ in degree p and q, and $\overline{a}$, $\overline{b}$ the respective classes; lift a and b to some elements $\widetilde{a} \in B^p$ and $\widetilde{b} \in B^q$, so that $\partial(\overline{a} \cdot \overline{b})$ is the class in $H^{p+q+1}I^\bullet$ of $d_B(\widetilde{a} \cdot \widetilde{b}) = d_B(\widetilde{a}) \cdot \widetilde{b} + (-1)^p \cdot \widetilde{a} \cdot d_B(\widetilde{b})$. Since $d_B(\widetilde{a})$ (resp. $d_B(\widetilde{b})$) represents the class of $\partial(\overline{a})$ (resp. of $\partial(\overline{b})$), the assertion follows from the explicit description of the bimodule structure on $I^\bullet[1]$ provided by remark 4.4.43(iv). $\square$

4.4.46. Let $B^\bullet$ and $C^\bullet$ be two differential graded A-algebras. We may endow the complex $B^\bullet \otimes_A C^\bullet$ with a structure of differential graded A-algebra, by the following rule :

$$(x_1 \otimes y_1) \cdot (x_2 \otimes y_2) := (-1)^{j_1 i_2} \cdot (x_1 x_2) \otimes (y_1 y_2) \qquad \text{for every } x_l \in B^{i_l},\ y_l \in C^{j_l}, \text{ with } l = 1, 2.$$

It is easily seen that, if both $B^\bullet$ and $C^\bullet$ are anti-commutative, the same holds for $B^\bullet \otimes_A C^\bullet$. In terms of morphisms of complexes, this multiplication law corresponds to the composition

$$(B^\bullet \otimes_A C^\bullet) \otimes_A (B^\bullet \otimes_A C^\bullet) \xrightarrow{\sim} (B^\bullet \otimes_A B^\bullet) \otimes_A (C^\bullet \otimes_A C^\bullet) \xrightarrow{\mu_B^\bullet \otimes_A \mu_C^\bullet} B^\bullet \otimes_A C^\bullet$$

where the first isomorphism is obtained by composing the associativity isomorphisms of example 4.1.8(ii) and the isomorphisms (4.1.9) that swap the tensor factors. Here $\mu_B^\bullet$ and $\mu_C^\bullet$ are the multiplication maps of $B^\bullet$ and $C^\bullet$.

Suppose now that both $B^\bullet$ and $C^\bullet$ are bounded above complexes; presumably, in this case there is a canonical way to define a differential graded algebra structure as well on (suitable representatives for)

$$D^\bullet := B^\bullet \overset{\mathbf{L}}{\otimes}_A C^\bullet$$

in such a way that this structure is well defined as an object of the derived category of A-$\mathbf{dga}$ (the latter should be, as usual, the localization of A-$\mathbf{dga}$, by the multiplicative system of quasi-isomorphisms). More modestly, we shall endow the graded A-module $H_\bullet D^\bullet$ with a natural structure of associative graded A-algebra, in such a way that the natural map

$$H_\bullet D^\bullet := \bigoplus_{i \in \mathbb{Z}} \mathrm{Tor}_i^A(B^\bullet, C^\bullet) \to H_\bullet(B^\bullet \otimes_A C^\bullet) \tag{4.4.47}$$

is a morphism of graded A-algebras. The multiplication of $H^\bullet D^\bullet$ is defined as the composition

$$H_i D^\bullet \otimes_A H_j D^\bullet \xrightarrow{\alpha} \mathrm{Tor}_{i+j}^A(B^\bullet \otimes_A B^\bullet, C^\bullet \otimes_A C^\bullet) \xrightarrow{\mu} H_{i+j} D^\bullet \qquad \text{for every } i, j \in \mathbb{Z}$$

where α is the bilinear pairing provided by (4.1.14), and with $\mu := \mathrm{Tor}_{i+j}^A(\mu_B^\bullet, \mu_C^\bullet)$.

Let us check first that the foregoing rule does define an associative multiplication on $H_\bullet D^\bullet$. To this aim, set $B_2^\bullet := B^\bullet \otimes_A B^\bullet$, $B_3^\bullet := B^\bullet \otimes_A B_2^\bullet$ and define likewise $C_2^\bullet$ and $C_3^\bullet$; a little diagram chase, together with (4.1.15), reduces to verifying the commutativity of the diagram

$$\begin{tikzcd}
\mathrm{Tor}_i^A(B_2^\bullet, C_2^\bullet) \otimes_A H_j D^\bullet \arrow[r, "\mu \otimes_A \mathbf{1}_{H_i D^\bullet}"] \arrow[d] & H_i D^\bullet \otimes_A H_j D^\bullet \arrow[d] & H_i D^\bullet \otimes_A \mathrm{Tor}_j^A(B_2^\bullet, C_2^\bullet) \arrow[l, "\mathbf{1}_{H_j D^\bullet} \otimes_A \mu"'] \arrow[d] \\
\mathrm{Tor}_{i+j}^A(B_3^\bullet, C_3^\bullet) \arrow[r, "\gamma"] & \mathrm{Tor}_{i+j}^A(B_2^\bullet, C_2^\bullet) & \mathrm{Tor}_{i+j}^A(B_3^\bullet, C_3^\bullet) \arrow[l, "\delta"']
\end{tikzcd}$$

whose vertical arrows are the bilinear pairings of (4.1.14), and with

$$\gamma := \mathrm{Tor}_{i+j}^A(\mu_B^\bullet \otimes_A \mathbf{1}_{B^\bullet}, \mu_C^\bullet \otimes_A \mathbf{1}_{C^\bullet}) \qquad \delta := \mathrm{Tor}_{i+j}^A(\mathbf{1}_{B^\bullet} \otimes_A \mu_B^\bullet, \mathbf{1}_{C^\bullet} \otimes_A \mu_C^\bullet).$$

We show the commutativity of the left subdiagram; the same argument applies to the right one. Unwinding the definitions, we come down to checking the commutativity, in $\mathsf{D}(A\text{-}\mathbf{Mod})$, of the diagram of complexes

$$\begin{tikzcd}
(P^\bullet_{B_2} \otimes_A C^\bullet_2) \otimes_A (P^\bullet_B \otimes_A C^\bullet) \arrow[r, "\sim"] \arrow[d, "\vartheta^\bullet"'] & (P^\bullet_{B_2} \otimes_A P^\bullet_B) \otimes_A C^\bullet_3 \arrow[r, "\varphi^\bullet_{12,3} \otimes_A \mathbf{1}_{C^\bullet_3}"] & P^\bullet_{B_3} \otimes_A C^\bullet_3 \arrow[d, "\eta^\bullet"] \\
(P^\bullet_B \otimes_A C^\bullet) \otimes_A (P^\bullet_B \otimes_A C^\bullet) \arrow[r, "\sim"] & (P^\bullet_B \otimes_A P^\bullet_B) \otimes_A C^\bullet_2 \arrow[r, "\varphi^\bullet_{1,23} \otimes_A \mathbf{1}_{C^\bullet_3}"] & P^\bullet_{B_2} \otimes_A C^\bullet_2
\end{tikzcd}$$

(notation of (4.1.12)), with $\vartheta^\bullet := (P^\bullet_{\mu_B} \otimes_A \mu^\bullet_C) \otimes_A \mathbf{1}_{P^\bullet_B \otimes_A C^\bullet}$ and $\eta^\bullet := P^\bullet_{\mu_B \otimes_A \mathbf{1}_B} \otimes_A (\mu^\bullet_C \otimes_A \mathbf{1}_{C^\bullet})$, and where $\varphi^\bullet_{12,3}$ and $\varphi^\bullet_{1,23}$ are as in (4.1.14) (and with the isomorphisms given by compositions of associativity and swapping isomorphisms). A direct inspection shows that this diagram commutes up to homotopy, as required.

Next, we check that (4.4.47) is a map of graded A-algebras. Unwinding the definitions, we see that – with the notation of (4.1.14) – the multiplication of $H_\bullet D^\bullet$ is the map on homology induced by the composition

$$(P^\bullet_B \otimes_A C^\bullet) \otimes_A (P^\bullet_B \otimes_A C^\bullet) \xrightarrow{\sim} (P^\bullet_B \otimes_A P^\bullet_B) \otimes_A C^\bullet_2 \to P^\bullet_{B_2} \otimes_A C^\bullet \to P^\bullet_B \otimes_A C$$

where the first isomorphism is again a composition of associativity isomorphisms and isomorphisms that swap the factors; the second map is $\varphi^\bullet_{12} \otimes_A \mu^\bullet_C$, where $\varphi^\bullet_{12} : P^\bullet_B \otimes_A P^\bullet_B \to P^\bullet_{B_2}$ is defined as in (4.1.14). The last map is $P^\bullet_{\mu_B} \otimes_A \mathbf{1}_{C^\bullet}$, where $P^\bullet_{\mu_B}$ is given by (4.1.12). Since the associativity and swapping isomorphisms are obviously natural in all their arguments, we come down to checking the commutativity, in $\mathsf{D}(A\text{-}\mathbf{Mod})$, of the diagram of complexes

$$\begin{tikzcd}
(P^\bullet_B \otimes_A P^\bullet_B) \otimes_A C^\bullet_2 \arrow[rr, "\varphi^\bullet_{12} \otimes_A \mu^\bullet_C"] \arrow[d, "(\rho^\bullet_B \otimes_A \rho^\bullet_B) \otimes_A \mathbf{1}_{C^\bullet_2}"'] & & P^\bullet_{B_2} \otimes_A C^\bullet \arrow[d, "P^\bullet_{\mu_B} \otimes_A \mathbf{1}_{C^\bullet}"] \\
B^\bullet_2 \otimes_A C^\bullet_2 \arrow[r, "\mu^\bullet_B \otimes_A \mu^\bullet_C"] & B^\bullet \otimes_A C^\bullet & P^\bullet_B \otimes_A C^\bullet. \arrow[l, "\rho^\bullet_B \otimes_A \mathbf{1}_{C^\bullet}"']
\end{tikzcd}$$

The latter is further reduced to the commutativity of

$$\begin{tikzcd}
P^\bullet_B \otimes_A P^\bullet_B \arrow[rr, "\varphi^\bullet_{12}"] \arrow[d, "\rho^\bullet_B \otimes_A \rho^\bullet_B"'] & & P^\bullet_{B_2} \arrow[d, "P^\bullet_{\mu_B}"] \\
B^\bullet_2 \arrow[r, "\mu^\bullet_B"] & B^\bullet & P^\bullet_B. \arrow[l, "\rho^\bullet_B"']
\end{tikzcd}$$

But a simple inspection shows that this diagram indeed commutes up to homotopy.

Lastly, we claim that if $B^\bullet$ and $C^\bullet$ are both anti-commutative, then the same holds for $H_\bullet D^\bullet$. Indeed, consider the diagram

$$\begin{tikzcd}
(P^\bullet_B \otimes_A P^\bullet_B) \otimes_A C^\bullet_2 \arrow[rrrr, "\sim"] \arrow[dr, "(\rho^\bullet_B \otimes_A \rho^\bullet_B) \otimes_A \mathbf{1}_{C^\bullet_2}"'] \arrow[ddd, "\varphi^\bullet_{12} \otimes_A \mu^\bullet_C"'] & & & & (P^\bullet_B \otimes_A P^\bullet_B) \otimes_A C^\bullet_2 \arrow[dl, "(\rho^\bullet_B \otimes_A \rho^\bullet_B) \otimes_A \mathbf{1}_{C^\bullet_2}"] \arrow[ddd, "\varphi^\bullet_{12} \otimes_A \mu^\bullet_C"] \\
& B^\bullet_2 \otimes_A C^\bullet_2 \arrow[rr, "\sim"] \arrow[dr, "\mu^\bullet_B \otimes_A \mu^\bullet_C"'] & & B^\bullet_2 \otimes_A C^\bullet_2 \arrow[dl, "\mu^\bullet_B \otimes_A \mu^\bullet_C"] & \\
& & B^\bullet \otimes_A C^\bullet & & \\
P^\bullet_{B_2} \otimes_A C^\bullet \arrow[rr, "P^\bullet_{\mu_B} \otimes_A \mathbf{1}_{C^\bullet}"] & & P^\bullet_B \otimes_A C^\bullet \arrow[u, "\rho^\bullet_B \otimes_A \mathbf{1}_{C^\bullet}"] & & P^\bullet_{B_2} \otimes_A C^\bullet. \arrow[ll, "P^\bullet_{\mu_B} \otimes_A \mathbf{1}_{C^\bullet}"']
\end{tikzcd}$$

(The upper isomorphism is obtained from the automorphism of $P_B^\bullet \otimes_A P_B^\bullet$ that swaps the two factors, and from the automorphism of $C_2^\bullet$ of the same type; likewise for the lower isomorphism.) The assumption on $B^\bullet$ and $C^\bullet$ says that the inner triangular subdiagram commutes, and the same holds – up to homotopy – for the upper and the left and right subdiagrams, by a simple inspection. Then also the outer rectangular subdiagram commutes up to homotopy, and the contention follows easily.

Example 4.4.48. An important example of differential graded algebra arises from the Koszul complex of (4.1.16). Indeed, let $\mathbf{f} := (f_1, \dots, f_r)$ be a finite sequence of elements of a ring A. We can endow $\mathbf{K}_\bullet(\mathbf{f})$ with a natural structure of strictly anti-commutative differential graded A-algebra, as follows. First, notice that there are natural isomorphisms

$$\mathbf{K}_i(\mathbf{f}) \xrightarrow{\sim} \Lambda_A^i(A^{\oplus r}) \qquad \text{for every } i = 0, \dots, r.$$

This is clear if $r = 1$, and for $r > 1$, such an isomorphism is established inductively, by means of the natural decomposition

$$\Lambda_A^i(L \oplus A) \xrightarrow{\sim} \Lambda_A^{i-1}(L) \oplus \Lambda_A^i(L) \qquad \text{for every } i \in \mathbb{N} \text{ and every } A\text{-module } L$$

(see [36, (4.3.18)]) and the corresponding decomposition

$$\mathbf{K}_i(\mathbf{f}) \xrightarrow{\sim} (\mathbf{K}_{i-1}(f_1, \dots, f_{r-1}) \otimes_A \mathbf{K}_1(f_r)) \oplus (\mathbf{K}_i(f_1, \dots, f_{r-1}) \otimes_A \mathbf{K}_0(f_r)).$$

Let $e_1, \dots, e_r$ denote the canonical basis of $A^{\oplus r}$; under this isomorphism, the differentials of the Koszul complex are identified with the operators $d_{\mathbf{f},\bullet}$ defined inductively as follows. We let $d_{\mathbf{f},1} : A^{\oplus r} \to A$ be the A-linear map given by the rule : $e_i \mapsto f_i$ for $i = 1, \dots, r$. If $i > 1$, set

$$d_{\mathbf{f},i}(e_{j_1} \wedge \cdots \wedge e_{j_i}) := f_{j_1} e_{j_2} \wedge \cdots \wedge e_{j_i} - e_{j_1} \wedge d_{\mathbf{f},i-1}(e_{j_2} \wedge \cdots \wedge e_{j-i})$$

for every sequence $(j_1, \dots, j_i) \in \{1, \dots, r\}^i$. It is easily seen that the complex $(\Lambda_A^\bullet(A^{\oplus r}), d_{\mathbf{f},\bullet})$ underlies a differential graded A-algebra, whose multiplication is given by usual the exterior product :

$$x \cdot y := x \wedge y \qquad \text{for every } x, y \in \Lambda_A^\bullet(A^{\oplus r}).$$

Notice as well that the isomorphism

$$\kappa_{\mathbf{f}} : \mathbf{K}_\bullet(\mathbf{f}) \xrightarrow{\sim} (\Lambda_A^\bullet(A^{\oplus r}), d_{\mathbf{f},\bullet})$$

thus obtained, is compatible with extensions of sequences : if $\mathbf{g} := (g_1, \dots, g_s)$ is any other sequence of elements of A, we have a commutative diagram

$$\begin{tikzcd}
\mathbf{K}_\bullet(\mathbf{f}) \otimes_A \mathbf{K}_\bullet(\mathbf{g}) \arrow[r, "\kappa_{\mathbf{f}} \otimes_A \kappa_{\mathbf{g}}"] \arrow[d] & (\Lambda_A^\bullet(A^{\oplus r}), d_{\mathbf{f},\bullet}) \otimes_A (\Lambda_A^\bullet(A^{\oplus s}), d_{\mathbf{g},\bullet}) \arrow[d] \\
\mathbf{K}_\bullet(\mathbf{f}, \mathbf{g}) \arrow[r, "\kappa_{\mathbf{f},\mathbf{g}}"] & (\Lambda_A^\bullet(A^{\oplus r+s}), d_{\mathbf{f},\mathbf{g},\bullet})
\end{tikzcd}$$

whose left vertical arrow comes directly from the definition of $\mathbf{K}_{\mathbf{f},\mathbf{g}}$, and whose right vertical arrow is given by [36, (4.3.18)]. Moreover, the right vertical arrow is even an isomorphism of differential graded A-algebras, if we endow the tensor product with the multiplication law as in (4.4.46).

4.4.49. Let A be a ring, $\mathbf{f} := (f_1, \dots, f_r)$ a regular sequence of elements of A; denote by J the ideal generated by $\mathbf{f}$, and set $A_0 := A/J$. We may regard the complex $A_0[0]$ as an (especially simple) differential graded A-algebra, with multiplication $\overline{\mu}^\bullet$ deduced from that of A_0, in the obvious way. We may then state :

Proposition 4.4.50. *In the situation of* (4.4.49)*, the* A_0*-module* J/J^2 *is free of rank* r*, and there exists a unique isomorphism of strictly anti-commutative graded* A*-algebras*

$$\Lambda^\bullet_{A_0}(J/J^2) \xrightarrow{\sim} H_\bullet(A_0[0] \overset{\mathbf{L}}{\otimes}_A A_0[0]) \tag{4.4.51}$$

which restricts, in degree 1*, to the natural identification*

$$\Lambda^1_{A_0}(J/J^2) = J/J^2 \xrightarrow{\sim} \mathrm{Tor}^A_1(A_0, A_0).$$

Proof. In this situation, proposition 4.1.21 says that the Koszul complex, with its natural augmentation, yields a resolution

$$\varepsilon^\bullet : \mathbf{K}_\bullet(\mathbf{f}) \to A_0[0]$$

by free A-modules. Hence, there is a unique isomorphism $\omega^\bullet : P^\bullet_{A_0} \xrightarrow{\sim} \mathbf{K}_\bullet(\mathbf{f})$ in $\mathsf{D}(A\text{-}\mathbf{Mod})$, whose composition with $\varepsilon^\bullet$ agrees with $\rho^\bullet_{A_0}$ (notation of (4.1.12)). Let us endow $\mathbf{K}_\bullet(\mathbf{f})$ with the differential graded A-algebra structure inherited via the isomorphism $\kappa_{\mathbf{f}}$ of example 4.4.48. Then $\varepsilon^\bullet$ is a map of differential graded A-algebras, we easily deduce a commutative diagram in $\mathsf{D}(A\text{-}\mathbf{Mod})$

$$\begin{tikzcd}
P^\bullet_{A_0} \otimes_A P^\bullet_{A_0} \arrow[r, "P^\bullet_{\overline{\mu}}"] \arrow[d, "\omega^\bullet \otimes_A \omega^\bullet"'] & P^\bullet_{A_0} \arrow[d, "\omega^\bullet"] \\
\mathbf{K}_\bullet(\mathbf{f}) \otimes_A \mathbf{K}_\bullet(\mathbf{f}) \arrow[r] & \mathbf{K}_\bullet(\mathbf{f})
\end{tikzcd}$$

whose bottom horizontal arrow is the multiplication map of $\mathbf{K}_\bullet(\mathbf{f})$, and where $P^\bullet_{\overline{\mu}}$ is defined as in (4.1.12). We conclude that $\omega^\bullet$ induces an isomorphism of anti-commutative graded A-algebras

$$H_\bullet(A_0[0] \overset{\mathbf{L}}{\otimes}_A A_0[0]) \xrightarrow{\sim} \mathbf{K}_\bullet(\mathbf{f}, A_0[0]) \xrightarrow{\sim} H_\bullet((\Lambda^\bullet_A(A^{\oplus r}), d_{\mathbf{f},\bullet}) \otimes_A A_0). \tag{4.4.52}$$

By simple inspection, we see that $d_{\mathbf{f},i} \otimes_A \mathbf{1}_{A_0} = 0$ for every $i \in \mathbb{Z}$, whence an isomorphism of strictly anti-commutative A-graded algebras :

$$H_\bullet((\Lambda^\bullet_A(A^{\oplus r}), d_{\mathbf{f},\bullet}) \otimes_A A_0) \xrightarrow{\sim} \Lambda^\bullet_{A_0}(A_0^{\oplus r}).$$

Combining with (4.4.52), we obtain in degree one a natural isomorphism

$$\Lambda^1_{A_0}(A_0^{\oplus r}) = A_0^{\oplus r} \xrightarrow{\sim} \mathrm{Tor}^A_1(A_0, A_0) \xrightarrow{\sim} J/J^2$$

which, finally, delivers the sought isomorphism of differential graded A-algebras. The uniqueness of (4.4.51) is clear, since the exterior algebra is generated by its degree one summand. ☐

Remark 4.4.53. (i) Simplicial A-algebras are another important source of differential graded algebras, thanks to the following construction. Let R be any simplicial A-algebra, $R_\bullet$ the associated chain complex of A-modules, and denote by $\mu_R : R \otimes_A R \to R$ the multiplication map of R. By considering the shuffle map for the bisimplicial A-module $R \boxtimes_A R$ (notation as in (4.2.53)), we deduce a natural map of complexes

$$\mu_{R_\bullet} : R_\bullet \otimes_A R_\bullet \xrightarrow{\mathsf{Sh}^{R\boxtimes_A R}_\bullet} (R \otimes_A R)_\bullet \xrightarrow{(\mu_R)_\bullet} R_\bullet$$

and taking into account propositions 4.2.54 and 4.2.57, it is easily seen that $(R_\bullet, \mu_{R_\bullet})$ is a strictly anti-commutative differential graded algebra. By remark 4.4.43(iii), we deduce that the graded A-module $H_\bullet R := \bigoplus_{p\in\mathbb{N}} H_p R$ is naturally an $\mathbb{N}$-graded associative unitary and strictly anti-commutative A-algebra.

(ii) Likewise, if M is any R-module, then we obtain on the associated chain complex $M_\bullet$ a natural structure of $R_\bullet$-bimodule, so that $H_\bullet M$ is naturally a graded $H_\bullet R$-bimodule.

(iii) Clearly, a morphism $\varphi : R \to S$ of simplicial A-algebras induces a morphism $\varphi_\bullet : R_\bullet \to S_\bullet$ of differential graded A-algebras, and a morphism $f : M \to N$ of R-modules induces a morphism $\varphi_\bullet : M_\bullet \to N_\bullet$ of $R_\bullet$-bimodules.

4.5. **Some homotopical algebra.** The methods of homotopical algebra allow to construct derived functors of non-additive functors, in a variety of situations. In this section, we present the basics of this theory, beginning with its cornerstone, the *standard resolution associated to a triple*, as explained in the following paragraph.

4.5.1. A *triple* $(\top, \eta, \mu)$ on a category $\mathscr{C}$ is the datum of a functor $\top : \mathscr{C} \to \mathscr{C}$ together with natural transformations :

$$\boldsymbol{\eta} : \mathbf{1}_{\mathscr{C}} \Rightarrow \top \qquad \boldsymbol{\mu} : \top \circ \top \Rightarrow \top$$

such that the following diagrams commute :

$$\begin{tikzcd}
(\top \circ \top) \circ \top \arrow[r, equal] \arrow[d, Rightarrow, "\boldsymbol{\mu} * \top"'] & \top \circ (\top \circ \top) \arrow[r, Rightarrow, "\top * \boldsymbol{\mu}"] & \top \circ \top \arrow[d, Rightarrow, "\boldsymbol{\mu}"] \\
\top \circ \top \arrow[rr, Rightarrow, "\boldsymbol{\mu}"] & & \top
\end{tikzcd}
\qquad
\begin{tikzcd}
\top \arrow[r, Rightarrow, "\top * \boldsymbol{\eta}"] \arrow[dr, equal, "\mathbf{1}_\top"'] & \top \circ \top \arrow[d, Rightarrow, "\boldsymbol{\mu}"] & \top \arrow[l, Rightarrow, "\boldsymbol{\eta} * \top"'] \arrow[dl, equal, "\mathbf{1}_\top"] \\
& \top .
\end{tikzcd}$$

Dually, a cotriple $(\perp, \boldsymbol{\varepsilon}, \boldsymbol{\delta})$ in a category $\mathscr{C}$ is a functor $\perp : \mathscr{C} \to \mathscr{C}$ together with natural transformations :

$$\boldsymbol{\varepsilon} : \perp \Rightarrow \mathbf{1}_{\mathscr{C}} \qquad \boldsymbol{\delta} : \perp \Rightarrow \perp \circ \perp$$

such that the following diagrams commute :

$$\begin{tikzcd}
\perp \arrow[rr, Rightarrow, "\boldsymbol{\delta}"] \arrow[d, Rightarrow, "\boldsymbol{\delta}"'] & & \perp \circ \perp \arrow[d, Rightarrow, "\boldsymbol{\delta} * \perp"] \\
\perp \circ \perp \arrow[r, Rightarrow, "\perp * \boldsymbol{\delta}"] & \perp \circ (\perp \circ \perp) \arrow[r, equal] & (\perp \circ \perp) \circ \perp
\end{tikzcd}
\qquad
\begin{tikzcd}
& \perp \arrow[dl, equal, "\mathbf{1}_\perp"'] \arrow[d, Rightarrow, "\boldsymbol{\delta}"] \arrow[dr, equal, "\mathbf{1}_\perp"] & \\
\perp & \perp \circ \perp \arrow[l, Rightarrow, "\perp * \boldsymbol{\varepsilon}"'] \arrow[r, Rightarrow, "\boldsymbol{\varepsilon} * \perp"] & \perp .
\end{tikzcd}$$

Notice that a cotriple in $\mathscr{C}$ is the same as a triple in $\mathscr{C}^o$.

4.5.2. A cotriple $\perp$ on $\mathscr{C}$ and an object A of $\mathscr{C}$ determine a simplicial object $\perp A[\bullet]$ in $\mathscr{C}$; namely, for every $n \in \mathbb{N}$ set $\perp A[n] := \perp^{n+1} A$, and define face and degeneracy operators :

$$\begin{aligned}
\partial_i &:= (\perp^i * \boldsymbol{\varepsilon} * \perp^{n-i})_A &&: \quad \perp A[n] \to \perp A[n-1] \\
\sigma_i &:= (\perp^i * \boldsymbol{\delta} * \perp^{n-i})_A &&: \quad \perp A[n] \to \perp A[n+1].
\end{aligned}$$

Using the foregoing commutative diagrams, one verifies easily that the simplicial identities (4.2.8) hold. Moreover, the morphism $\boldsymbol{\varepsilon}_A : \perp A \to A$ defines an augmentation $\perp A[\bullet] \to A$.

Dually, a triple $\top$ and an object A of $\mathscr{C}$ determine a cosimplicial object $\top A[\bullet] := \top^{\bullet+1} A$, such that

$$\begin{aligned}
\partial^i &:= (\top^i * \boldsymbol{\eta} * \top^{n-i})_A &&: \quad \top^n A \to \top^{n+1} A \\
\sigma^i &:= (\top^i * \boldsymbol{\mu} * \top^{n-i})_A &&: \quad \top^{n+2} A \to \top^{n+1} A
\end{aligned}$$

which is augmented by the morphism $\boldsymbol{\eta}_A : A \to \top A$. Clearly, the rule : $A \mapsto \perp A[\bullet]$ (resp. $A \mapsto \top A[\bullet]$) defines a functor

$$\mathscr{C} \to \widehat{s.}\mathscr{C} \qquad (\text{resp. } \mathscr{C} \to \widehat{c.}\mathscr{C}).$$

4.5.3. An adjoint pair (G, F) as in (1.1.8), with its unit $\boldsymbol{\eta}$ and counit $\boldsymbol{\varepsilon}$, determines a triple $(\top, \boldsymbol{\eta}, \boldsymbol{\mu})$, where :

$$\top := F \circ G : \mathscr{B} \to \mathscr{B} \qquad \boldsymbol{\mu} := F * \boldsymbol{\varepsilon} * G : \top \circ \top \Rightarrow \top$$

as well as a cotriple $(\perp, \boldsymbol{\varepsilon}, \boldsymbol{\delta})$, where :

$$\perp := G \circ F : \mathscr{A} \to \mathscr{A} \qquad \boldsymbol{\delta} := G * \boldsymbol{\eta} * F : \perp \Rightarrow \perp \circ \perp .$$

Indeed, the naturality of $\boldsymbol{\varepsilon}$ and $\boldsymbol{\eta}$ easily implies the commutativity of the diagrams of (4.5.1). The following proposition explains how to use these triples and cotriples to construct canonical resolutions.

Proposition 4.5.4. *In the situation of* (4.5.3)*, the following holds :*

(i) *For every* $A \in \mathrm{Ob}(\mathscr{A})$ *and* $B \in \mathrm{Ob}(\mathscr{B})$*, the augmented simplicial objects :*
$$\perp GB[\bullet] \xrightarrow{\boldsymbol{\varepsilon}_{GB}} GB \qquad F\perp A[\bullet] \xrightarrow{F*\boldsymbol{\varepsilon}_A} FA$$
are homotopically trivial (notation of (4.5.2)*).*

(ii) *Dually, the same holds for the augmented cosimplicial objects :*
$$GB \xrightarrow{G*\boldsymbol{\eta}_B} G\top B[\bullet] \qquad FA \xrightarrow{\boldsymbol{\eta}_{FA}} \top FA[\bullet].$$

Proof. Notice that
$$F\perp A[n] = \top FA[n] \qquad \text{for every } [n] \in \mathrm{Ob}(\Delta^\wedge).$$
Therefore, for every morphism $\varphi : [n] \to [m]$ in $\Delta^\wedge$, we have two morphisms
$$F\perp A[\varphi] : F\perp A[m] \to F\perp A[n] \qquad \top FA[\varphi] : F\perp A[n] \to F\perp A[m].$$
Now, recall that the augmentation ε_A defines a natural morphism
$$\perp A[\varepsilon^{\bullet+1}_{-1,0}] : \perp A[\bullet] \to s.A \qquad [n] \mapsto \perp A[\varepsilon^{n+1}_{-1,0}]$$
(notation of remark 4.2.11(iv)), and one sees that the system $(\top FA[\varepsilon^{n+1}_{-1,0}] \mid [n] \in \mathrm{Ob}(\Delta))$ defines a morphism
$$\top FA[\varepsilon^{\bullet+1}_{-1,0}] : s.FA \to F\perp A[\bullet]$$
which is right inverse to $F\perp A[\varepsilon^{\bullet+1}_{-1,0}]$. For every $n, k \in \mathbb{N}$ such that $k \leq n+1$, set
$$u_{n,k} := (\top FA[\varepsilon^k_{n-k,0}]) \circ (F\perp A[\varepsilon^k_{n-k,0}])$$
(notation of example 4.2.6(ii)).

Claim 4.5.5. The system $u_{\bullet\bullet}$ defines a homotopy from $\mathbf{1}_{F\perp A[\bullet]}$ to $(\top FA[\varepsilon^{\bullet+1}_{-1,0}]) \circ (F\perp A[\varepsilon^{\bullet+1}_{-1,0}])$ (see (4.2.14)).

Proof of the claim. By unwinding the definitions, we see that $F\perp A[\varepsilon^k_{n-k,0}]$ is the composition
$$F(\boldsymbol{\varepsilon}_{\perp^{n+1-k}A} \circ \cdots \circ \boldsymbol{\varepsilon}_{\perp^{n-1}A} \circ \boldsymbol{\varepsilon}_{\perp^n A}) : F\perp^{n+1}A \to F\perp^{n+1-k}A \qquad \text{for } k > 0$$
and $\top FA[\varepsilon^k_{n-k,0}]$ is the composition
$$\boldsymbol{\eta}_{F\perp^n A} \circ \boldsymbol{\eta}_{F\perp^{n-1}A} \circ \cdots \circ \boldsymbol{\eta}_{F\perp^{n+1-k}A} : F\perp^{n+1-k}A \to F\perp^{n+1}A \qquad \text{for } k > 0$$
and both equal $\mathbf{1}_{F\perp^{n+1}A}$ for $k = 0$. Now, since the face operator ∂_i of $F\perp A[n]$ is $F\perp^i(\boldsymbol{\varepsilon}_{\perp^{n-i}A})$ for every $i \leq n$, the naturality of $\boldsymbol{\eta}$ yields a commutative diagram
$$\begin{tikzcd} F\perp^n A \arrow[r, "\partial_{i-1}"] \arrow[d, "\boldsymbol{\eta}_{F\perp^n A}"'] & F\perp^{n-1}A \arrow[d, "\boldsymbol{\eta}_{F\perp^{n-1}A}"] \\ F\perp^{n+1}A \arrow[r, "\partial_i"] & F\perp^n A \end{tikzcd} \qquad \text{for every } i > 0 \text{ and every } n \in \mathbb{N}$$
whereas $\partial_0 \circ \boldsymbol{\eta}_{F\perp^n A} = \mathbf{1}_{F\perp^n A}$ for every $n \in \mathbb{N}$; whence the identities :
$$\partial_i \circ \top FA[\varepsilon^k_{n-k,0}] = \begin{cases} \top FA[\varepsilon^{k-1}_{n-k,0}] & \text{for } i < k \\ \top FA[\varepsilon^k_{n-k-1,0}] \circ \partial_{i-k} & \text{for } i \geq k. \end{cases}$$
On the other hand, we have the simplicial identities :
$$\varepsilon_i \circ \varepsilon^k_{n-k,0} = \begin{cases} \varepsilon^{k+1}_{n-k} & \text{for } i < k \\ \varepsilon^k_{n+1-k} \circ \varepsilon_{i-k} & \text{for } i \geq k. \end{cases}$$
Summing up, we conclude that
$$\partial_i \circ u_{n,k} = \begin{cases} u_{n-1,k-1} \circ \partial_i & \text{for } i < k \\ u_{n-1,k} \circ \partial_i & \text{for } i \geq k. \end{cases}$$

Arguing likewise, one deduces as well the corresponding commutation rule – spelled out in (4.2.14) – for the degeneracies σ_i, and the claim follows. $\Diamond$

Likewise, notice that

$$G\top B[n] = \perp GB[n] \qquad \text{for every } [n] \in \Delta^\wedge.$$

Therefore, for every morphism $\varphi : [n] \to [m]$ in $\Delta^\wedge$, we have as well two morphisms

$$\perp GB[\varphi] : \perp GB[m] \to \perp GB[n] \qquad G\top B[\varphi] : \perp GB[n] \to \perp GB[m]$$

and the system $(G\top B[\varepsilon^{n+1}_{-1,0}] \mid [n] \in \mathrm{Ob}(\Delta^\wedge))$ defines a morphism

$$G\top B[\varepsilon^{\bullet+1}_{-1,0}] : s.GB \to \perp GB[\bullet]$$

which is right inverse to the morphism $\perp GB[\varepsilon^{\bullet+1}_{-1,0}]$ deduced from the augmentation. Arguing as in the proof of claim 4.5.5, one checks that the rule

$$v_{n,k} := (G\top B[\varepsilon^{k\vee}_{n-k,0}]) \circ (\perp GB[\varepsilon^{k\vee}_{n-k,0}]) \qquad \text{for every } n, k \in \mathbb{N} \text{ such that } k \leq n+1$$

yields a homotopy v from $\mathbf{1}_{\perp GB[\bullet]}$ to $(G\top B[\varepsilon^{\bullet+1}_{-1,0}]) \circ (\perp GB[\varepsilon^{\bullet+1}_{-1,0}])$.

The dual statements admit the dual proof. $\square$

4.5.6. Now, denote by $\mathbf{Set}_\circ$ the category of *pointed sets*, whose objects are all the pairs (S, s) where S is a small set, and $s \in S$ is any element; the morphisms $f : (S, s) \to (S', s')$ are the mappings $f : S \to S'$ such that $f(s) = s'$. Next, let A be any ring, and consider the *pointed forgetful functor*

$$\mathsf{F} : A\text{-}\mathbf{Mod} \to \mathbf{Set}_\circ \qquad M \mapsto (M, 0_M)$$

(where $0_M \in M$ is the zero element of M). The functor F admits the left adjoint

$$\mathsf{L} : \mathbf{Set}_\circ \to A\text{-}\mathbf{Mod} \qquad (S, s) \mapsto A^{(S)}/As$$

(where $A^{(S)}$ denotes the free A-module with basis given by S, so $As \subset A^{(S)}$ is the direct summand generated by the basis element $s \in S$). According to proposition 4.5.4, the adjoint pair (L, F) yields a functor

$$\perp^A_\bullet : A\text{-}\mathbf{Mod} \to \widehat{s}.A\text{-}\mathbf{Mod}$$

into the category of augmented simplicial A-modules (notation of (4.2.10)), such that

$$\mathsf{F} \perp^A_\bullet M \to \mathsf{F}M$$

is a homotopically trivial augmented pointed simplicial set, for every A-module M. This functor is obtained by iterating the functor $\perp^A := \mathsf{L} \circ \mathsf{F}$, so in each degree it consists of a free A-module. We call $\perp^A_\bullet M$ the *standard free resolution of* M.

Remark 4.5.7. Notice that $\perp^A_\bullet 0 = s.0$, the constant simplicial A-module associated to the trivial A-module (see (4.2.4)). This is the reason why we prefer the adjoint pair (L, F), rather than the analogous pair considered in [56, I.1.5.5], arising from the forgetful functor from A-modules to non-pointed sets.

4.5.8. Let R be a simplicial A-algebra, and define the category R-$\mathbf{Mod}$ of R-modules, as in [36, §8.1]. Notice that any morphism $S \to R$ of simplicial rings induces a base change functor

$$S\text{-}\mathbf{Mod} \to R\text{-}\mathbf{Mod} \qquad (M[n] \mid n \in \mathbb{N}) \mapsto (R[n] \otimes_{S[n]} M[n] \mid n \in \mathbb{N})$$

which is left adjoint to the forgetful functor (details left to the reader).

Now, by applying the functors $\perp^{R[n]}_\bullet$ to the terms $M[n]$ of a given R-module M, we obtain an augmented simplicial R-module

$$\perp^R_\bullet M \to M \tag{4.5.9}$$

which amounts to a bisimplicial A-module, whose columns $\perp_\bullet^{R[n]} M[n] \to M[n]$ are augmented simplicial $R[n]$-modules, for every $n \in \mathbb{N}$. Likewise, the row $\perp_n^R M$ is an R-module, for every $n \in \mathbb{N}$.

Lemma 4.5.10. *In the situation of* (4.5.8), *let* $\varphi : M \to N$ *be any quasi-isomorphism of* R*-modules. Then the induced morphism*

$$\perp_n^R \varphi : \perp_n^R M \to \perp_n^R N$$

is a homotopy equivalence, for every $n \in \mathbb{N}$.

Proof. Set $\perp_{-1}^R M := M$, and notice the natural isomorphisms

$$\perp_n^R M \xrightarrow{\sim} R \otimes_{s.\mathbb{Z}} \perp^{s.\mathbb{Z}} \circ \perp_{n-1}^R M \qquad \text{for every } n \in \mathbb{N} \tag{4.5.11}$$

(where $s.\mathbb{Z}$ is the constant simplicial ring arising from $\mathbb{Z}$ (see (4.2.4)), which is the initial object in the category of simplicial rings). The assumption means that $\perp_{-1}^R \varphi$ is a quasi-isomorphism. However, (4.5.11) and Whitehead's theorem ([56, I.2.2.3]) imply that, if – for a given $n \in \mathbb{N}$ – the map $\perp_{n-1}^R \varphi$ is a quasi-isomorphism, then $\perp_n^R \varphi$ is a homotopy equivalence. We may thus conclude by a simple induction on n. $\square$

Example 4.5.12. Let M and N two R-modules. We may define two functors

$$M \overset{\ell}{\otimes}_R - \quad (\text{ resp. } - \overset{\ell}{\otimes}_R N\ ) \quad : \quad R\text{-}\mathbf{Mod} \to R\text{-}\mathbf{Mod}$$

by the rules :

$$P \mapsto M \otimes_R (\perp_\bullet^R P)^\Delta \quad (\text{ resp. } P \mapsto (\perp_\bullet^R P)^\Delta \otimes_R N\ ) \qquad \text{for every } R\text{-module } P$$

where Δ is the diagonal functor, from $\mathscr{C}$-bisimplicial to $\mathscr{C}$-simplicial objects (see (4.2.15)).

(i) We claim that these functors descend to the derived category. That is, suppose that $\varphi : P \to P'$ is a quasi-isomorphism; then $M \overset{\ell}{\otimes}_R \varphi$ and $\varphi \overset{\ell}{\otimes}_R N$ are both quasi-isomorphisms. Indeed, lemma 4.5.10 implies that the induced morphisms

$$M \otimes_R \perp_n^R \varphi : M \otimes_R \perp_n^R P \to M \otimes_R \perp_n^R P'$$

are quasi-isomorphisms, for every $n \in \mathbb{N}$, and likewise for $\varphi \otimes_R \perp_n^R N$, so the assertion follows from Eilenberg-Zilber's theorem 4.2.48(i).

(ii) Also, we claim that the notation $M \overset{\ell}{\otimes}_R N$ is unambiguous. That is, we may compute this object by applying the functor $M \overset{\ell}{\otimes}_R -$ to N, or by applying the functor $- \overset{\ell}{\otimes}_R N$ to M, and the resulting two R-modules are naturally isomorphic in $\mathsf{D}(R\text{-}\mathbf{Mod})$. Indeed, it suffices to check that the natural morphisms

$$M \otimes_R (\perp_\bullet^R N)^\Delta \xleftarrow{\alpha} (\perp_\bullet^R M)^\Delta \otimes_R (\perp_\bullet^R N)^\Delta \xrightarrow{\beta} (\perp_\bullet^R M)^\Delta \otimes_R N$$

induced by the augmentation (4.5.9), are both quasi-isomorphisms. We check this for α; the same argument shall apply to β. To ease notation, set $L := (\perp_\bullet^R N)^\Delta$. Then α is deduced by extracting the diagonal from the augmented simplicial R-module

$$\perp_\bullet^R M \otimes_R L \to M \otimes_R L \tag{4.5.13}$$

(so, the n-th column of $s.L$ is a constant and flat simplicial $R[n]$-module, for every $n \in \mathbb{N}$). Thus, we are reduced to checking that the columns of (4.5.13) are aspherical. However, for every $n \in \mathbb{N}$, the n-th column is the augmented simplicial $R[n]$-module $(\perp_\bullet^{R[n]} M[n]) \otimes_{R[n]} L[n]$; since $L[n]$ is flat, it then suffices to recall that the standard free resolution is aspherical.

(iii) The foregoing discussion yields a well defined functor

$$- \overset{\ell}{\otimes}_R - : \mathsf{D}(R\text{-}\mathbf{Mod}) \times \mathsf{D}(R\text{-}\mathbf{Mod}) \to \mathsf{D}(R\text{-}\mathbf{Mod})$$

called the *left derived tensor product*. The same method shall be applied hereafter to construct derived functors of certain non-additive functors.

Remark 4.5.14. Let M and N be as in example 4.5.12 and set $P := (\perp^R_\bullet M) \otimes_R s.N$ (this is a simplicial R-module; especially, it is a bisimplicial A-module); by Eilenberg-Zilber's theorem 4.2.48, the cochain complex associated to $M \overset{\ell}{\otimes}_R N$ is naturally isomorphic, in $\mathsf{D}(A\text{-}\mathbf{Mod})$ to the complex $\mathrm{Tot}(P^{\bullet\bullet})$, where $P^{\bullet\bullet}$ denotes the double complex associated to P. There follows a spectral sequence :

$$E^1_{pq} := \mathrm{Tor}_q^{R[p]}(M[p], N[p]) \Rightarrow H_{p+q}(M \overset{\ell}{\otimes}_R N)$$

whose differentials $d^1_{pq} : E^1_{pq} \to E^1_{p-1,q}$ are obtained as follows. For every integer $q \in \mathbb{N}$, let T_q be the R-module given by the rule : $T_q[p] := E^1_{pq}$, with face and degeneracy maps deduced from those of M, N and R, in the obvious way. Then d^1_{pq} is the differential in degree p of the chain complex $T_{q\bullet}$ associated to T_q (details left to the reader).

Example 4.5.15. (i) Another useful triple arises from the forgetful functor $A\text{-}\mathbf{Alg} \to \mathbf{Set}$ and its left adjoint, that attaches to any set S the free A-algebra $A[S]$. There results, for every A-algebra B, a standard simplicial resolution by free A-algebras

$$F_A(B) \to B.$$

Now, if R is again a simplicial A-algebra, and S any R-algebra, we can proceed as in the foregoing, to obtain a bisimplicial resolution $F_R(S) \to S$ whose columns $F_{R[n]}(S[n]) \to S[n]$ are augmented simplicial $R[n]$-algebras, for every $n \in \mathbb{N}$. A simple inspection shows that the proof of lemma 4.5.10 carries over – *mutatis mutandi* – to the present situation, hence a quasi-isomorphism $S \to S'$ of R-algebra induces a morphism $F_R(S)[n] \to F_R(S')[n]$ of R-algebras, which is a homotopy equivalence on the underlying R-modules, for every $n \in \mathbb{N}$.

(ii) We may then define a *derived tensor product* for R-algebras, proceeding as in example 4.5.12. Namely, if S and S' are any two R-algebras, we define two functors

$$S \overset{\ell}{\otimes}_R - \quad (\text{ resp. } - \overset{\ell}{\otimes}_R S' \text{ }) \quad : \quad R\text{-}\mathbf{Alg} \to R\text{-}\mathbf{Alg}$$

by the rules :

$$T \mapsto S \otimes_R F_R(T)^\Delta \quad (\text{ resp. } T \mapsto F_R(T)^\Delta \otimes_R N \text{ }) \qquad \text{for every } R\text{-algebra } T.$$

Arguing like in *loc.cit.* we see that both functors transform quasi-isomorphisms into quasi-isomorphisms, hence they descend to the derived category $\mathsf{D}(R\text{-}\mathbf{Alg})$, and moreover, the two functors are naturally isomorphic, so the notation $S \overset{\ell}{\otimes}_R S'$ is unambiguous : the detailed verification is left as an exercise for the reader.

4.5.16. Let $A\text{-}\mathbf{Alg}.\mathbf{Mod}$ be the category of all pairs (B, M), where B is any A-algebra, and M any B-module. The morphisms $(B, M) \to (B', M')$ are the pairs (f, φ), where $f : B \to B'$ is a morphism of A-algebras, and $\varphi : M \to f^*M'$ a B-linear map (here f^*M' denotes the B-module obtained from the B'-module M', by restriction of scalars along the map f). Suppose now that we have a commutative diagram of categories :

$$\begin{tikzcd} A\text{-}\mathbf{Alg}.\mathbf{Mod} \arrow[r, "T"] \arrow[d, "F"'] & \mathscr{C} \arrow[d, "f"] \\ A\text{-}\mathbf{Alg} \arrow[r, "g"] & \mathscr{B} \end{tikzcd}$$

such that :

- For every $X \in \mathrm{Ob}(B)$, the fibre $f^{-1}X$ is an abelian category whose filtered colimits are representable and exact.
- F is given by the rule $(B, M) \mapsto B$ for every $(B, M) \in \mathrm{Ob}(A\text{-}\mathbf{Alg}.\mathbf{Mod})$.
- For every A-algebra B, the restriction
$$T_B : F^{-1}B \to f^{-1}(gB)$$
of T commutes with filtered colimits, *i.e.*, if $((B, M_i) \mid i \in I)$ is any filtered system of objects of A-$\mathbf{Alg}.\mathbf{Mod}$ (over the same A-algebra B), then the induced morphism
$$\operatorname*{colim}_{i \in I} T(B, M_i) \to T(B, \operatorname*{colim}_{i \in I} M_i)$$
is an isomorphism.

The functors f and g extend to functors $s.f : s.\mathscr{C} \to s.\mathscr{B}$, respectively $s.g : s.A\text{-}\mathbf{Alg} \to s.\mathscr{B}$, and notice that – for R any simplicial A-algebra – T induces a functor

$$T_R : R\text{-}\mathbf{Mod} \to s.\mathscr{C}_{/R} := s.f^{-1}(s.gR) \qquad (M[n] \mid n \in \mathbb{N}) \mapsto (T(R[n], M[n]) \mid n \in \mathbb{N}).$$

For every R-module M, we set

$$LT_R(M) := T_R(\perp^R_\bullet M)^\Delta.$$

The augmentation $T_R(\perp^R_\bullet M) \to T_R M$ can be regarded as a morphism of bisimplicial objects

$$T_R(\perp^R_\bullet M) \to s.T_R M$$

(the columns of $s.T_R M$ are constant $\mathscr{C}$-simplicial objects), whence a morphism in $s.\mathscr{C}_{/R}$:

$$LT_R(M) \to s.T_R M^\Delta = T_R M \qquad \text{for every } R\text{-module } M. \tag{4.5.17}$$

In many applications, $s.\mathscr{C}_{/R}$ will be also an abelian category, but anyhow we can state the following :

Proposition 4.5.18. *With the notation of* (4.5.16)*, suppose that $\mathscr{C}$ is an abelian category. Then the following holds :*

(i) *If M is a flat R-module, then* (4.5.17) *is a quasi-isomorphism.*

(ii) *If $\varphi : M \to N$ is a quasi-isomorphism of R-modules, then the induced map*
$$LT_R(M) \to LT_R(N)$$
is a quasi-isomorphism.

Proof. (i): The assumption means that $M[n]$ is a flat $R[n]$-module for every $n \in \mathbb{N}$. Denote by C the unnormalized double complex associated to $T_R(\perp^R_\bullet M)$, and by C^Δ (resp. by D) the unnormalized complex associated to $LT_R(M)$ (resp. to $T_R M$). By Eilenberg-Zilber's theorem 4.2.48, we have a natural quasi-isomorphism

$$C^\Delta \to \mathrm{Tot}(C) \tag{4.5.19}$$

given by the Alexander-Whitney map of (4.2.33), and a simple inspection shows that the map $C^\Delta \to D$ induced by (4.5.17) factors through (4.5.19). Hence, it suffices to show that the map

$$\mathrm{Tot}(C) \to D$$

induced by the augmentation of $T_R(\perp^R_\bullet M)$, is a quasi-isomorphism, under the stated condition.

To this aim, it suffices to check that the augmented $\mathscr{C}$-simplicial object

$$T_R(\perp^{R[n]}_\bullet M[n]) \to T_R M[n] \tag{4.5.20}$$

is aspherical for every $n \in \mathbb{N}$. However, since $M[n]$ is a flat $R[n]$-module, it can be written as a filtered colimit of a system of free $R[n]$-modules ([57, Ch.I, Th.1.2]); on the other hand, the functor $\perp^{R[n]}_\bullet$ commutes with all filtered colimits, and the same holds for T_R, by assumption. Since the filtered colimits of the fibres of the functor f are exact, we are then reduced to checking the assertion in case $M[n]$ is a free $R[n]$-module; but in this case, (4.5.20) is even homotopically trivial, by virtue of proposition 4.5.4.

(ii): In light of Eilenberg-Zilber's theorem 4.2.48, it suffices to show that the induced map

$$T_R(\perp_n^R M) \to T_R(\perp_n^R N)$$

is a quasi-isomorphism for every $n \in \mathbb{N}$. In turns, this follows readily from lemma 4.5.10. $\square$

Remark 4.5.21. (i) The proof of proposition 4.5.18(i) applies as well to the derived tensor product; namely, for any two R-modules M and N there is a natural morphism of R-modules

$$M \overset{\ell}{\otimes}_R N \to M \otimes_R N$$

which is a quasi-isomorphism, if either M or N is a flat R-module (details left to the reader).

(ii) Especially, let S and S' be any two R-algebras. Then, since the standard free resolution $F_R(S)$ of example 4.5.15 is a flat simplicial R-module, (i) implies that the R-module underlying the R-algebra $S \overset{\ell}{\otimes}_R S'$, is naturally isomorphic, in $\mathsf{D}(R\text{-}\mathbf{Mod})$, to the derived tensor product of the R-modules underlying S and S'. In other words, the notation $- \overset{\ell}{\otimes}_R -$ is unambiguous, whether one refers to derived tensor products of algebras, or of their underlying modules.

(iii) Denote by $\sigma, \omega : R\text{-}\mathbf{Mod} \to R\text{-}\mathbf{Mod}$ respectively the suspension and loop functors ([56, I.3.2.1]), and recall that σ is left adjoint to ω. A simple inspection of the definitions yields natural identifications

$$(\sigma M) \otimes_R N \overset{\sim}{\to} \sigma(M \otimes_R N) \qquad (\omega M) \otimes_R N \overset{\sim}{\to} \omega(M \otimes_R N)$$

for every R-modules M and N. By the same token, it is clear that σ and ω transform flat R-modules into flat R-modules. In view of (i), we deduce natural isomorphisms

$$(\sigma M) \overset{\ell}{\otimes}_R N \overset{\sim}{\to} \sigma(M \overset{\ell}{\otimes}_R N) \qquad (\omega M) \overset{\ell}{\otimes}_R N \overset{\sim}{\to} \omega(M \overset{\ell}{\otimes}_R N) \qquad \text{in } \mathsf{D}(R\text{-}\mathbf{Mod})$$

for every R-modules M and N.

(iv) Let $f : N \to N'$ be any morphism of R-modules; in the same vein, we get a natural identification :

$$\mathrm{Cone}(M \otimes_R f) \overset{\sim}{\to} M \otimes_R \mathrm{Cone}\, f \qquad \text{for every } R\text{-module } M$$

(see [56, I.3.2.2] for the definition of the cone of a morphism of R-modules). It follows immediately that the functor $M \overset{\ell}{\otimes}_R - : \mathsf{D}(R\text{-}\mathbf{Mod}) \to \mathsf{D}(R\text{-}\mathbf{Mod})$ transforms distinguished triangles into distinguished triangles (see [56, I.3.2.2.4] for the definition of distinguished triangle in $\mathsf{D}(R\text{-}\mathbf{Mod})$). For future reference, let us also point out :

Lemma 4.5.22. *Let R be a simplicial A-algebra, X and Y two R-modules, $n, m \in \mathbb{N}$ two integers, and suppose that $H_i X = 0 = H_j Y$ for every $i < n$ and $j < m$. Then*

$$H_i(X \overset{\ell}{\otimes}_R Y) = 0 \qquad \textit{for every } i < n + m.$$

Proof. Set $\sigma^0 := \mathbf{1}_{R\text{-}\mathbf{Mod}}$, and define inductively $\sigma^k : R\text{-}\mathbf{Mod} \to R\text{-}\mathbf{Mod}$ by the rule : $\sigma^k := \sigma \circ \sigma^{k-1}$ for every $k > 0$. Define likewise ω^k, and notice that σ^k is left adjoint to ω^k, for every $k \in \mathbb{N}$ (remark 4.5.21(iii)). Let $f : \sigma^n \circ \omega^n X \to X$ denote the counit of adjunction, and notice that $\mathrm{Cone}\, f = 0$ in $\mathsf{D}(R\text{-}\mathbf{Mod})$. In view of remark 4.5.21(iv,v), we deduce that the natural morphisms

$$\sigma^n(\omega^n X \overset{\ell}{\otimes}_R Y) \to (\sigma^n \circ \omega^n X) \overset{\ell}{\otimes}_R Y \to X \overset{\ell}{\otimes}_R Y$$

are isomorphisms in $\mathsf{D}(R\text{-}\mathbf{Mod})$. Likewise, the natural morphism

$$\sigma^m(\omega^n X \overset{\ell}{\otimes}_R \omega^m Y) \to \omega^n X \overset{\ell}{\otimes}_R Y$$

is an isomorphism in $\mathsf{D}(R\text{-}\mathbf{Mod})$, so finally the same holds for the morphism

$$\sigma^{n+m}(\omega^n X \overset{\ell}{\otimes}_R \omega^m Y) \to X \overset{\ell}{\otimes}_R Y$$

whence the claim. □

4.5.23. In view of proposition 4.5.18, it is clear that, if $\mathscr{C}$ is an abelian category, LT_R yields a well defined functor on derived categories

$$LT_R : \mathsf{D}(R\text{-}\mathbf{Mod}) \to \mathsf{D}(s.\mathscr{C}_{/R}) \qquad M \mapsto LT_R(M)$$

(where $\mathsf{D}(s.\mathscr{C}_{/R})$ is the localization of $s.\mathscr{C}_{/R}$ with respect to the class of quasi-isomorphisms, and likewise for $\mathsf{D}(R\text{-}\mathbf{Mod})$: see [36, Def.8.1.3]). More generally, suppose that $\mathscr{C}$ is endowed with a functor

$$\Phi : \mathscr{C} \to \mathscr{A}$$

to an abelian category $\mathscr{A}$ (usually, this will be a forgetful functor of some sort), and denote by $\mathsf{D}_\Phi(s.\mathscr{C}_{/R})$ the localization of $s.\mathscr{C}_{/R}$ with respect to the system of its morphisms whose image under $s.\Phi$ are quasi-isomorphisms in $s.\mathscr{A}$. Then, since clearly

$$s.\Phi \circ LT_R = L(s.\Phi \circ T_R)$$

we see that LT_R descends to a well defined functor

$$LT_R : \mathsf{D}(R\text{-}\mathbf{Mod}) \to \mathsf{D}_\Phi(s.\mathscr{C}_{/R}).$$

We will need also the following slight refinement :

Corollary 4.5.24. *In the situation of proposition* 4.5.18*, the following holds :*

(i) *Let $\varphi : M \to N$ be a morphism of R-modules, and $n \in \mathbb{N}$ an integer such that*

$$H_i\varphi : H_iM \to H_iN$$

is an isomorphism, for every $i < n$. Then the same holds for the induced map

$$H_i(LT_R\varphi) : H_i(LT_RM) \to H_i(LT_RN).$$

(ii) *Suppose that $T(B, 0) = 0_{gB}$ (the initial and final object of $f^{-1}(gB)$) for every A-algebra B. Let $n \in \mathbb{N}$ be an integer, and M an R-module such that $H_iM = 0$ for every $i < n$. Then $H_i(LT_RM) = 0_{gR_i}$ for every $i < n$.*

Proof. (i): Denote by

$$\varphi_X : X \to \mathrm{cosk}_n X \qquad \text{for every } X \in \mathrm{Ob}(R\text{-}\mathbf{Mod})$$

the unit of adjunction (see (4.2.18)). Taking into account corollary 4.2.65(i), we have the following properties :

- $X[i] = \mathrm{cosk}_n X[i]$ and $\varphi[i]$ is the identity map of $X[i]$, for every $i \leq n$.
- $H_i(\mathrm{cosk}_n X) = 0$ for every $i \geq n$.

In view of proposition 4.5.18, we are then easily reduced to checking the assertion for the special where $N := \mathrm{cosk}_n M$, and $\varphi := \varphi_M$. However, since in this case $N[i] = M[i]$ for every $i \leq n$, it is clear that $\perp_\bullet^{R[i]} \varphi[i]$ is the identity automorphism of $\perp_\bullet^{R[i]} M[i]$, for every $i \leq n$. After applying the functor T_R, and extracting the diagonal, we see that the induced morphism $LT_RM \to LT_RN$ in $s.\mathscr{C}$ is given by the identity automorphism of $(LT_RM)[i]$, in every degree $i \leq n$, whence the contention.

(ii): In light of (i), we may assume that $M = s.0$ (the trivial R-module). In this case, the assertion follows easily from remark 4.5.7. □

4.5.25. In the situation of (4.5.16), take $\mathscr{C} := V\text{-}\mathbf{Alg}.\mathbf{Mod}$, $f := F$, and g the identity automorphism of $A\text{-}\mathbf{Alg}$. If $B \to B'$ is a map of A-algebras, and $(B,M) \in \mathrm{Ob}(\mathscr{C})$, we define $B' \otimes_B (B,M) := (B', B' \otimes_B M)$.

Corollary 4.5.26. *In the situation of* (4.5.25)*, suppose moreover that, for every flat morphism $B \to B'$ of A-algebras, the natural map*

$$B' \otimes_B T(B,M) \to T(B' \otimes_B (B,M))$$

is an isomorphism, for every $(B,M) \in \mathrm{Ob}(\mathscr{C})$. Then, for every flat morphism $\varphi : R \to S$ of simplicial V-algebras, the natural morphism

$$S \otimes_R LT_R M \to LT_S(S \otimes_R M) \tag{4.5.27}$$

is an isomorphism in $\mathsf{D}(S\text{-}\mathbf{Mod})$*, for every R-module M.*

Proof. (Recall that φ is flat if and only if $S[n]$ is a flat $R[n]$-algebra, for every $n \in \mathbb{N}$. The map of the proposition is deduced from the natural morphism $\perp^R_\bullet M \to \perp^S_\bullet (S \otimes_R M)$, given by functoriality of the standard free resolution.)

Let $\pi : \perp^R_\bullet M \to M$ be the standard free resolution of M; by the flatness assumption on S, the morphism $S \otimes_R \pi : S \otimes_R \perp^R_\bullet M \to S \otimes_R M$ is a free resolution of the S-module $S \otimes_R M$, whence a natural isomorphism

$$LT_S(S \otimes_R M) \xrightarrow{\sim} T_S((S \otimes_R \perp^R_\bullet M)^\Delta) \qquad \text{in } \mathsf{D}(S\text{-}\mathbf{Mod})$$

by proposition 4.5.18(i) and Eilenberg-Zilber's theorem 4.2.48. On the other hand, the assumptions on T yield a natural isomorphism

$$S \otimes_R LT_R(M) = S \otimes_R T_R((\perp^R_\bullet M)^\Delta) \xrightarrow{\sim} T_S((S \otimes_R \perp^R_\bullet M)^\Delta) \qquad \text{in } \mathsf{D}(S\text{-}\mathbf{Mod}).$$

By combining these isomorphisms, we get an isomorphism $S \otimes_R LT_R M \to LT_S(S \otimes_R M)$, and a simple inspection shows that the latter is realized by the natural map (4.5.27). $\square$

4.5.28. Let now R be any simplicial A-algebra, and $I \subset R$ any ideal, and set $R_0 := R/I$. The *Rees algebra of I* is the graded R-algebra

$$\mathsf{R}(R,I)^\bullet := \bigoplus_{p \in \mathbb{N}} I^p.$$

Notice the natural isomorphism of graded R_0-algebras

$$\mathsf{R}(R,I) \otimes_R R_0 \xrightarrow{\sim} \mathrm{gr}^\bullet_I R := \bigoplus_{p \in \mathbb{N}} I^p/I^{p+1}$$

as well as the exact sequence of $\mathsf{R}(R,I)$-modules

$$0 \to \mathrm{gr}^{\bullet+1}_I R \to \mathsf{R}(R,I) \otimes_R R/I^2 \to \mathrm{gr}^\bullet_I R \to 0$$

deduced from the natural projection $R/I^2 \to R_0$. Next, pick any quasi-isomorphism of R-algebras $P \to R_0$, with P a flat R-algebra; there follows an exact sequence of $P \otimes_R \mathsf{R}(R,I)$-modules

$$0 \to P \otimes_R \mathrm{gr}^{\bullet+1}_I R \to P \otimes_R \mathsf{R}(R,I) \otimes_R R/I^2 \xrightarrow{\beta} P \otimes_R \mathrm{gr}^\bullet_I R \to 0$$

and notice that β is actually a morphism of $P \otimes_R \mathsf{R}(R,I)$-algebras. According to remark 4.4.53(i), after forming the associated cochain complexes, we obtain an epimorphism $\beta^\bullet$ of differential graded $(P \otimes_R \mathsf{R}(R,I))^\bullet$-algebras, whose kernel $(P \otimes_R \mathrm{gr}^{\bullet+1}_I R)^\bullet$ is a two-sided ideal of $(P \otimes_R \mathsf{R}(R,I) \otimes_R R/I^2)^\bullet$. In this situation, we deduce a map of $H_\bullet(R_0 \overset{\ell}{\otimes}_R \mathsf{R}(P,I))$-modules

$$\delta : H_\bullet(R_0 \overset{\ell}{\otimes}_R \mathrm{gr}^\bullet_I R) \to H_\bullet(R_0 \overset{\ell}{\otimes}_R \mathrm{gr}^{\bullet+1}_I R)[1] \tag{4.5.29}$$

which is a graded derivation, according to lemma 4.4.45 (recall that here the shift $[1]$ refers to the homological grading; the notation $\mathrm{gr}^{\bullet+1}$ denotes also a shift in degrees, which however *does not* alter the signs of the scalar multiplication in the way prescribed by remark 4.4.43(v)).

4.5.30. Keep the situation of (4.5.28), and suppose furthermore that R is a constant A-algebra and I a constant ideal, *i.e.* $R = s.B$, and $I = s.J$ for some A-algebra B and some ideal $J \subset B$. We set $B_0 := B/J$ and

$$G^\bullet := R_0 \overset{\ell}{\otimes}_R \mathrm{gr}_I R^\bullet \qquad \Lambda_\bullet := \Lambda^\bullet_B(J/J^2) \qquad S^\bullet := \mathrm{Sym}^\bullet_B(J/J^2)$$

where the derived tensor product is formed in $\mathsf{D}(R\text{-}\mathbf{Alg})$, as in example 4.5.15, so $G^\bullet$ is represented by $P \otimes_R \mathrm{gr}^\bullet_I R$, and $H_\bullet G^\bullet$ is a bigraded A-algebra, strictly anti-commutative for the (homological) grading $H_\bullet$, and commutative for the (cohomological) grading deduced from the grading of $G^\bullet$ (see remark 4.4.53(i)). Also, $\Lambda_\bullet$ (resp. $S^\bullet$) is a strictly anti-commutative (resp. a commutative) graded B_0-algebra, and we use the homological (resp. cohomological) conventions for grading, so Λ_p (resp. S^p) is placed in degree $-p$ (resp. p). Moreover, a standard calculation yields a natural isomorphism of B_0-modules

$$H_0 G^0 \xrightarrow{\sim} B_0 \qquad H_1 G^0 \xrightarrow{\sim} \mathrm{Tor}^B_1(B_0, B_0) \xrightarrow{\sim} J/J^2$$

whence a natural map $\Lambda_\bullet \to H_\bullet G^0$ of strictly anti-commutative graded B_0-algebras, restricting to an isomorphism in degrees ≤ 1. On the other hand, we have a surjective map of commutative graded B_0-algebras

$$S^\bullet \to \mathrm{gr}^\bullet_J B := \bigoplus_{p \in \mathbb{N}} J^p/J^{p+1}$$

restricting as well to an isomorphism in degrees ≤ 1; since $P \otimes_R \mathrm{gr}^\bullet_I R$ is naturally a simplicial $\mathrm{gr}^\bullet_J B$-algebra, we obtain a natural map of bigraded $S^\bullet$-algebras

$$\omega^\bullet_\bullet : \Lambda_\bullet \otimes_B S^\bullet \to H_\bullet G^\bullet.$$

Now, $H_\bullet G^\bullet$ has been endowed with a derivation $\delta : H_\bullet G^\bullet \to H_\bullet G^{\bullet+1}[1]$ in (4.5.29); on the other hand, there exists on $\Lambda_\bullet \otimes_B S^\bullet$ a natural $S^\bullet$-linear graded derivation

$$\partial : \Lambda_\bullet \otimes_B S^\bullet \to \Lambda_\bullet \otimes_B S^{\bullet+1}[1]$$

as explained in [56, I.4.3.1.2]. We may then consider the diagram of $S^\bullet$-linear maps

$$\begin{tikzcd} \Lambda_\bullet \otimes_B S^\bullet \arrow[r, "\omega^\bullet_\bullet"] \arrow[d, "\partial"'] & H_\bullet G^\bullet \arrow[d, "\delta"] \\ \Lambda_\bullet \otimes_B S^{\bullet+1}[1] \arrow[r, "\omega^{\bullet+1}_\bullet[1]"] & H_\bullet G^{\bullet+1}[1]. \end{tikzcd} \tag{4.5.31}$$

Definition 4.5.32. (i) In the situation of (4.5.30), we say that the ideal J is *quasi-regular*, if the following conditions hold :

- The map $\omega^\bullet_\bullet$ restricts to an isomorphism $\omega^0_\bullet \xrightarrow{\sim} H_\bullet G^0$.
- The B_0-module J/J^2 is flat.

(ii) Let R be any simplicial A-algebra, and $I \subset R$ any ideal. We say that I is *quasi-regular*, if $I[n]$ is a quasi-regular ideal of $R[n]$, for every $n \in \mathbb{N}$.

Remark 4.5.33. (i) Suppose that $\underline{B} := (B_i \mid i \in I)$ is any filtered system of rings, and $\underline{J} := (J_i \subset B_i \mid i \in I)$ a filtered system of ideals, such that J_i is quasi-regular in B_i, for every $i \in I$. Denote by B (resp. J) the colimit of $\underline{B}$ (resp. of $\underline{J}$); then it is easily seen that J is quasi-regular in B.

(ii) Let B be any ring, and $C := B[X_i \mid i \in I]$ a free B-algebra (for any set I). Then the ideal $J := (X_i \mid i \in I) \subset C$ is quasi-regular. Indeed, (i) reduces to the case where I is a finite set, for which the assertion is a special case of the following :

Lemma 4.5.34. *In the situation of* (4.5.30)*, suppose that* J *is generated by a (finite) regular sequence of elements of* B*. Then* J *is quasi-regular.*

Proof. The assertion becomes a paraphrase of proposition 4.4.50, once we have identified the graded B-algebra $H_\bullet G^\bullet$ with the graded B-algebra $H_\bullet(B_0[0] \overset{\mathbf{L}}{\otimes}_B B_0[0])$. However, denote by $P^\bullet$ the cochain complex associated to the flat resolution $P \to s.B_0$; according to remark 4.4.53(i), $P^\bullet$ is naturally a differential graded B-algebra, and the induced map $\varepsilon^\bullet : P^\bullet \to B_0[0]$ is a morphism of differential graded B-algebras (for the multiplication law $\overline{\mu}^\bullet$ on $B_0[0]$ inherited from that of B_0). Then there exists in $\mathsf{D}(B\text{-}\mathbf{Mod})$ a unique isomorphism $\omega^\bullet : P^\bullet_{B_0} \to P^\bullet$ whose composition with $\varepsilon^\bullet$ agrees with $\rho^\bullet_{B_0}$ (notation of (4.1.12)), and moreover we get a commutative diagram in $\mathsf{D}(B\text{-}\mathbf{Mod})$:

$$\begin{tikzcd} P^\bullet_{B_0} \otimes_B P^\bullet_{B_0} \arrow[r, "P^\bullet_{\overline{\mu}}"] \arrow[d, "\omega^\bullet \otimes_B \omega^\bullet"'] & P^\bullet_{B_0} \arrow[d, "\omega^\bullet"] \\ P^\bullet \otimes_B P^\bullet \arrow[r] & P^\bullet \end{tikzcd}$$

whose bottom horizontal arrow is the multiplication map of $P^\bullet$, and where $P^\bullet_{\overline{\mu}}$ is defined as in (4.1.12). The assertion follows immediately. $\square$

Proposition 4.5.35. *In the situation of* (4.5.30)*, we have :*

(i) (4.5.31) *is a commutative diagram.*

(ii) *Suppose that* J *is a quasi-regular ideal. Then*

(a) $\omega^\bullet_\bullet$ *is an isomorphism.*

(b) *There exists a natural isomorphism of* B_0*-modules :*

$$\mathrm{Tor}_i^B(B_0, B/J^n) \overset{\sim}{\to} \mathrm{Coker}(\partial^{n-2}_{i+1} : \Lambda_{i+1} \otimes_B S^{n-2} \to \Lambda_i \otimes_B S^{n-1}) \qquad \textit{for every } n, i > 0.$$

Proof. (i): Since both derivations are $S^\bullet$-linear, it suffices to check that (4.5.31) commutes in (upper) degree 0. Morever, since the B-algebra $\Lambda_\bullet$ is generated by Λ_1, it suffices to check the commutativity of the diagram

(4.5.36)
$$\begin{tikzcd} \Lambda_1 \arrow[r, "\omega^0_1"] \arrow[d, "\partial^0_1"'] & H_1 G^0 \arrow[d, "\delta^0_1"] \\ S^1 \arrow[r, "\omega^1_0"] & H_0 G^1. \end{tikzcd}$$

However, according to [56, I.4.3.1.2], the map ∂^0_1 is just the identity of J/J^2. The map δ^0_1 is the boundary map arising – by the snake lemma – from the exact sequence of complexes

$$0 \to JP^\bullet/J^2P^\bullet \to P^\bullet/J^2P^\bullet \to P^\bullet/JP^\bullet \to 0$$

(where $P^\bullet$ denotes the cochain complex associated to the flat resolution $P \to s.B_0$). The same boundary map is used to define the isomorphism ω^0_1, and ω^1_0 is the natural isomorphism $J/J^2 \to H_0(P^\bullet/JP^\bullet)$. Summing up, the commutativity of (4.5.36) follows by simple inspection.

(ii.a): We prove, by induction on $n \in \mathbb{N}$, that every $\omega^n_\bullet$ is an isomorphism. For $n = 0, 1$ there is nothing to show, so suppose that $n > 1$, and the assertion is already known for every degree $< n$. In order to prove the assertion in degree n, it suffices to check that $\omega^n_0 : S^n \to \mathrm{gr}^n_J B$ is an isomorphism; indeed, in this case $\mathrm{gr}^n_J B$ is a flat B_0-module, and then a standard spectral sequence argument allows to conclude that ω^n_j is also an isomorphism, for every $j \in \mathbb{N}$.

Let $P \to R_0$ be a resolution as in (4.5.28). The I-adic filtration has finite length on P/I^kP, for every $k \in \mathbb{N}$, and therefore yields a convergent spectral sequence

$$E^1_{pq} \Rightarrow \mathrm{Tor}^B_{p+q}(B_0, B/J^k)$$

with

$$E^1_{pq} := \begin{cases} \mathrm{Tor}^B_{p+q}(B_0, \mathrm{gr}_J B^{-p}) & \text{for } p > -k \\ 0 & \text{otherwise} \end{cases}$$

whose differentials $E^1_{pq} \to E^1_{p-1,q}$ agree – by direct inspection – with the derivation δ^{-p}_{p+q}, whenever $p > 1 - k$. Especially, for $k = n + 1$, we get a complex

$$\Sigma \quad : \quad H_2 G^{n-2} \xrightarrow{\delta^{n-2}_2} H_1 G^{n-1} \xrightarrow{\delta^{n-1}_1} \mathrm{gr}^n_J B \to 0.$$

On the other hand, since J/J^2 is a flat B_0-module, the corresponding sequence

$$\Sigma' \quad : \quad \Lambda_2 \otimes_B S^{n-2} \xrightarrow{\partial} \Lambda_1 \otimes_B S^{n-1} \xrightarrow{\partial} S^n \to 0$$

is an exact complex (this is a segment of the Koszul complex of [56, I.4.3.1.7]). In light of (i), the maps ω^{n-2}_2 and ω^{n-1}_1 yield a morphism of complexes $\Sigma' \to \Sigma$, so we are reduced to checking that Σ is exact, and the inductive assumption already says that Σ is exact at the middle term $H_1 G^{n-1}$. Now notice that, since the Koszul complex is exact, the inductive assumption implies that $E^2_{pq} = 0$ for all (p, q) such that $p > 1 - n$ and $q > 0$; moreover, obviously we have $E^1_{pq} = 0$ for $p + q < 0$ or $p > 0$, therefore

$$E^\infty_{00} = E^1_{00} \qquad \text{and} \qquad E^\infty_{-n,n} = E^2_{-n,n} = \mathrm{Coker}\, \delta^{n-1}_1.$$

But a simple inspection shows that the natural map $H_0 P \to E^\infty_{00}$ is an isomorphism, so $E^\infty_{-n,n} = 0$, therefore δ^{n-1}_1 is surjective, and Σ is exact, as required.

(ii.b): We use the previous spectral sequence, for $k = n$. Taking into account (ii.a), and the exactness of the Koszul complex ([56, I.4.3.1.7]) we get

$$\begin{array}{ll} E^1_{pq} = 0 & \text{for } p + q < 0 \text{ or } p > 0 \text{ or } p \leq -n \\ E^2_{p,q} = 0 & \text{for every } (p, q) \text{ with } p > 1 - n \text{ and } q > 0 \\ E^2_{1-n,i+n-1} \xrightarrow{\sim} \mathrm{Coker}\, \partial^{n-2}_{i+1} & \text{for every } i \in \mathbb{N}. \end{array}$$

It follows that $E^r_{1-n+r,i+n-r} = 0$ for every $r \geq 2$ and every $i \in \mathbb{N}$; indeed, this is clear if $1 - n + r > 0$, and if the latter fails, we get $1 - n + r > 1 - n$ and $i + n - r > 0$, so the stated vanishing holds nevertheless. We deduce that $E^\infty_{1-n,i+n-1} = E^2_{1-n,i+n-1}$ for every $i \in \mathbb{N}$. Next, suppose that $i > 0$. Clearly we have $E^\infty_{p,i-p} = 0$ both for $p > 0$ and $p \leq -n$; and if $0 \geq p > 1 - n$, we get $i - p \geq i > 0$, so again $E^\infty_{p,i-p} = 0$. In conclusion, $E^\infty_{p,i-p} = 0$ except possibly for $p = 1 - n$, and the contention follows. □

The following is a well known theorem of Quillen, which is found in his typewritten notes “On the homology of commutative rings” that have been circulated since the late 60s, but have never been published.

Theorem 4.5.37. *Let R be any simplicial A-algebra, and $I \subset R$ a quasi-regular ideal such that $H_0 I = 0$. Then we have :*

$$H_i I^n = 0 \qquad \textit{for every } n \in \mathbb{N} \textit{ and every } i < n.$$

Proof. We argue by induction on $n \in \mathbb{N}$. For $n \leq 1$ there is nothing to prove, so suppose that $n > 1$, and that the assertion is already known for I^{n-1}. Set $R_k := R/I^{k+1}$ for every $k \in \mathbb{N}$, and let

$$\Lambda_\bullet \otimes_R S^\bullet := \Lambda^\bullet_R(I/I^2) \otimes_R \mathrm{Sym}^\bullet_R(I/I^2)$$

which is a bigraded R_0-module, endowed with the bigraded derivation $\partial^\bullet_\bullet$ such that $\partial^\bullet_\bullet[i]$ is the Koszul derivation of $\Lambda_\bullet \otimes_R S^\bullet[i]$, defined as in [56, I.4.3.1.2]. Set as well

$$C_i := \mathrm{Coker}\,(\partial^{n-3}_i : \Lambda_i \otimes_R S^{n-3} \to \Lambda_{i-1} \otimes_R S^{n-2}) \qquad \text{for every } i \in \mathbb{N}.$$

From the inductive assumption and lemma 4.5.22, we already know that

$$H_i(I \overset{\ell}{\otimes}_R I^{n-1}) = 0 \qquad \text{for every } i < n. \tag{4.5.38}$$

However, according to remark 4.5.14, there is a spectral sequence

$$E^1_{pq} := \mathrm{Tor}_q^{R[p]}(I[p], I[p]^{n-1}) \Rightarrow H_{p+q}(I \overset{\ell}{\otimes}_R I^{n-1})$$

and proposition 4.5.35(ii.b) yields natural isomorphisms

$$E^1_{pq} \xrightarrow{\sim} \mathrm{Tor}_{q+1}^{R[p]}(R_0[p], I[p]^{n-1}) \xrightarrow{\sim} \mathrm{Tor}_{q+2}^{R[p]}(R_0[p], R_{n-2}[p]) \xrightarrow{\sim} C_{q+3}[p] \qquad \text{for every } q > 0$$

under which, the differentials $E^1_{pq} \to E^1_{p-1,q}$ are identified with those of the chain complex associated to the R_0-module C_{q+3}. Now, since I/I^2 is a flat R_0-module, [56, I.4.3.1.7] says that the sequence of R_0-modules

$$\Sigma^i \quad : \quad 0 \to \Lambda_{n+i-3} \xrightarrow{\partial} \Lambda_{n+i-4} \otimes_R S^1 \to \cdots \to \Lambda_i \otimes_R S^{n-3} \xrightarrow{\partial} \Lambda_{i-1} \otimes_R S^{n-2} \to C_i \to 0$$

is an exact complex, for every $i > 0$. On the other hand, since $H_0 I/I^2 = 0$, (and again, since I/I^2 is a flat R_0-module), combining [56, I.4.3.2.1] and lemma 4.5.22, we see that

$$H_i(\Lambda_j \otimes_R S^{q-j}) = 0 \qquad \text{for every } q \in \mathbb{N}, \text{ every } j \leq q, \text{ and every } i < q.$$

We recall the following general

Claim 4.5.39. Let $\mathscr{A}$ be any abelian category, $d \in \mathbb{N}$ any integer, and consider an exact sequence

$$0 \to K_d \xrightarrow{f_d} K_{d-1} \to \cdots \to K_1 \xrightarrow{f_1} K_0 \to 0$$

of objects of $\mathsf{C}(\mathscr{A})$, such that $K_i = 0$ in $\mathsf{D}(\mathscr{A})$, for every $i = 1, \dots, d-1$. Then there is a natural isomorphism

$$K_0 \xrightarrow{\sim} K_d[d-1] \qquad \text{in } \mathsf{D}(\mathscr{A}).$$

Proof of the claim. For every $i = 0, \dots, d$, set $Z_i := \mathrm{Ker}\, f_i$. We have short exact sequences

$$0 \to Z_i \to K_i \to Z_{i-1} \to 0 \qquad \text{for } i = 1, \dots, d.$$

Since $K_i = 0$ in $\mathsf{D}(\mathscr{A})$, there follows natural isomorphisms $Z_{i-1} \xrightarrow{\sim} Z_i[1]$ for every $i = 1, \dots, d-1$. However, $Z_0 = K_0$ and $Z_{d-1} = K_d$, whence the claim. ◊

By applying claim 4.5.39 to the exact sequence $\mathrm{cosk}_{n+i-3}\Sigma^i$, we easily deduce that

$$H_j C_i = 0 \qquad \text{for every } i > 0 \text{ and every } j < n + i - 3$$

so finally $E^2_{pq} = 0$ for every $q > 0$ and every $p < n + q$, and clearly $E^1_{pq} = 0$ for every $q < 0$. Consequently, for every $i < n$ the term $E^\infty_{i-q,q} = 0$ vanishes for $q > 0$, and equals $E^2_{i,0}$ for $q = 0$. Lastly, a simple calculation shows that

$$E^2_{i,0} = H_i(I \otimes_R I^{n-1}) \qquad \text{for every } i \in \mathbb{N}.$$

Summing up, and taking into account 4.5.38, we conclude that $H_i(I \otimes_R I^{n-1}) = 0$ for every $i < n$, so we are reduced to checking :

Claim 4.5.40. The natural map $H_i(I \otimes_R I^{n-1}) \to H_i(I^n)$ is surjective, for every $i \leq n$.

Proof of the claim. Indeed, denote by K the kernel of the epimorphism $I \otimes_R I^{n-1} \to I^n$; as in the foregoing, we get natural isomorphisms

$$K[i] \xrightarrow{\sim} \mathrm{Tor}_1^{R[i]}(R_0[i], I[i]^{n-1}) \xrightarrow{\sim} \mathrm{Tor}_2^{R[i]}(R_0[i], R[i]/I[i]^{n-1}) \xrightarrow{\sim} C_3[i] \qquad \text{for every } i \in \mathbb{N}$$

that amount to an isomorphism $K \xrightarrow{\sim} C_3$ of R_0-modules. We have already seen that $H_i C_3 = 0$ for every $i < n$, whence the claim. □

4.6. **Witt vectors, Fontaine rings and divided power algebras.** We begin with a quick review of the ring of Witt vectors associated to an arbitrary ring; for the complete details we refer to [14, Ch.IX, §1]. Henceforth we let p be a fixed prime number. To start out, following [14, Ch.IX, §1, n.1] one defines, for every $n \in \mathbb{N}$, the *n-th Witt polynomial*

$$\boldsymbol{\omega}_n(X_0, \dots, X_n) := \sum_{i=0}^{n} p^i X_i^{p^{n-i}} \in \mathbb{Z}[X_0, \dots, X_n].$$

Notice the inductive relations :

$$\begin{aligned} \boldsymbol{\omega}_{n+1}(X_0, \dots, X_{n+1}) &= \boldsymbol{\omega}_n(X_0^p, \dots, X_n^p) + p^{n+1} X_{n+1} \\ &= X_0^{p^{n+1}} + p \cdot \boldsymbol{\omega}_n(X_1, \dots, X_{n+1}). \end{aligned} \tag{4.6.1}$$

Then one shows ([14, Ch.IX, §, n.3]) that there exist polynomials

$$S_n, P_n \in \mathbb{Z}[X_0, \dots, X_n, Y_0, \dots, Y_n] \quad I_n \in \mathbb{Z}[X_0, \dots, X_n] \quad F_n \in \mathbb{Z}[X_0, \dots, X_{n+1}]$$

characterized by the identities :

$$\begin{aligned} \boldsymbol{\omega}_n(S_0, \dots, S_n) &= \boldsymbol{\omega}_n(X_0, \dots, X_n) + \boldsymbol{\omega}_n(Y_0, \dots, Y_n) \\ \boldsymbol{\omega}_n(P_0, \dots, P_n) &= \boldsymbol{\omega}_n(X_0, \dots, X_n) \cdot \boldsymbol{\omega}_n(Y_0, \dots, Y_n) \\ \boldsymbol{\omega}_n(I_0, \dots, I_n) &= -\boldsymbol{\omega}_n(X_0, \dots, X_n) \\ \boldsymbol{\omega}_n(F_0, \dots, F_n) &= \boldsymbol{\omega}_{n+1}(X_0, \dots, X_{n+1}) \end{aligned}$$

for every $n \in \mathbb{N}$. A simple induction shows that :

$$F_n \equiv X_n^p \pmod{p\mathbb{Z}[X_0, \dots, X_{n+1}]} \qquad \text{for every } n \in \mathbb{N}. \tag{4.6.2}$$

(See [14, Ch.IX, §1, n.3, Exemple 4].)

4.6.3. Let now A be an arbitrary (commutative, unitary) ring. For every $\underline{a} := (a_n \mid n \in \mathbb{N})$, $\underline{b} := (b_n \mid n \in \mathbb{N})$ in $A^{\mathbb{N}}$, one sets :

$$\begin{aligned} S_A(\underline{a}, \underline{b}) &:= (S_n(a_0, \dots, a_n, b_0, \dots, b_n) \mid n \in \mathbb{N}) \\ P_A(\underline{a}, \underline{b}) &:= (P_n(a_0, \dots, a_n, b_0, \dots, b_n) \mid n \in \mathbb{N}) \\ I_A(\underline{a}) &:= (I_n(a_0, \dots, a_n) \mid n \in \mathbb{N}) \\ \boldsymbol{\omega}_A(\underline{a}) &:= (\boldsymbol{\omega}_n(a_0, \dots, a_n) \mid n \in \mathbb{N}). \end{aligned}$$

There follow identities :

$$\begin{aligned} \boldsymbol{\omega}_A(S_A(\underline{a}, \underline{b})) &= \boldsymbol{\omega}_A(\underline{a}) + \boldsymbol{\omega}_A(\underline{b}) \\ \boldsymbol{\omega}_A(P_A(\underline{a}, \underline{b})) &= \boldsymbol{\omega}_A(\underline{a}) \cdot \boldsymbol{\omega}_A(\underline{b}) \\ \boldsymbol{\omega}_A(I_A(\underline{a})) &= -\boldsymbol{\omega}_A(\underline{a}) \end{aligned}$$

for every $\underline{a}, \underline{b} \in A^{\mathbb{N}}$. With this notation one shows that the set $A^{\mathbb{N}}$, endowed with the addition law $S_A : A^{\mathbb{N}} \times A^{\mathbb{N}} \to A^{\mathbb{N}}$ and product law $P_A : A^{\mathbb{N}} \times A^{\mathbb{N}} \to A^{\mathbb{N}}$ is a commutative ring, whose zero element is the identically zero sequence $\underline{0}_A := (0, 0, \dots)$, and whose unit is the sequence $\underline{1}_A := (1, 0, 0, \dots)$ ([14, Ch.IX, §1, n.4, Th.1]). The resulting ring $(A^{\mathbb{N}}, S_A, P_A, \underline{0}_A, \underline{1}_A)$ is denoted $W(A)$, and called *the ring of Witt vectors associated to* A. Furthermore, let $\underline{a} \in W(A)$ be any element; then :

- The opposite $-\underline{a}$ (that is, with respect to the addition law S_A) is the element $I_A(\underline{a})$.
- For every $n \in \mathbb{N}$, the element $\boldsymbol{\omega}_n(\underline{a}) := \boldsymbol{\omega}_n(a_0, \dots, a_n) \in A$ is called the *n-th ghost component* of $\underline{a}$, and the map $\boldsymbol{\omega}_n : W(A) \to A : \underline{b} \mapsto \boldsymbol{\omega}_n(\underline{b})$ is a ring homomorphism.

Hence, the map $W(A) \to A^{\mathbb{N}} : \underline{a} \mapsto \boldsymbol{\omega}_A(\underline{a})$ is also a ring homomorphism (where $A^{\mathbb{N}}$ is endowed with termwise addition and multiplication). Henceforth, the addition and multiplication of elements $\underline{a}, \underline{b} \in W(A)$ shall be denoted simply in the usual way : $\underline{a} + \underline{b}$ and $\underline{a} \cdot \underline{b}$, and the neutral element of addition and multiplication shall be denoted respectively 0 and 1. The rule $A \mapsto W(A)$ is a functor from the category of rings to itself; indeed, if $\varphi : A \to B$ is any ring homomorphism, one obtains a ring homomorphism $W(\varphi) : W(A) \to W(B)$ by setting $\underline{a} \mapsto (\varphi(a_n) \mid n \in \mathbb{N})$. In other words, $W(\varphi)$ is induced by the map of sets $\varphi^{\mathbb{N}} : A^{\mathbb{N}} \to B^{\mathbb{N}}$, and moreover one has the identity :

$$\varphi^{\mathbb{N}} \circ \boldsymbol{\omega}_A = \boldsymbol{\omega}_B \circ W(\varphi).$$

4.6.4. Next, one defines two mappings on elements of $W(A)$, by the rule :

$$F_A(\underline{a}) := (F_n(a_0, \dots, a_{n+1}) \mid n \in \mathbb{N})$$
$$V_A(\underline{a}) := (0, a_0, a_1, \dots).$$

F_A and V_A are often called, respectively, the *Frobenius* and *Verschiebung* maps ([14, Ch.IX, §1, n.5]). Let also $f_A : A^{\mathbb{N}} \to A^{\mathbb{N}}$ be the ring endomorphism given by the rule $\underline{a} \mapsto (a_1, a_2, \dots)$, and $v_A : A^{\mathbb{N}} \to A^{\mathbb{N}}$ the endomorphism of the additive group of $A^{\mathbb{N}}$ defined by $\underline{a} \mapsto (0, pa_0, pa_1, \dots)$. Then the basic identity characterizing $(F_n \mid n \in \mathbb{N})$ can be written in the form :

$$\boldsymbol{\omega}_A \circ F_A = f_A \circ \boldsymbol{\omega}_A. \tag{4.6.5}$$

On the other hand, (4.6.1) yields the identity :

$$\boldsymbol{\omega}_A \circ V_A = v_A \circ \boldsymbol{\omega}_A.$$

Let $\varphi : A \to B$ be a ring homomorphism; directly from the definitions we obtain :

$$W(\varphi) \circ F_A = F_B \circ W(\varphi) \qquad W(\varphi) \circ V_A = V_B \circ W(\varphi). \tag{4.6.6}$$

The map F_A is an endomorphism of the ring $W(A)$, and V_A is an endomorphism of the additive group underlying $W(A)$ ([14, Ch.IX, §1, n.5, Prop.3]). Moreover one has the identities :

$$\begin{aligned} F_A \circ V_A &= p \cdot \mathbf{1}_{W(A)} \\ V_A(\underline{a} \cdot F_A(\underline{b})) &= V_A(\underline{a}) \cdot \underline{b} \quad \text{for every } \underline{a}, \underline{b} \in W(A) \end{aligned} \tag{4.6.7}$$

and F_A is a lifting of the Frobenius endomorphism of $W(A)/pW(A)$, *i.e.* :

$$F_A(\underline{a}) \equiv \underline{a}^p \pmod{pW(A)} \qquad \text{for every } \underline{a} \in W(A).$$

It follows easily from (4.6.7) that $V_m(A) := \operatorname{Im} V_A^m$ is an ideal of $W(A)$ for every $m \in \mathbb{N}$. The filtration $(V_m(A) \mid m \in \mathbb{N})$ defines a linear topology $\mathscr{T}$ on $W(A)$. This topology can be described explicitly, due to the identity :

$$\underline{a} = (a_0, \dots, a_{m-1}, 0, \dots) + V_A^m \circ f_A^m(\underline{a}) \quad \text{for every } \underline{a} \in W(A). \tag{4.6.8}$$

(See [14, Ch.IX, §1, n.6, Lemme 4] : here the addition is taken in the additive group underlying $W(A)$.) Indeed, (4.6.8) implies that $\mathscr{T}$ agrees with the product topology on $A^{\mathbb{N}}$, regarded as the infinite countable power of the discrete space A. Hence $(W(A), \mathscr{T})$ is a complete and separated topological ring.

4.6.9. The projection $\boldsymbol{\omega}_0$ admits a useful set-theoretic splitting, the *Teichmüller mapping* :

$$\tau_A : A \to W(A) \qquad a \mapsto (a, 0, 0, \dots).$$

For every $a \in A$, the element $\tau_A(a)$ is known as the *Teichmüller representative* of a. Notice that:

$$\boldsymbol{\omega}_n \circ \tau_A(a) = a^{p^n} \qquad \text{for every } n \in \mathbb{N}.$$

Moreover, τ_A is a multiplicative map, *i.e.* the following identity holds :

$$\tau_A(a) \cdot \tau_A(b) = \tau_A(a \cdot b) \qquad \text{for every } a, b \in A$$

(see [14, Ch.IX, §1, n.6, Prop.4(a)]). Furthermore, (4.6.8) implies the identity :

$$\underline{a} = \sum_{n=0}^{\infty} V_A^n(\tau_A(a_n)) \qquad \text{for every } \underline{a} \in W(A) \tag{4.6.10}$$

(where the convergence of the series is relative to the topology $\mathscr{T}$: see [14, Ch.IX, §1, n.6, Prop.4(b)]).

4.6.11. For every $n \in \mathbb{N}$ we set :

$$W_n(A) := W(A)/V_n(A)$$

(the *n-truncated Witt vectors* of A). This defines an inverse system of rings $(W_n(A) \mid n \in \mathbb{N})$, whose limit is $W(A)$. Clearly the ghost components descend to well-defined maps :

$$\overline{\boldsymbol{\omega}}_m : W_n(A) \to A \qquad \text{for every } m < n.$$

Especially, the map $\overline{\boldsymbol{\omega}}_0 : W_1(A) \to A$ is an isomorphism.

4.6.12. Let now A be a ring of characteristic p. In this case (4.6.2) yields the identity :

$$F_A(\underline{a}) = (a_n^p \mid n \in \mathbb{N}) \qquad \text{for every } \underline{a} := (a_n \mid n \in \mathbb{N}) \in W(A). \tag{4.6.13}$$

As an immediate consequence we get :

$$p \cdot \underline{a} = V_A \circ F_A(\underline{a}) = F_A \circ V_A(\underline{a}) = (0, a_0^p, a_1^p, \dots) \qquad \text{for every } \underline{a} \in W(A). \tag{4.6.14}$$

For such a ring A, the p-adic topology on $W(A)$ agrees with the $V_1(A)$-adic topology, and both are finer than the topology $\mathscr{T}$. Moreover, $W(A)$ is complete for the p-adic topology ([14, Ch.IX, §1, n.8, Prop.6(b)]). Furthermore, let $\Phi_A : A \to A$ be the Frobenius endomorphism $a \mapsto a^p$; (4.6.13) also implies the identity :

$$F_A \circ \tau_A = \tau_A \circ \Phi_A. \tag{4.6.15}$$

4.6.16. Suppose additionally that A is *perfect*, *i.e.* that Φ_A is an automorphism on A. It follows from (4.6.13) that F_A is an automorphism on $W(A)$. Hence, in view of (4.6.7) and (4.6.14) :

$$p^n \cdot W(A) = V_n(A) \qquad \text{for every } n \in \mathbb{N}.$$

So, the topology $\mathscr{T}$ agrees with the p-adic topology; more precisely, one sees that the p-adic filtration coincides with the filtration $(V_n(A) \mid n \in \mathbb{N})$ and with the $V_1(A)$-adic filtration. Especially, the 0-th ghost component descends to an isomorphism

$$\overline{\boldsymbol{\omega}}_0 : W(A)/pW(A) \xrightarrow{\sim} A.$$

Also, (4.6.10) can be written in the form :

$$\underline{a} = \sum_{n=0}^{\infty} p^n \cdot \tau_A(a_n^{p^{-n}}) \qquad \text{for every } \underline{a} \in W(A). \tag{4.6.17}$$

The next few results are proposed as a set of exercises in [14, Ch.IX, pp.70-71]. For the sake of completeness, we sketch the proofs.

4.6.18. Let A be a ring, and $(J_n \mid n \in \mathbb{N})$ a decreasing sequence of ideals of A, such that $pJ_n + J_n^p \subset J_{n+1}$ for every $n \in \mathbb{N}$. Let $\pi_n : A \to A/J_n$ and $\pi'_n : A/J_n \to A/J_0$ be the natural projections. Let also R be a ring of characteristic p, and $\overline{\varphi} : R \to A/J_0$ a ring homomorphism. We say that a map of sets $\varphi_n : R \to A/J_n$ is a *lifting of* $\overline{\varphi}$ if $\varphi_n(x^p) = \varphi_n(x)^p$ for every $x \in R$, and $\pi'_n \circ \varphi_n = \overline{\varphi}$.

Lemma 4.6.19. *In the situation of* (4.6.18)*, the following holds :*

(i) *If φ_n and φ'_n are two liftings of $\overline{\varphi}$, then φ_n and φ'_n agree on R^{p^n}.*

(ii) *If R is a perfect ring, then there exists a unique lifting $\varphi_n : R \to A/J_n$ of $\overline{\varphi}$. We have $\varphi_n(1) = 1$ and $\varphi_n(xy) = \varphi_n(x) \cdot \varphi_n(y)$ for every $x, y \in R$.*

(iii) *If R is perfect and A is complete and separated for the topology defined by the filtration $(J_n \mid n \in \mathbb{N})$, then there exists a unique map of sets $\varphi : R \to A$ such that $\pi_0 \circ \varphi = \overline{\varphi}$ and $\varphi(x^p) = \varphi(x)^p$ for every $x \in R$. Moreover, $\varphi(1) = 1$ and $\varphi(xy) = \varphi(x) \cdot \varphi(y)$ for every $x, y \in R$. If furthermore, A is a ring of characteristic p, then φ is a ring homomorphism.*

Proof. (i) is an easy consequence of the following :

Claim 4.6.20. Let $x, y \in R$; if $x \equiv y \pmod{J_0}$, then $x^{p^n} \equiv y^{p^n} \pmod{J_n}$ for every $n \in \mathbb{N}$.

Proof of the claim. It is shown by an easy induction on $n \in \mathbb{N}$, which we leave to the reader to work out. ◊

(ii): The uniqueness follows from (i). For the existence, pick any map of sets $\psi : R \to A$ such that $\pi_0 \circ \psi = \overline{\varphi}$. We let :

$$\varphi_n(x) := \pi_n \circ \psi(x^{p^{-n}})^{p^n} \qquad \text{for every } x \in R.$$

Using claim 4.6.20 one verifies easily that φ_n does not depend on the choice of ψ. Especially, define $\psi' : R \to A$ by the rule : $x \mapsto \psi(x^{p^{-1}})^p$ for every $x \in R$. Clearly $\pi_0 \circ \psi' = \overline{\varphi}$ as well, hence :

$$\varphi_n(x^p) = \pi_n \circ \psi'(x^{p^{1-n}})^{p^n} = \pi_n \circ \psi(x^{p^{-n}})^{p^{n+1}} = \varphi_n(x)^p \qquad \text{for every } x \in R$$

as claimed. If we choose ψ so that $\psi(1) = 1$, we obtain $\varphi_n(1) = 1$. Finally, since $\overline{\varphi}$ is a ring homomorphism we have $\psi(x) \cdot \psi(y) \equiv \psi(xy) \pmod{J_0}$ for every $x, y \in R$. Hence

$$\psi(x)^{p^n} \cdot \psi(y)^{p^n} \equiv \psi(xy)^{p^n} \pmod{J_n}$$

(again by claim 4.6.20) and finally $\varphi_n(x^{p^n}) \cdot \varphi_n(y^{p^n}) = \psi((xy)^{p^n})$, which implies the last stated identity, since R is perfect.

(iii): The existence and uniqueness of φ follow from (ii). It remains only to show that $\varphi(x) + \varphi(y) = \varphi(x + y)$ in case A is a ring of characteristic p. It suffices to check the latter identity on the projections onto A/J_n, for every $n \in \mathbb{N}$, in which case one argues analogously to the foregoing proof of the multiplicative property for φ_n : the details shall be left to the reader. □

Proposition 4.6.21. *In the situation of* (4.6.18)*, suppose that R is a perfect ring and A is complete and separated for the topology defined by the filtration $(J_n \mid n \in \mathbb{N})$. Then the following holds :*

(i) *For every $n \in \mathbb{N}$ there exists a unique ring homomorphism $v_n : W_{n+1}(A/J_0) \to A/J_n$ such that the following diagram commutes :*

$$\begin{array}{ccc} W_{n+1}(A) & \xrightarrow{\boldsymbol{\omega}_n} & A \\ {\scriptstyle W_{n+1}(\pi_0)}\downarrow & & \downarrow{\scriptstyle \pi_n} \\ W_{n+1}(A/J_0) & \xrightarrow{v_n} & A/J_n. \end{array}$$

(ii) *Let* $u_n := v_n \circ W_{n+1}(\overline{\varphi}) \circ F_R^{-n}$*, where* $F_R : W_{n+1}(R) \to W_{n+1}(R)$ *is induced by the Frobenius automorphism of* $W(R)$*. Then, for every* $n \in \mathbb{N}$ *the following diagram commutes :*

$$\begin{tikzcd} W_{n+2}(R) \arrow[r, "u_{n+1}"] \arrow[d] & A/J_{n+1} \arrow[d, "\vartheta"] \\ W_{n+1}(R) \arrow[r, "u_n"] & A/J_n \end{tikzcd}$$

where the vertical maps are the natural projections.

(iii) *There exists a unique ring homomorphism* u *such that the following diagram commutes:*

$$\begin{tikzcd} W(R) \arrow[r, "u"] \arrow[d, "\boldsymbol{\omega}_0"'] & A \arrow[d, "\pi_0"] \\ R \arrow[r, "\overline{\varphi}"] & A/J_0. \end{tikzcd}$$

Furthermore, u *is continuous for the* p*-adic topology on* $W(R)$*, and for every* $\underline{a} := (a_n \mid n \in \mathbb{N}) \in W(R)$ *we have :*

$$u(\underline{a}) = \sum_{n=0}^{\infty} p^n \cdot \varphi(a_n^{p^{-n}}) \tag{4.6.22}$$

where $\varphi : R \to A$ *is the unique map of sets characterized as in lemma* 4.6.19(iii).

Proof. (i): We have to check that $\boldsymbol{\omega}_n(a_0, \dots, a_n) \in J_n$ whenever $a_0, \dots, a_n \in J_0$, which is clear from the definition of the Witt polynomial $\boldsymbol{\omega}_n(X_0, \dots, X_n)$.

(ii): Since F_R is an isomorphism, it boils down to verifying :

Claim 4.6.23. $\vartheta \circ v_{n+1}(\overline{\varphi}(a_0), \dots, \overline{\varphi}(a_{n+1})) = v_n(\overline{\varphi}(F_0(a_0, a_1)), \dots, \overline{\varphi}(F_n(a_0, \dots, a_{n+1})))$ for every $\underline{a} := (a_0, \dots, a_{n+1}) \in W_{n+1}(A)$.

Proof of the claim. Directly from (4.6.6) we derive :

$$(\overline{\varphi}(F_0(a_0, a_1)), \dots, \overline{\varphi}(F_n(a_0, \dots, a_{n+1}))) = (F_0(\overline{\varphi}(a_0), \overline{\varphi}(a_1)), \dots, F_n(\overline{\varphi}(a_0), \dots, \overline{\varphi}(a_n)))$$

for every $\underline{a} \in W_{n+1}(A)$. Hence, it suffices to show that

$$\vartheta \circ v_{n+1}(b_0, \dots, b_{n+1}) = v_n(F_0(b_0, b_1), \dots, F_n(b_0, \dots, b_{n+1}))$$

for every $(b_0, \dots, b_{n+1}) \in W_{n+1}(A/J_0)$. The latter identity follows from (4.6.5). $\Diamond$

(iii): The existence of u follows from (ii); indeed, it suffices to take $u := \lim_{n \in \mathbb{N}} u_n$. With this definition, it is also clear that u is continuous for the topology $\mathscr{T}$ of $W(R)$; however, since R is perfect, the latter coincides with the p-adic topology (see (4.6.12)). Next we remark that $\pi_0 \circ u \circ \tau_R = \overline{\varphi} \circ \boldsymbol{\omega}_0 \circ \tau_R = \overline{\varphi}$, hence :

$$u \circ \tau_R = \varphi \tag{4.6.24}$$

by lemma 4.6.19(iii). Identity (4.6.22) follows from (4.6.17), (4.6.24) and the continuity of u. $\square$

4.6.25. Let now R be any ring; we let $(A_n; \varphi_n : A_{n+1} \to A_n \mid n \in \mathbb{N})$ be the inverse system of rings such that $A_n := R/pR$ and φ_n is the Frobenius endomorphism $\Phi_R : R/pR \to R/pR$ for every $n \in \mathbb{N}$. We set :

$$\mathbf{E}(R)^+ := \lim_{n \in \mathbb{N}} A_n.$$

So $\mathbf{E}(R)^+$ is the set of all sequences $(a_n \mid n \in \mathbb{N})$ of elements $a_n \in R/pR$ such that $a_n = a_{n+1}^p$ for every $n \in \mathbb{N}$. We denote by $\mathscr{T}_{\mathbf{E}}$ the pro-discrete topology on $\mathbf{E}(R)^+$ obtained as the inverse limit of the discrete topologies on the rings A_n. As an immediate consequence of the definition, $\mathbf{E}(R)^+$ is a perfect ring. Let $\Phi_{\mathbf{E}(R)^+}$ be the Frobenius automorphism; notice that :

$$\Phi^{-1}_{\mathbf{E}(R)^+}(a_n \mid n \in \mathbb{N}) = (a_{n+1} \mid n \in \mathbb{N}) \qquad \text{for every } (a_n \mid n \in \mathbb{N}) \in \mathbf{E}(R)^+.$$

The projection to A_0 defines a natural ring homomorphism

$$\overline{u}_R : \mathbf{E}(R)^+ \to R/pR$$

and a basis of open neighborhoods of $0 \in \mathbf{E}(R)^+$ is given by the descending family of ideals :

$$\mathrm{Ker}(\mathbf{E}(R)^+ \to A_n) = \mathrm{Ker}(\overline{u}_R \circ \Phi^{-n}_{\mathbf{E}(R)^+}) \qquad \text{for every } n \in \mathbb{N}.$$

4.6.26. Suppose now that $(R, |\cdot|_R)$ is a valuation ring with value group Γ_R, residue characteristic p and generic characteristic 0. Set $\Gamma_R^{p^\infty} := \bigcap_{n\in\mathbb{N}} \Gamma_R^{p^n}$, and let $\Gamma_{\mathbf{E}(R)} \subset \Gamma_R^{p^\infty}$ be the convex subgroup consisting of all $\gamma \in \Gamma_R^{p^\infty}$ such that $|p|_R^{-n} > \gamma > |p|_R^n$ for every sufficiently large $n \in \mathbb{N}$. We define a mapping

$$|\cdot|_{\mathbf{E}(R)} : \mathbf{E}(R)^+ \to \Gamma_{\mathbf{E}(R)} \cup \{0\}$$

as follows. Let $\underline{a} := (a_n \mid n \in \mathbb{N}) \in \mathbf{E}(R)^+$ be any element, and for every $n \in \mathbb{N}$ choose a representative $\widetilde{a}_n \in R$ for the class $a_n \in R/pR$. If $|\widetilde{a}_n|_R \leq |p|_R$ for every $n \in \mathbb{N}$, then we set $|\underline{a}|_{\mathbf{E}(R)} := 0$; if there exists $n \in \mathbb{N}$ such that $|\widetilde{a}_n| > |p|_R$, then we set $|\underline{a}|_{\mathbf{E}(R)} := |\widetilde{a}_n|_R^{p^n}$. One verifies easily that the definition is independent of all the choices : the details shall be left to the reader.

Lemma 4.6.27. (i) *In the situation of* (4.6.26), *the pair* $(\mathbf{E}(R)^+, |\cdot|_{\mathbf{E}(R)})$ *is a valuation ring.*

(ii) *Especially, if R is deeply ramified of rank one, then $\Gamma_R = \Gamma_{\mathbf{E}(R)}$ is the value group of* $\mathbf{E}(R)^+$.

Proof. (i): One checks easily that $|\underline{a} \cdot \underline{b}|_{\mathbf{E}(R)} = |\underline{a}|_{\mathbf{E}(R)} \cdot |\underline{b}|_{\mathbf{E}(R)}$ for every $\underline{a}, \underline{b} \in \mathbf{E}(R)^+$, as well as the inequality : $|\underline{a} + \underline{b}|_{\mathbf{E}(R)} \leq \max(|\underline{a}|_{\mathbf{E}(R)}, |\underline{b}|_{\mathbf{E}(R)})$. It is also clear from the definition that $|\underline{a}|_{\mathbf{E}(R)} = 0$ if and only if $\underline{a} = 0$. Next, let us show that $\mathbf{E}(R)^+$ is a domain. Indeed, suppose that $\underline{a} := (a_n \mid n \in \mathbb{N})$ and $\underline{b} := (b_n \mid n \in \mathbb{N})$ are two non-zero elements such that $\underline{a} \cdot \underline{b} = 0$; we may assume that $0 < |\underline{a}|_{\mathbf{E}(R)} \leq |\underline{b}|_{\mathbf{E}(R)}$, in which case $|\widetilde{a}_n|_R^{p^n} \leq |\widetilde{b}_n|_R^{p^n}$ for every sufficiently large $n \in \mathbb{N}$, and every choice of representative $\widetilde{a}_n \in R$ (resp. $\widetilde{b}_n \in R$) for a_n (resp. for b_n). By assumption we have $|\widetilde{a}_n \cdot \widetilde{b}_n|_R \leq |p|_R$ for every $n \in \mathbb{N}$, whence $|\widetilde{a}_n|_R^2 \leq |p|_R$ for sufficiently large $n \in \mathbb{N}$, hence $|\widetilde{a}_n^p|_R \leq |p|_R$, consequently $\underline{a}^p = 0$, and since $\mathbf{E}(R)^+$ is perfect, we deduce that $\underline{a} = 0$, a contradiction. To conclude the proof of (i), it then suffices to show :

Claim 4.6.28. $|\underline{a}|_{\mathbf{E}(R)} \leq |\underline{b}|_{\mathbf{E}(R)}$ if and only if $\underline{a} \cdot \mathbf{E}(R)^+ \subset \underline{b} \cdot \mathbf{E}(R)^+$.

Proof of the claim. We may assume that $0 < |\underline{a}|_{\mathbf{E}(R)} \leq |\underline{b}|_{\mathbf{E}(R)}$, and we need to show that $\underline{b}$ divides $\underline{a}$ in $\mathbf{E}(R)^+$. Choose as usual representatives $\widetilde{a}_n$ and $\widetilde{b}_n$ for a_n and b_n, for every $n \in \mathbb{N}$; we may assume that $|\widetilde{b}_n| \geq |p|$ for every $n \in \mathbb{N}$, and then directly from the definitions, it is clear that $|\widetilde{a}_n|_R \leq |\widetilde{b}_n|_R$, hence there exist $n_0 \in \mathbb{N}$, $x_n \in R$ such that $0 \neq a_n = x_n \cdot b_n$ in R/pR for every $n \geq n_0$. We compute : $a_n = a_{n+1}^p = x_{n+1}^p \cdot b_{n+1}^p = x_{n+1}^p \cdot b_n$, hence $(x_{n+1}^p - x_n) \cdot b_n = 0$ for every $n \geq n_0$. For every $n \in \mathbb{N}$, let $y_n := x_{n+1}^p - x_n$; notice that $b_n^2 \neq 0$ and $y_n \cdot b_n = 0$ in R/pR whenever $n \geq n_0$, which implies especially that y_n does not divide b_n in R/pR. It follows that b_n divides y_n when $n \geq n_0$, and therefore $y_n^2 = 0$ for all $n \geq n_0$. In turn, this shows that $x_{n+1}^{p^2} = x_n^p$ for every $n \geq n_0$, so that, up to replacing finitely many terms, the sequence $\underline{x} := (x_{n+1}^p \mid n \in \mathbb{N})$ is a well-defined element of $\mathbf{E}(R)^+$. Clearly $\underline{a} \cdot \underline{x} = \underline{b}$, which is the sought assertion. $\Diamond$

(ii): If R is deeply ramified of rank one, then the Frobenius endomorphism Φ_R is surjective ([36, Prop.6.6.6]), so that $\Gamma_R = \Gamma_R^{p^\infty} = \Gamma_{\mathbf{E}}$, and moreover the map $\overline{u}_R : \mathbf{E}(R)^+ \to R/pR$ is obviously surjective, so that $|\mathbf{E}(R)^+|_{\mathbf{E}(R)} = \Gamma_R^+ \cup \{0\}$. $\square$

4.6.29. We resume the general situation of (4.6.25), and we define :

$$\mathbf{A}(R)^+ := W(\mathbf{E}(R)^+)$$

where $W(-)$ denotes the ring of Witt vectors as in (4.6.3). According to (4.6.16), $\mathbf{A}(R)^+$ is complete and separated for the p-adic topology; moreover we see from (4.6.14) that p is a regular element of $\mathbf{A}(R)^+$, and the ghost component $\boldsymbol{\omega}_0$ descends to an isomorphism

$$\overline{\boldsymbol{\omega}}_0 : \mathbf{A}(R)^+/p\mathbf{A}(R)^+ \xrightarrow{\sim} \mathbf{E}(R)^+.$$

The map $\boldsymbol{\omega}_0$ also admits a set-theoretic multiplicative splitting, the Teichmüller mapping of (4.6.9), which will be here denoted :

$$\tau_R : \mathbf{E}(R)^+ \to \mathbf{A}(R)^+.$$

Furthermore, let $R^\wedge$ denote the p-adic completion of R; by proposition 4.6.21(iii), the composition $\overline{u}_R \circ \boldsymbol{\omega}_0$ lifts to a unique map

$$u_R : \mathbf{A}(R)^+ \to R^\wedge$$

which is continuous for the p-adic topologies. Finally, $\mathbf{A}(R)^+$ is endowed with an automorphism, which we shall denote by $\boldsymbol{\sigma}_R : \mathbf{A}(R)^+ \to \mathbf{A}(R)^+$, which lifts the Frobenius map of $\mathbf{E}(R)^+$, *i.e.* such that the horizontal arrows of the diagram :

$$\begin{tikzcd} \mathbf{A}(R)^+ \arrow[r, "\boldsymbol{\sigma}_R"] \arrow[d, shift right, "\boldsymbol{\omega}_0"'] & \mathbf{A}(R)^+ \arrow[d, shift right, "\boldsymbol{\omega}_0"'] \\ \mathbf{E}(R)^+ \arrow[r, "\Phi_{\mathbf{E}(R)^+}"] \arrow[u, shift right, "\tau_R"'] & \mathbf{E}(R)^+ \arrow[u, shift right, "\tau_R"'] \end{tikzcd} \qquad (4.6.30)$$

commute with the downward arrows (see (4.6.16)). The horizontal arrows commute also with the upward ones, due to (4.6.15).

4.6.31. Let $\underline{a} := (a_n \mid n \in \mathbb{N})$ be any element of $\mathbf{E}(R)^+$, and for every $n \in \mathbb{N}$ choose a representative $\widetilde{a}_n \in R$ for the class a_n. In view of (4.6.22) and by inspecting the proof of lemma 4.6.19 we deduce easily the following identity

$$u_R \circ \tau_R(\underline{a}) = \lim_{n\to\infty} \widetilde{a}_n^{p^n} \qquad (4.6.32)$$

where the convergence is relative to the p-adic topology of $R^\wedge$.

Proposition 4.6.33. *With the notation of* (4.6.29)*, suppose moreover that there exist* $\pi, a \in R$ *such that* π *is regular,* a *is invertible,* $p = \pi^p \cdot a$*, and* Φ_R *induces an isomorphism :*

$$R/\pi R \xrightarrow{\sim} R/pR \quad : \quad x \bmod \pi R \mapsto x^p \bmod pR.$$

Then the following holds :

(i) u_R *and* $\overline{u}_R$ *are surjections.*
(ii) *There exists a regular element* $\vartheta \in \mathbf{A}(R)^+$ *such that :*

$$\operatorname{Ker} u_R = \vartheta \cdot \mathbf{A}(R)^+.$$

Proof. (i): Indeed, if Φ_R is surjective, the same holds for $\overline{u}_R$, and then the claim follows easily from [61, Th.8.4].

(ii): The surjectivity of $\overline{u}_R$ says especially that we may find $\underline{\pi} := (\pi_n \mid n \in \mathbb{N}) \in \mathbf{E}(R)^+$ such that π_0 is the image of π in R/pR.

Claim 4.6.34. (i) π_n^p generates $\operatorname{Ker}(\Phi_R^n : R/pR \to R/pR)$ for every $n \in \mathbb{N}$.

(ii) Let $f_n \in R$ be any lifting of π_n; then there exists an invertible element $a_n \in R^\wedge$ such that $f_n^{p^{n+1}} = a_n \cdot p$.

(iii) $\underline{\pi}^p$ is a regular element of $\mathbf{E}(R)^+$, and generates $\mathrm{Ker}\,\overline{u}_R$.

Proof of the claim. (i): We argue by induction on $n \in \mathbb{N}$. We have $\pi_0^p = 0$, so the claim is clear for $n = 0$. For $n = 1$, the assertion is equivalent to our assumption on π. Next, suppose that the assertion is known for some $n \geq 1$, and let $x \in R/pR$ such that $\Phi_R^{n+1}(x) = 0$; hence $\Phi_R(x) = y \cdot \pi_n^p$ for some $y \in R/pR$. Since Φ_R is surjective, we may write $y = z^p$ for some $z \in R/pR$, therefore $(x - z \cdot \pi_n)^p = 0$, and consequently $x - z \cdot \pi_n \in \pi_0(R/pR)$, so $x \in \pi_n(R/pR) = \pi_{n+1}^p(R/pR)$, as stated.

(ii): By assumption, $f_n^{p^n} - \pi \in pR$, hence $f_n^{p^{n+1}} \equiv a^{-1}p \pmod{\pi pR}$, in other words, $f_n^{p^{n+1}} = p(a^{-1} + \pi b)$ for some $b \in R$. Since $R^\wedge$ is complete and separated for the π-adic topology, the element $a_n := a^{-1} + \pi b$ is invertible in $R^\wedge$; whence the contention.

(iii): In view of (i), for every $n \in \mathbb{N}$ we have a ladder of short exact sequences :

$$\begin{tikzcd}
0 \arrow[r] & (\pi_{n+1}^p) \arrow[r] \arrow[d, "\Phi_R"'] & R/pR \arrow[r, "\Phi_R^{n+1}"] \arrow[d, "\Phi_R"'] & R/pR \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & (\pi_n^p) \arrow[r] & R/pR \arrow[r, "\Phi_R^n"] & R/pR \arrow[r] & 0
\end{tikzcd}$$

and since all the vertical arrows are surjections, we deduce a short exact sequence ([75, Lemma 3.5.3]) :

$$0 \to \lim_{n\in\mathbb{N}} (\pi_n^p) \xrightarrow{j} \mathbf{E}(R)^+ \xrightarrow{\overline{u}_R} R/pR \to 0.$$

Next, choose f_n as in (ii); since p is regular in R, it is also regular in $R^\wedge$, and then the same holds for f_n. It follows easily that the kernel of the multiplication map $\mu_n : R/pR \to R/pR : x \mapsto \pi_n^p x$ is the ideal $I_n := (\pi_n^{p^{n+1}-p})$, whence a ladder of short exact sequences :

$$\begin{tikzcd}
0 \arrow[r] & I_n \arrow[r] \arrow[d, "\Phi_R"'] & R/pR \arrow[r, "\mu_n"] \arrow[d, "\Phi_R"'] & (\pi_n^p) \arrow[r] \arrow[d, "\Phi_R"] & 0 \\
0 \arrow[r] & I_{n-1} \arrow[r] & R/pR \arrow[r, "\mu_{n-1}"] & (\pi_{n-1}^p) \arrow[r] & 0
\end{tikzcd}$$

for every $n > 0$. Notice that $p^{n+1} - p \geq p^n$, hence $\pi_{n-1}^{p^{n+1}-p} = 0$ for every $n > 0$, so the inverse system $(I_n \mid n \in \mathbb{N})$ is essentially zero, and the system of maps $(\mu_n \mid n \in \mathbb{N})$ defines an isomorphism ([75, Prop.3.5.7]) :

$$\mu : \mathbf{E}(R)^+ \xrightarrow{\sim} \lim_{n\in\mathbb{N}} (\pi_n^p).$$

By inspecting the construction, one sees that the composition $j \circ \mu : \mathbf{E}(R)^+ \to \mathbf{E}(R)^+$ is none else than the map : $\underline{x} \mapsto \underline{x} \cdot \underline{\pi}^p$. Both assertions follow immediately. ◇

Claim 4.6.35. $\boldsymbol{\omega}_0$ restricts to a surjection $\mathrm{Ker}\,u_R \to \mathrm{Ker}\,\overline{u}_R$.

Proof of the claim. By the snake lemma, we are reduced to showing that u_R restricts to a surjection $\mathrm{Ker}\,\boldsymbol{\omega}_0 \to pR^\wedge$, which is clear since $\mathrm{Ker}\,\boldsymbol{\omega}_0 = p\mathbf{A}(R)^+$ and u_R is surjective. ◇

From [61, Th.8.4] and claims 4.6.35, 4.6.34(iii) we deduce that every $\vartheta \in \mathrm{Ker}\,u_R \cap \boldsymbol{\omega}_0^{-1}(\underline{\pi}^p)$ generates $\mathrm{Ker}\,u_R$. Furthermore, claim 4.6.34(iii) also says that the sequence (p, ϑ) is regular on $\mathbf{A}(R)^+$. Since $\mathbf{A}(R)^+$ is p-adically separated, the sequence (ϑ, p) is regular as well (corollary 4.1.22); especially ϑ is regular, as stated. □

Example 4.6.36. Let R be a ring such that p is regular in R, and Φ_R is surjective. Suppose also that we have a compatible system of elements $u_n := p^{1/p^n} \in R^\wedge$ (so that $u_{n+1}^p = u_n$ for every $n \in \mathbb{N}$). In this case, an element ϑ as provided by proposition 4.6.33 can be chosen explicitly. Indeed, the sequence $\underline{a} := (p, p^{1/p}, p^{1/p^2}, \dots)$ defines an element of $\mathbf{E}(R)^+$ such that $u_R \circ \tau_R(\underline{a}) = p$ (see (4.6.32)), and by inspecting the proof of the proposition, we deduce that the element

$$\vartheta := p - \tau_R(p, p^{1/p}, p^{1/p^2}, \dots)$$

is a generator of $\operatorname{Ker} u_R$.

Definition 4.6.37. Let $\mathbf{m}, \mathbf{n} : \mathbb{Z} \to \mathbb{N}$ be any two maps of sets.

(i) The *support* of $\mathbf{n}$ is the subset :

$$\operatorname{Supp} \mathbf{n} := \{i \in \mathbb{Z} \mid \mathbf{n}(i) \neq 0\}.$$

(ii) For every $t \in \mathbb{Z}$, we let $\mathbf{n}[t] : \mathbb{Z} \to \mathbb{N}$ be the mapping defined by the rule: $\mathbf{n}[t](k) := \mathbf{n}(k+t)$ for every $k \in \mathbb{Z}$.

(iii) We define a partial ordering on the set of all mappings $\mathbb{Z} \to \mathbb{N}$ by declaring that $\mathbf{n} \geq \mathbf{m}$ if and only if $\mathbf{n}(k) \geq \mathbf{m}(k)$ for every $k \in \mathbb{Z}$.

4.6.38. Keep the assumptions of proposition 4.6.33, and let $\mathbf{n} : \mathbb{Z} \to \mathbb{N}$ be any mapping with finite support. We set :

$$\vartheta_{\mathbf{n}} := \prod_{i \in \operatorname{Supp} \mathbf{n}} \boldsymbol{\sigma}_R^i(\vartheta^{\mathbf{n}(i)})$$

where $\vartheta \in \mathbf{A}(R)^+$ is – as in proposition 4.6.33(ii) – any generator of $\operatorname{Ker} u_R$, and $\boldsymbol{\sigma}_R$ is the automorphism introduced in (4.6.29). For every such $\mathbf{n}$, the element $\vartheta_{\mathbf{n}}$ is regular in $\mathbf{A}(R)^+$, and if $\mathbf{n} \geq \mathbf{m}$, then $\vartheta_{\mathbf{m}}$ divides $\vartheta_{\mathbf{n}}$. We endow $\mathbf{A}(R)^+$ with the linear topology $\mathscr{T}_{\mathbf{A}}$ which admits the family of ideals $(\vartheta_{\mathbf{n}} \cdot \mathbf{A}(R)^+ \mid \mathbf{n} : \mathbb{Z} \to \mathbb{N})$ as a fundamental system of open neighborhoods of $0 \in \mathbf{A}(R)^+$.

Theorem 4.6.39. *In the situation of* (4.6.38), *the following holds :*

(i) *The topological ring* $(\mathbf{A}(R)^+, \mathscr{T}_{\mathbf{A}})$ *is complete and separated.*

(ii) *The homomorphism of topological rings* $\boldsymbol{\omega}_0 : (\mathbf{A}(R)^+, \mathscr{T}_{\mathbf{A}}) \to (\mathbf{E}(R)^+, \mathscr{T}_{\mathbf{E}})$ *is a quotient map.*

Proof. We resume the notation of the proof of proposition 4.6.33; especially, the image of ϑ in $\mathbf{E}(R)^+$ is the element $\underline{\pi}^p$, and $\underline{\pi}$ lifts the image of π in R/pR.

Claim 4.6.40. (i) The topology $\mathscr{T}_{\mathbf{E}}$ on $\mathbf{E}(R)^+$ agrees with the $\underline{\pi}$-adic topology.

(ii) The sequences $(p, \vartheta_{\mathbf{n}})$ and $(\vartheta_{\mathbf{n}}, p)$ are regular in $\mathbf{A}(R)^+$ for every $\mathbf{n} : \mathbb{Z} \to \mathbb{N}$ of finite support.

Proof of the claim. Clearly :

$$\boldsymbol{\omega}_0(\vartheta_{\mathbf{n}}) = \underline{\pi}^{\sum_{i \in \mathbb{Z}} \mathbf{n}(i) \cdot p^{i+1}}$$

and then the regularity of $(p, \vartheta_{\mathbf{n}})$ follows easily from claim 4.6.34(iii), which also implies that :

$$\underline{\pi}^{p^{n+1}} \cdot \mathbf{E}(R)^+ = \operatorname{Ker} (\overline{u}_R \circ \Phi_{\mathbf{E}(R)^+}^{-n}) \qquad \text{for every } n \in \mathbb{N}$$

which, in turn, yields (i). To show that also the sequence $(\vartheta_{\mathbf{n}}, p)$ is regular, it suffices to invoke corollary 4.1.22. $\Diamond$

Claim 4.6.40(i) already implies assertion (ii). For every $n \in \mathbb{N}$, let $\mathbf{A}_n^{+\wedge}$ denote the completion of $\mathbf{A}_n^+ := \mathbf{A}(R)^+/p^{n+1}\mathbf{A}(R)^+$ for the topology $\mathscr{T}_{\mathbf{A}_n}$, defined as the unique topology such that the surjection $(\mathbf{A}(R)^+, \mathscr{T}_{\mathbf{A}}) \to (\mathbf{A}_n^+, \mathscr{T}_{\mathbf{A}_n})$ is a quotient map. For every $n > 0$ we have a natural commutative diagram :

$$\begin{tikzcd} 0 \arrow[r] & \mathbf{A}_0^+ \arrow[r] \arrow[d] & \mathbf{A}_n^+ \arrow[r] \arrow[d] & \mathbf{A}_{n-1}^+ \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathbf{A}_0^{+\wedge} \arrow[r] & \mathbf{A}_n^{+\wedge} \arrow[r] & \mathbf{A}_{n-1}^{+\wedge} \arrow[r] & 0 \end{tikzcd} \tag{4.6.41}$$

whose vertical arrows are the completion maps. Since p is regular in $\mathbf{A}(R)^+$, the top row of (4.6.41) is short exact.

Claim 4.6.42. The bottom row of (4.6.41) is short exact as well.

Proof of the claim. It suffices to show that the topology $\mathscr{T}_{\mathbf{A}_0}$ on $\mathbf{A}_0^+$ agrees with the subspace topology induced from $\mathbf{A}_n^+$ ([61, Th.8.1]). This boils down to verifying the equality of ideals of $\mathbf{A}(R)^+$:

$$(\vartheta_{\mathbf{n}}) \cap (p^k) = (\vartheta_{\mathbf{n}} \cdot p^k) \tag{4.6.43}$$

for every $k \in \mathbb{N}$ and every $\mathbf{n} : \mathbb{Z} \to \mathbb{N}$ as in (4.6.38). The latter follows easily from claim 4.6.40(ii) and [61, Th.16.1]. $\diamondsuit$

Claim 4.6.44. The topological ring $(\mathbf{A}_n^+, \mathscr{T}_{\mathbf{A}_n})$ is separated and complete for every $n \in \mathbb{N}$.

Proof of the claim. We shall argue by induction on $n \in \mathbb{N}$. For $n = 0$ the assertion reduces to claim 4.6.40(i) (and the fact that $\mathscr{T}_{\mathbf{E}}$ is obviously a complete and separated topology on $\mathbf{E}(R)^+$). Suppose that $n > 0$ and that the assertion is known for every integer $< n$. This means that the right-most and left-most vertical arrows of (4.6.41) are isomorphisms. Then claim 4.6.42 implies that the middle arrow is bijective as well, as required. $\diamondsuit$

Claim 4.6.45. For every $\mathbf{n} : \mathbb{Z} \to \mathbb{N}$ as in (4.6.38), the ring $\mathbf{A}(R)^+/\vartheta_{\mathbf{n}}\mathbf{A}(R)^+$ is complete and separated for the p-adic topology.

Proof of the claim. Since $\mathbf{A}(R)^+$ is complete for the p-adic topology, [61, Th.8.1(ii)] reduces to showing that the p-adic topology on $\mathbf{A}(R)^+$ induces the p-adic topology on its submodule $\vartheta_{\mathbf{n}}\mathbf{A}(R)^+$. The latter assertion follows again from (4.6.43). $\diamondsuit$

To conclude the proof of (i) we may now compute :

$$\begin{aligned} \lim_{\mathbf{n}:\mathbb{Z}\to\mathbb{N}} \mathbf{A}(R)^+/\vartheta_{\mathbf{n}}\mathbf{A}(R)^+ &\simeq \lim_{\mathbf{n}:\mathbb{Z}\to\mathbb{N}} \lim_{k\in\mathbb{N}} \mathbf{A}_k^+/\vartheta_{\mathbf{n}}\mathbf{A}_k^+ && \text{(by claim 4.6.45)} \\ &\simeq \lim_{k\in\mathbb{N}} \lim_{\mathbf{n}:\mathbb{Z}\to\mathbb{N}} \mathbf{A}_k^+/\vartheta_{\mathbf{n}}\mathbf{A}_k^+ \\ &\simeq \lim_{k\in\mathbb{N}} \mathbf{A}_k^+ && \text{(by claim 4.6.44)} \\ &\simeq \mathbf{A}(R)^+ \end{aligned}$$

which is the contention. $\square$

We conclude this section with a review of the theory of divided power modules and algebras.

Definition 4.6.46. Let A be a ring, $k \in \mathbb{N}$ an integer, and $\underline{M} := (M_1, \dots, M_k)$ an object of the abelian category $(A\text{-}\mathbf{Mod})^k$, *i.e.* any sequence of A-modules. Let also Q be another A-module, and $\underline{d} := (d_1, \dots, d_k) \in \mathbb{N}^{\oplus k}$ any sequence of k integers.

(i) We denote by $F_{\underline{M}}, G_Q : A\text{-}\mathbf{Alg} \to \mathbf{Set}$ the functors given respectively by the rules :

$$B \mapsto (M_1 \times \cdots \times M_k) \otimes_A B \quad \text{and} \quad B \mapsto Q \otimes_A B \qquad \text{for every } A\text{-algebra } B.$$

Also, let $M_i^\vee := \mathrm{Hom}_A(M_i, A)$ for every $i = 1, \dots, k$, and set :

$$\mathrm{Sym}_A^\bullet(\underline{M}^\vee) := \bigotimes_{i=1}^{k} \mathrm{Sym}_A^\bullet(M_i^\vee)$$

which we view as a $\mathbb{N}^{\oplus k}$-graded A-algebra, with the grading given by the rule :

$$\mathrm{Sym}_A^{\underline{n}}(\underline{M}^\vee) := \bigotimes_{i=1}^{k} \mathrm{Sym}_A^{n_i}(M_i^\vee) \qquad \text{for every } \underline{n} := (n_1, \dots, n_k) \in \mathbb{N}^{\oplus k}.$$

(ii) A *homogeneous multipolynomial law* of degree $\underline{d}$ on the sequence of A-modules $\underline{M}$, with values in Q, is a natural transformation $\lambda : F_{\underline{M}} \to G_Q$, which we write also

$$\lambda : \underline{M} \rightsquigarrow Q$$

such that the following holds. For every A-algebra B, every $(x_1, \dots, x_k) \in F_{\underline{M}}(B)$ and every $(b_1, \dots, b_k) \in B^k$, we have

$$\lambda_B(b_1 x_1, \dots, b_k x_k) = b_1^{d_1} \cdots b_k^{d_k} \cdot \lambda_B(x_1, \dots, x_k).$$

We denote by $\mathrm{Pol}_A^{\underline{d}}(\underline{M}, Q)$ the set of all homogeneous multipolynomial laws $\underline{M} \rightsquigarrow Q$ of degree $\underline{d}$.

Remark 4.6.47. (i) Let $\underline{f} := (f_i : M_i' \to M_i \mid i = 1, \dots, k)$ be a sequence of A-linear maps, and set $\underline{M}' := (M_1', \dots, M_k')$; let also $g : Q \to Q'$ be another A-linear map. Suppose that $\lambda : \underline{M} \rightsquigarrow Q$ is a homogeneous multipolynomial law of degree d; then we obtain a law

$$g \circ \lambda \circ \underline{f} : \underline{M}' \rightsquigarrow Q'$$

of the same type, by the rule :

$$B \mapsto (g \otimes_A B) \circ \lambda_B \circ ((f_1 \times \cdots \times f_k) \otimes_A B) : (M_1' \times \cdots \times M_k') \otimes_A B \to Q' \otimes_A B$$

for every A-algebra B. This shows that the rule $(\underline{M}, Q) \mapsto \mathrm{Pol}_A^{\underline{d}}(\underline{M}, Q)$ is functorial in both arguments. However, it is not *a priori* obvious that this functor takes values in (essentially) small sets. The latter assertion will nevertheless be proven shortly.

(ii) Suppose that $\underline{M}' := (M_1', \dots, M_{k'}')$ is another sequence of A-modules, Q' another A-module, and $\underline{d}' := (d_1', \dots, d_{k'}') \in \mathbb{N}^{\oplus k'}$ another sequence of integers. If we have two homogeneous polynomial laws $\lambda : \underline{M} \rightsquigarrow Q$ and $\lambda' : \underline{M}' \rightsquigarrow Q'$ of degrees respectively $\underline{d}$ and $\underline{d}'$, we can form the tensor product

$$\lambda \otimes \lambda' : (\underline{M}, \underline{M}') \rightsquigarrow Q \otimes_A Q'$$

which is the homogeneous multipolynomial law of degree $(\underline{d}, \underline{d}')$ given by the rule :

$$(\lambda \otimes \lambda')_B(x_1, \dots, x_k, x_1', \dots, x_{k'}') := \lambda_B(x_1, \dots, x_k) \otimes \lambda_B'(x_1', \dots, x_{k'}')$$

for every A-algebra B and every $(x_1, \dots, x_{k'}') \in (M_1 \times \cdots \times M_{k'}') \otimes_A B$.

(iii) Let B be any A-algebra; if $\lambda : \underline{M} \rightsquigarrow Q$ is a law as in (i), then the restriction of λ to B-algebras yields a law $\lambda_{/B} : \underline{M} \otimes_A B \to Q \otimes_B$. In this way, we obtain a natural mapping

$$\mathrm{Pol}_A^{\underline{d}}(\underline{M}, Q) \to \mathrm{Pol}_B^{\underline{d}}(\underline{M} \otimes_A B, Q \otimes_A B). \tag{4.6.48}$$

Lemma 4.6.49. *In the situation of definition* 4.6.46*, suppose that the A-module M_i is free of finite rank, for every $i = 1, \dots, k$. Then, there exists a natural bijection*

$$\mathrm{Pol}_A^{\underline{d}}(\underline{M}, Q) \xrightarrow{\sim} Q \otimes_A \mathrm{Sym}_A^{\underline{d}}(\underline{M}^\vee).$$

Proof. Indeed, fix bases $(e_{i1}, \dots, e_{ir_i})$ for each A-module M_i, and denote by $(e^*_{i1}, \dots, e^*_{ir_i})$ the dual basis of $M_i^\vee$. For every $i = 1, \dots, k$ there is a natural (injective) map of A-algebras

$$\mathrm{Sym}^\bullet_A(M_i^\vee) \to \mathrm{Sym}^\bullet_A(\underline{M}^\vee) \quad : \quad s \mapsto 1 \otimes \cdots \otimes s \otimes \cdots \otimes 1$$

which allows to view naturally the e^*_{ij} as elements of $\mathrm{Sym}^\bullet_A(\underline{M}^\vee)$. Suppose that $\lambda : \underline{M} \rightsquigarrow Q$ is a given homogeneous multipolynomial law of degree $\underline{d}$, and set

$$P_\lambda(e^*_{ij}) := \lambda_{\mathrm{Sym}^\bullet_A(\underline{M}^\vee)} \left(\sum_{j=1}^{r_1} e_{1j} \otimes e^*_{1j}, \dots, \sum_{j=1}^{r_k} e_{kj} \otimes e^*_{kj} \right) \in Q \otimes_A \mathrm{Sym}^\bullet_A(\underline{M}^\vee).$$

Notice that the terms in parenthesis are elements of $M_i \otimes_A M_i^\vee \subset M_i \otimes_A \mathrm{Sym}^\bullet_A(M_i^\vee)$ that do not depend on the chosen bases, since they correspond to the identity automorphisms of M_i, under the natural identification $M_i \otimes_A M_i^\vee \xrightarrow{\sim} \mathrm{End}_A(M_i)$. Now, let B be any A-algebra, and $(b_{ij} \mid i = 1, \dots, k;\ j = 1, \dots, r_i)$ any sequence of elements of B. By considering the unique map of A-algebras $\mathrm{Sym}^\bullet_A(\underline{M}) \to B$ given by the rule : $e^*_{ij} \mapsto b_{ij}$ for every $i = 1, \dots, k$ and every $j = 1, \dots, r_i$, it is easily seen that

$$\lambda_B \left(\sum_{j=1}^{r_1} b_{1j} e_{1j}, \dots, \sum_{j=1}^{r_k} b_{kj} e_{kj} \right) = P_\lambda(b_{ij}) \in Q \otimes_A B.$$

In other words, λ is completely determined by the polynomial $P_\lambda(e^*_{ij})$ with coefficients in Q.

Next, set $B := \mathrm{Sym}^\bullet_A(\underline{M}^\vee)[Y_1, \dots, Y_k]$; the homogeneity condition on λ implies that

$$P_\lambda(Y_i e^*_{ij}) = \lambda_B \left(Y_1 \cdot \sum_{j=1}^{r_1} e_{1j} \otimes e^*_{1j}, \dots, Y_k \cdot \sum_{j=1}^{r_k} e_{kj} \otimes e^*_{kj} \right) = Y_1^{d_1} \cdots Y_k^{d_k} \cdot P(e^*_{ij})$$

which means that $P_\lambda \in Q \otimes_A \mathrm{Sym}^{\underline{d}}_A(\underline{M}^\vee)$. Conversely, it is clear that any element of $Q \otimes_A \mathrm{Sym}^{\underline{d}}_A(\underline{M}^\vee)$ yields a homogeneous law of degree $\underline{d}$, whence the lemma. $\square$

Remark 4.6.50. (i) Lemma 4.6.49 shows especially that, if $M_1, \dots, M_k$ are free A-modules of finite rank, the rule $Q \mapsto \mathrm{Pol}^{\underline{d}}_A(\underline{M}, Q)$ yields a functor with values in essentially small sets, and it is clear that the sum of two homogeneous multipolynomial laws of degree $\underline{d}$ is still a law of the same type; likewise, if we multiply such a law by an element of A, we get a new law of the same type. Up to isomorphism, we may therefore assume that this is a functor $A\text{-}\mathbf{Mod} \to A\text{-}\mathbf{Mod}$.

(ii) Next, set

$$\Gamma^{\underline{d}}_A(\underline{M}) := (\mathrm{Sym}^{\underline{d}}_A \underline{M}^\vee)^\vee.$$

The lemma can be rephrased by saying that, under the stated assumptions, the A-module $\Gamma^{\underline{d}}_A(\underline{M})$ represents the functor

$$A\text{-}\mathbf{Mod} \to A\text{-}\mathbf{Mod} \quad : \quad Q \mapsto \mathrm{Pol}^{\underline{d}}_A(\underline{M}, Q).$$

(iii) The rule $(A, M) \mapsto \Gamma^{\underline{d}}_A(\underline{M})$ is functorial for sequences $\underline{M}$ of free A-modules of finite rank. Indeed, let $\underline{\varphi} : \underline{M} \to \underline{N}$ be a morphism of such sequences. For every degree $\underline{d}$ of length k, the induced map $\mathrm{Pol}^{\underline{d}}_A(\underline{N}, Q) \to \mathrm{Pol}^{\underline{d}}_A(\underline{M}, Q)$ corresponds to a map

$$\mathrm{Sym}^{\underline{d}}_A(\underline{N}^\vee) \otimes_A Q \to \mathrm{Sym}^{\underline{d}}_A(\underline{M}^\vee) \otimes_A Q \tag{4.6.51}$$

that can be worked out as follows. Pick bases $(e_{ij} \mid j = 1, \dots, r_i)$ for N_i and $(e_{ij} \mid j = 1, \dots, s_i)$ for M_i, for every $i = 1, \dots, k$, and denote as usual by $(e^*_{ij} \mid j = 1, \dots, r_i)$, $(f^*_{ij} \mid j = 1, \dots, s_i)$ the respective dual bases. Say that $\lambda : N \rightsquigarrow Q$ is a given homogeneous law of degree $\underline{d}$, and let

$P_\lambda(e^*_{ij}) \in \mathrm{Sym}^d_A(\underline{N}^\vee)\otimes_A Q$ be the corresponding homogeneous polynomial. By construction, the polynomial $P_{\lambda\circ\underline{\varphi}} \in \mathrm{Sym}^d_A(\underline{M}^\vee)\otimes_A Q$ corresponding to $\lambda\circ\varphi$ is calculated as

$$\begin{aligned}
P_{\lambda\circ\underline{\varphi}} &:= (\lambda\circ\underline{\varphi})_{\mathrm{Sym}^\bullet_A(\underline{M}^\vee)}\left(\sum_{j=1}^{s_i} f_{ij}\otimes f^*_{ij} \mid i=1,\dots,k\right) \\
&= \lambda_{\mathrm{Sym}^\bullet_A(\underline{M}^\vee)}\left(\sum_{j=1}^{s_i} \varphi_i(f_{ij})\otimes f^*_{ij} \mid i=1,\dots,k\right) \\
&= \lambda_{\mathrm{Sym}^\bullet_A(\underline{M}^\vee)}\left(\sum_{j=1}^{s_i} e_{ij}\otimes \varphi_i^\vee(e^*_{ij}) \mid i=1,\dots,k\right) \\
&= P_\lambda(\varphi_i^\vee(e^*_{ij})).
\end{aligned}$$

In other words, (4.6.51) equals $\mathrm{Sym}^d_A(\underline{\varphi}^\vee)\otimes_A \mathbf{1}_Q$, so if we let

$$\Gamma^d_A(\underline{\varphi}) := \mathrm{Sym}^d_A(\underline{\varphi}^\vee)^\vee$$

we obtain :

- a functor $\Gamma^d_A(-)$ from the full subcategory $(A\text{-}\mathbf{Mod}^o)^k_{\mathrm{ff}}$ of $(A\text{-}\mathbf{Mod}^o)^k$ consisting of all sequences of free A-modules of finite rank, into the category of A-modules
- for every A-module Q, a well defined isomorphism of functors on this subcategory

$$(Q,\underline{M})\mapsto(\mathrm{Pol}^d_A(\underline{M},Q)\xrightarrow{\sim}\mathrm{Hom}_A(\Gamma^d_A\underline{M},Q)\quad:\quad A\text{-}\mathbf{Mod}\times(A\text{-}\mathbf{Mod}^o)^k_{\mathrm{ff}}\to A\text{-}\mathbf{Mod}$$

The following result extends the above observations to arbitrary sequences $\underline{M}$ of A-modules.

Theorem 4.6.52. *Let $\underline{M}$ be any sequence as in definition* 4.6.46(i). *We have :*

(i) $\mathrm{Pol}^d_A(\underline{M},Q)$ *is an essentially small set, for every A-module Q.*
(ii) *There exists a functor*

$$\Gamma^d_A : (A\text{-}\mathbf{Mod})^k \to A\text{-}\mathbf{Mod} \tag{4.6.53}$$

with a natural isomorphism of A-module valued functors

$$\mathrm{Pol}^d_A(-,Q)\xrightarrow{\sim}\mathrm{Hom}_A(\Gamma^d_A(-),Q)\qquad \textit{for every } A\textit{-module } Q.$$

Proof. Suppose first that all the A-modules $M_1,\dots,M_k$ are finitely presented. Then, for every $i=1,\dots,k$, pick a surjective A-linear map $f_i : L_i\to M_i$, with L_i free of finite rank. Next, let $g_i : L_i\oplus L_i\to M_i$ be the map given by the rule : $(l,l')\mapsto f_i(l-l')$ for every $(l,l')\in L_i\oplus L_i$; since M_i is finitely presented, $\mathrm{Ker}\, g_i$ is finitely generated, for every $i=1,\dots,k$, so we may find a surjective A-linear map $L'_i\to\mathrm{Ker}\, g_i$. By composing with the inclusion $\mathrm{Ker}\, g_i\to L_i\oplus L_i$ and the two projections $L_i\oplus L_i\to L_i$, we deduce a presentation of M_i :

$$\mathrm{Coker}\,(h^1_i,h^2_i : L'_i \rightrightarrows L_i)\xrightarrow{\sim} M_i \tag{4.6.54}$$

such that the induced map

$$L'_i\to L_i\times_{M_i} L_i$$

is surjective. Notice that the latter condition is stable under arbitrary base changes $A\to B$. Set $\underline{L} := (L_1,\dots,L_k)$, $\underline{L}' := (L'_1,\dots,L'_k)$ and $\underline{h}^j := (h^j_1,\dots,h^j_k)$ for $j=1,2$.

Claim 4.6.55. The presentation (4.6.54) induces a presentation of sets :

$$\mathrm{Pol}^d_A(\underline{M},Q)\xrightarrow{\sim}\ \mathrm{Ker}\,(\mathrm{Pol}^d_A(\underline{L},Q)\rightrightarrows\mathrm{Pol}^d_A(\underline{L}',Q))\qquad \text{for every } A\text{-module } Q.$$

Proof of the claim. Indeed, suppose that $\lambda : \underline{L} \rightsquigarrow Q$ is a given homogeneous multipolynomial law of degree $\underline{d}$, such that $\lambda \circ \underline{h}^1 = \lambda \circ \underline{h}^2$ (notation of remark 4.6.47(i)). Since the induced map $L'_i \otimes_A B \to (L_i \otimes_A B) \times_{M \otimes_A B} (L_i \otimes_A B)$ is surjective for every $i = 1, \dots, k$, it follows that λ_B descends (uniquely) to a map $\mu_B : (M_1 \times \cdots \times M_k) \otimes_A B \to Q \otimes_A B$. An easy inspection shows that the resulting rule : $B \mapsto \mu_B$ is a homogeneous multipolynomial law of degree $\underline{d}$. $\diamondsuit$

From claim 4.6.55 and lemma 4.6.49, it follows easily that the theorem holds for $\underline{M}$, with

$$\Gamma^{\underline{d}}_A(\underline{M}) := \mathrm{Coker}\,(\Gamma^{\underline{d}}_A(\underline{h}^1), \Gamma^{\underline{d}}_A(\underline{h}^2) : \Gamma^{\underline{d}}_A(\underline{L}') \rightrightarrows \Gamma^{\underline{d}}_A(\underline{L}))$$

(details left to the reader). Lastly, let $\underline{M}$ be an arbitrary sequence. We may then find a filtered system of sequences $(\underline{M}_\sigma \mid \sigma \in \Sigma)$ consisting of finitely presented A-modules and with A-linear transition maps, whose colimit is isomorphic to $\underline{M}$. Since such colimits are preserved by arbitrary base changes, and commute with the forgetful functor to sets, it follows easily that the system induces an isomorphism of sets :

$$\mathrm{Pol}^{\underline{d}}_A(\underline{M}, Q) \xrightarrow{\sim} \lim_{\sigma \in \Sigma} \mathrm{Pol}^{\underline{d}}_A(\underline{M}_\sigma, Q) \qquad \text{for every } A\text{-module } Q$$

(details left to the reader). In view of the foregoing, it then follows that the theorem holds for $\underline{M}$, with

$$\Gamma^{\underline{d}}_A(\underline{M}) := \operatorname*{colim}_{\sigma \in \Sigma} \Gamma^{\underline{d}}_A(\underline{M}_\sigma).$$

Again, we invite the reader to spell out the details. $\square$

4.6.56. The identity map of $\Gamma^{\underline{d}}_A \underline{M}$ corresponds to a homogeneous multipolynomial law

$$\boldsymbol{\lambda}^{\underline{d}}_{\underline{M}} : \underline{M} \rightsquigarrow \Gamma^{\underline{d}}_A \underline{M}$$

such that every other law $\underline{M} \rightsquigarrow Q$ homogeneous of the same degree, factors uniquely through $\boldsymbol{\lambda}^{\underline{d}}_{\underline{M}}$ and an A-linear map $\Gamma^{\underline{d}}_A \underline{M} \to Q$ (in the sense explained in remark 4.6.47(i)). Especially, if $\underline{M}' \in \mathrm{Ob}((A\text{-}\mathbf{Mod})^{k'})$ is another sequence, and $\underline{d}' \in \mathbb{N}^{\oplus k'}$ any sequence of integers, the tensor product $\boldsymbol{\lambda}^{\underline{d}}_{\underline{M}} \otimes \boldsymbol{\lambda}^{\underline{d}'}_{\underline{M}'}$ (remark 4.6.47(ii)) factors uniquely through $\boldsymbol{\lambda}^{\underline{d},\underline{d}'}_{\underline{M},\underline{M}'}$ and an A-linear map

$$\Gamma^{\underline{d},\underline{d}'}_A(\underline{M}, \underline{M}') \to \Gamma^{\underline{d}}_A(\underline{M}) \otimes_A \Gamma^{\underline{d}'}_A(\underline{M}'). \tag{4.6.57}$$

Furthermore, notice that, in the situation of remark 4.6.47(iii), the mapping (4.6.48) is clearly A-linear; hence, we get an induced B-linear map

$$B \otimes_A \Gamma^{\underline{d}}_A(\underline{M}) \to \Gamma^{\underline{d}}_B(B \otimes_A \underline{M}) \tag{4.6.58}$$

for every $\underline{M}$ and $\underline{d}$ as above, and every A-algebra B.

Corollary 4.6.59. *With the notation of* (4.6.56)*, we have :*

(i) *The functor* (4.6.53) *commutes with all filtered colimits.*
(ii) *The map* (4.6.57) *is an isomorphism of A-modules, for every $\underline{M}$, $\underline{M}'$, $\underline{d}$ and $\underline{d}'$.*
(iii) *For every A-algebra B, the map* (4.6.58) *is an isomorphism of B-modules.*
(iv) *If the A-modules $M_1, \dots, M_k$ are all free (resp. finitely presented, resp. finitely generated, resp. flat), then the same holds for $\Gamma^{\underline{d}}_A(\underline{M})$.*

Proof. (i): It suffices to check that the natural map

$$\mathrm{Pol}^{\underline{d}}_A(\operatorname*{colim}_{\sigma \in \Sigma} \underline{M}_\sigma, Q) \to \lim_{\sigma \in \Sigma} \mathrm{Pol}^{\underline{d}}_A(\underline{M}_\sigma, Q)$$

is an isomorphism, for every filtered system of A-modules $(\underline{M}_\sigma \mid \sigma \in \Sigma)$ and every A-module Q; the latter assertion is obvious (details left to the reader).

(iv): The assertion concerning free modules of finite rank, and finitely generated or finitely presented modules follows directly from the explicit construction in lemma 4.6.49 and theorem 4.6.52. The assertion for flat modules follows from the assertion for free modules of finite rank and from (i), by means of Lazard's theorem [57, Ch.I, Th.1.2]. Moreover, from remark 4.6.50(iii), we see that, if $\varphi : \underline{M} \to \underline{N}$ is a morphism of sequences of free A-modules of finite rank, such that φ_i is a split injection for every $i = 1, \dots, n$, then $\Gamma^d_A(\varphi)$ is also a split injective map of free A-modules; from this, and from (i), it follows easily that Γ^d_A transforms sequences of free A-modules (of possibly infinite rank) into free A-modules : details left to the reader.

(iii): In light of (i), it is easily seen that the natural transformation (4.6.57) commutes with arbitrary filtered colimits of A-modules, hence we are reduced to the case where $\underline{M}$ is a sequence of finitely presented A-modules. Next, claim 4.6.55 shows that a presentation (4.6.54) yields a commutative diagram

$$\begin{tikzcd}
\Gamma^d_A(\underline{L}') \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \Gamma^d_A(\underline{L}) \arrow[r] \arrow[d] & \Gamma^d_A(\underline{M}) \arrow[r] \arrow[d] & 0 \\
\Gamma^d_B(B \otimes_A \underline{L}') \arrow[r, shift left] \arrow[r, shift right] & \Gamma^d_B(B \otimes_A \underline{L}) \arrow[r] & \Gamma^d_B(B \otimes_A \underline{M}) \arrow[r] & 0
\end{tikzcd}$$

with exact rows. Therefore, we are further reduced to the case where $\underline{M}$ is a sequence of free A-modules of finite rank, in which case the assertion follows by a simple inspection of the proof of lemma 4.6.49.

(ii): Let $t : \Gamma^{d,d'}_A(\underline{M}, \underline{M}') \to Q$ be an A-linear map. To t we associate an A-linear map

$$t^* : \Gamma^d_A(\underline{M}) \otimes_A \Gamma^{d'}_A(\underline{M}') \to Q$$

as follows. Set $\tau := t \circ \boldsymbol{\lambda}^{d,d'}_{\underline{M},\underline{M}'} : (\underline{M}, \underline{M}') \rightsquigarrow Q$ (notation of remark 4.6.47(i)). To every A-algebra B, the law τ associates the mapping

$$B \otimes_A \underline{M} \to \mathrm{Pol}^{d'}_B(B \otimes_A \underline{M}', B \otimes_A Q) \qquad x \mapsto \tau_{B,x}$$

where $(\tau_{B,x})_C(y) := \tau_C(1 \otimes x, y)$ for every $x \in B \otimes_A \underline{M}$, every B-algebra C and every $y \in C \otimes_A \underline{M}'$. In light of (iii), the latter is the same as a mapping

$$B \otimes_A \underline{M} \to \mathrm{Hom}_B(B \otimes_A \Gamma^{d'}_A(\underline{M}'), B \otimes_A Q) \qquad x \mapsto t_{B,x}$$

and notice that, for every $x \in B \otimes_A \underline{M}$, the B-linear map $t_{B,x}$ is characterized by the identity

$$t_{B,x}(\boldsymbol{\lambda}^{d'}_{\underline{M}',B}(y)) = (\tau_{B,x})_B(y) \qquad \text{for every } y \in B \otimes_A \underline{M}'. \tag{4.6.60}$$

We deduce a compatible system of mappings

$$B \otimes_A \Gamma^{d'}_A(\underline{M}') \to \mathrm{Pol}^d_B(B \otimes_A \underline{M}, B \otimes_A Q) \quad : \quad y \mapsto \tau^*_{B,y} \qquad \text{for every } A\text{-algebra } B$$

where $(\tau^*_{B,y})_C(x) := t_{C,x}(1 \otimes y)$ for every B-algebra C, every $x \in C \otimes_A \underline{M}$, and every $y \in B \otimes_A \Gamma^{d'}_A(\underline{M})$. The C-linearity of $t_{C,x}$ implies that these latter mappings are B-linear (for every A-algebra B), and then they are the same as a compatible system of B-linear maps

$$B \otimes_A \Gamma^{d'}_A(\underline{M}') \to \mathrm{Hom}_B(B \otimes_A \Gamma^d_A(\underline{M}), B \otimes_A Q) \quad : \quad y \mapsto t^*_{B,y} \tag{4.6.61}$$

for every A-algebra B. Notice that, for every $y \in B \otimes_A \Gamma^{d'}_A(\underline{M}')$, the B-linear map $t^*_{B,y}$ is characterized by the identity :

$$t^*_{B,y}(\boldsymbol{\lambda}^d_{\underline{M},B}(x)) = (\tau^*_{B,y})_B(x) \qquad \text{for every } x \in B \otimes_A \underline{M}. \tag{4.6.62}$$

Obviously, the datum of a compatible system (4.6.61) is the same as that of a map t^* as sought.

Claim 4.6.63. $t^* \circ (4.6.57) = t$.

Proof of the claim. It suffices to show that $t^* \circ$ (4.6.57) $\circ \boldsymbol{\lambda}^{\underline{d},\underline{d}'}_{\underline{M},\underline{M}'} = \tau$. However, recall that (4.6.57)$\circ \boldsymbol{\lambda}^{\underline{d},\underline{d}'}_{\underline{M},\underline{M}'} = \boldsymbol{\lambda}^{\underline{d}}_{\underline{M}} \otimes \boldsymbol{\lambda}^{\underline{d}'}_{\underline{M}'}$, so we are reduced to checking that $t^* \circ (\boldsymbol{\lambda}^{\underline{d}}_{\underline{M}} \otimes \boldsymbol{\lambda}^{\underline{d}'}_{\underline{M}'}) = \tau$. Hence, let B be any A-algebra, and $m \in B \otimes_A \underline{M}$, $m' \in B \otimes_A \underline{M}'$ any two elements. To ease notation, set $x := \boldsymbol{\lambda}^{\underline{d}}_{\underline{M}}(m)$ and $y := \boldsymbol{\lambda}^{\underline{d}'}_{\underline{M}'}(m')$. We compute :

$$\begin{aligned}(t^* \circ (\boldsymbol{\lambda}^{\underline{d}}_{\underline{M}} \otimes \boldsymbol{\lambda}^{\underline{d}'}_{\underline{M}'}))_B(m, m') &= (B \otimes_A t^*)(x \otimes y) \\ &= t^*_{B,y}(x) \\ &= (\tau^*_{B,y})_B(m) && \text{(by (4.6.62))} \\ &= t_{B,m}(y) \\ &= (\tau_{B,m})_B(m') && \text{(by (4.6.60))} \\ &= \tau_B(m, m')\end{aligned}$$

as claimed. $\diamondsuit$

Especially, if we let t be the identity map of $\Gamma^{\underline{d},\underline{d}'}_A(\underline{M}, \underline{M}')$, claim 4.6.63 shows that the resulting t^* is a left inverse for (4.6.57). Now, if all the modules $M_1, \dots, M'_{k'}$ are free of finite rank, then lemma 4.6.49 shows that both of the A-modules appearing in (4.6.57) are free and have the same rank, so the assertion follows, in this case.

Next, suppose that the modules of the two sequences are finitely presented; in this case, we argue by descending induction on the number f of free A-modules appearing in the two sequences. If $f = k + k'$, the assertion has just been shown. Suppose that $f < k + k'$, and the assertion is already known for all pairs of sequences containing at least $f + 1$ free modules. After permutation of the sequences, we may assume that M_i is not free, for some index $i \le k$. Then, pick a presentation of M_i as in (4.6.54), and let $\underline{N}'$ (resp. $\underline{N}''$) be the sequences obtained from $\underline{M}$ by replacing M_i with L_i (resp. with L'_i); by naturality of (4.6.57), the maps h^1_i and h^2_i induce a commutative diagram with exact rows :

$$\begin{array}{ccccccc}\Gamma^{\underline{d}}_A(\underline{N}'', \underline{M}') & \rightrightarrows & \Gamma^{\underline{d}}_A(\underline{N}', \underline{M}') & \longrightarrow & \Gamma^{\underline{d}}_A(\underline{M}, \underline{M}') & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \Gamma^{\underline{d}}_A(\underline{N}'') \otimes_A \Gamma^{\underline{d}}_A(\underline{M}') & \rightrightarrows & \Gamma^{\underline{d}}_A(\underline{N}') \otimes_A \Gamma^{\underline{d}}_A(\underline{M}') & \longrightarrow & \Gamma^{\underline{d}}_A(\underline{M}) \otimes_A \Gamma^{\underline{d}}_A(\underline{M}') & \longrightarrow & 0\end{array}$$

so it suffices to prove the assertion for the pair of sequences (N', M') and (N'', M'); but the latter contain $f + 1$ free A-modules, so we conclude by induction.

Lastly, if $\underline{M}$ and $\underline{N}$ are arbitrary sequences, we may present them as filtered colimits of sequences of finitely presented A-modules; since tensor products commute with such colimits, we are done, by the foregoing case (details left to the reader). $\square$

Remark 4.6.64. (i) Let $\underline{M}$ and $\underline{d}$ be as in (4.6.56), suppose that all the A-modules M_i are free of finite rank, and set $S(\underline{M}) := \mathrm{Sym}^{\bullet}_A(\underline{M}^\vee)$. Let Q be another A-module, and $\lambda : \underline{M} \rightsquigarrow Q$ a homogeneous multipolinomial law of degree $\underline{d}$. From lemma 4.6.49 we may extract the following calculation for the A-linear map $\lambda^\dagger : \Gamma^{\underline{d}}_A(\underline{M}) \to Q$ corresponding to λ. Set

$$P_\lambda := \lambda_{S(\underline{M})}(\mathbf{1}_{M_1}, \dots, \mathbf{1}_{M_k}) \in \Gamma^{\underline{d}}_A(\underline{M})^\vee \otimes_A Q.$$

Then $\lambda^\dagger$ is the image of P_λ under the natural identification $\Gamma^{\underline{d}}_A(\underline{M})^\vee \otimes_A Q \xrightarrow{\sim} \mathrm{Hom}_A(\Gamma^{\underline{d}}_A(\underline{M}), Q)$.

(ii) Let $\underline{M}$, $\underline{M}'$, $\underline{d}$ and $\underline{d}'$ be as in (4.6.56), suppose that all the A-modules M_i and M'_i are free of finite rank, and to ease notation let

$$\boldsymbol{\lambda} := \boldsymbol{\lambda}^{\underline{d}}_{\underline{M}} \qquad \boldsymbol{\lambda}' := \boldsymbol{\lambda}^{\underline{d}'}_{\underline{M}'} \qquad \lambda := \boldsymbol{\lambda} \otimes \boldsymbol{\lambda}'.$$

By (i), we compute

$$P_\lambda = \boldsymbol{\lambda}_{S(\underline{M},\underline{M}')}(\mathbf{1}_{M_1},\dots,\mathbf{1}_{M_k}) \otimes \boldsymbol{\lambda}'_{S(\underline{M},\underline{M}')}(\mathbf{1}_{M'_1},\dots,\mathbf{1}_{M'_{k'}}).$$

However, $\boldsymbol{\lambda}_{S(\underline{M},\underline{M}')}(\mathbf{1}_{M_1},\dots,\mathbf{1}_{M_k})$ is the image of $P_{\boldsymbol{\lambda}}$ under the A-linear induced by the A-algebra homomorphism $S(\underline{M}) \to S(\underline{M},\underline{M}') = S(\underline{M}) \otimes_A S(\underline{M}')$ given by the rule $s \mapsto s \otimes 1$, for every $s \in S(\underline{M})$. Likewise we determine $\boldsymbol{\lambda}'_{S(\underline{M},\underline{M}')}(\mathbf{1}_{M'_1},\dots,\mathbf{1}_{M'_{k'}})$, and we conclude that $P_\lambda = P_{\boldsymbol{\lambda}} \otimes P_{\boldsymbol{\lambda}'}$, whence $\lambda^\dagger = \boldsymbol{\lambda}^\dagger \otimes \boldsymbol{\lambda}^\dagger$, under the natural identification

$$\mathrm{End}_A(\Gamma_A^{\underline{d},\underline{d}'}(\underline{M},\underline{M}')) \xrightarrow{\sim} \mathrm{End}_A(\Gamma_A^{\underline{d}}(\underline{M})) \otimes_A \mathrm{End}_A(\Gamma_A^{\underline{d}'}(\underline{M}')).$$

But by definition, $(\boldsymbol{\lambda}_{\underline{M}}^{\underline{d}})^\dagger$ is the identity automorphism of $\Gamma_A^{\underline{d}}(\underline{M})$, for every sequence $\underline{M}$, and every degree $\underline{d}$. We see therefore that :

$$(\boldsymbol{\lambda}_{\underline{M}}^{\underline{d}} \otimes \boldsymbol{\lambda}_{\underline{M}'}^{\underline{d}'})^\dagger = (\boldsymbol{\lambda}_{\underline{M},\underline{M}'}^{\underline{d},\underline{d}'})^\dagger$$

which translates as saying that the A-linear map (4.6.57) is none else than the transpose of the natural isomorphism

$$\mathrm{Sym}_A^{\underline{d}}(\underline{M}^\vee) \otimes_A \mathrm{Sym}_A^{\underline{d}'}(\underline{M}'^\vee) \xrightarrow{\sim} \mathrm{Sym}_A^{\underline{d},\underline{d}'}(\underline{M}^\vee,\underline{M}'^\vee).$$

This gives an alternative way to prove corollary 4.6.59(ii), in the case of free A-modules.

4.6.65. Let $\underline{M}$ and $\underline{M}'$ be two sequences of A-modules, of the same length k, and $\underline{e}$, $\underline{e}'$ two degrees of length k. If $\lambda : (\underline{M},\underline{M}') \rightsquigarrow Q$ is any homogeneous law of degree $(\underline{e},\underline{e}')$, then we can also regard λ as a homogeneous multipolynomial law of degree $\underline{e}+\underline{e}'$ on the sequence $\underline{M}\oplus\underline{M}'$ of length k. In this way, we obtain a natural A-linear mapping

$$\mathrm{Pol}_A^{\underline{e},\underline{e}'}((\underline{M},\underline{M}'),Q) \to \mathrm{Pol}_A^{\underline{e}+\underline{e}'}(\underline{M}\oplus\underline{M}',Q) \qquad \text{for every } A\text{-module } Q \tag{4.6.66}$$

which, by transposition, corresponds to a unique A-linear map

$$\Gamma_A^{\underline{e}+\underline{e}'}(\underline{M}\oplus\underline{M}') \to \Gamma_A^{\underline{e},\underline{e}'}(\underline{M},\underline{M}'). \tag{4.6.67}$$

Fix $\underline{d} \in \mathbb{N}^{\oplus k}$; in view of corollary 4.6.59(ii), we see that the sum of the maps (4.6.67) for all pairs $(\underline{e},\underline{e}')$ with $\underline{e}+\underline{e}' = \underline{d}$, is a well defined A-linear map

$$\Delta_{\underline{M},\underline{M}'}^{\underline{d}} := \bigoplus_{\underline{e}+\underline{e}'=\underline{d}} \Delta_{\underline{M},\underline{M}'}^{\underline{e},\underline{e}'} : \Gamma_A^{\underline{d}}(\underline{M}\oplus\underline{M}') \to \bigoplus_{\underline{e}+\underline{e}'=\underline{d}} \Gamma_A^{\underline{e}}(\underline{M}) \otimes_A \Gamma_A^{\underline{e}'}(\underline{M}').$$

Proposition 4.6.68. *The map $\Delta_{\underline{M},\underline{M}'}^{\underline{d}}$ is an isomorphism, for every $\underline{M}$, $\underline{M}'$, and every degree $\underline{d}$.*

Proof. Arguing as in the proof of corollary 4.6.59(ii), we are easily reduced to the case where $\underline{M}$ and $\underline{M}'$ are sequences of free A-modules of finite rank. In this case, (4.6.67) can be computed as in remark 4.6.64(i); namely, it corresponds to the element

$$P \in \mathrm{Sym}_A^{\underline{e}+\underline{e}'}((\underline{M}\oplus\underline{M}')^\vee) \otimes_A \mathrm{Sym}_A^{\underline{e},\underline{e}'}(\underline{M}^\vee,\underline{M}'^\vee)^\vee$$

determined as follows. Take $Q := \Gamma_A^{\underline{e},\underline{e}'}(\underline{M},\underline{M}')$, and let $\lambda : \underline{M}\oplus\underline{M}' \rightsquigarrow \Gamma_A^{\underline{e},\underline{e}'}(\underline{M},\underline{M}')$ be the image of $\boldsymbol{\lambda}_{\underline{M},\underline{M}'}^{\underline{e},\underline{e}'}$ under (4.6.66); then

$$P = \lambda_{S(\underline{M}\oplus\underline{M}')}(\mathbf{1}_{M_1\oplus M'_1},\dots,\mathbf{1}_{M_k\oplus M'_k}).$$

Thus, we come down to evaluating the image of $\mathbf{1}_{M_i\oplus M'_i}$ under the natural identification

$$(M_i\oplus M'_i)\otimes_A S(M_i\oplus M'_i) \xrightarrow{\sim} (M_i \otimes_A S(M_i\oplus M'_i)) \times (M'_i \otimes_A S(M_i\oplus M'_i)).$$

for every $i = 1,\dots,k$. We easily find that $\mathbf{1}_{M_i\oplus M'_i} \mapsto (p_i,p'_i)$ under this identification, where $p_i : M_i\oplus M'_i \to M_i$ and $p'_i : M_i\oplus M'_i \to M'_i$ are the natural projections; so finally

$$P = (\boldsymbol{\lambda}_{\underline{M},\underline{M}'}^{\underline{e},\underline{e}'})_{S(\underline{M}\oplus\underline{M}')}(p_1,\dots,p_k,p'_1,\dots,p'_k). \tag{4.6.69}$$

Pick a basis $(e_{i1},\dots,e_{ir_i})$ (resp. $(e'_{i1},\dots,e'_{ir'_i})$) for each M_i (resp. M'_i), let $(e^*_{i1},\dots,e^*_{ir_i})$ (resp. $(e'^*_{i1},\dots,e'^*_{ir'_i})$)) be the dual basis, and for every $=1,\dots,k$ and every $j=1,\dots,r_i$ (resp. $j=1,\dots,r'_i$) denote by ε_{ij} (resp. ε'_{ij}) the element of $(M_i\oplus M'_i)^\vee$ that agrees with e^*_{ij} on M_i and that vanishes on M'_i (resp. that agrees with e'^*_{ij} on M'_i and vanishes on M_i). Notice that

$$p_i=\sum_{j=1}^{r_i}\varepsilon^*_{ij}\otimes e_{ij}\qquad p'_i=\sum_{j=1}^{r'_i}\varepsilon'^*_{ij}\otimes e'_{ij}\qquad\text{for every } i=1,\dots,k.$$

With this notation, it follows that the right hand-side of (4.6.69) is the image of

$$(\boldsymbol{\lambda}^{\underline{e},\underline{e}'}_{\underline{M},\underline{M}'})_{S(\underline{M},\underline{M}')}(\mathbf{1}_{M_1},\dots,\mathbf{1}_{M'_k})=\mathbf{1}_{\mathrm{Sym}^{\underline{e},\underline{e}'}_A(\underline{M},\underline{M}')}$$

under the natural map

$$S(\underline{M},\underline{M}')\to S(\underline{M}\oplus\underline{M}')\quad:\quad e^*_{ij}\mapsto\varepsilon^*_{ij}\quad e'^*_{ij}\mapsto\varepsilon'^*_{ij}. \tag{4.6.70}$$

In other words, the sought P is none else than the restriction of (4.6.70) to the direct factor $\mathrm{Sym}^{\underline{e},\underline{e}'}_A(\underline{M}^\vee,\underline{M}'^\vee)$, and this restriction is an isomorphism onto a direct factor of the A-module $\mathrm{Sym}^{\underline{e}+\underline{e}'}_A(\underline{M}\oplus\underline{M}')$. By transposition, we see that (4.6.67) is identified with the projection onto a natural direct factor of $\Gamma^{\underline{e}+\underline{e}'}_A(\underline{M}\oplus\underline{M}')$, and a simple inspection shows that the sum of all these projections is the identity map of the latter A-module, whence the proposition. $\square$

Remark 4.6.71. (i) Let $\underline{M},\underline{M}'$, Q and λ be as in (4.6.65). The image of λ under (4.6.66) is characterized as the unique law $\varphi:\underline{M}\oplus\underline{M}'\rightsquigarrow Q$ of degree $\underline{e}+\underline{e}'$ such that $\varphi(m,m')=\lambda_B(m,m')$ for every A-algebra B and every $(m,m')\in B\otimes_A(\underline{M}\oplus\underline{M}')$. In turns, φ corresponds to the unique A-linear map $f:\Gamma^{\underline{e}+\underline{e}'}_A(\underline{M}\oplus\underline{M}')\to Q$ such that $(B\otimes_A f)(\boldsymbol{\lambda}^{\underline{e}+\underline{e}'}_{\underline{M}\oplus\underline{M}'})_B(m,m')=\lambda_B(m,m')$ for every B and (m,m') as above. Especially, if we take $\lambda:=\lambda^{\underline{e},\underline{e}'}_{\underline{M},\underline{M}'}$, we see that (4.6.67) is characterized as the unique A-linear map such that

$$(\boldsymbol{\lambda}^{\underline{e}+\underline{e}'}_{\underline{M}\oplus\underline{M}'})_B(m,m')\mapsto(\boldsymbol{\lambda}^{\underline{e},\underline{e}'}_{\underline{M},\underline{M}'})_B(m,m')$$

for every B and (m,m') as above. Lastly, it follows that $\Delta^{\underline{e},\underline{e}'}_{\underline{M},\underline{M}'}$ is characterized as the unique A-linear map such that

$$(\boldsymbol{\lambda}^{\underline{e}+\underline{e}'}_{\underline{M}\oplus\underline{M}'})_B(m,m')\mapsto(\boldsymbol{\lambda}^{\underline{e}}_{\underline{M}})_B(m)\otimes(\lambda^{\underline{e}'}_{\underline{M}'})_B(m')$$

for every A-algebra B and every $(m,m')\in B\otimes_A(\underline{M}\oplus\underline{M}')$.

(ii) In the same vein, let $f:\underline{M}\to\underline{N}$ be any morphism in $(A\text{-}\mathbf{Mod})^a$, and $\underline{d}\in\mathbb{N}^{\oplus k}$ any degree. Then we see that the induced map

$$\Gamma^{\underline{d}}_A(f):\Gamma^{\underline{d}}_A(\underline{M})\to\Gamma^{\underline{d}}_A(\underline{N})$$

is characterized as the unique A-linear map such that

$$(\boldsymbol{\lambda}^{\underline{d}}_{\underline{M}})_B(m)\mapsto(\boldsymbol{\lambda}^{\underline{d}}_{\underline{N}})_B(f(m))$$

for every A-algebra B, and every $m\in B\otimes_A\underline{M}$ (details left to the reader).

4.6.72. Let $\underline{M}$, $\underline{M}'$, $\underline{M}''$ be three sequences of A-modules of length k, and $\underline{d}$, $\underline{d}'$, $\underline{d}''$ three degrees, also of lengths k. We obtain a diagram of A-linear maps :

$$\begin{tikzcd}
\Gamma^{\underline{e}+\underline{e}'+\underline{e}''}_A(\underline{M}\oplus\underline{M}'\oplus\underline{M}'') \arrow[r, "\Delta^{\underline{e},\underline{e}'+\underline{e}''}_{\underline{M},\underline{M}'\oplus\underline{M}''}"] \arrow[d, "\Delta^{\underline{e}+\underline{e}',\underline{e}''}_{\underline{M}\oplus\underline{M}',\underline{M}''}"'] & \Gamma^{\underline{e}}_A(\underline{M})\otimes_A\Gamma^{\underline{e}'+\underline{e}''}_A(\underline{M}'\oplus\underline{M}'') \arrow[d, "\Gamma^{\underline{e}}_A(\underline{M})\otimes_A\Delta^{\underline{e}',\underline{e}''}_{\underline{M}',\underline{M}''}"] \\
\Gamma^{\underline{e}+\underline{e}'}_A(\underline{M}\oplus\underline{M}')\otimes_A\Gamma^{\underline{e}''}_A(\underline{M}'') \arrow[r, "\Delta^{\underline{e},\underline{e}'}_{\underline{M},\underline{M}'}\otimes_A\Gamma^{\underline{e}''}_A(\underline{M}'')"] & \Gamma^{\underline{e}}_A(\underline{M})\otimes_A\Gamma^{\underline{e}'}_A(\underline{M}')\otimes_A\Gamma^{\underline{e}''}_A(\underline{M}'').
\end{tikzcd}$$

Proposition 4.6.73. *The diagram of* (4.6.72) *commutes.*

Proof. By remark 4.6.71(i), the map $\Delta^{\underline{e},\underline{e}'+\underline{e}''}_{\underline{M},\underline{M}'\oplus\underline{M}''}$ is characterized as the unique one such that

$$(\boldsymbol{\lambda}^{\underline{e}+\underline{e}'+\underline{e}''}_{\underline{M}\oplus\underline{M}'})_B(m,m',m'') \mapsto (\boldsymbol{\lambda}^{\underline{e}}_{\underline{M}})_B(m)\otimes(\boldsymbol{\lambda}^{\underline{e}'+\underline{e}''}_{\underline{M}\oplus\underline{M}'})_B(m',m'')$$

for every A-algebra B and every $(m,m',m'')\in B\otimes_A(\underline{M}\oplus\underline{M}'\oplus\underline{M}'')$, and $\Delta^{\underline{e}',\underline{e}''}_{\underline{M}',\underline{M}''}$ is the unique A-linear map such that

$$(\boldsymbol{\lambda}^{\underline{e}'+\underline{e}''}_{\underline{M}'\oplus\underline{M}''})_B(m',m'') \mapsto (\boldsymbol{\lambda}^{\underline{e}'}_{\underline{M}'})_B(m')\otimes(\lambda^{\underline{e}''}_{\underline{M}''})_B(m'')$$

for every A-algebra B and every $(m',m'')\in B\otimes_A(\underline{M}'\oplus\underline{M}'')$. Therefore, the composition of the top horizontal and right vertical arrows in the diagram is the unique A-linear map such that

$$(\boldsymbol{\lambda}^{\underline{e}+\underline{e}'+\underline{e}''}_{\underline{M}\oplus\underline{M}'})_B(m,m',m'') \mapsto (\boldsymbol{\lambda}^{\underline{e}}_{\underline{M}})_B(m)\otimes(\boldsymbol{\lambda}^{\underline{e}'}_{\underline{M}'})_B(m')\otimes(\lambda^{\underline{e}''}_{\underline{M}''})_B(m'')$$

for every B and (m,m',m'') as above. The same calculation can be carried out for the composition of the other two arrows, and clearly one obtains the same result. $\square$

4.6.74. Let $\underline{M}$ be any sequence of A-modules of length k, and denote by $s:\underline{M}\oplus\underline{M}\to\underline{M}$ the addition map : $(m,m')\mapsto m+m'$ for every $m,m'\in\underline{M}$. The map s induces A-linear maps

$$\mu^{\underline{e},\underline{e}'}_{\underline{M}}:\Gamma^{\underline{e}}_A(\underline{M})\otimes_A\Gamma^{\underline{e}'}(\underline{M})\xrightarrow{\delta^{\underline{e},\underline{e}'}}\Gamma^{\underline{e}+\underline{e}'}_A(\underline{M}\oplus\underline{M})\xrightarrow{\Gamma^{\underline{e}+\underline{e}'}_A(s)}\Gamma^{\underline{d}}_A(\underline{M})\qquad\text{for every }\underline{e},\underline{e}'\in\mathbb{N}^{\oplus k}$$

where $\delta^{\underline{e},\underline{e}'}$ is the natural inclusion map obtained by restricting the inverse of $\Delta^{\underline{e}+\underline{e}'}_{\underline{M},\underline{M}}$ to the direct factor $\Gamma^{\underline{e}}_A(\underline{M})\otimes_A\Gamma^{\underline{e}'}(\underline{M})$. In light of remark 4.6.71(i,ii), the map $\mu^{\underline{e},\underline{e}'}_{\underline{M}}$ is characterized as the unique A-linear map such that

$$(\boldsymbol{\lambda}^{\underline{e}}_{\underline{M}})_B(m)\otimes(\boldsymbol{\lambda}^{\underline{e}'}_{\underline{M}})_B(m')\mapsto(\boldsymbol{\lambda}^{\underline{e}+\underline{e}'}_{\underline{M}})_B(m+m') \tag{4.6.75}$$

for every A-algebra B and every $(m,m')\in B\otimes_A(\underline{M}\oplus\underline{M})$. Using this characterization, it is easily seen that the diagram

$$\begin{tikzcd}
\Gamma^{\underline{e}}_A(\underline{M})\otimes_A\Gamma^{\underline{e}'}_A(\underline{M})\otimes_A\Gamma^{\underline{e}''}_A(\underline{M}) \arrow[r, "\mu^{\underline{e},\underline{e}'}_{\underline{M}}\otimes_A\Gamma^{\underline{e}''}_A(\underline{M})"] \arrow[d, "\Gamma^{\underline{e}}_A(\underline{M})\otimes_A\mu^{\underline{e}',\underline{e}''}_{\underline{M}}"'] & \Gamma^{\underline{e}+\underline{e}'}_A(\underline{M})\otimes_A\Gamma^{\underline{e}''}_A(\underline{M}) \arrow[d, "\mu^{\underline{e}+\underline{e}',\underline{e}''}_{\underline{M}}"] \\
\Gamma^{\underline{e}}_A(\underline{M})\otimes_A\Gamma^{\underline{e}'+\underline{e}''}_A(\underline{M}) \arrow[r, "\mu^{\underline{e},\underline{e}'+\underline{e}''}_{\underline{M}}"] & \Gamma^{\underline{e}+\underline{e}'+\underline{e}''}_A(\underline{M})
\end{tikzcd}$$

commutes, for every $\underline{e},\underline{e}',\underline{e}''\in\mathbb{N}^{\oplus k}$. The foregoing shows that the maps $\mu^{\bullet\bullet}_{\underline{M}}$ define on

$$\Gamma^{\bullet}_A(\underline{M}):=\bigoplus_{\underline{d}\in\mathbb{N}^{\oplus k}}\Gamma^{\underline{d}}_A(\underline{M}).$$

a natural structure of associative $\mathbb{N}^{\oplus k}$-graded A-algebra. Moreover, (4.6.75) also easily implies that this algebra is commutative. We call $\Gamma^{\bullet}_A(\underline{M})$ the *divided power envelope* of $\underline{M}$. It is customary to use the notation

$$m^{[\underline{d}]}:=(\boldsymbol{\lambda}^{\underline{d}}_{\underline{M}})_B(m)\in B\otimes_A\underline{M}$$

for every A-algebra B, every $m\in B\otimes_A\underline{M}$, and every $\underline{d}\in\mathbb{N}^{\oplus k}$. Then, proposition 4.6.68 and a simple inspection of the definitions yield the identity

$$(a+b)^{[\underline{d}]}=\sum_{\underline{e}+\underline{e}'=\underline{d}}a^{[\underline{e}]}\cdot b^{[\underline{e}']} \tag{4.6.76}$$

for every A-algebra B and every $a, b \in B \otimes_A \underline{M}$. Clearly, the above construction yields a well defined functor

$$(A\text{-}\mathbf{Mod})^k \to A\text{-}\mathbf{Alg} \quad : \quad M \mapsto \Gamma^\bullet_A(\underline{M}) \tag{4.6.77}$$

and notice that, for every morphism $f : \underline{M} \to \underline{N}$ of sequences, we have

$$(\Gamma^\bullet_A f)(a^{[\underline{d}]}) = (fa)^{[\underline{d}]} = (\Gamma^\bullet_A f(a))^{[\underline{d}]}$$

for every degree $\underline{d}$, every A-algebra B, and every $a \in B \otimes_A \underline{M}$.

Theorem 4.6.78. *With the notation of* (4.6.74)*, we have :*

(i) *For every A-algebra B, there exists a natural isomorphism of $\mathbb{N}^{\oplus k}$-graded algebras :*

$$B \otimes_A \Gamma^\bullet_A(\underline{M}) \xrightarrow{\sim} \Gamma^\bullet_B(B \otimes_A M).$$

(ii) *The functor* (4.6.77) *commutes with filtered colimits.*

(iii) *For any two sequences of A-modules $\underline{M}$ and $\underline{M}'$ of the same length, there exists a natural isomorphism of graded A-algebras :*

$$\Gamma^\bullet_A(\underline{M} \oplus \underline{M}') \xrightarrow{\sim} \Gamma^\bullet_A(\underline{M}) \otimes_A \Gamma^\bullet_A(\underline{M}')$$

(with the $\mathbb{N}^{\oplus k}$-grading of the tensor product defined in the obvious way).

Proof. (i) and (ii) follow immediately from corollary 4.6.59(i,iii), and (iii) follows directly from proposition 4.6.68. □

4.6.79. Consider the special case of a sequence $\underline{M}$ consisting of a single A-module M. From (4.6.76), by evaluating the expression $(2m)^{[d]} = 2^d \cdot m^{[d]}$, a simple induction on d shows that

$$d! \cdot m^{[d]} = m^d \qquad \text{in } \Gamma^\bullet_A(M), \text{ for every } m \in M. \tag{4.6.80}$$

Corollary 4.6.81. *In the situation of* (4.6.79)*, we have*

$$m^{[i]} \cdot m^{[j]} = \binom{i+j}{i} \cdot m^{[i+j]} \qquad \textit{in } \Gamma^\bullet_A(M)\textit{, for every } m \in M \textit{ and every } i, j \in \mathbb{N}.$$

Proof. By considering the A-linear map $f : A \to M$ such that $f(1) = m$, we reduce to the case where $M = A$. By considering the unique ring homomorphism $\mathbb{Z} \to A$ we reduce further – in light of theorem 4.6.78(i) – to the case where $A = \mathbb{Z}$ and $M = \mathbb{Z}$. In this case, $\Gamma_\mathbb{Z}(\mathbb{Z})$ is a torsion-free $\mathbb{Z}$-module, by lemma 4.6.49, so the identity follows easily from (4.6.80). □

4.7. **Regular rings.** In section 9.6, we shall need a criterion for the regularity of suitable extensions of regular local rings. Such a criterion is stated incorrectly in [30, Ch.0, Th.22.5.4]; our first task is to supply a corrected version of *loc.cit.*

4.7.1. Consider a local ring A, with maximal ideal $\mathfrak{m}_A$, and residue field $k_A := A/\mathfrak{m}_A$ of characteristic $p > 0$. From [30, Ch.0, Th.20.5.12(i)] we get a complex of k_A-vector spaces

$$0 \to \mathfrak{m}_A/(\mathfrak{m}_A^2 + pA) \xrightarrow{d_A} \Omega^1_{A/\mathbb{Z}} \otimes_A k_A \to \Omega^1_{k_A/\mathbb{Z}} \to 0. \tag{4.7.2}$$

Lemma 4.7.3. *The complex* (4.7.2) *is exact.*

Proof. Set $(A_0, \mathfrak{m}_{A_0}) := (A/pA, \mathfrak{m}_A/pA)$ and $\mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$; the induced sequence of ring homomorphisms $\mathbb{F}_p \to A_0 \to k_A$ yields a distinguished triangle ([56, Ch.II, §2.1.2.1]) :

$$\mathbb{L}_{A_0/\mathbb{F}_p} \overset{\mathbf{L}}{\otimes}_{A_0} k_A \to \mathbb{L}_{k_A/\mathbb{F}_p} \to \mathbb{L}_{k_A/A_0} \to \mathbb{L}_{A_0/\mathbb{F}_p} \overset{\mathbf{L}}{\otimes}_{A_0} k_A[1].$$

However, [36, Th.6.5.12(ii)] says that $H_i\mathbb{L}_{k_A/\mathbb{F}_p} = 0$ for every $i > 0$ (one applies *loc.cit.* to the valued field $(k_A, |\cdot|)$, where $|\cdot|$ is the trivial valuation; more directly, one may observe that k_A

is the colimit of a filtered system of smooth $\mathbb{F}_p$-algebras, since $\mathbb{F}_p$ is perfect, and then recall that the functor $\mathbb{L}$ commutes with filtered colimits). We deduce a short exact sequence

$$0 \to H_1\mathbb{L}_{k_A/A_0} \to \Omega^1_{A_0/\mathbb{F}_p} \otimes_{A_0} k_A \to \Omega^1_{k_A/\mathbb{F}_p} \to 0$$

which, under the natural identifications

$$\Omega^1_{k_A/\mathbb{Z}} \xrightarrow{\sim} \Omega^1_{k_A/\mathbb{F}_p} \qquad \Omega^1_{A/\mathbb{Z}} \otimes_A k_A \xrightarrow{\sim} \Omega^1_{A_0/\mathbb{F}_p} \otimes_{A_0} k_A \qquad H_1\mathbb{L}_{k_A/A_0} \xrightarrow{\sim} \mathfrak{m}_{A_0}/\mathfrak{m}^2_{A_0}$$

([56, Ch.III, Cor.1.2.8.1]) becomes (4.7.2), up to replacing d_A by $-d_A$ ([56, Ch.III, Prop.1.2.9]; details left to the reader). □

4.7.4. Keep the situation of (4.7.1), and define

$$A_2 := A \times_{k_A} W_2(k_A)$$

where $W_2(k_A)$ is the ring of 2-truncated Witt vectors of k_A, as in (4.6.11), which is augmented over k_A, via the ghost component map $\overline{\omega}_0 : W_2(k_A) \to k_A$. Recall that the addition (resp. multiplication) law of $W_2(k_A)$ are given by polynomials $S_0(X_0, Y_0)$ and $S_1(X_0, X_1, Y_0, Y_1)$ (resp. $P_0(X_0, Y_0)$ and $P_1(X_0, X_1, Y_0, Y_1)$) uniquely determined by the identities detailed in (4.6); after a simple calculation, we find :

$$S_0 = X_0 + Y_0 \qquad S_1 = X_1 + Y_1 - \sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} X_0^i Y_0^{p-i}$$

$$P_0 = X_0 Y_0 \qquad P_1 = X_0^p Y_1 + X_1 Y_0^p + pX_1Y_1.$$

Notice that $V_1(k_A) := \operatorname{Ker}\overline{\omega}_0$ is an ideal of $W_2(k_A)$ such that $V_1(k_A)^2 = 0$; especially, it is naturally a k_A-vector space, with addition (resp. scalar multiplication) given by the rule : $(0, a) + (0, b) := (0, a + b)$ (resp. $x \cdot (0, a) := (x, 0) \cdot (0, a) = (0, x_0^p \cdot a)$) for every $a, b, x \in k_A$. In other words, the map

$$V_1(k_A) \to k_A^{1/p} \qquad (0, a) \mapsto a^{1/p} \tag{4.7.5}$$

is an isomorphism of k_A-vector spaces, and we also see that $V_1(k_A)$ is a one-dimensional $k_A^{1/p}$-vector space, with scalar multiplication given by the rule : $a\cdot(0, b) := (0, a^p b)$ for every $a \in k_A^{1/p}$ and $b \in k_A$. By construction, we have a natural exact sequence of A_2-modules :

$$0 \to V_1(k_A) \to A_2 \xrightarrow{\pi} A \to 0. \tag{4.7.6}$$

Especially, A_2 is a local ring, and π is a local ring homomorphism inducing an isomorphism on residue fields.

4.7.7. It is easily seen that the rule $(A, \mathfrak{m}_A) \mapsto A_2$ defines a functor on the category $\mathbf{Local}$ of local rings and local ring homomorphisms, to the category of (commutative, unitary) rings. Hence, let us set

$$(A, \mathfrak{m}_A) \mapsto \overline{\Omega}_A := \Omega^1_{A_2/\mathbb{Z}} \otimes_{A_2} A.$$

It follows that the rule $(A, \mathfrak{m}_A) \mapsto (A, \overline{\Omega}_A)$ yields a functor $\mathbf{Local} \to \mathbf{Alg}.\mathbf{Mod}$ to the category whose objects are the pairs (B, M) where B is a ring, and M is a B-module (the morphisms $(B, M) \to (B', M')$ are the pairs (φ, ψ) where $\varphi : B \to B'$ is a ring homomorphism, and $\psi : B' \otimes_B M \to M$ is a B'-linear map : cp. [36, Def.2.5.22(ii)]).

In view of (4.7.6) and [30, Ch.0, Th.20.5.12(i)] we get an exact sequence of A-modules

$$V_1(k_A) \to \overline{\Omega}_A \to \Omega^1_{A/\mathbb{Z}} \to 0$$

whence, after tensoring with k_A, a sequence of k_A-vector spaces

$$0 \to V_1(k_A) \xrightarrow{j} \overline{\Omega}_A \otimes_A k_A \xrightarrow{\rho} \Omega^1_{A/\mathbb{Z}} \otimes_A k_A \to 0 \tag{4.7.8}$$

which is right exact by construction.

Proposition 4.7.9. *With the notation of* (4.7.7), *we have :*

(i) *If* $p \notin \mathfrak{m}_A^2$, *then the sequence* (4.7.8) *is exact.*

(ii) *If* $p \in \mathfrak{m}_A^2$, *then* $\mathrm{Ker}\, j = \{(0, a^p) \mid a \in k_A\}$. *Especially, the isomorphism* (4.7.5) *identifies* $\mathrm{Ker}\, j$ *with the subfield* k_A *of* $k_A^{1/p}$.

Proof. In view of lemma 4.7.3, the kernel of ρ is naturally identified with the kernel of the natural map

$$\mathfrak{m}_{A_2}/(\mathfrak{m}_{A_2}^2 + pA_2) \to \mathfrak{m}_A/(\mathfrak{m}_A^2 + pA).$$

On the other hand, it is easily seen that $\mathfrak{m}_{A_2} = \mathfrak{m}_A \oplus V_1(k_A)$, and under this identification, the multiplication law of A_2 restricts on $\mathfrak{m}_{A_2}$ to the mapping given by the rule : $(a, b) \cdot (a', b') := (aa', 0)$ for every $a, a' \in \mathfrak{m}_A$ and $b, b' \in V_1(k_A)$. There follows a natural isomorphism

$$\mathfrak{m}_{A_2}/\mathfrak{m}_{A_2}^2 \xrightarrow{\sim} (\mathfrak{m}_A/\mathfrak{m}_A^2) \oplus V_1(k_A) \tag{4.7.10}$$

which identifies the map $\mathfrak{m}_{A_2}/\mathfrak{m}_{A_2}^2 \to \mathfrak{m}_A/\mathfrak{m}_A^2$ deduced from π (notation of (4.7.6)), with the projection on the first factor. Moreover, by inspecting the definitions, we easily get a commutative diagram

$$\begin{tikzcd}
V_1(k_A) \arrow[r, "j"] \arrow[d] & \overline{\Omega}_A \otimes_A k_A \\
\mathfrak{m}_{A_2}/\mathfrak{m}_{A_2}^2 \arrow[r] & \mathfrak{m}_{A_2}/(\mathfrak{m}_{A_2}^2 + pA_2) \arrow[u, "d_{A_2}"']
\end{tikzcd}$$

whose bottom horizotal arrow is the quotient map, and whose left vertical arrow is the inclusion map of the direct summand $V_1(k_A)$ resulting from (4.7.10).

The map $A_2 \to \mathfrak{m}_{A_2}/\mathfrak{m}_{A_2}^2$ given by the rule : $a \mapsto p\overline{a} := pa \pmod{\mathfrak{m}_{A_2}^2}$ factors (uniquely) through a k_A-linear map

$$t_{A_2} : k_A \to \mathfrak{m}_{A_2}/\mathfrak{m}_{A_2}^2 \qquad a \pmod{\mathfrak{m}_A} \mapsto pa \pmod{\mathfrak{m}_{A_2}^2}$$

and likewise we may define a map $t_A : k_A \to \mathfrak{m}_A/\mathfrak{m}_A^2$. The snake lemma then gives an induced map $\partial : \mathrm{Ker}\, t_A \to V_1(k_A)$, and in view of the foregoing, the proposition follows from :

Claim 4.7.11. (i) If $p \notin \mathfrak{m}_A^2$, then t_A is injective.

(ii) If $p \in \mathfrak{m}_A^2$, then $\mathrm{Im}\, \partial = \{(0, a^p) \mid a \in k_A\}$.

Proof of the claim. (i) is obvious.

(ii): By virtue of (4.6.7), we have $p = (p, (0, 1))$ in A_2, and if $(a, y) \in A_2$ is any element, then $p \cdot (a, y) = (pa, (0, a^p))$, so in case $p \in \mathfrak{m}_A^2$, the map t_{A_2} is given by the rule :

$$a \mapsto (pa, (0, \overline{a}^p)) = (0, (0, \overline{a}^p)) \in (\mathfrak{m}_A/\mathfrak{m}_A^2) \oplus V_1(k_A) \qquad \text{for every } a \in A.$$

By the same token, in this case t_A is the zero map. The claim follows straightforwardly. □

4.7.12. Keep the notation of (4.7.7), and assume that $p \notin \mathfrak{m}_A^2$, so (4.7.8) is a k_A-extension of $\Omega^1_{A/\mathbb{Z}} \otimes_A k_A$ by $V_1(k_A)$, by virtue of proposition 4.7.9; hence, (4.7.8)$\otimes_{k_A} k_A^{1/p}$ is a $k_A^{1/p}$-extension of the corresponding $k_A^{1/p}$-vector spaces. Recall now that $V_1(k_A)$ is naturally a $k_A^{1/p}$-vector space of dimension one, and let $\psi : V_1(k_A) \otimes_{k_A} k_A^{1/p} \to V_1(k_A)$ be the scalar multiplication. By push out along ψ, we obtain therefore an extension $\psi *$ (4.7.8) $\otimes_{k_A} k_A^{1/p}$ fitting into a commutative ladder with exact rows :

$$\begin{tikzcd}
0 \arrow[r] & V_1(k_A) \otimes_{k_A} k_A^{1/p} \arrow[r] \arrow[d, "\psi"'] & \overline{\Omega}_A \otimes_A k_A^{1/p} \arrow[r] \arrow[d, "\boldsymbol{\psi}"'] & \Omega^1_{A/\mathbb{Z}} \otimes_A k_A^{1/p} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & V_1(k_A) \arrow[r, "\mathbf{j}"] & \boldsymbol{\Omega}_A \arrow[r] & \Omega^1_{A/\mathbb{Z}} \otimes_A k_A^{1/p} \arrow[r] & 0
\end{tikzcd}$$

(see [36, §2.5.5]). We consider now the mapping :

$$\boldsymbol{d} : A \to \boldsymbol{\Omega}_A \qquad a \mapsto \boldsymbol{\psi}(d(a, \tau_{k_A}(\overline{a})) \otimes 1) \qquad \text{for every } a \in A$$

where :

- $\overline{a} \in k_A$ is the image of a in k_A
- τ_{k_A} is the Teichmüller mapping (see (4.6.9)), so $\tau_{k_A}(\overline{a}) = (\overline{a}, 0) \in W_2(k_A)$
- $d : A_2 \to \Omega_{A_2/\mathbb{Z}}$ is the universal derivation of A_2.

Since τ_{k_A} is multiplicative, the map $\boldsymbol{d}$ satisfies Leibniz's rule, *i.e.* we have

$$\boldsymbol{d}(ab) = \overline{a} \cdot \boldsymbol{d}(b) + \overline{b} \cdot \boldsymbol{d}(a) \qquad \text{for every } a, b \in A.$$

However, $\boldsymbol{d}$ is not quite a derivation, since additivity fails for τ_{k_A}, hence also for $\boldsymbol{d}$. Instead, recalling that $(p-1)! = -1$ in $\mathbb{F}_p$, we get the identity :

$$\begin{aligned}\tau_{k_A}(\overline{a} + \overline{b}) &= \tau_{k_A}(\overline{a}) + \tau_{k_A}(\overline{b}) - \sum_{i=1}^{p-1} \left(0, \frac{\overline{a}^i}{i!} \cdot \frac{\overline{b}^{p-i}}{(p-i)!}\right) \\ &= \tau_{k_A}(\overline{a}) + \tau_{k_A}(\overline{b}) - \sum_{i=1}^{p-1} \frac{\overline{a}^{i/p}}{i!} \cdot \frac{\overline{b}^{1-i/p}}{(p-i)!} \cdot p\end{aligned}$$

for every $\overline{a}, \overline{b} \in k_A$. On the other hand, notice that

$$\begin{aligned}\boldsymbol{\psi}(d(0, x \cdot p) \otimes 1) &= \mathbf{j} \circ \psi(x \cdot p \otimes 1) \\ &= x \cdot \mathbf{j} \circ \psi(p \otimes 1) \\ &= x \cdot \boldsymbol{\psi}(d(0, p) \otimes 1) \\ &= x \cdot \boldsymbol{\psi}(d(p - (p, 0)) \otimes 1) \\ &= -x \cdot \boldsymbol{d}(p)\end{aligned}$$

for every $x \in k_A^{1/p}$, whence :

$$\boldsymbol{d}(a+b) = \boldsymbol{d}(a) + \boldsymbol{d}(b) + \sum_{i=1}^{p-1} \frac{\overline{a}^{i/p}}{i!} \cdot \frac{\overline{b}^{1-i/p}}{(p-i)!} \cdot \boldsymbol{d}(p) \qquad \text{for every } a, b \in A.$$

4.7.13. Especially, notice we do have $\boldsymbol{d}(a+b) = \boldsymbol{d}(a) + \boldsymbol{d}(b)$ in case either a or b lies in $\mathfrak{m}_A$. Hence, $\boldsymbol{d}$ restricts to an additive map $\mathfrak{m}_A \to \boldsymbol{\Omega}_A$, and Leibniz's rule implies that the latter descends to a k_A-linear map

$$\overline{\boldsymbol{d}} : \mathfrak{m}_A/\mathfrak{m}_A^2 \to \boldsymbol{\Omega}_A.$$

Proposition 4.7.14. *In the situation of* (4.7.1), *suppose that* $p \notin \mathfrak{m}_A^2$. *Then there exists a natural exact sequence of* $k_A^{1/p}$*-vector spaces :*

$$0 \to \mathfrak{m}_A/\mathfrak{m}_A^2 \otimes_{k_A} k_A^{1/p} \xrightarrow{\overline{\boldsymbol{d}} \otimes k_A^{1/p}} \boldsymbol{\Omega}_A \to \Omega^1_{k_A/\mathbb{Z}} \otimes_{k_A} k_A^{1/p} \to 0.$$

Proof. By inspecting the constructions in (4.7.12), we obtain a commutative ladder of k_A-vector spaces, with exact rows :

$$\begin{tikzcd} 0 \arrow[r] & pA/(\mathfrak{m}_A^2 \cap pA) \arrow[r] \arrow[d, "i"'] & \mathfrak{m}_A/\mathfrak{m}_A^2 \arrow[r] \arrow[d, "\overline{\boldsymbol{d}}"] & \mathfrak{m}_A/(\mathfrak{m}_A^2 + pA) \arrow[r] \arrow[d, "d_A"] & 0 \\ 0 \arrow[r] & V_1(k_A) \arrow[r, "\mathbf{j}"] & \boldsymbol{\Omega}_A \arrow[r] & \Omega^1_{A/\mathbb{Z}} \otimes_A k_A^{1/p} \arrow[r] & 0 \end{tikzcd}$$

such that $d_A \otimes_{k_A} k_B$ is injective with cokernel naturally isomorphic to $\Omega^1_{k_A/\mathbb{Z}} \otimes_{k_A} k_A^{1/p}$ (lemma 4.7.3). By the snake lemma, it then suffices to show that $i \otimes_{k_A} k_A^{1/p}$ is an isomorphism; but since both the source and target of the latter map are one-dimensional $k_A^{1/p}$-vector spaces, we come down to checking that i is not the zero map. But a simple inspection yields $i(p) = (0,1)$, as required. □

4.7.15. Keep the notation of (4.7.1), and let now $f_1, \dots, f_n$ be a finite sequence of elements of A, and $e_1, \dots, e_n$ a sequence of integers such that $e_i > 1$ for every $i = 1, \dots, n$. Set

$$C := A[T_1, \dots, T_n]/(T_1^{e_1} - f_1, \dots, T_n^{e_n} - f_n).$$

Fix a prime ideal $\mathfrak{n} \subset C$ such that $\mathfrak{n} \cap A = \mathfrak{m}_A$, and let $B := C_\mathfrak{n}$. So the induced map $A \to B$ is a local ring homomorphism; we denote by $\mathfrak{m}_B$ the maximal ideal of B, and set $k_B := B/\mathfrak{m}_B$. Also, define the integer ν as follows :

- If $p \in \mathfrak{m}_A^2$, then $\nu := \dim_{k_A} E$, where $E \subset \Omega^1_{A/\mathbb{Z}} \otimes_A k_A$ is the k_A-vector space spanned by $df_1, \dots, df_n$.
- If $p \notin \mathfrak{m}_A^2$, then $\nu := \dim_{k_A^{1/p}} E$, where $E \subset \mathbf{\Omega}_A$ is the $k_A^{1/p}$-vector space spanned by $\boldsymbol{d}(f_1), \dots, \boldsymbol{d}(f_n)$.

Theorem 4.7.16. *In the situation of* (4.7.15), *suppose moreover that :*

(a) $f_i \in \mathfrak{m}_A$, *for every* $i \leq n$ *such that* p *does not divide* e_i.
(b) $\mathfrak{m}_A/\mathfrak{m}_A^2$ *is a finite dimensional* k_A*-vector space.*

Then $\mathfrak{m}_B/\mathfrak{m}_B^2$ *is a finite dimensional* k_B*-vector space, and we have :*

$$\dim_{k_B} \mathfrak{m}_B/\mathfrak{m}_B^2 = n + \dim_{k_A} \mathfrak{m}_A/\mathfrak{m}_A^2 - \nu.$$

Proof. Let us begin with the following general result :

Claim 4.7.17. Let K be any field, and L a field extension of K of finite type. Then

$$\dim_L \Omega^1_{L/K} - \dim_L H_1\mathbb{L}_{L/K} = \mathrm{tr.deg}[L : K].$$

Proof of the claim. This is known as *Cartier's identity*, and a proof is given in [30, Ch.0, Th.21.7.1]. We present a proof via the cotangent complex formalism. Suppose first that K is of finite type over its prime field K_0 (so K_0 is either $\mathbb{Q}$ or a finite field of prime order). In this case, notice that $H_1\mathbb{L}_{L/K_0} = 0$ (cp. the proof of lemma 4.7.3); then the transitivity triangle for the sequence of maps $K_0 \to K \to L$ ([56, Ch.II, §2.1.2.1]) yields an exact sequence

$$0 \to H_1\mathbb{L}_{L/K} \to \Omega^1_{K/K_0} \otimes_K L \to \Omega^1_{L/K_0} \to \Omega^1_{L/K} \to 0$$

and the claim follows easily, after one remarks that $\dim_K \Omega^1_{K/K_0} = \mathrm{tr.deg}[K : K_0]$, and likewise for Ω^1_{L/K_0}. Next, let K be an arbitrary field, and write K as the union of the filtered family $(K_\lambda \mid \lambda \in \Lambda)$ of its subfields that are finitely generated over K_0. Choose elements $x_1, \dots, x_t \in L$ such that $K(x_1, \dots, x_t) = L$. Let $\varphi : K[X_1, \dots, X_t] \to L$ be the map of K-algebras given by the rule : $X_i \mapsto x_i$ for $i = 1, \dots, t$. Then $I := \mathrm{Ker}\,\varphi$ is a finitely generated ideal, so we may find $\lambda \in \Lambda$ and an ideal $I_\lambda \subset K_\lambda[X_1, \dots, X_t]$ such that $I = I_\lambda \otimes_{K_\lambda} K$ (details left to the reader). Denote by L_λ the field of fractions of $K_\lambda[T_1, \dots, T_t]/I_\lambda$; it is easily seen that $L_\lambda \otimes_{K_\lambda} K$ is an integral domain, and its field of fractions is a K-algebra naturally isomorphic to L. Especially, we have $\mathrm{tr.deg}[L_\lambda : K_\lambda] = \mathrm{tr.deg}[L : K]$, and on the other hand, there is a natural isomorphism in $\mathsf{D}^-(L\text{-}\mathbf{Mod})$:

$$\mathbb{L}_{L_\lambda/K_\lambda} \otimes_{L_\lambda} L \xrightarrow{\sim} \mathbb{L}_{L/K}$$

([56, Ch.II, Prop.2.2.1, Cor.2.3.1.1]). Then the sought identity for the extension $K \subset L$ follows from the same identity for the extension $K_\lambda \subset L_\lambda$. The latter is already known, by the previous case. ◊

Let $\mathfrak{q} \subset A[T_1, \dots, T_n]$ be the preimage of $\mathfrak{n}$, set $R := A[T_1, \dots, T_n]_\mathfrak{q}$, and denote by $\mathfrak{p}$ the maximal ideal of R. We prove first the following special case :

Claim 4.7.18. (i) $\dim_{k_B} \mathfrak{p}/\mathfrak{p}^2 = n + \dim_{k_A} \mathfrak{m}_A/\mathfrak{m}_A^2$.

(ii) The natural map $\gamma : \mathfrak{m}_A/\mathfrak{m}_A^2 \to \mathfrak{p}/\mathfrak{p}^2$ is injective.

Proof of the claim. (i): According to [36, Th.6.5.12(i)], we have $H_i\mathbb{L}_{k_B/k_A} = 0$ for every $i > 1$ (we apply *loc.cit.* to the extension $(k_A, |\cdot|_{k_A}) \subset (k_B, |\cdot|_{k_B})$ of valued fields with trivial valuations); then the transitivity triangle for the cotangent complex relative to the maps $A \to k_A \to k_B$ yields a short exact sequence of k_B-vector spaces :

$$0 \to H_1\mathbb{L}_{k_A/A} \otimes_{k_A} k_B \xrightarrow{\alpha} H_1\mathbb{L}_{k_B/A} \to H_1\mathbb{L}_{k_B/k_A} \to 0$$

([56, Ch.II, §2.1.2.1]). Likewise, since $H_i\mathbb{L}_{R/A} = 0$ for every $i > 1$ ([56, Ch.II, Cor.1.2.6.3]), the sequence of maps $A \to R \to k_B$ yields an exact sequence of k_B-vector spaces :

$$0 \to H_1\mathbb{L}_{k_B/A} \xrightarrow{\beta} H_1\mathbb{L}_{k_B/R} \to \Omega^1_{R/A} \otimes_R k_B \to \Omega_{k_B/A} \to 0.$$

However, we have natural isomorphisms

$$H_1\mathbb{L}_{k_A/A} \xrightarrow{\sim} \mathfrak{m}_A/\mathfrak{m}_A^2 \qquad H_1\mathbb{L}_{k_B/R} \xrightarrow{\sim} \mathfrak{p}/\mathfrak{p}^2$$

([56, Ch.III, Cor.1.2.8.1]), and clearly $\dim_{k_B} \Omega_{R/A} \otimes_R k_B = n$. Thus :

$$\dim_{k_A} \mathfrak{m}_A/\mathfrak{m}_A^2 = \dim_{k_B} H_1\mathbb{L}_{k_B/A} - \dim_{k_B} H_1\mathbb{L}_{k_B/k_A}$$
$$\dim_{k_B} \mathfrak{p}/\mathfrak{p}^2 = \dim_{k_B} H_1\mathbb{L}_{k_B/A} + n - \dim_{k_B} \Omega_{k_B/k_A}.$$

Taking into account the identity

$$\dim_{k_B} H_1\mathbb{L}_{k_B/k_A} = \dim_{k_B} \Omega_{k_B/k_A}$$

provided by claim 4.7.17, the assertion follows.

(ii): Notice that the composition of α and β yields an injective map $\mathfrak{m}_A/\mathfrak{m}_A^2 \to \mathfrak{p}/\mathfrak{p}^2$ so it suffices to check that this composition equals γ. However, let

$$\Sigma \quad : \quad H_1\mathbb{L}_{k_B/R} \xrightarrow{d} H_0 k_B \otimes_R \mathbb{L}_{R/A}$$

be the boundary map of the transitivity triangle

(4.7.19) $$k_B \otimes_R \mathbb{L}_{R/A} \to \mathbb{L}_{k_B/A} \to \mathbb{L}_{k_B/R} \to k_B \otimes_R \sigma\mathbb{L}_{R/A}$$

arising from the sequence $A \to R \to k_B$; we regard Σ as a complex placed in degrees $[-1, 0]$, and then it is clear that the triangle $\tau_{\geq -1}$(4.7.19) is naturally isomorphic in $\mathsf{D}(k_B\text{-}\mathbf{Mod})$ to the triangle

(4.7.20) $$k_B \otimes_R H_0\mathbb{L}_{R/A}[0] \to \Sigma \to H_1\mathbb{L}_{k_B/R}[1] \to k_B \otimes_R H_0\mathbb{L}_{R/A}[1].$$

According to [56, III.1.2.9.1], the complex Σ is naturally isomorphic to the complex

$$\Theta \quad : \quad \mathfrak{p}/\mathfrak{p}^2 \xrightarrow{-d_{R/A}} k_B \otimes_R \Omega^1_{R/A}$$

(placed in degrees $[-1, 0]$) where $d_{R/A}$ is induced by the universal derivation $R \to \Omega^1_{R/A}$. Likewise, we have natural isomorphisms

(4.7.21) $$\tau_{\geq -1}\mathbb{L}_{k_B/R} \xrightarrow{\sim} \mathfrak{p}/\mathfrak{p}^2[1] \qquad \tau_{\geq -1}\mathbb{L}_{R/A} \xrightarrow{\sim} \Omega^1_{R/A}[0]$$

and under these identifications, 4.7.20 is the obvious triangle deduced from Θ. Especially, the map β is naturally identified with the inclusion map $\mathrm{Ker}\, d_{R/A} \to \mathfrak{p}/\mathfrak{p}^2$.

Likewise, there exists a natural identification

(4.7.22) $$\tau_{\geq -1}\mathbb{L}_{k_A/A} \xrightarrow{\sim} \mathfrak{m}_A/\mathfrak{m}_A^2[1] \qquad \text{in } \mathsf{D}(k_A\text{-}\mathbf{Mod})$$

as well as a natural map of complexes

(4.7.23) $$k_B \otimes_{k_A} (\mathfrak{m}_A/\mathfrak{m}_A^2)[1] \to \Theta$$

deduced from γ. To conclude, it suffices to check that the map

(4.7.24) $$\tau_{\geq -1} k_B \otimes_A \mathbb{L}_{k_A/A} \to \tau_{\geq -1}\mathbb{L}_{k_B/A}$$

coming from the transitivity triangle for the sequence $A \to k_A \to k_B$, corresponds to the morphism (4.7.23), under the identification (4.7.22) and the previous identification of $\tau_{\geq -1}\mathbb{L}_{k_B/A}$ with Θ. To this aim, it suffices to compare the maps obtained by applying to these two morphisms the functor $\mathrm{Ext}^1_{k_B}(-, M[0])$, for arbitrary k_B-modules M. Now, recall that there exists a natural isomorphism

$$\mathrm{Ext}^1_{k_B}(\tau_{\geq -1}\mathbb{L}_{k_B/R}, M[0]) \xrightarrow{\sim} \mathrm{Exal}_R(k_B, M) \qquad \text{for every } k_B\text{-module } M.$$

As explained in [56, III.1.2.8], under the identification (4.7.21), this becomes the following k_B-linear isomorphism

(4.7.25) $$\mathrm{Hom}_{k_B}(\mathfrak{p}/\mathfrak{p}^2, M) \xrightarrow{\sim} \mathrm{Exal}_R(k_B, M) \qquad \varphi \mapsto \varphi * U$$

(notation of [36, §2.5.5]), where

$$U \quad : \quad 0 \to \mathfrak{p}/\mathfrak{p}^2 \to R/\mathfrak{p}^2 \to k_B \to 0$$

is the natural extension. There is a natural surjection

$$\mathrm{Hom}_{k_B}(\mathfrak{p}/\mathfrak{p}^2, M) \to \mathrm{Ext}^1_{k_B}(\Theta, M[0])$$

and the foregoing implies that the natural isomorphism

$$\mathrm{Ext}^1_{k_B}(\tau_{\geq -1}\mathbb{L}_{k_B/A}, M[0]) \xrightarrow{\sim} \mathrm{Exal}_A(k_B, M)$$

is also realized as in (4.7.25) : given a class c in $\mathrm{Ext}^1_{k_B}(\Theta, M[0])$, take an arbitrary representative $\varphi : \mathfrak{p}/\mathfrak{p}^2 \to M$, and the correspondence associates to c the class of the push out $\varphi * U$.

By the same token, we have a natural isomorphism

$$\mathrm{Ext}^1_{k_B}(\tau_{\geq -1} k_B \otimes_{k_A} \mathbb{L}_{k_A/A}, M(0]) \xrightarrow{\sim} \mathrm{Exal}_A(k_A, M) \qquad \text{for every } k_B\text{-module } M$$

and on the one hand, the map $\mathrm{Ext}^1_{k_B}((4.7.24), M[0])$ is identified naturally with the map

$$\mathrm{Exal}_A(k_B, M) \to \mathrm{Exal}_A(k_A, M)$$

given by pull back along the inclusion map $j : k_A \to k_B$. On the other hand, by [56, III.1.2.8], the identification (4.7.21) induces the isomorphism

$$\mathrm{Hom}_{k_A}(\mathfrak{m}_A/\mathfrak{m}_A^2, M) \xrightarrow{\sim} \mathrm{Exal}_A(k_A, M) \qquad \psi \mapsto \psi * U'$$

where

$$U' \quad : \quad 0 \to \mathfrak{m}_A/\mathfrak{m}_A^2 \to A/\mathfrak{m}_A^2 \to k_A \to 0$$

is the natural extension. So finally, the assertion boils down to the identity

$$(\varphi \circ \gamma) * U' = \varphi * U * j \qquad \text{in } \mathrm{Exal}_A(k_A, M)$$

for every k_B-module M and every $\varphi : \mathfrak{p}/\mathfrak{p}^2 \to M$. We leave the verification as an exercise for the reader. $\diamondsuit$

Now, let $F \subset \mathfrak{p}/\mathfrak{p}^2$ be the k_B-vector space spanned by $T_1^{e_1} - f_1, \dots, T_n^{e_n} - f_n$. Clearly, we have a short exact sequence

$$0 \to F \to \mathfrak{p}/\mathfrak{p}^2 \to \mathfrak{m}_B/\mathfrak{m}_B^2 \to 0.$$

Suppose first that $p \notin \mathfrak{m}_A^2$, in which case $p \notin \mathfrak{p}^2$, by claim 4.7.18(ii), hence $\mathbf{\Omega}_R$ is well defined. On the other hand, notice that $\boldsymbol{d}(T_i^{e_i}) = \boldsymbol{d}(T_i^{e_i} - f_i) + \boldsymbol{d}(f_i)$ in $\mathbf{\Omega}_R$, since $T_i^{e_i} - f_i \in \mathfrak{p}$. Now, if e_i is a multiple of p, Leibniz's rule yields $\boldsymbol{d}(T_i^{e_i}) = e_i \cdot T_i^{e_i - 1}\boldsymbol{d}(T_i) = 0$. If e_i is not a multiple of p, we have $f_i \in \mathfrak{m}_A$ by assumption; hence $T_i \in \mathfrak{p}$ and therefore $\boldsymbol{d}(T_i^{e_i}) = 0$ again, since $e_i > 1$. In either case, we find

$$\boldsymbol{d}(f_i) = -\overline{\boldsymbol{d}}(T_i^{e_i} - f_i) \qquad \text{in } \mathbf{\Omega}_R, \text{ for every } i = 1, \dots, n.$$

In view of proposition 4.7.14, it follows that $\dim_{k_B} \mathfrak{m}_B/\mathfrak{m}_B^2 = \dim_{k_B} \mathfrak{p}/\mathfrak{p}^2 - \dim_{k_B^{1/p}} E'$, where $E' \subset \boldsymbol{\Omega}_R$ is the $k_B^{1/p}$-vector space spanned by $\boldsymbol{d}(f_1), \dots, \boldsymbol{d}(f_n)$. Taking into account claim 4.7.18(i), it then suffices to remark :

Claim 4.7.26. With the foregoing notation, we have :

(i) The natural map $\Omega^1_{A/\mathbb{Z}} \otimes_A R \to \Omega^1_{R/\mathbb{Z}}$ is injective, and its image is a direct summand of $\Omega^1_{R/\mathbb{Z}}$.

(ii) If $p \notin \mathfrak{m}_A$, the natural map $\boldsymbol{\Omega}_A \otimes_{k_A^{1/p}} k_B^{1/p} \to \boldsymbol{\Omega}_R$ is injective.

Proof of the claim. (i) is a standard calculation (more precisely, the complement of $\Omega^1_{A/\mathbb{Z}} \otimes_A R$ in $\Omega^1_{R/\mathbb{Z}}$ is the free R-module generated by $dT_1, \dots, dT_n$).

(ii): By inspecting the constructions, we get a natural commutative ladder of $k_B^{1/p}$-vector spaces

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V_1(k_A) \otimes_{k_A^{1/p}} k_B^{1/p} & \longrightarrow & \boldsymbol{\Omega}_A \otimes_{k_A^{1/p}} k_B^{1/p} & \longrightarrow & \Omega^1_{A/\mathbb{Z}} \otimes_A k_B^{1/p} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & V_1(k_B) & \longrightarrow & \boldsymbol{\Omega}_R & \longrightarrow & \Omega^1_{R/\mathbb{Z}} \otimes_R k_B^{1/p} & \longrightarrow & 0 \end{array}$$

whose central vertical arrow is the map of the claim. However, it is easily seen that the left vertical arrow is an isomorphism, so the assertion follows from (i). ◊

Lastly, suppose $p \in \mathfrak{m}_A^2$, so that $p \in \mathfrak{p}^2$ as well. Arguing as in the foregoing case, we see that $\dim_{k_B} F$ equals the dimension of the k_B-vector subspace of $\Omega^1_{R/\mathbb{Z}} \otimes_R k_B$ spanned by $df_1, \dots, df_n$, and in view of claim 4.7.26(i), the latter equals ν, whence the contention. □

Corollary 4.7.27. *In the situation of theorem* 4.7.16*, the following conditions are equivalent :*

(a) *A is a regular local ring, and $\nu = n$.*

(b) *B is a regular local ring.*

Proof. Suppose first that (b) holds. Since the map $A \to B$ is faithfully flat, it is easily seen that A is noetherian, and then [30, Ch.0, Prop.17.3.3(i)] shows already that A is a regular local ring. Moreover, C is clearly a finite A-algebra, therefore $\dim_{k_B} \mathfrak{m}_B/\mathfrak{m}_B^2 = \dim B \leq \dim A = \dim_{k_A} \mathfrak{m}_A/\mathfrak{m}_A^2$. Theorem 4.7.16 then implies that $\nu = n$, so (a) holds.

Next, suppose that (a) holds. We apply theorem 4.7.16 as in the foregoing, to deduce that $\dim B = \dim A = \dim_{k_B} \mathfrak{m}_B/\mathfrak{m}_B^2$, whence (b) : details left to the reader. □

Remark 4.7.28. (i) Keep the notation of corollary 4.7.27. In [30, Ch.0, Th.22.5.4] it is asserted that condition (b) is equivalent to the following :

(a’) A is regular and the space $E \subset \Omega^1_{A/\mathbb{Z}} \otimes_A k_A$ spanned by $df_1, \dots, df_n$ has dimension n.

When $p \in \mathfrak{m}_A^2$, this condition (a’) agrees with our condition (a), so in this case of course we do have (a’)⇔(b). However, the latter equivalence fails in general, in case $p \notin \mathfrak{m}_A$: the mistake is found in [30, Ch.0, Rem.22.4.8], which is false. The implication (a’)⇒(b) does remain true in all cases : this is easily deduced from theorem 4.7.16, since the image of $\boldsymbol{d}(f)$ in $\Omega^1_{A/\mathbb{Z}} \otimes_{k_A} k_A^{1/p}$ agrees with df, for every $f \in A$. The proof of *loc.cit.* is correct for $p \in \mathfrak{m}_A^2$, which is the only case that is used in the proof of corollary 4.7.27.

(ii) Moreover, in the situation of corollary 4.7.27, suppose that B is a regular local ring. Then we have :

- If $p \notin \mathfrak{m}_B^2$, then the image of the sequence $df_1^{1/e_1}, \dots, df_n^{1/e_n}$ in $\Omega^1_{B/\mathbb{Z}} \otimes_B k_B$ spans a k_B-vector space of dimension n.

- If $p \in \mathfrak{m}_B^2$, then the image of the sequence $\boldsymbol{d}(f_1^{1/e_1}), \dots, \boldsymbol{d}(f_n^{1/e_n})$ in $\boldsymbol{\Omega}_B$ spans a $k_B^{1/p}$-vector space of dimension n.

Indeed, consider the ring $C := B[T_1, \dots, T_n]/(T_1^p - f_1^{1/e_1}, \dots, T_n^p - f_n^{1/e_n})$. Then, corollary 4.7.27 applies to the extension $A \subset C$, the sequence $(f_1, \dots, f_n)$, and the sequence of integers $(pe_1, \dots, pe_n)$, so C is a regular local ring. But the same corollary applies as well to the extension $B \subset C$, the sequence $(f_1^{1/e_1}, \dots, f_n^{1/e_n})$, and the sequence of integers $(p, \dots, p)$, and yields the assertion.

(iii) Let us say that the sequence $(f_1, \dots, f_n)$ is *maximal in* A if the following holds :

- If $p \in \mathfrak{m}_A^2$, then $(df_1, \dots, df_r)$ is a basis of the k_A-vector space $\Omega^1_{A/\mathbb{Z}} \otimes_A k_A$.
- If $p \notin \mathfrak{m}_A^2$, then $(\boldsymbol{d}(f_1), \dots, \boldsymbol{d}(f_r))$ is a basis of the $k_A^{1/p}$-vector space $\boldsymbol{\Omega}_A$.

Then, in the situation of (ii), we claim that the sequence $(f_1, \dots, f_n)$ is maximal in A if and only if the sequence $(f_1^{1/e_1}, \dots, f_n^{1/e_n})$ is maximal in B. Indeed, under the current assumptions we have $\dim_{k_A} \Omega^1_{k_A/\mathbb{Z}} = \dim_{k_B} \Omega^1_{k_B/\mathbb{Z}}$, since these integers are equal to the transcendence degree of k_A (and k_B) over $\mathbb{F}_p$, and on the other hand $\dim_{k_A} \mathfrak{m}_A/\mathfrak{m}_A^2 = \dim_{k_B} \mathfrak{m}_B/\mathfrak{m}_B^2$, since A and B are regular local rings of the same dimension; then the assertion follows from (ii), lemma 4.7.3 and proposition 4.7.14 (details left to the reader).

4.7.29. Let $p > 0$ be a prime integer, and A an $\mathbb{F}_p$-algebra. Denote by $\Phi_A : A \to A$ the *Frobenius endomorphism* of A, given by the rule : $a \mapsto a^p$ for every $a \in A$. For every A-module M, we let $M_{(\Phi)}$ be the A-module obtained from M via restriction of scalars along the map Φ_A (that is, $a \cdot m := a^p m$ for every $a \in A$ and $m \in M$). Notice that Φ_A is an A-linear map $A \to A_{(\Phi)}$. Theorem 4.7.30, and part (i) of the following theorem 4.8.42 are due to E.Kunz.

Theorem 4.7.30. *Let A be a noetherian local $\mathbb{F}_p$-algebra. Then the following conditions are equivalent :*

(i) *A is regular.*
(ii) *Φ_A is a flat ring homomorphism.*
(iii) *There exists $n > 0$ such that Φ_A^n is a flat ring homomorphism.*

Proof. (i)⇒(ii): Let $A^\wedge$ be the completion of A, and $f : A \to A^\wedge$ the natural map. Clearly

$$f \circ \Phi_A = \Phi_{A^\wedge} \circ f.$$

Since f is faithfully flat, it follows that Φ_A is flat if and only if the same holds for $\Phi_{A^\wedge}$, so we may replace A by $A^\wedge$, and assume from start that A is complete, hence $A = k[[T_1, \dots, T_d]]$, for a field k of characteristic p, and $d = \dim A$ ([30, Ch.0, Th.19.6.4]). Then, it is easily seen that

$$\Phi_A(A) = A^p = k^p[[T_1^p, \dots, T_d^p]].$$

Set $B := k[[T_1^p, \dots, T_d^p]]$; the ring A is a free B-module (of rank p^d), hence it suffices to check that the inclusion map $A^p \to B$ is flat. However, denote by $\mathfrak{m}$ the maximal ideal of A^p; clearly B is an $\mathfrak{m}$-adically ideal-separated A-module (see [61, p.174, Def.]), hence it suffices to check that $B/\mathfrak{m}^k B$ is a flat $A^p/\mathfrak{m}^k$-module for every $k > 0$ ([61, Th.22.3]). The latter is clear, since $k[T_1^p, \dots, T_d^p]$ is a flat $k^p[T_1^p, \dots, T_d^p]$-module.

(ii)⇒(iii) is obvious.

(iii)⇒(i): Notice first that $\mathrm{Spec}\,\Phi_A^n$ is the identity map on the topological space underlying $\mathrm{Spec}\,A$; especially, Φ_A^n is flat if and only if it is faithfully flat, and the latter condition implies that Φ_A^n is injective. We easily deduce that if (iii) holds, then A is reduced. Now, consider quite generally, any finite system $x_\bullet := (x_1, \dots, x_t)$ of elements of A, and let $I \subset A$ be the ideal generated by $x_\bullet$; we shall say that $x_\bullet$ is a system of *independent* elements, if I/I^2 is a free A/I-module of rank n. We show first the following :

Claim 4.7.31. Let $y, z, x_2, \dots, x_t$ be a family of elements of A, such that $x_\bullet := (yz, x_2, \dots, x_t)$ is a system of independent elements, and denote by $J \subset A$ the ideal generated by $x_\bullet$. We have :

(i) $(y, x_2, \dots, x_t)$ is a system of independent elements of A.

(ii) If $\mathrm{length}_A A/J$ is finite, then

$$\mathrm{length}_A A/J = \mathrm{length}_A A/(y, x_2, \dots, x_t) + \mathrm{length}_A A/(z, x_2, \dots, x_t).$$

Proof of the claim. (i): Suppose $a_1 y + a_2 x_2 + \cdots + a_t x_t = 0$ is a linear relation with $a_1, \dots, a_t \in A$, and let $I \subset A$ be the ideal generated by $y, x_2, \dots, x_t$. We have to show that $a_1, \dots, a_t \in I$. However, as $a_1 yz + a_2 z x_2 + \cdots + a_t z x_t = 0$, it follows by assumption, that a_1 lies in $J \subset I$. Write $a_1 = b_1 yz + b_2 x_2 + \cdots + b_t x_t$; then

$$b_1 y^2 z + (a_2 + b_2 y) x_2 + \cdots + (a_t + b_t y) x_t = 0.$$

Therefore $a_i + b_i y \in J$ for $i = 2, \dots, t$, and therefore $a_2, \dots, a_t \in I$, as required.

(ii): It suffices to show that the natural map of A-modules :

$$A/(z, x_2, \cdots, x_t) \to I/J \qquad a \mapsto ay + J$$

is an isomorphism. However, the surjectivity is immediate. To show the injectivity, suppose that $ay \in J$, *i.e.* $ay = b_1 yz + b_2 x_2 + \cdots + b_t x_t$ for some $b_1, \dots, b_t \in A$; we deduce that $(b_1 z - a) yz + b_2 z x_2 + \cdots + b_t z x_t = 0$, hence $a - b_1 z \in J$ by assumption, so a lies in the ideal generated by $z, x_2, \dots, x_t$, as required. $\diamondsuit$

Now, set $q := p^n$, and $A_\nu := A^{q^\nu} \subset A$ for every integer $\nu > 0$; pick a minimal system $x_\bullet := (x_1, \dots, x_t)$ of generators of the maximal ideal $\mathfrak{m}_A$ of A, and notice that, since A is reduced, $\Phi_A^{n\nu}$ induces an isomorphism $A \to A_\nu$, hence $x_\bullet^{(\nu)} := (x_1^{q^\nu}, \dots, x_t^{q^\nu})$ is a minimal system of generators for the maximal ideal $\mathfrak{m}_\nu$ of A_ν. Set as well $I_\nu := \mathfrak{m}_\nu A$; since the inclusion map $A_\nu \to A$ is flat by assumption for every $\nu > 0$, we have a natural isomorphism of A-modules

$$(\mathfrak{m}_\nu/\mathfrak{m}_\nu^2) \otimes_{A_\nu} A \xrightarrow{\sim} I_\nu/I_\nu^2.$$

On the other hand, set $k_\nu := A_\nu/\mathfrak{m}_\nu$; by Nakayama's lemma, $\dim_{k_\nu} \mathfrak{m}_\nu/\mathfrak{m}_\nu^2 = t$, so I_ν/I_ν^2 is a free A-module of rank t, *i.e.* $x_\bullet^{(\nu)}$ is an independent system of elements of A. From claim 4.7.31(ii) and a simple induction, we deduce that

$$\mathrm{length}_A A/I_\nu = \mathrm{length}_{A^\wedge} A^\wedge/I_\nu A^\wedge = q^{\nu t} \qquad \text{for every } \nu > 0 \tag{4.7.32}$$

(where the first equality holds, since I_ν is an open ideal in the $\mathfrak{m}_A$-adic topology of A). According to [30, Ch.0, Th.19.9.8] (and its proof), $A^\wedge$ contains a field isomorphic to $k_0 := A/\mathfrak{m}_A$, and the inclusion map $k_0 \to A^\wedge$ extends to a surjective ring homomorphism $k_0[[X_1, \dots, X_t]] \to A^\wedge$, such that $X_i \mapsto x_i$ for $i = 1, \dots, t$. Denote by J the kernel of this surjection; in view of (4.7.32), we have

$$\mathrm{length}_{A^\wedge} k_0[[X_1, \dots, X_t]]/(J, X_1^{q^\nu}, \dots, X_t^{q^\nu}) = q^{\nu t}$$

which means that $J \subset (X_1^{q^\nu}, \dots, X_t^{q^\nu})$ for every $\nu > 0$. We conclude that $J = 0$, and $A^\wedge = k_0[[X_1, \dots, X_t]]$ is regular, so the same holds for A. $\square$

The last result of this section is a characterization of regular local rings via the cotangent complex, borrowed from [1], which shall be used in the following section on excellent rings.

Lemma 4.7.33. *Let A be a ring, $(a_1, \dots, a_n)$ a regular sequence of elements of A, that generates an ideal $I \subset A$, and set $A_0 := A/I$. Then there is a natural isomorphism*

$$\mathbb{L}_{A_0/A} \xrightarrow{\sim} I/I^2[1] \qquad \textit{in } \mathsf{D}(A_0\text{-}\mathbf{Mod})$$

and I/I^2 is a free A_0-module of rank n.

Proof. Notice that $H_0\mathbb{L}_{A_0/A} = 0$, and there is a natural isomorphism $H_1\mathbb{L}_{A_0/A} \xrightarrow{\sim} I/I^2$ ([56, Ch.III, Cor.1.2.8.1]). There follows a natural morphism $\mathbb{L}_{A_0/A} \to I/I^2[1]$, and we shall show more precisely, that this morphism is an isomorphism. We proceed by induction on n. Hence, suppose first that $n = 1$, set $a := a_1$, and $B := A[T]$, the free polynomial A-algebra in one variable; define a map of A-algebras $B \to A$ by the rule $T \mapsto a$. Set also $B_0 := B/TB$ (so B_0 is isomorphic to A). Since a is regular, it is easily seen that the natural morphism

$$B_0 \overset{\mathbf{L}}{\otimes}_B A \to B_0 \otimes_B A = A_0$$

is an isomorphism in $\mathsf{D}(B\text{-}\mathbf{Mod})$. It follows that the induced morphism $\mathbb{L}_{B_0/B} \otimes_{B_0} A_0 \to \mathbb{L}_{A_0/A}$ is an isomorphism in $\mathsf{D}(A_0\text{-}\mathbf{Mod})$ ([56, Ch.II, Prop.2.2.1]), so it suffices to check the assertion for the ring B and its regular element T. However, the sequence of ring homomorphisms $A \to B \to B_0$ induces a distinguished triangle ([56, Ch.II, Prop.2.1.2])

$$\mathbb{L}_{B/A} \otimes_B B_0 \to \mathbb{L}_{B_0/A} \to \mathbb{L}_{B_0/B} \to \mathbb{L}_{B/A} \otimes_B B_0[1]$$

and since clearly $\mathbb{L}_{B_0/A} \simeq 0$ in $\mathsf{D}(B_0\text{-}\mathbf{Mod})$, and $\mathbb{L}_{B/A} \simeq \Omega^1_{B/A}[0] \simeq B[0]$ ([56, Ch.II, Prop.1.2.4.4]), the assertion follows (details left to the reader). Next, suppose that $n > 1$, and that the assertion is already known for regular sequences of length $< n$. Denote by $I' \subset A$ the ideal generated by $a_1, \dots, a_{n-1}$, and set $A' := A/I'$. There follows a sequence of ring homomorphisms $A \to A' \to A_0$, and the inductive assumption implies that the natural morphisms

$$\mathbb{L}_{A'/A} \to I'/I'^2[1] \qquad \mathbb{L}_{A_0/A'} \to I/(I^2 + I')[1]$$

are isomorphisms, and I'/I'^2 (resp. $I/(I^2 + I')$) is a free A'-module (resp. A_0-module) of rank $n - 1$ (resp. of rank 1). Then the assertion follows easily, by inspecting the distinguished triangle

$$\mathbb{L}_{A'/A} \otimes_{A'} A_0 \to \mathbb{L}_{A_0/A} \to \mathbb{L}_{A_0/A'} \to \mathbb{L}_{A'/A} \otimes_{A'} A_0[1]$$

given again by [56, Ch.II, Prop.2.1.2] (details left to the reader). $\square$

Proposition 4.7.34. *Let A be a local noetherian ring, $a \in A$ a non-invertible element, and set $A_0 := A/aA$. The following conditions are equivalent :*

(a) *a is a regular element of A.*
(b) *$H_2\mathbb{L}_{A_0/A} = 0$, and aA/a^2A is a free A_0-module of rank one.*

Proof. For any ring R, any non-invertible element $x \in R$, and any $n \in \mathbb{N}$, set $R_n := R/x^{n+1}R$, and consider the R_0-linear map

$$\beta_{x,n} : R_0 \to x^n R/x^{n+1}R$$

induced by multiplication by x^n. We remark :

Claim 4.7.35. Suppose that R is a noetherian local ring, denote by κ_R the residue field of R, and let $n > 0$ be any given integer. The following conditions are equivalent :

(c) $\beta_{x,n}$ is an isomorphism.
(d) $x^n \neq 0$ and $\mathrm{Tor}_1^{R_0}(x^n R/x^{n+1}R, \kappa_R) = 0$.
(e) $x^n \neq 0$ and the surjection $R_0 \to \kappa_R$ induces a surjective map

$$H_2(\mathbb{L}_{R_{n-1}/R} \otimes_{R_{n-1}} R_0) \to H_2(\mathbb{L}_{R_{n-1}/R} \otimes_{R_{n-1}} \kappa_R).$$

Proof of the claim. It is easily seen that (c)⇒(d).

Conversely, if (d) holds, notice that $x^n R/x^{n+1}R \neq 0$, since $\bigcap_{n\in\mathbb{N}} x^n R = 0$. Hence $\kappa_R \otimes_R \beta_{x,n}$ is an isomorphism of one-dimensional κ_R-vector spaces. On the other hand, under assumption (d), the natural map $\kappa_R \otimes_R \mathrm{Ker}\,\beta_{x,n} \to \mathrm{Ker}(\kappa_R \otimes_R \beta_{x,n})$ is an isomorphism. By Nakayama's lemma, we conclude that $\mathrm{Ker}\,\beta_{x,n} = 0$, *i.e.* (c) holds.

Next, recall the natural isomorphism of R_0-modules

$$H_1(\mathbb{L}_{R_{n-1}/R} \otimes_{R_{n-1}} M) \xrightarrow{\sim} x^n R/x^{2n} R \otimes_{R_{n-1}} M \xrightarrow{\sim} x^n R/x^{n+1} R \otimes_{R_0} M$$

for every R_0-module M ([56, Ch.III, Cor.1.2.8.1]). Denote by $\mathfrak{m}_0$ the kernel of the surjection $R_0 \to \kappa_R$; there follows a left exact sequence

$$0 \to \mathrm{Tor}_1^{R_0}(x^n R/x^{n+1} R, \kappa_R) \to H_1(\mathbb{L}_{R_{n-1}/R} \otimes_{R_{n-1}} \mathfrak{m}_0) \to H_1(\mathbb{L}_{R_{n-1}/R} \otimes_{R_{n-1}} R_0)$$

which shows that (d)⇔(e) (details left to the reader). ◊

Claim 4.7.36. In the situation of claim 4.7.35, the following conditions are equivalent :

(f) x is a regular element of R.

(g) $\beta_{x,n}$ is an isomorphism, for every $n \in \mathbb{N}$.

Proof of the claim. If (f) holds, x^n is a regular element of R for every $n > 0$, and then (g) follows easily. Conversely, assume (g), and suppose that $yx = 0$ for some $y \in R$; we claim that $y \in x^n R$ for every $n \in \mathbb{N}$. We argue by induction on n : for $n = 0$, there is nothing to prove. Suppose that we have already obtained a factorization $y = x^n z$ for some $z \in R$. Then $x^{n+1} z = 0$, so the class of z in R_0 lies in $\mathrm{Ker}\,\beta_{n+1}$, hence this class must vanish, *i.e.* $z \in xR$, and therefore $y \in x^{n+1} R$. Since $\bigcap_{n\in\mathbb{N}} x^n R = 0$, we deduce that $y = 0$, whence (f). ◊

Claim 4.7.37. Let $f : R \to R'$ be any ring homomorphism, and set $x' := f(x)$. Suppose that $\beta_{x,n}$ and $\beta_{x',n}$ are both isomorphisms, for some integer $n \in \mathbb{N}$, and set $R'_n := R'/x'^{n+1} R'$. Then the induced morphism

$$\mathbb{L}_{R_0/R_n} \otimes_{R_0} R'_0 \to \mathbb{L}_{R'_0/R'_n}$$

is an isomorphism in $\mathsf{D}(R'_0\text{-}\mathbf{Mod})$.

Proof of the claim. If $n = 0$, there is nothing to prove, hence assume that $n > 0$. We remark that, for every $i = 0, \dots, n$, the complex

$$R_n \xrightarrow{x^{i+1}} R_n \xrightarrow{x^{n-i}} R_n$$

is exact. Indeed, for $i = 0$, this results immediately from the assumption that $\mathrm{Ker}\,\beta_{n,x} = 0$. Suppose that $i > 0$, and that the assertion is already known for $i-1$; then, if $yx^{n-i} \in x^{n+1} R$ for some $y \in R$, the inductive hypothesis yields $y \in x^i R$, so say that $y = x^i u$ and $yx^{n-i} = zx^{n+1}$ for some $u, z \in R$. It follows that $x^n(u - zx) = 0$, and then $u - zx \in xR$, again since $\mathrm{Ker}\,\beta_{x,n} = 0$; thus, $u \in xR$, and $y \in x^{i+1} R$, as asserted.

We deduce that the R_n-module R_0 admits a free resolution

$$\Sigma \quad : \quad \cdots \to R_n \xrightarrow{x} R_n \xrightarrow{x^n} R_n \xrightarrow{x} R_n \to R_0.$$

The same argument applies to R' and its element x', and yields a corresponding free resolution Σ' of the R'_n-module R'_0. A simple inspection shows that $\Sigma' \otimes_{R_n} R'_n = \Sigma$, *i.e.* the natural morphism $R_0 \overset{\mathbf{L}}{\otimes}_{R_n} R'_n \to R'_0$ is an isomorphism in $\mathsf{D}(R'_0\text{-}\mathbf{Mod})$. The claim then follows from [56, Ch.II, Prop.2.2.1]. ◊

With these preliminaries, we may now return to the situation of the proposition : first, lemma 4.7.33 says that (a)⇒(b). For the converse, we shall apply the criterion of claim 4.7.36 : namely, we shall show, by induction on n, that $\beta_{a,n} : A_0 \to a^n A/a^{n+1} A$ is bijective for every $n \in \mathbb{N}$.

For $n = 0$, there is nothing to prove. Assume that $n > 0$, and that the assertion is already known for $n-1$. Let $\mathfrak{m} \subset A[T]$ be the (unique) maximal ideal containing T, and set $B := A_\mathfrak{m}$. We let $f : B \to A$ be the map of A-algebras given by the rule : $T \mapsto a$. Define as usual $B_n := B/T^{n+1} B$ and $A_n := A/a^{n+1} A$ for every $n \in \mathbb{N}$. Clearly $\beta_{T,n-1} : B_0 \to T^{n-1} B/T^n B$ is bijective, and the same holds for $\beta_{a,n-1}$, by inductive assumption. Then, claim 4.7.37 says that the induced morphism $\mathbb{L}_{B_0/B_{n-1}} \otimes_{B_0} A_0 \to \mathbb{L}_{A_0/A_{n-1}}$ is an isomorphism in $\mathsf{D}(A_0\text{-}\mathbf{Mod})$.

Denote by κ the residue field of A and B, and notice as well that $H_2(\mathbb{L}_{A_0/A} \otimes_{A_0} \kappa) = 0$, by virtue of (b). Consequently, the commutative diagram of ring homomorphisms

$$\begin{array}{ccccc} B & \longrightarrow & B_{n-1} & \longrightarrow & B_0 \\ \downarrow & & \downarrow & & \downarrow \\ A & \longrightarrow & A_{n-1} & \longrightarrow & A_0 \end{array}$$

induces a commutative ladder with exact rows ([56, Ch.II, Prop.2.1.2]) :

$$\begin{array}{ccccc} H_3(\mathbb{L}_{B_0/B_{n-1}} \otimes_{B_0} \kappa) & \longrightarrow & H_2(\mathbb{L}_{B_{n-1}/B} \otimes_{B_{n-1}} \kappa) & \longrightarrow & H_2(\mathbb{L}_{B_0/B} \otimes_{B_0} \kappa) \\ \downarrow & & \downarrow & & \downarrow \\ H_3(\mathbb{L}_{A_0/A_{n-1}} \otimes_{A_0} \kappa) & \longrightarrow & H_2(\mathbb{L}_{A_{n-1}/A} \otimes_{A_{n-1}} \kappa) & \longrightarrow & 0 \end{array}$$

whose left vertical arrow is an isomorphism. It follows that the central vertical arrow is surjective. Consider now the commutative diagram

$$(4.7.38) \qquad \begin{array}{ccc} H_2(\mathbb{L}_{B_{n-1}/B} \otimes_{B_{n-1}} B_0) & \longrightarrow & H_2(\mathbb{L}_{B_{n-1}/B} \otimes_{B_{n-1}} \kappa) \\ \downarrow & & \downarrow \\ H_2(\mathbb{L}_{A_{n-1}/A} \otimes_{A_{n-1}} A_0) & \longrightarrow & H_2(\mathbb{L}_{A_{n-1}/A} \otimes_{A_{n-1}} \kappa) \end{array}$$

induced by the maps $B_0 \to A_0 \to \kappa$. We have just seen that the right vertical arrow of (4.7.38) is surjective, and the same holds for its top horizontal arrow, in light of claim 4.7.35. Thus, finally, the bottom horizontal arrow is surjective as well, so $\beta_{a,n}$ is an isomorphism (claim 4.7.35), and the proposition is proved. □

Theorem 4.7.39. *Let A be a noetherian local ring, $I \subset A$ an ideal, and set $A_0 := A/I$. The following conditions are equivalent :*

(a) *Every minimal system of generators of I is a regular sequence of elements of A.*
(b) *I is generated by a regular sequence of A.*
(c) *The natural morphism $\mathbb{L}_{A_0/A} \to I/I^2[1]$ is an isomorphism in $\mathsf{D}(A_0\text{-}\mathbf{Mod})$, and I/I^2 is a flat A_0-module.*
(d) *$H_2\mathbb{L}_{A_0/A} = 0$, and I/I^2 is a flat A_0-module.*

Proof. Clearly (a)⇒(b) and (c)⇒(d); also, lemma 4.7.33 shows that (b)⇒(c).

(d)⇒(a): Let $(a_1, \dots, a_n)$ be a minimal system of generators for I; recall that the length n of the sequence equals $\dim_\kappa I \otimes_A \kappa$, where κ denotes the residue field of A. We shall argue by induction on n. For $n = 0$, there is nothing to show, and the case $n = 1$ is covered by proposition 4.7.34. Set $B := A/a_1A$ and $J := IB$. Assumption (d) implies that $H_2(\mathbb{L}_{A_0/A} \otimes_{A_0} \kappa) = 0$, therefore the sequence of ring homomorphisms $A \to B \to A_0$ induces an exact sequence

$$0 \to H_2(\mathbb{L}_{A_0/B} \otimes_{A_0} \kappa) \to H_1(\mathbb{L}_{B/A} \otimes_B \kappa) \to I \otimes_A \kappa \to J \otimes_B \kappa \to 0$$

([56, Ch.II, Prop.2.1.2 and Ch.III, Cor.1.2.8.1]). However, clearly J admits a generating system of length $n-1$, hence $n' := \dim_\kappa J \otimes_B \kappa < n$. On the other hand, $\dim_\kappa H_1(\mathbb{L}_{B/A} \otimes_B \kappa) = 1$, so we have necessarily $n' = n-1$ and $H_2(\mathbb{L}_{A_0/B} \otimes_{A_0} \kappa) = 0$. The latter means that $H_2\mathbb{L}_{A_0/B} = 0$ and J/J^2 is a flat B-module. By inductive assumption, we deduce that the sequence $(\overline{a}_2, \dots, \overline{a}_n)$ of the images in B of $(a_2, \dots, a_n)$, is regular. By virtue of lemma 4.7.33, it follows that $H_3(\mathbb{L}_{A_0/B} \otimes_{A_0} \kappa) = 0$, whence a left exact sequence

$$0 \to H_2(\mathbb{L}_{B/A} \otimes_B \kappa) \to H_2(\mathbb{L}_{A_0/A} \otimes_{A_0} \kappa) = 0$$

obtained by applying again [56, Ch.II, Prop.2.1.2 and Ch.III, Cor.1.2.81] to the sequence $A \to B \to A_0$. Thus, $H_2\mathbb{L}_{B/A} = 0$ and a_1A/a_1^2A is a flat B-module, so a_1 is a regular element (proposition 4.7.34) and finally, $(a_1, \dots, a_n)$ is a regular sequence, as required. $\square$

Corollary 4.7.40. *Let A be a local noetherian ring, with maximal ideal $\mathfrak{m}$, and residue field κ. Then the following conditions are equivalent :*

(a) *A is regular.*
(b) *The natural morphism $\mathbb{L}_{\kappa/A} \to \mathfrak{m}/\mathfrak{m}^2[1]$ is an isomorphism.*
(c) *$H_2\mathbb{L}_{\kappa/A} = 0$.*

Proof. It follows immediately, by invoking theorem 4.7.39 with $I := \mathfrak{m}$, and lemma 4.7.33. $\square$

4.8. **Excellent rings.** Recall that a morphism of schemes $f : X \to Y$ is called *regular*, if it is flat, and for every $y \in Y$, the fibre $f^{-1}(y)$ is locally noetherian and regular ([31, Ch.IV, Déf.6.8.1]).

Lemma 4.8.1. *Let $f : X \to Y$ and $g : Y \to Z$ be two morphisms of locally noetherian schemes. We have :*

(i) *If f and g are regular, then the same holds for $g \circ f$.*
(ii) *If $g \circ f$ is regular, and f is faithfully flat, then g is regular.*

Proof. (i): Clearly $h := g \circ f$ is flat. Let $z \in Z$ be any point, and K any finite extension of $\kappa(z)$. Set

$$X' := h^{-1}(z) \times_{\kappa(z)} K \qquad \text{and} \qquad Y' := g^{-1}(z) \times_{\kappa(z)} K.$$

It is easily seen that the induced morphism $f' : X' \to Y'$ is regular. Moreover, for every $y \in Y'$, the local ring $\mathscr{O}_{Y',y'}$ is regular, since g is regular. Then the assertion follows from :

Claim 4.8.2. Let $A \to B$ be a flat and local ring homomorphism of local noetherian rings. Denote by $\mathfrak{m}_A \subset A$ the maximal ideal, and suppose that both A and $B_0 := B/\mathfrak{m}_AB$ are regular. Then B is regular.

Proof of the claim. On the one hand, $\dim B = \dim A + \dim B_0$ ([30, Ch.IV, Cor.6.1.2]). On the other hand, let $a_1, \dots, a_n$ be a minimal generating system for $\mathfrak{m}_A$, and $b_1, \dots, b_m$ a system of elements of the maximal ideal $\mathfrak{m}_B$ of B, whose images in B_0 is a minimal generating system for $\mathfrak{m}_B/\mathfrak{m}_AB$. By Nakayama's lemma, it is easily seen that the system $a_1, \dots, a_n, b_1, \dots, b_m$ generates the ideal $\mathfrak{m}_B$. Since A and B_0 are regular, $n = \dim A$ and $m = \dim B$, so $n + m = \dim B$, and the claim follows. $\diamondsuit$

(ii): Clearly g is flat. Then the assertion follows easily from [30, Ch.0, Prop.17.3.3(i)] : details left to the reader. $\square$

Definition 4.8.3. Let A be a noetherian ring.

(i) We say that A is a *G-ring*, if the formal fibres of $\operatorname{Spec} A$ are geometrically regular, *i.e.* for every $\mathfrak{p} \in \operatorname{Spec} A$, the natural morphism $\operatorname{Spec} A^\wedge_\mathfrak{p} \to \operatorname{Spec} A_\mathfrak{p}$ from the spectrum of the $\mathfrak{p}$-adic completion of A, is regular : see [31, Ch.IV, §7.3.13].
(ii) We say that A is *quasi-excellent*, if A is a G-ring, and moreover the following holds. For every prime ideal $\mathfrak{p} \subset A$, and every finite radicial extension K' of the field of fractions K of $B := A/\mathfrak{p}$, there exists a finite B-subalgebra B' of K' such that the field of fractions of B' is K', and the *regular locus* of $\operatorname{Spec} B'$ is an open subset (the latter is the set of all prime ideals $\mathfrak{q} \subset B'$ such that $B'_\mathfrak{q}$ is a regular ring).
(iii) We say that A is a *Nagata ring*, if the following holds. For every $\mathfrak{p} \in \operatorname{Spec} A$ and every finite field extension $\kappa(\mathfrak{p}) \subset L$, the integral closure of $A/\mathfrak{p}$ in L is a finite A-module. (The rings enjoying this latter property are called *universally japanese* in [30, Ch.0, Déf.23.1.1].)

(iv) We say that A is *universally catenarian* if every A-algebra B of finite type is catenarian, *i.e.* any two saturated chains $(\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_n)$, $(\mathfrak{q}_0 \subset \cdots \subset \mathfrak{q}_m)$ of prime ideals of B, with $\mathfrak{p}_0 = \mathfrak{q}_0$ and $\mathfrak{p}_n = \mathfrak{q}_m$, have the same length (so $n = m$) ([31, Ch.IV, Déf.5.6.2]).

(v) We say that A is *excellent*, if it is quasi-excellent and universally catenarian ([31, Ch.IV, Déf.7.8.2]).

Lemma 4.8.4. *Let A be a noetherian ring.*

(i) *If A is quasi-excellent, then A is a Nagata ring.*
(ii) *If A is a G-ring, then every quotient and every localization of A is a G-ring.*
(iii) *Suppose that the natural morphism* $\mathrm{Spec}\, A^\wedge_\mathfrak{m} \to \mathrm{Spec}\, A_\mathfrak{m}$ *is regular for every maximal ideal* $\mathfrak{m} \subset A$. *Then A is a G-ring.*
(iv) *If A is a local G-ring, then A is quasi-excellent.*
(v) *If A is a complete local ring, then A is excellent.*

Proof. (i): This is [31, Ch.IV, Cor.7.7.3].

(ii): The assertion for localizations is obvious. Next, if $I \subset A$ is any ideal, and $\mathfrak{p} \subset A$ any prime ideal containing I, then $(A/I)^\wedge_\mathfrak{p} = A^\wedge_\mathfrak{p}/IA^\wedge_\mathfrak{p}$, from which it is immediate that A/I is a G-ring, if the same holds for A.

(iv) follows from [31, Ch.IV, Th.6.12.7, Prop.7.3.18, Th.7.4.4(ii)].

(v): In light of (iv), it suffices to remark that every complete noetherian local ring is universally catenarian ([31, Ch.IV, Prop.5.6.4] and [30, Ch.0, Th.19.8.8(i)]) and is a G-ring ([30, Ch.0, Th.22.3.3, Th.22.5.8, and Prop.19.3.5(iii)]).

(iii): Let us remark, more generally :

Claim 4.8.5. Let $\varphi : A \to B$ be a faithfully flat ring homomorphism of noetherian rings, such that $f := \mathrm{Spec}\,\varphi$ is regular. If B is a G-ring, the same holds for A.

Proof of the claim. In light of (ii), we easily reduce to the case where both A and B are local, φ is a local ring homomorphism, and it suffices to show that the natural morphism $\pi_A : \mathrm{Spec}\, A^\wedge \to \mathrm{Spec}\, A$ is regular (where $A^\wedge$ is the completion of A). Consider the commutative diagram :

$$
\begin{tikzcd}
\mathrm{Spec}\, B^\wedge \arrow[r, "f^\wedge"] \arrow[d, "\pi_B"'] & \mathrm{Spec}\, A^\wedge \arrow[d, "\pi_A"] \\
\mathrm{Spec}\, B \arrow[r, "f"] & \mathrm{Spec}\, A.
\end{tikzcd}
\tag{4.8.6}
$$

By assumption, π_B is a regular morphism; Then the same holds for $f \circ \pi_B = \pi_A \circ f^\wedge$ (lemma 4.8.1(i)). However, it is easily seen that the induced map $\varphi^\wedge : A^\wedge \to B^\wedge$ is still a local ring homomorphism, hence $f^\wedge$ is faithfully flat, so the claim follows from lemma 4.8.1(ii). ◊

Now, in order to prove (iii), it suffices to check that $A_\mathfrak{m}$ is a G-ring for every maximal ideal $\mathfrak{m} \subset A$. In view of our assumption, the latter assertion follows from claim 4.8.5 and (v). □

Definition 4.8.7. Let A be any topological ring, whose topology is linear; we shall consider :

- For any A-algebra C, the category $\mathsf{Exal}_A(C)$ whose objects are all short exact sequences of A-modules

$$\Sigma \quad : \quad 0 \to M \to E \xrightarrow{\psi} C \to 0 \tag{4.8.8}$$

such that E is an A-algebra (with A-module structure given by the structure map $A \to E$), and ψ is a map of A-algebras. The morphisms in $\mathsf{Exal}_A(C)$ are the commutative ladders of A-modules :

(4.8.9)
$$\begin{tikzcd} 0 \arrow[r] & M' \arrow[r] \arrow[d] & E' \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & M'' \arrow[r] & E'' \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

where g is a map of A-algebras.

- For any topological A-algebra C, whose topology is linear, the category $\mathsf{Exaltop}_A(C)$ whose objects are the short exact sequences of A-modules (4.8.8) such that E is a topological A-algebra whose topology is linear (and again, with A-module structure given by the structure map $A \to E$), and ψ is a continuous and open map of topological A-algebras, whose kernel M is a discrete topological space, for the topology induced from E. The morphisms in $\mathsf{Exaltop}_A(C)$ are the commutative ladders (4.8.9) such that g is a continuous map of topological A-algebras.

4.8.10. Now, let A be as in definition 4.8.7, C any topological A-algebra whose topology is linear, $\varphi : A \to C$ the structure morphism, and $(I_\lambda \mid \lambda \in \Lambda)$ (resp. $(J_{\lambda'} \mid \lambda' \in \Lambda')$) a filtered system of open ideals of A (resp. of C), which is a fundamental system of open neighborhoods of 0. Let $\Lambda'' \subset \Lambda \times \Lambda'$ be the subset of all (λ, λ') such that $\varphi(I_\lambda) \subset J_{\lambda'}$; the set Λ'' is partially ordered, by declaring that $(\lambda, \lambda') \leq (\mu, \mu')$ if and only if $I_\mu \subset I_\lambda$ and $J_{\mu'} \subset J_{\lambda'}$. Set $A_\lambda := A/I_\lambda$ for every $\lambda \in \Lambda$ (resp. $C_{\lambda'} := C/J_{\lambda'}$ for every $\lambda' \in \Lambda'$); for $(\lambda, \lambda'), (\mu, \mu') \in \Lambda''$ with $(\lambda, \lambda') \leq (\mu, \mu')$, the surjection $\pi_{\mu'\lambda'} : C_{\mu'} \to C_{\lambda'}$ induces a functor

$$\mathsf{Exal}_{A_\lambda}(C_{\lambda'}) \to \mathsf{Exal}_{A_\mu}(C_{\mu'}) \qquad \Sigma \mapsto \Sigma * \pi_{\mu'\lambda'}$$

(see [36, §2.5.5]) and clearly the rule $(\lambda, \lambda') \mapsto \mathsf{Exal}_{A_\lambda}(C_{\lambda'})$ yields a pseudo-functor

$$\mathsf{E} : (\Lambda'', \leq) \to \mathbf{Cat}.$$

Moreover, we have a pseudo-cocone :

$$\mathsf{E} \Rightarrow \mathsf{Exaltop}_A(C) \tag{4.8.11}$$

defined as follows. To any $(\lambda, \lambda') \in \Lambda''$ and any object $\Sigma_{\lambda,\lambda'} : 0 \to M \to E_{\lambda'} \to C_{\lambda'} \to 0$ of $\mathsf{Exal}_{A_\lambda}(C_{\lambda'})$, one assigns the extension (4.8.8) obtained by pulling back $\Sigma_{\lambda,\lambda'}$ along the projection $\pi_{\lambda'} : C \to C_{\lambda'}$. Let $\beta : E \to E_{\lambda'}$ be the induced projection; we endow E with the linear topology defined by the fundamental system of all open ideals of the form $\psi^{-1}J \cap \beta^{-1}J'$ where J (resp. J') ranges over the set of open ideals of C (resp. over the set of all ideals of $E_{\lambda'}$). With this topology, it is easily seen that both ψ and the induced structure map $A \to E$ are continuous ring homomorphisms. Moreover, if $J \subset J_{\lambda'}$, then $\psi^{-1}J \cap \beta^{-1}J' = J \times (J' \cap M)$, from which it follows that ψ is an open map. Furthermore, $M \cap \beta^{-1}0 = 0$, which shows that M is discrete, for the topology induced by the inclusion map $M \to E$. Summing up, we have associated to $\Sigma_{\lambda,\lambda'}$ a well defined object $\Sigma_{\lambda,\lambda'} * \pi_{\lambda'}$ of $\mathsf{Exaltop}_A(C)$, and it is easily seen that the rule $\Sigma_{\lambda,\lambda'} \mapsto \Sigma_{\lambda,\lambda'} * \pi_{\lambda'}$ is functorial in $\Sigma_{\lambda,\lambda'}$, and for $(\lambda, \lambda) \leq (\mu, \mu')$ in Λ'', there is a natural isomorphism in $\mathsf{Exaltop}_A(C)$:

$$\Sigma_{\lambda,\lambda'} * \pi_{\lambda'} \xrightarrow{\sim} (\Sigma_{\lambda,\lambda'} * \pi_{\mu'\lambda'}) * \pi_{\mu'}$$

(details left to the reader).

Lemma 4.8.12. *The pseudo-cocone* (4.8.11) *induces an equivalence of categories :*

$$\beta : 2\text{-}\operatorname*{colim}_{\Lambda''} \mathsf{E} \xrightarrow{\sim} \mathsf{Exaltop}_A(C).$$

Proof. Let Σ as in (4.8.8) be any object of $\mathsf{Exaltop}_A(C)$. By assumption, there exists an open ideal $J \subset E$ such that $M \cap J = 0$. Since ψ is an open map, there exists $\lambda' \in \Lambda'$ such that $J_{\lambda'} \subset \psi(J)$, and after replacing J by $J \cap \psi^{-1}J_{\lambda'}$, we may assume that $\psi(J) = J_{\lambda'}$. Likewise, if

$\varphi_E : A \to E$ is the structure morphism, there exists $\lambda \in \Lambda$ such that $I_\lambda \subset \varphi_E^{-1} J$, and it follows that the induced extension

$$\Sigma_{\lambda,\lambda'} : 0 \to M \to E/J \to C_{\lambda'} \to 0$$

is an object of $\mathsf{Exal}_{A_\lambda}(C_{\lambda'})$. We notice :

Claim 4.8.13. There exists a natural isomorphism $\Sigma \stackrel{\sim}{\to} \Sigma_{\lambda,\lambda'} * \pi_{\lambda'}$ in $\mathsf{Exaltop}_A(C)$.

Proof of the claim. (i): By construction, ψ and the projection $\pi_J : E \to E/J$ define a unique morphism $\gamma : E \to (E/J) \times_{C_{\lambda'}} C$ of A-algebras, restricting to the identity map on M (which is an ideal in both of these A-algebras). It is clear that γ is an isomorphism, and therefore it yields a natural isomorphism $\Sigma \stackrel{\sim}{\to} \Sigma_{\lambda,\lambda'} * \pi_{\lambda'}$ in $\mathsf{Exal}_A(C)$. It remains to check that γ is continuous and open. For the continuity, it suffices to remark that, for every ideal $I \subset E/J$ and every $\mu' \in \Lambda'$, the ideals $\gamma^{-1}(I \times_{C_{\lambda'}} C) = \pi_J^{-1} I$ and $\gamma^{-1}(E/J \times_{C_{\lambda'}} J_{\mu'}) = \psi^{-1} J_{\mu'}$ are open in E, which is obvious, since J is an open ideal and ψ is continuous. Lastly, let $I \subset E$ be any open ideal such that $I \subset J \cap \psi^{-1} J_{\lambda'}$; since ψ is an open map, it is easily seen that $\gamma(I) = 0 \times \psi(I)$ is an open ideal of $(E/J) \times_{C_{\lambda'}} C$, so γ is open. $\diamondsuit$

From claim 4.8.13 we see already that β is essentially surjective. It also follows easily that β is full. Indeed, consider any morphism $s : \Sigma' \to \Sigma''$ of $\mathsf{Exaltop}_A(C)$ as in (4.8.9), and pick an open ideal $J'' \subset E''$ with $J \cap M'' = 0$; set $J' := g^{-1} J''$, and notice that $J' \cap M' = 0$. Moreover, if the image of J'' in C equals $J_{\lambda'}$ for some $\lambda' \in \Lambda'$, then clearly the same holds for the image of J' in C. Therefore, in this case the foregoing construction yields objects $\Sigma'_{\lambda,\lambda'}$ and $\Sigma''_{\lambda,\lambda'}$ of $\mathsf{Exal}_{A_\lambda}(C_{\lambda'})$ (for a suitable $\lambda \in \Lambda$), whose middle terms are respectively E'/J' and E''/J'', and s descends to a morphism $s_{\lambda'} : \Sigma'_{\lambda,\lambda'} \to \Sigma''_{\lambda,\lambda'}$, whose middle term is the map $g_{\lambda'} : E'/J' \to E''/J''$ induced by g. By inspecting the proof of claim 4.8.13, we deduce a commutative diagram

$$\begin{array}{ccc} E' & \xrightarrow{\sim} & (E'/J') \times_{C_{\lambda'}} C \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g_{\lambda'} \times_{C_{\lambda'}} C} \\ E'' & \xrightarrow{\sim} & (E''/J'') \times_{C_{\lambda'}} C \end{array}$$

whose horizontal arrows are the maps that define the isomorphisms $\Sigma' \stackrel{\sim}{\to} \Sigma'_{\lambda,\lambda'} * \pi_{\lambda'}$ and $\Sigma'' \stackrel{\sim}{\to} \Sigma''_{\lambda,\lambda'} * \pi_{\lambda'}$ in $\mathsf{Exaltop}_A(C)$. It follows easily that $s_\lambda * \pi_{\lambda'} = s$, whence the assertion. Lastly, the faithfulness of β is immediate, since the projections $\pi_{\lambda'}$ are surjective maps. $\square$

4.8.14. Let A be as in definition 4.8.7, and C any A-algebra (resp. any topological A-algebra); we denote by

$$\mathsf{nilExal}_A(C) \qquad (\text{ resp. } \mathsf{nilExaltop}_A(C)\ )$$

the full subcategory of $\mathsf{Exal}_A(C)$ (resp. of $\mathsf{Exaltop}_A(C)$) whose objects are the *nilpotent extensions* of C, *i.e.* those extensions (4.8.8), where M is a nilpotent ideal of E. Moreover, in the situation of (4.8.10), clearly E restricts to a pseudo-functor

$$\mathsf{nilE} : (\Lambda'', \leq) \to \mathbf{Cat} \qquad (\lambda, \lambda') \mapsto \mathsf{nilExal}_{A_\lambda}(C_{\lambda'}).$$

Also, (4.8.11) restricts to a pseudo-cocone on nilE, and lemma 4.8.12 immediately implies an equivalence of categories

$$2\text{-}\operatorname*{colim}_{\Lambda''} \mathsf{nilE} \stackrel{\sim}{\to} \mathsf{nilExaltop}_A(C). \tag{4.8.15}$$

Proposition 4.8.16. *Let A and C be as in* (4.8.10). *The following holds :*

(i) *Suppose that* $J^2_{\lambda'}$ *is open in* C*, for every* $\lambda' \in \Lambda'$*. Then the forgetful functor*

$$\mathsf{nilExaltop}_A(C) \to \mathsf{nilExal}_A(C) \tag{4.8.17}$$

is fully faithful.

(ii) *Suppose additionally, that :*

(a) C *is a noetherian ring, and* $I \subset C$ *is an ideal such that the topology of* C *agrees with the* I*-preadic topology.*

(b) I^2_λ *is open in* A*, for every* $\lambda \in \Lambda$*.*

Then the essential image of (4.8.17) *is the (full) subcategory of all nilpotent extensions* (4.8.8) *such that the* C*-module* M/M^2 *is annihilated by a power of* I*.*

Proof. (i): The functor is obviously faithful, and in light of (4.8.15), we come down to the following situation. We have a commutative ladder of extensions of A-algebras

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d] & E \arrow[r, "\psi"] \arrow[d, "g"] & C \arrow[r] \arrow[d, "\pi_{\lambda'}"] & 0 \\ 0 \arrow[r] & N \arrow[r] & E' \arrow[r] & C_{\lambda'} \arrow[r] & 0 \end{tikzcd}$$

for some $\lambda' \in \Lambda'$, whose top (resp. bottom) row is an object of $\mathsf{nilExaltop}_A(C)$ (resp. of $\mathsf{nilExal}_{A_\lambda}(C_{\lambda'})$, for some $\lambda \in \Lambda$), and we need to show that the kernel of g contains an open ideal. To this aim, we remark :

Claim 4.8.18. For any open ideal $J \subset E$, the ideal J^2 is open as well.

Proof of the claim. We may assume that $J \cap M = 0$. We have $I := \psi(J^2) = \psi(J)^2$, so the assumption in (i) say that I is open in C, and therefore $\psi^{-1}I = J^2 \oplus M$ is open in E, so finally $J \cap \psi^{-1}I = J^2$ is open in E, as stated. $\Diamond$

Now, set $J := \psi^{-1}J_{\lambda'}$; then J is an open ideal of E, and clearly $g(I) \subset N$. Say that $N^k = 0$; then $g(I^k) = 0$, and I^k is an open ideal of E, by claim 4.8.18, whence the contention.

(ii): It is easily seen that, for every extension (4.8.8) in the essential image of (4.8.17), we must have $I^k(M/M^2) = 0$ for every sufficiently large $k \geq 0$ (details left to the reader). Conversely, consider a nilpotent extension Σ as in (4.8.8), with $I^k(M/M^2) = 0$ and $M^t = 0$ for some $k, t \in \mathbb{N}$. Pick a finite system $\mathbf{f} := (f_1, \dots, f_r)$ of elements of E whose images in B form a system of generators for I^k; notice that the ideal $(\mathbf{f}^n)B$ is open in B, and $(\mathbf{f}^n)M = 0$ for every $n \geq t$. There follows an inverse system of exact sequences

$$H_1(\mathbf{f}^n, B) \to M \xrightarrow{\beta_n} E/(\mathbf{f}^n)E \to B/(\mathbf{f}^n)B \to 0 \qquad \text{for every } n \geq t$$

(notation of (4.1.16)) with transition maps induced by the morphisms $\boldsymbol{\varphi}_\mathbf{f}$ of (4.1.23). As B is noetherian, lemma 4.1.30 and remark 4.1.31 imply that the system $(H_1(\mathbf{f}^n, B) \mid n \in \mathbb{N})$ is essentially zero. Therefore, the same holds for the inverse system $(\mathrm{Ker}\, \beta_n \mid n > 0)$. However, the transition map $\mathrm{Ker}\, \beta_{n+1} \to \mathrm{Ker}\, \beta_n$ is obviously injective for every $n \geq t$, so we conclude that β_n is injective for some sufficiently large integer n. For such n, we obtain a nilpotent extension

$$0 \to M \to E/(\mathbf{f}^n)E \to B/(\mathbf{f}^n)B \to 0. \tag{4.8.19}$$

Lastly, let $\varphi : A \to B$ be the structure map, and pick $\lambda \in \Lambda$ such that $I_\lambda \subset \varphi^{-1}I^k$; it is easily seen that $I^t_\lambda M = 0$ and $\varphi(I^s_\lambda) \subset (\mathbf{f}^n)B$ for $s \in \mathbb{N}$ large enough. Therefore, some sufficiently large power of I_λ annihilates $E/(\mathbf{f}^n)E$; under our assumption (b), such power of I_λ contains another open ideal I_μ, so (4.8.19) is an object of $\mathsf{nilExal}_{A_\mu}(B/(\mathbf{f}^n)B)$ whose image in $\mathsf{nilExaltop}_A(B)$ agrees with Σ. $\square$

Proposition 4.8.20. *Let A be a topological ring (whose topology is linear), B a noetherian A-algebra, $I \subset B$ an ideal, and suppose that :*

(i) *The structure map $A \to B$ is continuous for the I-adic topology on B.*
(ii) *For every open ideal $J \subset A$, the ideal J^2 is also open.*

Then the following conditions are equivalent :

(a) *B (with its I-adic topology) is a formally smooth A-algebra.*
(b) *$\Omega^1_{B/A} \otimes_B B/I$ is a projective B/I-module, and $H_1(\mathbb{L}_{B/A} \otimes_B B/I) = 0$.*

Proof. More generally, let A and C be as in (4.8.10), and M a *discrete* C-module, *i.e.* a C-module annihilated by an open ideal; we denote by $\mathrm{Exaltop}_A(C, M)$ the C-module of square zero topological A-algebra extensions of C by M. Likewise, for every $(\lambda, \lambda') \in \Lambda''$, and every $C_{\lambda'}$-module M, let $\mathrm{Exal}_{A_\lambda}(C_{\lambda'}, M)$ be the $C_{\lambda'}$-module of isomorphism classes of square zero A_λ-algebra extensions of $C_{\lambda'}$ by M. For $(\lambda, \lambda') \leq (\mu, \mu')$, we get a natural map of $C_{\mu'}$-modules

(4.8.21) $$\mathrm{Exal}_{A_\lambda}(C_{\lambda'}, M) \to \mathrm{Exal}_{A_\mu}(C_{\mu'}, M) \qquad \text{for every } C_{\lambda'}\text{-module } M$$

and (4.8.15) implies a natural isomorphism :

$$\operatorname*{colim}_{(\lambda,\lambda') \in \Lambda''} \mathrm{Exal}_{A_\lambda}(C_{\lambda'}, M) \xrightarrow{\sim} \mathrm{Exaltop}_A(C, M)$$

which is well defined for every discrete C-module M. By inspecting the definitions, and taking into account [30, Ch.0, Prop.19.4.3], it is easily seen that C is a formally smooth A-algebra if and only if $\mathrm{Exaltop}_A(C, M) = 0$ for every such discrete C-module M. We also have a natural C-linear map :

(4.8.22) $$\mathrm{Exaltop}_A(C, M) \to \mathrm{Exal}_A(C, M) \qquad \text{for every discrete } C\text{-module } M$$

and proposition 4.8.16(i) shows that (4.8.22) is an injective map, provided $J^2_{\lambda'}$ is an open ideal of C, for every $\lambda' \in \Lambda'$. Moreover, for $C := B$ (with its I-adic topology), the map (4.8.22) is an isomorphism, by virtue of proposition 4.8.16(ii).

Now, recall the natural isomorphism of C_λ-modules ([56, Ch.III, Th.1.2.3])

$$\mathrm{Exal}_A(B, M) \xrightarrow{\sim} \mathrm{Ext}^1_B(\mathbb{L}_{B/A}, M) \qquad \text{for every } B\text{-module } M.$$

Combining with the foregoing, we deduce a natural B-linear isomorphism

$$\mathrm{Exaltop}_A(B, M) \xrightarrow{\sim} \mathrm{Ext}^1_B(\mathbb{L}_{B/A}, M)$$

for every B-module M annihilated by some ideal I^k. Summing up, B is a formally smooth A-algebra if and only if $\mathrm{Ext}^1_B(\mathbb{L}_{B/A}, M)$ vanishes for every discrete B-module M. Set $B_0 := B/I$; by considering the I-adic filtration on a given M, a standard argument shows that the latter condition holds if and only if it holds for every B_0-module M, and in turns, this is equivalent to the vanishing of $\mathrm{Ext}^1_{B_0}(\mathbb{L}_{B/A} \otimes_B B_0, M)$ for every B_0-module M. This last condition is equivalent to (b), as stated. □

Proposition 4.8.23. *Let A be a ring, B a noetherian A-algebra of finite (Krull) dimension, $n \in \mathbb{N}$ an integer, and suppose that $H_k(\mathbb{L}_{B/A} \otimes_B \kappa(\mathfrak{p})) = 0$ for every prime ideal $\mathfrak{p} \subset B$ and every $k = n, \dots, n + \dim B$. Then $H_n(\mathbb{L}_{B/A} \otimes_B M) = 0$ for every B-module M.*

Proof. Let us start out with the following more general :

Claim 4.8.24. Let A be a ring, B a noetherian A-algebra, $\mathfrak{p} \in \operatorname{Spec} B$ a prime ideal, $n \in \mathbb{N}$ an integer, and suppose that the following two conditions hold :

(a) $H_n(\mathbb{L}_{B/A} \otimes_B \kappa(\mathfrak{p})) = 0$.
(b) $H_{n+1}(\mathbb{L}_{B/A} \otimes_B B/\mathfrak{q}) = 0$ for every proper specialization $\mathfrak{q}$ of $\mathfrak{p}$ in $\operatorname{Spec} B$.

Then, $H_n(\mathbb{L}_{B/A} \otimes_B B/\mathfrak{p}) = 0$.

Proof of the claim. Let $b \in B \setminus \mathfrak{p}$ be any element, and set $M := B/(\mathfrak{p} + bB)$. Since B is noetherian, M admits a finite filtration $M_0 \subset M_1 \subset \cdots \subset M_n := M$ such that, for every $i = 0, \dots, n-1$, the subquotient M_{i+1}/M_i is isomorphic to $B/\mathfrak{q}$, for some proper specialization $\mathfrak{q}$ of $\mathfrak{p}$ ([61, Th.6.4]). From (b), and a simple induction, we deduce that $H_{n+1}(\mathbb{L}_{B/A} \otimes_B M) = 0$. Whence, by considering the short exact sequence of B-modules

$$0 \to B/\mathfrak{p} \xrightarrow{b} B/\mathfrak{p} \to M \to 0$$

we see that scalar multiplication by b is an injective map on the B-module $H_n(\mathbb{L}_{B/A} \otimes_B B/\mathfrak{p})$. Since this holds for every $b \in B \setminus \mathfrak{p}$, we conclude that the natural map

$$H_n(\mathbb{L}_{B/A} \otimes_B B/\mathfrak{p}) \to H_n(\mathbb{L}_{B/A} \otimes_B B/\mathfrak{p}) \otimes_B B_\mathfrak{p} = H_n(\mathbb{L}_{B/A} \otimes_B \kappa(\mathfrak{p}))$$

is injective. Then the assertion follows from (a). $\Diamond$

Now, let $\mathfrak{p} \subset B$ be any prime ideal; the assumption, together with claim 4.8.24 and a simple induction on $d := \dim B/\mathfrak{p}$, shows that

$$H_k(\mathbb{L}_{B/A} \otimes_B B/\mathfrak{p}) = 0 \qquad \text{for every } k = n, \dots, n + \dim B - d.$$

For any B-module M, set $H(M) := H_n(\mathbb{L}_{B/A} \otimes_B M)$. Especially, we get $H(B/\mathfrak{p}) = 0$, for any prime ideal $\mathfrak{p} \subset B$. Since B is noetherian, it follows easily that $H(M) = 0$ for any B-module M of finite type (details left to the reader). Next, if M is arbitrary, we may write it as the union of the filtered family $(M_i \mid i \in I)$ of its submodules of finite type; since $H(M)$ is the colimit of the induced system $(H(M_i) \mid i \in I)$, we see that $H(M) = 0$, as sought. $\square$

Corollary 4.8.25. *Let $A \to B$ be a homomorphism of noetherian rings. Then the following conditions are equivalent :*

(a) $\Omega^1_{B/A}$ *is a flat B-module, and $H_i\mathbb{L}_{B/A} = 0$ for every $i > 0$.*
(b) $\Omega^1_{B/A}$ *is a flat B-module, and $H_1\mathbb{L}_{B/A} = 0$.*
(c) *The induced morphism of schemes $f : \operatorname{Spec} B \to \operatorname{Spec} A$ is regular.*
(d) $H_1(\mathbb{L}_{B/A} \otimes_B \kappa(x)) = 0$ *for every $x \in \operatorname{Spec} B$.*

Proof. Let $x \in X := \operatorname{Spec} B$ be any point; according to [30, Ch.0, Th.19.7.1], the following conditions are equivalent :

(e) the map on stalks $\mathscr{O}_{Y,f(x)} \to \mathscr{O}_{X,x}$ is formally smooth for the preadic topologies defined by the maximal ideals.
(f) f is flat at the point x, and the $\kappa(f(x))$-algebra $\mathscr{O}_{f^{-1}f(x),x}$ is geometrically regular.

On the other hand, by virtue of proposition 4.8.20, condition (e) is equivalent to the vanishing of $H_1(\mathbb{L}_{B/A} \otimes_B \kappa(x))$, whence (b)⇒(c)⇔(d). It remains to check that (d)⇒(a). However, assume (d); taking into account proposition 4.8.23, and arguing by induction on i, we easily show that (a) will follow, provided

$$H_i(\mathbb{L}_{B/A} \otimes_B \kappa(x)) = 0 \qquad \text{for every } x \in X \text{ and every } i > 1.$$

For every $x \in X$, set $B_x := \mathscr{O}_{X,x} \otimes_A \kappa(f(x))$; since B is A-flat, the latter holds if and only if

$$H_i(\mathbb{L}_{B_x/\kappa(f(x))} \otimes_B \kappa(x)) = 0 \qquad \text{for every } x \in X \text{ and every } i > 1$$

([56, Ch.II, Prop.2.2.1 and Ch.III, Cor.2.3.1.1]). Hence, we may replace A by $\kappa(f(x))$, and B by B_x, and assume from start that A is a field and B is a local geometrically regular A-algebra with residue field κ_B, and it remains to check that $H_i(\mathbb{L}_{B/A} \otimes_B \kappa_B) = 0$ for every $i > 1$. However, in view of corollary 4.7.40, the sequence of ring homomorphisms $A \to B \to \kappa_B$ yields an isomorphism

$$H_i(\mathbb{L}_{B/A} \otimes_B \kappa_B) \xrightarrow{\sim} H_i\mathbb{L}_{\kappa_B/A} \qquad \text{for every } i > 1$$

([56, Ch.II, Prop.2.1.2]) so we conclude by the following general :

Claim 4.8.26. Let $K \subset E$ be any extension of fields. Then $H_i\mathbb{L}_{E/K} = 0$ for every $i > 1$.

Proof of the claim. By [56, Ch.II, (1.2.3.4)], we may reduce to the case where E is a finitely generated extension of K, say $E = K(a_1, \dots, a_n)$. We proceed by induction on n. If $n = 1$, then E is either an algebraic extension or a purely transcendental extension of K. In the latter case, the assertion is immediate ([56, Ch.II, Prop.1.2.4.4 and Ch.III, Cor.2.3.1.1]). For the case of an algebraic extension, the assertion is a special case of [36, Th.6.3.32(i)] (we apply *loc.cit.* to the valued field $(K, |\cdot|)$ with trivial valuation $|\cdot|$). Lastly, if $n > 1$, set $L := K(a_1, \dots, a_{n-1})$. By inductive assumption we have $H_i\mathbb{L}_{L/K} = H_i\mathbb{L}_{E/L} = 0$ for $i > 1$; on the other hand, there is a distinguished triangle ([56, Ch.II, Prop.2.1.2])

$$\mathbb{L}_{L/K} \otimes_L E \to \mathbb{L}_{E/K} \to \mathbb{L}_{E/L} \to \mathbb{L}_{L/K} \otimes_L E[1] \qquad \text{in } \mathsf{D}(E\text{-}\mathbf{Mod}).$$

The sought vanishing follows immediately, $\square$

Corollary 4.8.27. *Let A be a local noetherian ring, $A^\wedge$ the completion of A. Then the following conditions are equivalent :*

(a) *A is quasi-excellent.*
(b) *$\Omega^1_{A^\wedge/A}$ is a flat $A^\wedge$-module, and $H_1\mathbb{L}_{A^\wedge/A} = 0$.*
(c) *$\Omega^1_{A^\wedge/A}$ is a flat $A^\wedge$-module, and $H_i\mathbb{L}_{A^\wedge/A} = 0$ for every $i > 0$.*

Proof. Taking into account lemma 4.8.4(iii), this is a special case of corollary 4.8.25. $\square$

Proposition 4.8.28. *Let $p > 0$ be a prime integer, A a regular local and excellent $\mathbb{F}_p$-algebra, B a local noetherian A-algebra, $\mathfrak{m}_B$ the maximal ideal of B, and M a B-module of finite type. Then the B-module $\Omega^1_{A/\mathbb{F}_p} \otimes_A M$ is separated for the $\mathfrak{m}_B$-preadic topology.*

Proof. Let $\varphi : A \to B$ be the structure morphism, and set $\mathfrak{p} := \varphi^{-1}\mathfrak{m}_B$; the localization $A_\mathfrak{p}$ is still excellent (lemma 4.8.4(ii,iv)) and regular, and clearly $\Omega^1_{A_\mathfrak{p}/\mathbb{F}_p} \otimes_{A_\mathfrak{p}} M = \Omega^1_{A/\mathbb{F}_p} \otimes_A M$, hence we may replace A by $A_\mathfrak{p}$, and assume that φ is local. Let $A^\wedge$ and $B^\wedge$ be the completions of A and B, set $M^\wedge := B^\wedge \otimes_B M$, and notice that both of the natural maps $\mathbb{F}_p \to A$ and $A \to A^\wedge$ are regular. In view of corollary 4.8.25, it follows that both of the natural B-linear maps

$$\Omega^1_{A/\mathbb{F}_p} \otimes_A M \to \Omega^1_{A/\mathbb{F}_p} \otimes_A M^\wedge \qquad \Omega^1_{A/\mathbb{F}_p} \otimes_A M \to \Omega^1_{A^\wedge/\mathbb{F}_p} \otimes_{A^\wedge} M^\wedge$$

are injective (to see the injectivity of the second map, one applies the transitivity triangle arising from the sequence of ring homomorphisms $\mathbb{F}_p \to A \to A^\wedge$: details left to the reader). Thus, we may assume that A is complete. Now, for every ring R, and every integer $m \in \mathbb{N}$, set

$$R_{(m)} := R[[T_1, \dots, T_m]] \qquad R_{\langle m \rangle} := R[[T_1^p, \dots, T_m^p]] \subset R_{(m)}. \tag{4.8.29}$$

With this notation, we have an isomorphism $A \xrightarrow{\sim} \kappa_{(d)}$ of $\mathbb{F}_p$-algebras, where κ is the residue field of A, and $d := \dim A$ ([30, Ch.0, Th.19.6.4]).

Claim 4.8.30. Let K be any field of characteristic p, and $m \in \mathbb{N}$ any integer. We have :

(i) There exists a cofiltered system $(K^\lambda \mid \lambda \in \Lambda)$ of subfields of K such that $[K : K^\lambda]$ is finite for every $\lambda \in \Lambda$, and $\bigcap_{\lambda \in \Lambda} K^\lambda = K^p$.
(ii) For every system $(K^\lambda \mid \lambda \in \Lambda)$ fulfilling the condition of (i), the following holds :
  (a) $\Omega^1_{K_{(m)}/K^\lambda_{\langle m \rangle}}$ is a free $K_{(m)}$-module of finite rank, for every $\lambda \in \Lambda$, and the rule
  $$M \mapsto \Omega_m(M) := \lim_{\lambda \in \Lambda} (\Omega^1_{K_{(m)}/K^\lambda_{\langle m \rangle}} \otimes_{K_{(m)}} M)$$
  defines an exact functor $K_{(m)}\text{-}\mathbf{Mod} \to K_{(m)}\text{-}\mathbf{Mod}$.
  (b) The natural map
  $$\eta_m(M) : \Omega^1_{K_{(m)}/\mathbb{F}_p} \otimes_{K_{(m)}} M \to \Omega_m(M)$$
  is injective for every $K_{(m)}$-module M.

(c) Let F (resp. F^λ) denote the field of fractions of $K_{(m)}$ (resp. of $K^\lambda_{\langle m\rangle}$, for every $\lambda \in \Lambda$); then $\bigcap_{\lambda\in\Lambda} F^\lambda = F^p$.

Proof of the claim. (i) and (ii.c) follow from [30, Ch.0, Prop.21.8.8] (and its proof).

(ii.a): For given $\lambda \in \Lambda$, say that $x_1, \dots, x_r$ is a p-basis of K over K^λ; then it is easily seen that $x_1, \dots, x_r, T_1, \dots, T_m$ is a p-basis of $K_{(m)}$ over $K^\lambda_{\langle m\rangle}$ (see [30, Ch.0, Déf.21.1.9]). According to [30, Ch.0, Cor.21.2.5], it follows that $\Omega^1_{K_{(m)}/K^\lambda_{\langle m\rangle}}$ is the free $K_{(m)}$-module of finite type with basis $dx_1, \dots, dx_r, dT_1, \dots, dT_m$. Moreover, say that $K^\mu \subset K^\lambda$, and let $x_{r+1}, \dots, x_s$ be a p-basis of K^λ over K^μ; then $x_1, \dots, x_s$ is a p-basis of K over K^μ ([30, Ch.0, Lemme 21.1.10]), so the induced map

$$\Omega^1_{K_{(m)}/K^\mu_{\langle m\rangle}} \to \Omega^1_{K_{(m)}/K^\lambda_{\langle m\rangle}}$$

is a projection onto a direct factor, and the assertion follows easily.

(ii.b): We are easily reduced to the case where M is a $K_{(m)}$-module of finite type, and in light of (ii.a), we may further assume that M is a cyclic $K_{(m)}$-module. Next, we remark that, due to (ii.c), the natural map

$$\Omega^1_{F/\mathbb{F}_p} \to \lim_{\lambda\in\Lambda} \Omega^1_{F/F^\lambda}$$

is injective ([30, Ch.0, Th.21.8.3]); in other words, $\eta_m(F)$ is injective. In order to show the injectivity of $\eta_m(M)$, it then suffices to check that the functor $M \mapsto \Omega^1_{K_{(m)}/\mathbb{F}_p} \otimes_{K_{(m)}} M$ is exact. The latter holds by virtue of corollary 4.8.25, since the (unique) morphism of schemes $\mathrm{Spec}\, K_{(m)} \to \mathrm{Spec}\, \mathbb{F}_p$ is obviously regular. This completes the proof for $m = 0$. Suppose now that $m > 0$, and that the injectivity of $\eta_n(M)$ is already known for every $n < m$ and every $K_{(n)}$-module M. By the foregoing, it remains to check that $\eta_m(K_{(m)}/I)$ is injective, for every non-zero ideal $I \subset K_{(m)}$. Pick any non-zero $f \in I$, and set $R := K_{(m-1)}$; according to [14, Ch.VII, n.7, Lemme 3] and [14, Ch.VII, n.8, Prop.6], there exist an automorphism σ of the ring $K_{(m)}$, and elements $g \in R[T_m]$, $u \in K^\times_{(m)}$ such that $\sigma(f) = u\cdot g$, and $g = T^d_m + a_1T^{d-1}_m + \cdots + a_d$ for some $d \geq 0$ and certain elements $a_1, \dots, a_d$ of the maximal ideal of R. However, set $M' := K_{(m)}/\sigma(I)$; in view of the commutative diagram of $\mathbb{F}_p$-modules :

$$\begin{CD}
\Omega^1_{K_{(m)}/\mathbb{F}_p} \otimes_{K_{(m)}} M @>{\eta_m(M)}>> \Omega_m(M) \\
@VVV @VVV \\
\Omega^1_{K_{(m)}/\mathbb{F}_p} \otimes_{K_{(m)}} M' @>{\eta_m(M')}>> \Omega_m(M')
\end{CD}$$

(whose vertical arrows are induced by σ) we see that $\eta_m(M)$ is injective, if and only if the same holds for $\eta_m(M')$. Hence, we may replace I by $\sigma(I)$, and assume that $g \in I$. In this case, set also $R^\lambda := K^\lambda_{\langle m-1\rangle}$ for every $\lambda \in \Lambda$, and notice that the natural maps

$$R[T_m]/g^pR[T_m] \to K_{(m)}/g^pK_{(m)} \qquad R^\lambda[T^p_m]/g^pR^\lambda[T^p_m] \to K^\lambda_{\langle m\rangle}/g^pK^\lambda_{\langle m\rangle}$$

are bijective; there follows a commutative diagram of $K_{(m)}$-modules :

$$\begin{CD}
\Omega^1_{R[T_m]/\mathbb{F}_p} \otimes_{R[T_m]} M @>>> \Omega^1_{K_{(m)}/\mathbb{F}_p} \otimes_{K_{(m)}} M \\
@V{\alpha^\lambda \otimes_{R[T_m]} M}VV @VV{\eta^\lambda_m \otimes_{K_{(m)}} M}V \\
\Omega^1_{R[T_m]/R^\lambda} \otimes_{R[T_m]} M @>>> \Omega^1_{K_{(m)}/K^\lambda_{\langle m\rangle}} \otimes_{K_{(m)}} M
\end{CD}$$

whose horizontal arrows are isomorphisms, and $\eta_m(M) = \lim_{\lambda\in\Lambda} \eta_m^\lambda \otimes_{K_{(m)}} M$. On the other hand, for every $\lambda \in \Lambda$ we have a commutative ladder of $R[T_m]$-modules with exact rows :

$$\Sigma_\lambda \quad : \quad \begin{tikzcd} 0 \arrow[r] & \Omega^1_{R/\mathbb{F}_p} \otimes_R R[T_m] \arrow[r] \arrow[d, "\eta^\lambda_{m-1}\otimes_R R[T_m]"'] & \Omega^1_{R[T_m]/\mathbb{F}_p} \arrow[r] \arrow[d, "\alpha^\lambda"'] & \Omega^1_{R[T_m]/R} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \Omega^1_{R/R^\lambda} \otimes_R R[T_m] \arrow[r] & \Omega^1_{R[T_m]/R^\lambda} \arrow[r] & \Omega^1_{R[T_m]/R} \arrow[r] & 0 \end{tikzcd}$$

and notice that the rows of $\Sigma_\lambda \otimes_{R[T_m]} M$ are still short exact, for every $\lambda \in \Lambda$. By inductive assumption, $\lim_{\lambda\in\Lambda} \eta^\lambda_{m-1} \otimes_R M$ is an injective map; we deduce that the same holds for $\lim_{\lambda\in\Lambda} \alpha^\lambda \otimes_{R[T]} M$, and the claim follows. $\Diamond$

Take $K := \kappa$, and pick any cofiltered system $(K^\lambda \mid \lambda \in \Lambda)$ as provided by claim 4.8.30(i); in light of claim 4.8.30(ii.b), it now suffices to show that the resulting $\Omega_d(M)$ is a separated B-module, for every noetherian $\kappa_{(d)}$-algebra B and every B-module M of finite type. However, $\Omega_d(M)$ is a submodule of $\prod_{\lambda\in\Lambda}(\Omega^1_{K_{(m)}/K^\lambda_{\langle m\rangle}} \otimes_{K_{(m)}} M)$, hence we are reduced to checking that each direct factor of the latter B-module is separated. But in view of claim 4.8.30(ii.a), we see that each such factor is a finite direct sum of copies of M, so finally we come down to the assertion that M is separated for the $\mathfrak{m}_B$-adic topology, which is well known. $\Box$

Theorem 4.8.31. *Let $\varphi : A \to B$ be a local ring homomorphism of local noetherian rings. Suppose that A is quasi-excellent, and φ is formally smooth for the preadic topologies defined by the maximal ideals. Then* $\mathrm{Spec}\,\varphi$ *is regular.*

Proof. This is the main result of [2]. We begin with the following general remark :

Claim 4.8.32. Let R be a local noetherian ring, $\mathfrak{m}_R \subset R$ the maximal ideal, $H : R\text{-}\mathbf{Mod} \to R\text{-}\mathbf{Mod}$ an additive functor, and suppose that

(a) H is R-linear, *i.e.* $H(t \cdot \mathbf{1}_M) = t \cdot H(\mathbf{1}_M)$ for every R-module M, and every $t \in R$.
(b) H is *semi-exact*, *i.e.* for any short exact sequence $0 \to M' \to M \to M'' \to 0$ of R-modules, the induced sequence $H(M') \to H(M) \to H(M'')$ is exact.
(c) H commutes with filtered colimits.
(d) $H(M)$ is separated for the $\mathfrak{m}_R$-adic topology, for every R-module M of finite type.
(e) $H(R/\mathfrak{m}_R) = 0$.

Then $H(M) = 0$ for every R-module M.

Proof of the claim. Since H commutes with filtered colimits, it suffices to show that $H(M) = 0$ for every finitely generated R-module M, and since H is semi-exact, a simple induction reduces further to the case where M is a cyclic R-module. Let now $\mathscr{F}$ be the family of all ideals I of R such that $H(R/I) \neq 0$, and suppose, by way of contradiction, that $\mathscr{F} \neq \varnothing$; pick a maximal element J of $\mathscr{F}$, and set $M := R/J$. By assumption, $J \neq \mathfrak{m}_R$; thus, let $t \in R$ be a non-invertible element with $t \notin J$; we get an exact sequence

$$H(M) \xrightarrow{t} H(M) \to H(B_{(m+n)}/(J + tB_{(m+n)}))$$

whose third term vanishes, by the maximality of J. On the other hand, $\bigcap_{n\in\mathbb{N}} t^n H(M) = 0$, since $H(M)$ is separated. Thus $H(M) = 0$, contradicting the choice of J, and the claim follows. $\Diamond$

Denote by κ the residue field of A, and for every $m, n \in \mathbb{N}$, let $\varphi_{m,n} : A_{(m)} \to B_{(m+n)}$ be the composition of $\varphi_{(m)} : A_{(m)} \to B_{(m)}$ (the T-adic completion of $\varphi \otimes_A A[T_1, \dots, T_m]$) with the natural inclusion map $B_{(m)} \to B_{(m+n)}$ (notation of (4.8.29)).

Claim 4.8.33. In the situation of the theorem, suppose furthermore that A is either a field or a complete discrete valuation ring of mixed characteristic. Then, the morphism $\mathrm{Spec}\,\varphi_{m,n}$ is regular for every $m, n \in \mathbb{N}$.

Proof of the claim. Set

$$H(M) := H_1(\mathbb{L}_{B_{(m+n)}/A_{(m)}} \otimes_{B_{(m+n)}} M).$$

We shall consider separately three different cases :

• Suppose first that A is a field of characteristic $p > 0$. According to corollary 4.8.25, it suffices to show that $H(M)$ vanishes for every $B_{(m+n)}$-module M. Notice that the natural map $\mathbb{F}_p \to B_{(m+n)}$ is regular; from the distinguished triangle ([56, Ch.II, Prop.2.1.2])

$$\mathbb{L}_{A_{(m)}/\mathbb{F}_p} \otimes_{A_{(m)}} B_{(m+n)} \to \mathbb{L}_{B_{(m+n)}/\mathbb{F}_p} \to \mathbb{L}_{B_{(m+n)}/A_{(m)}} \to \mathbb{L}_{A_{(m)}/\mathbb{F}_p} \otimes_{A_{(m)}} B_{(m+n)}[1]$$

and corollary 4.8.25, we deduce an injective $B_{(m+n)}$-linear map

$$H(M) \to \Omega^1_{A_{(m)}/\mathbb{F}_p} \otimes_{A_{(m)}} M$$

from which it follows that, if M is a $B_{(m+n)}$-module of finite type, $H(M)$ is a separated $B_{(m+n)}$-module, for the preadic topology defined by the maximal ideal of $B_{(m+n)}$ (proposition 4.8.28). Since φ is formally smooth, $\varphi_{m,n}$ is also formally smooth for the preadic topologies defined by the maximal ideals of $A_{(m)}$ and $B_{(m+n)}$; from proposition 4.8.20, we see that $H(M) = 0$, if M is the residue field of $B_{(m+n)}$. Then the assertion follows from claim 4.8.32.

• Next, suppose that A is either a field of characteristic zero, or a complete discrete valuation ring of mixed characteristic (so either $\kappa = A$, or else κ is a field of positive characteristic). According to corollary 4.8.25, it suffices to show that $H(M)$ vanishes for $M = \kappa(\mathfrak{q})$, where $\mathfrak{q} \subset B_{(m+n)}$ is any prime ideal. Fix such $\mathfrak{q}$, and set $\mathfrak{p} := \mathfrak{q} \cap A_{(m)}$; if $\mathfrak{p} = 0$, then M is a K-algebra, where K is the field of fractions of $A_{(m)}$; now, K is a field of characteristic zero, and $B' := B_{(m+n)} \otimes_{A_{(m)}} K$ is a regular local K-algebra, so the induced morphism $\mathrm{Spec}\,B' \to \mathrm{Spec}\,K$ is regular; since $H(M) = H_1(\mathbb{L}_{B'/K} \otimes_{B'} M)$, the assertion follows from corollary 4.8.25. Notice that this argument applies especially to the case where $m = 0$; for the general case, we argue by induction on m. Hence, suppose that $n \in \mathbb{N}$, $m > 0$, and that the assertion is already known for $\varphi_{n,m-1}$.

Consider first the case where A is a discrete valuation ring, and $\mathfrak{p}$ contains the maximal ideal of A, and set $\overline{B} := B \otimes_A \kappa$. Since $\varphi_{m,n}$ is flat, and since M is a $\overline{B}_{(m+n)}$-module, we have a natural isomorphism

$$H(M) \xrightarrow{\sim} H_1(\mathbb{L}_{\overline{B}_{(m+n)}/\kappa_{(m)}} \otimes_{\overline{B}_{(m+n)}} M).$$

Then the sought vanishing follows from the foregoing, since $\overline{B}$ is a formally smooth κ-algebra (for the preadic topology of its maximal ideal).

Lastly, suppose that either A is a field, or $\mathfrak{p}$ does not contain the maximal ideal of A (and $\mathfrak{p} \neq 0$). In either of these two cases, we may find $f \in \mathfrak{p}$ whose image in $\kappa_{(m)}$ is not zero. According to [14, Ch.VII, n.7, Lemme 3] and [14, Ch.VII, n.8, Prop.6], we may find an automorphism σ of the A-algebra $A_{(m)}$ and elements $g \in A_{(m-1)}[T_m]$, $u \in A^\times_{(m)}$ such that $\sigma(f) = u \cdot g$, and $g = T^d_m + a_1 T^{d-1}_m + \cdots + a_d$ for some $d \geq 0$ and certain elements $a_1, \dots, a_d$ of the maximal ideal of $A_{(m-1)}$. Denote by $\sigma' : B_{(m)} \xrightarrow{\sim} B_{(m)}$ the T-adic completion of $\sigma \otimes_A B$, and let $\sigma_B : B_{(m+n)} \xrightarrow{\sim} B_{(m+n)}$ be the T-adically continuous automorphism that restricts to σ' on $B_{(m)}$, and such that $\sigma_B(T_i) = T_i$ for $i = m+1, \dots, m+n$. Set $M' := B_{(m+n)}/\sigma_B(\mathfrak{q})$; by construction, we have a commutative diagram of A-algebras

$$\begin{tikzcd} A_{(m)} \arrow[r, "\varphi_{m,n}"] \arrow[d, "\sigma"'] & B_{(m+n)} \arrow[d, "\sigma_B"] \\ A_{(m)} \arrow[r, "\varphi_{m,n}"] & B_{(m+n)} \end{tikzcd}$$

inducing an isomorphism

$$\mathbb{L}_{B_{(m+n)}/A_{(n)}} \otimes_{B_{(m+n)}} M \xrightarrow{\sim} \mathbb{L}_{B_{(m+n)}/A_{(n)}} \otimes_{B_{(m+n)}} M' \qquad \text{in } \mathsf{D}(A\text{-}\mathbf{Mod}).$$

Thus, we may replace M by M', and assume from start that $g \in \mathfrak{p}$. In this case, set $B' := B[[T_{m+1}, \dots, T_{m+n}]]$, and notice that both of the natural maps

$$A_{(m-1)}[T_m]/gA_{(m-1)}[T_m] \to A_{(m)}/gA_{(m)} \qquad B'_{(m-1)}[T_m]/gB'_{(m-1)}[T_m] \to B_{(m+n)}/gB_{(m+n)}$$

are isomorphisms. Since both $\varphi_{m,n}$ and the map $A_{(m-1)}[T_m] \to B'_{(m-1)}[T_m]$ induced by φ are flat ring homomorphisms, there follows a natural isomorphism of $B_{(m+n)}$-modules :

$$\begin{aligned} \mathbb{L}_{B_{(m+n)}/A_{(n)}} \otimes_{B_{(m+n)}} M &\xrightarrow{\sim} \mathbb{L}_{B'_{(m-1)}[T_m]/A_{(m-1)}[T_m]} \otimes_{B'_{(m-1)}[T_m]} M \\ &\xrightarrow{\sim} \mathbb{L}_{B'_{(m-1)}/A_{(m-1)}} \otimes_{B'_{(m-1)}} M \end{aligned}$$

([56, Ch.II, Prop.2.2.1]). However, the resulting map $A_{(m-1)} \to B'_{(m-1)}$ is none else than $\varphi_{m-1,n}$, up to a relabeling of the variables; the vanishing of $H(M)$ then follows from the inductive assumption (and from corollary 4.8.25). $\Diamond$

Claim 4.8.34. In the situation of the theorem, suppose furthermore that A and B are complete, and let M be a B-module of finite type. Then $H_1(\mathbb{L}_{B/A} \otimes_B M)$ is a B-module of finite type.

Proof of the claim. Let $\overline{f} : A_0 \to \kappa$ be a surjective ring homomorphism, with A_0 a Cohen ring ([30, Ch.0, Th.19.8.6(ii)]), and set $\overline{B} := B \otimes_A \kappa$; in light of [30, Ch.0, Lemme 19.7.1.3], there exists a flat local, complete and noetherian A_0-algebra B_0 fitting into a cocartesian diagram :

$$\begin{tikzcd} A_0 \arrow[r, "\psi"] \arrow[d, "\overline{f}"'] & B_0 \arrow[d, "\overline{f}_B"] \\ \kappa \arrow[r] & \overline{B}. \end{tikzcd}$$

Denote by κ_B the residue field of B; there follow natural isomorphisms of B-modules :

$$H_1(\mathbb{L}_{B/A} \otimes_B \kappa_B) \xrightarrow{\sim} H_1(\mathbb{L}_{\overline{B}/\kappa} \otimes_{\overline{B}} \kappa_B) \xrightarrow{\sim} H_1(\mathbb{L}_{B_0/A_0} \otimes_{B_0} \kappa_B)$$

([56, Ch.II, Prop.2.2.1]) which, according to proposition 4.8.20, imply that ψ is formally smooth (for the preadic topologies defined by the maximal ideals). By [30, Ch.0, Th.19.8.6(i)], $\overline{f}$ lifts to a ring homomorphism $f : A_0 \to A$, whence a commutative diagram

$$\begin{tikzcd} A_0 \arrow[r, "\psi"] \arrow[d, "\varphi\circ f"'] & B_0 \arrow[d, "\overline{f}_B"] \\ B \arrow[r] & \overline{B}. \end{tikzcd}$$

Then, by [30, Ch.0, Cor.19.3.11], the map $\overline{f}_B$ lifts to a ring homomorphism $f_B : B_0 \to B$. Notice now that both f and f_B are local maps and induce isomorphisms on the residue fields; it follows easily that, for suitable $m, n \in \mathbb{N}$, they extend to surjective maps g and g_B fitting into a commutative diagram

$$\begin{tikzcd} A_{0,(m)} \arrow[r, "\psi_{(m,n)}"] \arrow[d, "g"'] & B_{0,(m+n)} \arrow[d, "g_B"] \\ A \arrow[r, "\varphi"] & B \end{tikzcd}$$

whence exact sequences ([56, Ch.II, Prop.2.1.2])

$$H_1(\mathbb{L}_{B_{0,(m+n)}/A_{0,(m)}} \otimes_{B_{0,(m+n)}} M) \to H_1(\mathbb{L}_{B/A_{0,(m)}} \otimes_B M) \to H_1(\mathbb{L}_{B_{0,(m+n)}/B} \otimes_{B_{0,(m+n)}} M)$$
$$H_1(\mathbb{L}_{B/A_{0,(m)}} \otimes_B M) \to H_1(\mathbb{L}_{B/A} \otimes_B M) \to \Omega^1_{A/A_{0,(m)}} \otimes_A M = 0.$$

However, claim 4.8.33 applies to ψ, and together with corollary 4.8.25, it implies that the first module of the first of these sequences vanishes; on the other hand, it is easily seen that the third B-module of the same sequence is finitely generated, so the same holds for the middle term. By inspecting the second exact sequence, the claim follows. $\diamondsuit$

We may now conclude the proof of the theorem : let $A^\wedge$ and $B^\wedge$ be the completions of A and B; since φ is formally smooth, the same holds for its completion $\varphi^\wedge : A^\wedge \to B^\wedge$. First, we show that $\mathrm{Spec}\,\varphi^\wedge$ is regular; to this aim, it suffices to check that $H^\wedge(M) := H_1(\mathbb{L}_{B^\wedge/A^\wedge} \otimes_{B^\wedge} M)$ vanishes for every $B^\wedge$-module M (corollary 4.8.25). However, we know already that $H^\wedge(M)$ is a $B^\wedge$-module of finite type, if the same holds for M (claim 4.8.34); especially, for such M, $H^\wedge(M)$ is separated for the adic topology of $B^\wedge$ defined by the maximal ideal. Moreover, $H^\wedge(M) = 0$, if M is the residue field of $B^\wedge$ (proposition 4.8.20). Then the assertion follows from claim 4.8.32. Lastly, the natural morphism $\mathrm{Spec}\,A^\wedge \to \mathrm{Spec}\,A$ is regular by assumption, hence the same holds for the induced morphism $\mathrm{Spec}\,B^\wedge \to \mathrm{Spec}\,A$ (lemma 4.8.1(i)). Finally, since $B^\wedge$ is a faithfully flat B-algebra, we conclude that $\mathrm{Spec}\,\varphi$ is regular, by virtue of lemma 4.8.1(ii). $\square$

Proposition 4.8.35. *Let A be a noetherian local ring, and $\varphi : A \to B$ an ind-étale local ring homomorphism. Then :*

(i) *A is quasi-excellent if and only if the same holds for B.*
(ii) *If A is excellent, the same holds for B.*

Proof. Set $f := \mathrm{Spec}\,\varphi$, and $f^\wedge := \mathrm{Spec}\,\varphi^\wedge$, where $\varphi^\wedge : A^\wedge \to B^\wedge$ is the map of complete local rings obtained from φ. Notice first that, for every $y \in \mathrm{Spec}\,A$, and every $x \in f^{-1}(y)$, the stalk $\mathscr{O}_{f^{-1}y,x}$ is an ind-étale $\kappa(y)$-algebra, *i.e.* is a separable algebraic extension of $\kappa(y)$, hence $f^{-1}(y)$ is geometrically regular, and therefore f is regular and faithfully flat. Moreover, f is the limit of a cofiltered system of formally étale morphisms, hence it is formally étale; *a fortiori*, B is also formally smooth over A for the preadic topologies, so $B^\wedge$ is formally smooth over $A^\wedge$ for the preadic topologies, and consequently $f^\wedge$ is a regular morphism (theorem 4.8.31).

(i): If B is quasi-excellent, claim 4.8.5 and lemma 4.8.4(iv) imply that A is quasi-excellent. Next, assume that A is quasi-excellent; we have a commutative diagram (4.8.6), in which π_A is a regular morphism, and then the same holds for $\pi_A \circ f^\wedge = f \circ \pi_B$ (lemma 4.8.1(i)). Now, let $y \in \mathrm{Spec}\,B$ be any point, and set $z := f(y)$; we know that $(f \circ \pi_B)^{-1}(z)$ is a geometrically regular affine scheme, so write it as the spectrum of a noetherian ring C. The stalk $E := \mathscr{O}_{f^{-1}(z),y}$ is a separable algebraic field extension of $\kappa(z)$, hence $\pi_B^{-1}(y) \simeq \mathrm{Spec}\,C \otimes_B E$, the spectrum of a localization of C, so it is again geometrically regular, *i.e.* π_B is a regular morphism, and we conclude by lemma 4.8.4(iii,iv).

(ii): By (i), it remains only to show that B is universally catenarian, provided the same holds for A. To this aim, it suffices to apply [33, Ch.IV, Lemma 18.7.5.1]. $\square$

Proposition 4.8.36. *In the situation of* (4.7.29)*, suppose that Φ_A is a finite ring homomorphism, and for every prime ideal $\mathfrak{p} \subset A$, set $\kappa(\mathfrak{p}) := A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$ (the residue field of the point $\mathfrak{p} \in \operatorname{Spec} A$). Then*

$$\dim A_\mathfrak{p}/\mathfrak{q}A_\mathfrak{p} = \dim_{\kappa(\mathfrak{p})} \Omega^1_{\kappa(\mathfrak{p})/\mathbb{F}_p} - \dim_{\kappa(\mathfrak{q})} \Omega^1_{\kappa(\mathfrak{q})/\mathbb{F}_p}$$

for every pair of prime ideals $\mathfrak{q} \subset \mathfrak{p} \subset A$.

Proof. To begin with, we notice :

Claim 4.8.37. Suppose that Φ_A is a finite map. We have :

(i) For every A-algebra B of essentially finite type, Φ_B is finite as well.
(ii) Suppose moreover, that A is local, and let $A^\wedge$ the completion of A. Then $\Phi_{A^\wedge} = \mathbf{1}_{A^\wedge} \otimes_A \Phi_A$ is finite, and $\Omega^1_{A^\wedge/\mathbb{F}_p} = A^\wedge \otimes_A \Omega^1_{A/\mathbb{F}_p}$.
(iii) If A is local and reduced, then the same holds for $A^\wedge$.
(iv) Suppose moreover, that A is a local integral domain, and denote by K (resp. κ) the field of fractions (resp. the residue field) of A. Then

$$\dim_K \Omega^1_{K/\mathbb{F}_p} = \dim_\kappa \Omega^1_{\kappa/\mathbb{F}_p} + \dim A. \tag{4.8.38}$$

Proof of the claim. (i): Suppose first that $B = A[X]$; then it is easily seen that $\Phi_B = \Phi_A \otimes_{\mathbb{F}_p} \Phi_{\mathbb{F}_p[X]}$, whence the contention, in this case. By an easy induction, we deduce that the claim holds as well for any free polynomial A-algebra of finite type. Next, let $I \subset A$ be any ideal; since $\Phi_A(I) \subset I$, it is easily seen that $\Phi_{A/I} = \Phi_A \otimes_A A/I$, so $\Phi_{A/I}$ is finite, and therefore the assertion holds for any A-algebra of finite type. Lastly, let $S \subset A$ be any multiplicative subset; since $\Phi_A(S) \subset S$, it is easily seen that

$$\Phi_{S^{-1}A} = S^{-1}\Phi_A \tag{4.8.39}$$

and the assertion follows.

(ii): Denote by $\mathfrak{m}_A$ the maximal ideal of A, and set $B := A_{(\Phi)}$; by assumption, $\Phi_A : A \to B$ is a finite A-linear map, hence its $\mathfrak{m}_A$-adic completion $(\Phi_A)^\wedge : A^\wedge \to B^\wedge$ equals $\mathbf{1}_{A^\wedge} \otimes_A \Phi_A$. Since $\Phi_A(\mathfrak{m}_A) = \mathfrak{m}_A^p$, we have a natural $A^\wedge$-linear identification on $\mathfrak{m}_A$-adic completions :

$$B^\wedge \xrightarrow{\sim} (A^\wedge)_{(\Phi)} \tag{4.8.40}$$

and under this identification, $(\Phi_A)^\wedge = \Phi_{A^\wedge}$, whence the first assertion of (ii). Next, we notice that (4.8.40) yields a natural identification $\Omega^1_{A^\wedge/\mathbb{F}_p} = \Omega^1_{B^\wedge/\mathbb{F}_p}$, and on the other hand, the sequence of ring homomorphisms

$$\mathbb{F}_p \to A^\wedge \xrightarrow{\Phi_{A^\wedge}} B^\wedge$$

yields natural identifications :

$$\Omega^1_{B^\wedge/\mathbb{F}_p} = \Omega^1_{B^\wedge/A^\wedge} = A^\wedge \otimes_A \Omega^1_{B/A} = A^\wedge \otimes_A \Omega^1_{B/\mathbb{F}_p} = B^\wedge \otimes_B \Omega^1_{B/\mathbb{F}_p} = A^\wedge \otimes_A \Omega^1_{A/\mathbb{F}_p}$$

where the last equality is induced by (4.8.40) and the identity map $A \xrightarrow{\sim} B$. This completes the proof of the second assertion.

(iii): Since A is reduced, Φ_A is injective, and then the same holds for its completion $(\Phi_A)^\wedge$. But we have seen that the latter is naturally identified with $\Phi_{A^\wedge}$, whence the claim.

(iv): Notice that $\Omega^1_{K/\mathbb{F}_p}$ is a K-vector space of finite dimension, since its dimension equals $[K : K^p]$ ([30, Ch.0, Th.21.4.5]), and the latter is finite, by (i); the same argument applies to the κ-vector space $\Omega^1_{\kappa/\mathbb{F}_p}$. Denote by $\delta(A)$ the difference between the left and right hand-side of (4.8.38); we have to show that $\delta(A) = 0$. Let $\mathfrak{p} \subset A^\wedge$ be any minimal prime ideal, so that $\dim A^\wedge/\mathfrak{p} = d := \dim A$. In view of (iii), $L := (A^\wedge)_\mathfrak{p}$ is a field; taking into account (ii), we get

$$\Omega^1_{L/\mathbb{F}_p} = L \otimes_{A^\wedge} \Omega^1_{A^\wedge/\mathbb{F}_p} = L \otimes_A \Omega^1_{A/\mathbb{F}_p} = L \otimes_K \Omega^1_{K/\mathbb{F}_p}.$$

Hence $\dim_L \Omega^1_{L/\mathbb{F}_p} = \dim_K \Omega^1_{K/\mathbb{F}_p}$, so we see that

$$(4.8.41) \qquad \delta(A) = \delta(A^\wedge/\mathfrak{p}) \qquad \text{for any minimal prime ideal } \mathfrak{p} \subset A^\wedge.$$

We may then replace A by $A^\wedge/\mathfrak{p}$, after which we may assume that A is a complete local domain. In this case, A contains a field mapping isomorphically onto κ, and there is a finite map of κ-algebras $A' := \kappa[[T_1, \dots, T_d]] \to A$ ([61, Th.29.4(iii)]). Denote by K' the field of fractions of A', and notice that $[K : K'] = [K : K^p] \cdot [K^p : K'] = [K : K^p] \cdot [K^p : K'^p]$; on the other hand, Φ_K induces an isomorphism $K \xrightarrow{\sim} K^p$, hence have as well $[K : K'] = [K^p : K'^p]$, so $[K : K^p] = [K' : K'^p]$ and finally :

$$\dim_{K'} \Omega^1_{K'/\mathbb{F}_p} = \dim_K \Omega^1_{K/\mathbb{F}_p}$$

([30, Ch.0, Cor.21.2.5]). Since $\dim A' = \dim A$, we conclude that $\delta(A) = \delta(A')$. Hence, it suffices to check that $\delta(A') = 0$. Furthermore, set $B := \kappa[T_1, \dots, T_d]$, and let $\mathfrak{m} \subset B$ be the maximal ideal generated by $T_1, \dots, T_d$; we have $A' = B^\wedge_\mathfrak{m}$, so (4.8.41) further reduces to showing that $\delta(B_\mathfrak{m}) = 0$, which shall be left as an exercise for the reader. ◊

Now, in view of claim 4.8.37(i), we may replace A by $A_\mathfrak{p}/\mathfrak{q}A_\mathfrak{p}$, and assume from start that A is a local domain, that $\mathfrak{q} = 0$ and that $\mathfrak{p}$ is the maximal ideal; in this case, the sought identity is given by claim 4.8.37(iv). □

Theorem 4.8.42. *With the notation of* (4.7.29)*, suppose that A is noetherian. We have :*

(i) *If Φ_A is a finite ring homomorphism, then A is excellent.*
(ii) *Conversely, suppose that A is a local Nagata ring, with residue field k, and that $[k : k^p]$ is finite. Then Φ_A is a finite ring homomorphism.*

Proof. (i): We reproduce the proof from [60, Appendix, Th.108].

Claim 4.8.43. If Φ_A is finite, then A is quasi-excellent.

Proof of the claim. We check first the openness condition for the regular loci. To this aim, in light of claim 4.8.37(i), we may assume that A is an integral domain, and it suffices to prove that the regular locus of $\operatorname{Spec} A$ is an open subset. Set $B := A_{(\Phi)}$ (notation of (4.7.29)), and let $\mathfrak{p} \subset A$ be any prime ideal; by theorem 4.7.30 and (4.8.39), the ring $A_\mathfrak{p}$ is regular if and only if $(\Phi_A)_\mathfrak{p}$ is a flat ring homomorphism, if and only if $B_\mathfrak{p}$ is a flat $A_\mathfrak{p}$-module. Since, by assumption, B is a finite A-module, the latter holds if and only if $B_\mathfrak{p}$ is a free $A_\mathfrak{p}$-module. Then, the assertion follows from [61, Th.4.10].

Next, we show that A is a G-ring. By claim 4.8.37(i), we may assume that A is local. From claim 4.8.37(ii) we see that the natural map

$$A_{(\Phi)} \overset{\mathbf{L}}{\otimes}_A A^\wedge \to (A^\wedge)_{(\Phi)}$$

is an isomorphism in $\mathsf{D}(A\text{-}\mathbf{Mod})$. Then the assertion follows from [36, Lemma 6.5.13(i)] and corollary 4.8.27. ◊

Now, it follows easily from proposition 4.8.36 and claim 4.8.37(i), that A is universally catenarian; combining with claim 4.8.43, we obtain (i).

(ii): The assertion to prove is that $A_{(\Phi)}$ is a finite A-module. To this aim, denote by $I \subset A$ the nilradical ideal; we remark :

Claim 4.8.44. It suffices to show that $(A/I)_{(\Phi)}$ is a finite A-module.

Proof of the claim. Indeed, suppose that $(A/I)_{(\Phi)}$ is a finite A-module; we shall deduce, by induction on s, that $(A/I^s)_{(\Phi)}$ is a finite A-module, for every $s > 0$. Since A is noetherian, we have $I^t = 0$ for $t > 0$ large enough, so the claim will follow. The assertion for $s = 1$ is our assumption. Suppose therefore that $s > 1$, and that we already know the assertion for $s - 1$. Notice that I^s/I^{s-1} is a finite A/I-module, hence a finite $(A/I)_{(\Phi)}$-module, by our assumption. Since $(A/I^{s-1})_{(\Phi)}$ is a finite A-module, we deduce easily that the assertion holds for s, as required. $\diamondsuit$

In view of claim 4.8.44, we may replace A by A/I (which is obviously still a Nagata ring), and assume from start that A is reduced. In this case, let $\{\mathfrak{p}_1, \dots, \mathfrak{p}_t\}$ be the set of minimal prime ideals of A; we have a commutative diagram of ring homomorphisms :

$$\begin{tikzcd} A \arrow[r, "\Phi_A"] \arrow[d] & A \arrow[d] \\ \prod_{i=1}^t A/\mathfrak{p}_i \arrow[r, "\prod_{i=1}^t \Phi_{A/\mathfrak{p}_i}"] & \prod_{i=1}^t A/\mathfrak{p}_i \end{tikzcd}$$

whose vertical arrows are injective and finite maps. Suppose now that $(A/\mathfrak{p}_i)_{(\Phi)}$ is a finite $A/\mathfrak{p}_i$-module for every $i = 1, \dots, t$. Then $\prod_{i=1}^t (A/\mathfrak{p}_i)_{(\Phi)}$ is a finite A-module, and therefore the same holds for its A-submodule $A_{(\Phi)}$. Thus, we are reduced to checking the sought assertion for each of the quotients $A/\mathfrak{p}_i$ (which are still Nagata rings), and hence we may assume from start that A is a domain. In this case, denote by K the field of fractions of A; by the definition of Nagata ring, we are further reduced to checking that $\Phi_K : K \to K$ is a finite field extension, *i.e.* that $[K : K^p]$ is finite. Denote $A^\wedge$ the completion of A; we remark :

Claim 4.8.45. (i) The endomorphism $\Phi_{A^\wedge}$ is a finite map.

(ii) $\operatorname{Spec} K \otimes_A A^\wedge$ is a geometrically reduced K-scheme.

Proof of the claim. (i): Indeed, $A^\wedge$ is a quotient of a power series ring $B := k[[T_1, \dots, T_r]]$ ([30, Ch.0, Th.19.8.8(i)]), hence it suffices to show that Φ_B is finite. However, $B^p = k^p[[T_1^p, \dots, T_r^p]]$, and $[k : k^p]$ is finite by assumption, whence the claim.

(ii): This is [31, Ch.IV, Th.7.6.4]. $\diamondsuit$

Let $\mathfrak{p} \subset A^\wedge$ be any minimal prime ideal; it follows from claim 4.8.45(ii) that $L := A_\mathfrak{p}$ is a separable field extension of K ([31, Ch.IV, Prop.4.6.1]), so the natural map

$$\Omega^1_{K/\mathbb{F}_p} \otimes_K L \to \Omega^1_{L/\mathbb{Z}}$$

is injective ([30, Ch.0, Cor.20.6.19(i)]); on the other hand, claim 4.8.45(i) implies that $[L : L^p]$ is finite, *i.e.* $\Omega^1_{L/\mathbb{F}_p}$ is a finite-dimensional L-vector space ([30, Ch.0, Th.21.4.5]). We conclude that $\Omega^1_{K/\mathbb{F}_p}$ is a finite-dimensional K-vector space, whence the contention, again by *loc.cit.* $\square$

5. Local cohomology

5.1. **Cohomology in a ringed space.**

Definition 5.1.1. Let X be a topological space, $\mathscr{F}$ a sheaf on X. We say that $\mathscr{F}$ is *qc-flabby* if the restriction map $\mathscr{F}(V) \to \mathscr{F}(U)$ is a surjection whenever $U \subset V$ are quasi-compact open subsets of X.

Lemma 5.1.2. *Suppose that the topological space X fulfills the following two conditions :*

(a) *X is quasi-separated (*i.e.* the intersection of any two quasi-compact open subsets of X is quasi-compact).*

(b) *X admits a basis of quasi-compact open subsets.*

Let $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ be a short exact sequence of abelian sheaves on X. Then :

(i) *If $\mathscr{F}'$ is qc-flabby, the induced sequence*

$$0 \to \mathscr{F}'(U) \to \mathscr{F}(U) \to \mathscr{F}''(U) \to 0$$

is short exact for every quasi-compact open subset $U \subset X$.

(ii) *If both $\mathscr{F}$ and $\mathscr{F}'$ are qc-flabby, the same holds for $\mathscr{F}''$.*

Proof. (i) is a variant of [39, Ch.II, Th.3.1.2]. Indeed, we may replace X by U, and thereby assume that X is quasi-compact. Then we have to check that every section $s'' \in \mathscr{F}''(X)$ is the image of an element of $\mathscr{F}(X)$. However, we can find a finite covering $(U_i \mid i = 1, \dots, n)$ of X, consisting of quasi-compact open subsets, such that $s''_{|U_i}$ is the image of a section $s_i \in \mathscr{F}(U_i)$. For every $k \leq n$, let $V_k := U_1 \cup \cdots \cup U_k$; we show by induction on k that $s''_{|V_k}$ is in the image of $\mathscr{F}(V_k)$; the lemma will follow for $k = n$. For $k = 1$ there is nothing to prove. Suppose that $k > 1$ and that the assertion is known for all $j < k$; hence we can find $t \in \mathscr{F}(V_{k-1})$ whose image is $s''_{|V_{k-1}}$. The difference $u := (t - s_k)_{|U_k \cap V_{k-1}}$ lies in the image of $\mathscr{F}'(U_k \cap V_{k-1})$; since $U_k \cap V_{k-1}$ is quasi-compact and $\mathscr{F}'$ is qc-flabby, u extends to a section of $\mathscr{F}'(U_k)$. We can then replace s_k by $s_k + u$, and assume that s_k and t agree on $U_k \cap V_{k-1}$, whence a section on $V_k = U_k \cup V_{k-1}$ with the sought property. Assertion (ii) is left to the reader. □

Lemma 5.1.3. *Let $F : \mathscr{C} \to \mathscr{C}'$ be a covariant additive left exact functor between abelian categories. Suppose that $\mathscr{C}$ has enough injectives, and let $\mathscr{M} \subset \mathscr{C}$ be a subcategory such that :*

(a) *For every $A \in \mathrm{Ob}(\mathscr{C})$, there exists a monomorphism $A \to M$, with $M \in \mathrm{Ob}(\mathscr{M})$.*
(b) *Every $A \in \mathrm{Ob}(\mathscr{C})$ isomorphic to a direct factor of an object of $\mathscr{M}$, is an object of $\mathscr{M}$.*
(c) *For every short exact sequence*
$$0 \to M' \to M \to M'' \to 0$$
in $\mathscr{C}$ with M' and M in $\mathscr{M}$, the object M'' is also in $\mathscr{M}$, and the sequence
$$0 \to F(M') \to F(M) \to F(M'') \to 0$$
is short exact.

Then, every injective object of $\mathscr{C}$ is in $\mathscr{M}$, and we have $R^pF(M) = 0$ for every $M \in \mathrm{Ob}(\mathscr{M})$ and every $p > 0$.

Proof. This is [43, Lemme 3.3.1]. For the convenience of the reader we sketch the easy proof. First, if I is an injective object of $\mathscr{C}$, then by (a) we can find a monomorphism $I \to M$ with M in $\mathscr{M}$; hence I is a direct summand of M, so it is in $\mathscr{M}$, by (b). Next, let $M \in \mathscr{M}$, and choose an injective resolution $M \to I^\bullet$ in $\mathscr{C}$. For every $i \in \mathbb{N}$ let $Z^i := \mathrm{Ker}(d^i : I^i \to I^{i+1})$; in order to prove that $R^pF(M) = 0$ for every $p > 0$, we have to show that the sequences $0 \to F(Z^i) \to F(I^i) \to F(Z^{i+1}) \to 0$ are short exact for every $i \in \mathbb{N}$. In turns, this will follow from (c), once we know that Z^i and I^i are in $\mathscr{M}$. This is already established for I^i, and for Z^i it is derived easily from (c), via induction on i. □

5.1.4. Let X be a topological space. To X we attach the site $\mathscr{X}$ whose objects are all the families $\mathfrak{U} := (U_i \mid i \in I)$ of open subsets of X (*i.e.* all the covering families – indexed by a set I taken within a fixed universe – of all the open subsets of X). The morphisms $\beta : \mathfrak{V} := (V_j \mid j \in J) \to \mathfrak{U}$ in $\mathscr{X}$ are the *refinements*, *i.e.* the maps $\beta : J \to I$ such that $V_j \subset U_{\beta(j)}$ for every $j \in J$. The covering $\mathfrak{U}$ determines a space $T(\mathfrak{U}) := \amalg_{i \in I} U_i$, the disjoint union of the open subsets U_i, endowed with the inductive limit topology. A morphism $\beta : \mathfrak{V} \to \mathfrak{U}$ is covering for the topology of $\mathscr{X}$ if and only if the induced map $T(\mathfrak{V}) \to T(\mathfrak{U})$ is surjective, and a family $(\mathfrak{V}_i \to \mathfrak{U} \mid i \in I)$ is covering if and only if $\mathfrak{V}_i \to \mathfrak{U}$ is covering for at least one $i \in I$. Notice that all fibre products in $\mathscr{X}$ are representable.

5.1.5. For any object $\mathfrak{V}$ of $\mathscr{X}$, let $\mathscr{C}_\mathfrak{V}$ be the full subcategory of $\mathscr{X}/\mathfrak{V}$ consisting of all covering morphisms $\mathfrak{U} \to \mathfrak{V}$ (notation of (1.1.2)). For any morphism $\mathfrak{V} \to \mathfrak{W}$ of $\mathscr{X}$, we have a natural functor $\mathscr{C}_\mathfrak{W} \to \mathscr{C}_\mathfrak{V} : \mathfrak{U} \mapsto \mathfrak{U} \times_\mathfrak{W} \mathfrak{V}$, and the rule $\mathfrak{V} \mapsto \mathscr{C}_\mathfrak{V}$ is therefore a pseudo-functor $\mathscr{X}^o \to \mathbf{Cat}$ from the opposite of $\mathscr{X}$ to the 2-category of categories. To any object $\mathfrak{U} \to \mathfrak{V}$ of $\mathscr{C}_\mathfrak{V}$ we attach a simplicial object $\mathfrak{U}_\bullet$ of $\mathscr{X}$, as follows ([4, Exp.V, §1.10]). For every $n \in \mathbb{N}$, let $\mathfrak{U}_n := \mathfrak{U} \times_\mathfrak{V} \cdots \times_\mathfrak{V} \mathfrak{U}$, the $(n+1)$-th power of $\mathfrak{U}$; the face morphisms $\partial_i : \mathfrak{U}_n \to \mathfrak{U}_{n-1}$ (for all $n \geq 1$ and $i = 0, \dots, n$) are the natural projections.

5.1.6. Let $\mathscr{F}$ be a presheaf on $\mathscr{X}$; the *cosimplicial Čech complex* associated to $\mathscr{F}$ and a covering morphism $\mathfrak{U} \to \mathfrak{V}$, is defined as usual by setting :

$$C^n(\mathfrak{U}, \mathscr{F}) := \mathscr{F}(\mathfrak{U}_n)$$

with coface maps $\partial^i := \mathscr{F}(\partial_i) : C^{n-1}(\mathfrak{U}, \mathscr{F}) \to C^n(\mathfrak{U}, \mathscr{F})$ (for all $n \geq 1$ and $i = 0, \dots, n$). It extends naturally to an *augmented cosimplicial Čech complex* :

$$C^\bullet_{\mathrm{aug}}(\mathfrak{U}, \mathscr{F}) := \mathrm{Cone}(\mathscr{F}(X)[0] \to C^\bullet(\mathfrak{U}, \mathscr{F})[-1]).$$

One defines a functor

$$F_{\mathfrak{V}} : \mathscr{C}^o_{\mathfrak{V}} \to \mathbf{Set} \qquad (\mathfrak{U} \to \mathfrak{V}) \mapsto \mathrm{Equal}(C^0(\mathfrak{U}, \mathscr{F}) \underset{\partial^1}{\overset{\partial^0}{\rightrightarrows}} C^1(\mathfrak{U}, \mathscr{F}))\ .$$

Since $\mathscr{C}_{\mathfrak{V}}$ is usually not a cofiltered category, the colimit over $\mathscr{C}^o_{\mathfrak{V}}$ is in general not well behaved (*e.g.* not exact). However, suppose that $\mathfrak{U}$ and $\mathfrak{U}'$ are two coverings of $\mathfrak{V}$, and $\varphi, \psi : \mathfrak{U} \to \mathfrak{U}'$ are two morphisms of coverings; it is well known that the induced maps of cosimplicial objects

$$C^\bullet(\varphi, \mathscr{F}), C^\bullet(\psi, \mathscr{F}) : C^\bullet(\mathfrak{U}', \mathscr{F}) \to C^\bullet(\mathfrak{U}, \mathscr{F})$$

are homotopic, hence they induce the same map $F_{\mathfrak{V}}(\mathfrak{U}') \to F_{\mathfrak{V}}(\mathfrak{U})$. Hence, the functor $F_{\mathfrak{V}}$ factors through the filtered partially ordered set $\mathscr{D}_{\mathfrak{V}}$ obtained as follows. We define a transitive relation $\leq$ on the coverings of $\mathfrak{V}$, by the rule :

$$\mathfrak{U} \leq \mathfrak{U}' \quad \text{if and only if there exists a refinement } \mathfrak{U}' \to \mathfrak{U} \text{ in } \mathscr{C}_{\mathfrak{V}}.$$

Then $\mathscr{D}_{\mathfrak{V}}$ is the quotient of $\mathrm{Ob}(\mathscr{C}_{\mathfrak{V}})$ such that $\leq$ descends to an ordering on $\mathscr{D}_{\mathfrak{V}}$ (in the sense of [16, Ch.III, §1, n.1]); one checks easily that the resulting partially ordered set $(\mathscr{D}_{\mathfrak{V}}, \leq)$ is filtered. One can then introduce the functor

$$\mathrm{Hom}_{\mathbf{Cat}}(\mathscr{X}^o, \mathbf{Set}) \to \mathrm{Hom}_{\mathbf{Cat}}(\mathscr{X}^o, \mathbf{Set}) \qquad \mathscr{F} \mapsto \mathscr{F}^+$$

on the category of presheaves on $\mathscr{X}$, by the rule :

$$\mathscr{F}^+(\mathfrak{V}) := \operatorname*{colim}_{\mathfrak{U} \in \mathscr{D}_{\mathfrak{V}}} F_{\mathfrak{V}}(\mathfrak{U}). \tag{5.1.7}$$

It is well known that $\mathscr{F}^+$ is a separated presheaf, and if $\mathscr{F}$ is separated, then $\mathscr{F}^+$ is a sheaf. Hence $\mathscr{F}^{++}$ is a sheaf for every presheaf $\mathscr{F}$, and it easy to verify that the functor $\mathscr{F} \mapsto \mathscr{F}^{++}$ is left adjoint to the inclusion functor

$$\iota : \mathscr{X}^\sim \to \mathrm{Hom}_{\mathbf{Cat}}(\mathscr{X}^o, \mathbf{Set}) \tag{5.1.8}$$

from the category of sheaves on $\mathscr{X}$ to the category of presheaves.

5.1.9. Notice that every presheaf $\mathscr{F}$ on X extends canonically to a presheaf on $\mathscr{X}$; namely for every covering $\mathfrak{U} := (U_i \mid i \in I)$ one lets $\mathscr{F}'(\mathfrak{U}) := \prod_{i \in I} \mathscr{F}(U_i)$; moreover, the functor $\mathscr{F} \mapsto \mathscr{F}'$ commutes with the sheafification functors $\mathscr{F} \mapsto \mathscr{F}^{++}$ (on X and on $\mathscr{X}$). Furthermore, one has a natural identification : $\Gamma(\mathscr{X}, \mathscr{F}') = \Gamma(X, \mathscr{F})$.

The following lemma generalizes [39, Th.3.10.1].

Lemma 5.1.10. *Let X be a topological space fulfilling conditions* (a) *and* (b) *of lemma* 5.1.2. *Then the following holds :*

(i) *For every filtered system $\underline{\mathscr{F}} := (\mathscr{F}_\lambda \mid \lambda \in \Lambda)$ of sheaves on X, and every quasi-compact subset $U \subset X$, the natural map :*

$$\operatorname*{colim}_{\lambda \in \Lambda} \Gamma(U, \mathscr{F}_\lambda) \to \Gamma(U, \operatorname*{colim}_{\lambda \in \Lambda} \mathscr{F}_\lambda)$$

is bijective.

(ii) *If moreover, $\underline{\mathscr{F}}$ is a system of abelian sheaves, then the natural morphisms*

$$\operatorname*{colim}_{\lambda\in\Lambda} R^i\Gamma(U,\mathscr{F}_\lambda) \to R^i\Gamma(U,\operatorname*{colim}_{\lambda\in\Lambda}\mathscr{F}_\lambda)$$

are isomorphisms for every $i\in\mathbb{N}$ and every quasi-compact open subset $U\subset X$.

Proof. (i): In view of (5.1.9), we regard $\underline{\mathscr{F}}$ as a system of sheaves on $\mathscr{X}$.

We shall say that a covering $\mathfrak{U} := (U_i \mid i\in I)$ is quasi-compact, if it consists of finitely many quasi-compact open subsets U_i. Let $\mathscr{P} := \operatorname*{colim}_{\lambda\in\Lambda}\iota(\mathscr{F}_\lambda)$. Clearly $\operatorname*{colim}_{\lambda\in\Lambda}\mathscr{F}_\lambda = \mathscr{P}^{++}$.

Claim 5.1.11. For every quasi-compact covering $\mathfrak{V}$, the natural map

$$\mathscr{P}(\mathfrak{V}) \to \mathscr{P}^+(\mathfrak{V})$$

is a bijection.

Proof of the claim. Let $\mathfrak{U}\to\mathfrak{V}$ be any covering morphism. If $\mathfrak{V}$ is quasi-compact, condition (b) of lemma 5.1.2 implies that we can find a further refinement $\mathfrak{W}\to\mathfrak{U}$ with $\mathfrak{W}$ quasi-compact, hence in (5.1.7) we may replace the partially ordered set $\mathscr{D}_{\mathfrak{V}}$ by its full cofinal subset consisting of the equivalence classes of the quasi-compact coverings. Let $\mathfrak{U}\to\mathfrak{V}$ be such a quasi-compact refinement; say that $\mathfrak{U} = (U_i \mid i = 1,\dots,n)$. Since filtered colimits are exact, we deduce:

$$C^n(\mathfrak{U},\mathscr{P}) = \operatorname*{colim}_{\lambda\in\Lambda} C^n(\mathfrak{U},\mathscr{F}_\lambda) \qquad \text{for every } n\in\mathbb{N}.$$

Finally, since equalizers are finite limits, they commute as well with filtered colimits, so the claim follows easily. ◊

In view of claim 5.1.11, assertion (i) is equivalent to :

Claim 5.1.12. The natural map

$$\mathscr{P}^+(\mathfrak{V}) \to \mathscr{P}^{++}(\mathfrak{V})$$

is a bijection for every quasi-compact covering $\mathfrak{V}$.

Proof of the claim. We compute $\mathscr{P}^{++}(\mathfrak{V})$ by means of (5.1.7), applied to the presheaf $\mathscr{F} := \mathscr{P}^+$; arguing as in the proof of claim 5.1.11 we may restrict the colimit to the cofinal subset of $\mathscr{D}_{\mathfrak{V}}$ consisting of the equivalence classes of all quasi-compact objects. Let $\mathfrak{U}\to\mathfrak{V}$ be such a quasi-compact covering; since X is quasi-separated, also $\mathfrak{U}\times_{\mathfrak{V}}\mathfrak{U}$ is quasi-compact, hence $\mathscr{P}^+(\mathfrak{U}) = \mathscr{P}(\mathfrak{U})$ and $\mathscr{P}^+(\mathfrak{U}\times_{\mathfrak{V}}\mathfrak{U}) = \mathscr{P}(\mathfrak{U}\times_{\mathfrak{V}}\mathfrak{U})$, again by claim 5.1.11. And again, the claim follows easily from the fact that equalizers commute with filtered colimits. ◊

(ii): Suppose that $\underline{\mathscr{F}}$ is a system of abelian sheaves on X. For every $\lambda\in\Lambda$ we choose an injective resolution $\mathscr{F}_\lambda\to\mathscr{I}^\bullet_\lambda$. This choice can be made functorially, so that the filtered system $\underline{\mathscr{F}}$ extends to a filtered system $(\mathscr{F}_\lambda\to\mathscr{I}^\bullet_\lambda \mid \lambda\in\Lambda)$ of complexes of abelian sheaves. Clearly $\operatorname*{colim}_{\lambda\in\Lambda}\mathscr{I}^\bullet_\lambda$ is a resolution of $\operatorname*{colim}_{\lambda\in\Lambda}\mathscr{F}_\lambda$. Moreover :

Claim 5.1.13. $\operatorname*{colim}_{\lambda\in\Lambda}\mathscr{I}^\bullet_\lambda$ is a complex of qc-flabby sheaves.

Proof of the claim. Indeed, in view of (i) we have

$$\Gamma(U,\operatorname*{colim}_{\lambda\in\Lambda}\mathscr{I}^n_\lambda) = \operatorname*{colim}_{\lambda\in\Lambda}\Gamma(U,\mathscr{I}^n_\lambda)$$

for every $n\in\mathbb{N}$ and every quasi-compact open subset $U\subset X$. Then the claim follows from the well known fact that injective sheaves are flabby (hence qc-flabby). ◊

One sees easily that if U is quasi-compact, the functor $\Gamma(U,-)$ and the class $\mathscr{M}$ of qc-flabby sheaves fulfill conditions (a) and (b) of lemma 5.1.3, and (c) holds as well, in light of lemma 5.1.2; therefore qc-flabby sheaves are acyclic for $\Gamma(U,-)$ and assertion (ii) follows easily from claim 5.1.13. □

5.1.14. Let Λ be a small cofiltered category with final object $0 \in \mathrm{Ob}(\Lambda)$, and X_0 a quasi-separated scheme; we consider the datum consisting of:

- Two cofiltered systems $\underline{X} := (X_\lambda \mid \lambda \in \Lambda)$ and $\underline{Y} := (Y_\lambda \mid \lambda \in \Lambda)$ of X_0-schemes, with affine transition morphisms :
$$\varphi_u : X_\mu \to X_\lambda \qquad \psi_u : Y_\mu \to Y_\lambda \qquad \text{for every morphism } u : \mu \to \lambda \text{ in } \Lambda.$$
- A cofiltered system of sheaves $\underline{\mathscr{F}} := (\mathscr{F}_\lambda \mid \lambda \in \Lambda)$ where $\mathscr{F}_\lambda$ is a sheaf on Y_λ for every $\lambda \in \mathrm{Ob}(\Lambda)$, with transition maps :
$$\psi_u^{-1}\mathscr{F}_\lambda \to \mathscr{F}_\mu \qquad \text{for every morphism } u : \mu \to \lambda \text{ in } \Lambda.$$
- A system $\underline{g} := (g_\lambda : Y_\lambda \to X_\lambda \mid \lambda \in \Lambda)$ of quasi-compact and quasi-separated morphisms, such that $g_\lambda \circ \psi_u = \varphi_u \circ g_\mu$ for every morphism $u : \mu \to \lambda$.

By [32, Ch.IV, Prop.8.2.3] the (inverse) limits of the systems $\underline{X}$, $\underline{Y}$ and $\underline{g}$ are representable by X_0-schemes and morphisms
$$X_\infty := \lim_{\lambda \in \Lambda} X_\lambda \qquad Y_\infty := \lim_{\lambda \in \Lambda} Y_\lambda \qquad g_\infty := \lim_{\lambda \in \Lambda} g_\lambda : Y_\infty \to X_\infty$$
and the natural morphisms
$$\varphi_\lambda : X_\infty \to X_\lambda \qquad \psi_\lambda : Y_\infty \to Y_\lambda.$$
are affine for every $\lambda \in \Lambda$. Moreover, $\underline{\mathscr{F}}$ induces a sheaf on Y_∞ :
$$\mathscr{F}_\infty := \operatorname*{colim}_{\lambda \in \Lambda} \psi_\lambda^{-1}\mathscr{F}_\lambda.$$

Proposition 5.1.15. *In the situation of* (5.1.14)*, the following holds :*

(i) *Suppose that X_0 is quasi-compact. Then the natural map*
$$\operatorname*{colim}_{\lambda \in \Lambda} \Gamma(Y_\lambda, \mathscr{F}_\lambda) \to \Gamma(Y_\infty, \mathscr{F}_\infty)$$
is a bijection.

(ii) *If $\underline{\mathscr{F}}$ is a system of abelian sheaves, then the natural morphisms :*
$$\operatorname*{colim}_{\lambda \in \Lambda} \varphi_\lambda^{-1} R^i g_{\lambda *}\mathscr{F}_\lambda \to R^i g_{\infty *}\mathscr{F}_\infty$$
are isomorphisms for every $i \in \mathbb{N}$.

Proof. (i): Arguing as in the proof of lemma 5.1.10(i), we regard each $\mathscr{F}_\lambda$ as a sheaf on the site T_λ attached to Y_λ by (5.1.4); similarly $\mathscr{F}_\infty$ is a sheaf on the site T_∞ obtained from Y_∞. The assumption implies that Y_∞, as well as Y_λ for every $\lambda \in \Lambda$, are quasi-compact and quasi-separated. Then lemma 5.1.10(i) shows that the natural map :
$$\operatorname*{colim}_{\lambda \in \Lambda} \Gamma(Y_\infty, \psi_\lambda^{-1}\mathscr{F}_\lambda) \to \Gamma(Y_\infty, \mathscr{F}_\infty)$$
is a bijection. Let us set $\mathscr{G}_\lambda := \psi_\lambda^{-1}\iota(\mathscr{F}_\lambda)$ (notation of (5.1.8)). Quite generally, the presheaf pull-back of a separated presheaf – under a map of topological spaces – is separated; hence $\mathscr{G}_\lambda$ is a separated presheaf, so the global sections of its sheafification can be computed via the Čech complex : in the notation of (5.1.6), we have
$$\begin{aligned}\Gamma(Y_\infty, \mathscr{F}_\infty) &= \operatorname*{colim}_{\lambda \in \Lambda} \operatorname*{colim}_{\mathfrak{U}_\infty \in \mathscr{D}_{Y_\infty}} \mathrm{Equal}(C^0(\mathfrak{U}_\infty, \mathscr{G}_\lambda) \rightrightarrows C^1(\mathfrak{U}_\infty, \mathscr{G}_\lambda)) \\ &= \operatorname*{colim}_{\mathfrak{U}_\infty \in \mathscr{D}_{Y_\infty}} \operatorname*{colim}_{\lambda \in \Lambda} \mathrm{Equal}(C^0(\mathfrak{U}_\infty, \mathscr{G}_\lambda) \rightrightarrows C^1(\mathfrak{U}_\infty, \mathscr{G}_\lambda)) .\end{aligned}$$
However, arguing as in the proof of claim 5.1.11, we see that we may replace the filtered set $\mathscr{D}_{Y_\infty}$ by its cofinal system of quasi-compact coverings. If $\mathfrak{U}_\infty \to Y_\infty$ is such a quasi-compact covering, we can find $\lambda \in \mathrm{Ob}(\Lambda)$, and a quasi-compact covering $\mathfrak{U}_\lambda \to U_\lambda$ such that $U_\lambda \subset Y_\lambda$ is a quasi-compact open subset, and $\mathfrak{U}_\infty = \mathfrak{U}_\lambda \times_{Y_\lambda} Y_\infty$ ([32, Ch.IV, Cor.8.2.11]). Up to

replacing λ by some $\lambda' \geq \lambda$, we can achieve that $U_\lambda = Y_\lambda$, and moreover if $\mathfrak{U}_\mu \to Y_\mu$ is any other covering with the same property, then we can find $\rho \in \Lambda$ with $\rho \geq \mu, \lambda$ and such that $\mathfrak{U}_\lambda \times_{Y_\lambda} Y_\rho = \mathfrak{U}_\mu \times_{Y_\mu} Y_\rho$ ([32, Ch.IV, Cor.8.3.5]). Likewise, if $\mathfrak{V} \to Y_\lambda$ is a covering such that $\mathfrak{V} \times_{Y_\lambda} Y_\infty = \mathfrak{U}_\infty \times_{Y_\infty} \mathfrak{U}_\infty$, then, up to replacing λ by a larger index, we have $\mathfrak{V} = \mathfrak{U}_\lambda \times_{Y_\lambda} \mathfrak{U}_\lambda$. Hence, let us fix any such covering $\mathfrak{U}_{\lambda_0} \to Y_{\lambda_0}$ and set $\mathfrak{U}_\mu := \mathfrak{U}_{\lambda_0} \times_{Y_{\lambda_0}} Y_\mu$ for every $\mu \geq \lambda_0$; it follows easily that

$$\begin{aligned}\operatorname*{colim}_{\lambda\in\Lambda} \mathrm{Equal}(C^0(\mathfrak{U}_\infty, \mathscr{G}_\lambda) \rightrightarrows C^1(\mathfrak{U}_\infty, \mathscr{G}_\lambda)) &= \operatorname*{colim}_{\mu\geq\lambda_0} \mathrm{Equal}(C^0(\mathfrak{U}_\mu, \mathscr{F}_\mu) \rightrightarrows C^1(\mathfrak{U}_\mu, \mathscr{F}_\mu)) \\ &= \operatorname*{colim}_{\mu\geq\lambda_0} \Gamma(Y_\mu, \mathscr{F}_\mu).\end{aligned}$$

whence (i). To show (ii), we choose for each $\lambda \in \Lambda$ an injective resolution $\mathscr{F}_\lambda \to \mathscr{I}^\bullet_\lambda$, having care to construct $\mathscr{I}^\bullet_\lambda$ functorially, so that $\underline{\mathscr{F}}$ extends to a compatible system of complexes $(\mathscr{F}_\lambda \to \mathscr{I}^\bullet_\lambda \mid \lambda \in \Lambda)$.

Claim 5.1.16. $\mathscr{I}^\bullet_\infty := \operatorname*{colim}_{\lambda\in\Lambda} \psi_\lambda^{-1} \mathscr{I}^\bullet_\lambda$ is a complex of qc-flabby sheaves.

Proof of the claim. Indeed, let $U_\infty \subset V_\infty$ be any two quasi-compact open subset of Y_∞; we need to show that the restriction map

$$\Gamma(V_\infty, \mathscr{I}^\bullet_\infty) \to \Gamma(U_\infty, \mathscr{I}^\bullet_\infty)$$

is onto. By [32, Ch.IV, Cor.8.2.11] we can find $\lambda \in \Lambda$ and quasi-compact open subsets $U_\lambda \subset V_\lambda \subset Y_\lambda$ such that $U_\infty = \psi_\lambda^{-1} U_\lambda$, and likewise for V_∞. Let us set $U_u := \psi_u^{-1} U_\lambda$ for every $u \in \mathrm{Ob}(\Lambda/\lambda)$, and likewise define V_u. Then, up to replacing Λ by Λ/λ, we may assume that $U_\mu \subset V_\mu$ are defined for every $\mu \in \Lambda$. By (i) the natural map

$$\operatorname*{colim}_{\lambda\in\Lambda} \Gamma(U_\lambda, \mathscr{I}^n_\lambda) \to \Gamma(U_\infty, \mathscr{I}^n_\infty)$$

is bijective for every $n \in \mathbb{N}$, and likewise for V_∞. Since an injective sheaf is flabby, the claim follows easily. ◊

Using the assumptions and lemma 5.1.2, one checks easily that the functor $g_{\infty *}$ and the class $\mathscr{M}$ of qc-flabby sheaves on Y_∞ fulfill conditions (a)–(c) of lemma 5.1.3. Therefore we get a natural isomorphism in $\mathsf{D}(\mathbb{Z}_{X_\infty}\text{-}\mathbf{Mod})$:

$$g_{\infty *} \mathscr{I}^\bullet_\infty \xrightarrow{\sim} R g_{\infty *} \mathscr{F}_\infty.$$

To conclude, let $V_\infty \to X_\infty$ be any quasi-compact open immersion, which as usual we see as the limit of a system $(V_\lambda \to X_\lambda \mid \lambda \in \Lambda)$ of quasi-compact open immersions; it then suffices to show that the natural map

$$\operatorname*{colim}_{\lambda\in\Lambda} \Gamma(V_\lambda, \mathscr{I}^\bullet_\lambda) \to \Gamma(V_\infty, \mathscr{I}^\bullet_\infty)$$

is an isomorphism of complexes. The latter assertion holds by (i). □

Corollary 5.1.17. *Let $f : Y \to X$ be a quasi-compact and quasi-separated morphism of schemes, $\mathscr{F}$ a flat quasi-coherent $\mathscr{O}_X$-module, and $\mathscr{G}$ any $\mathscr{O}_Y$-module. Then the natural map*

$$\mathscr{F} \otimes_{\mathscr{O}_X} Rf_* \mathscr{G} \to Rf_*(f^* \mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{G}) \tag{5.1.18}$$

is an isomorphism.

Proof. The assertion is local on X, hence we may assume that $X = \operatorname{Spec} A$ for some ring A, and $\mathscr{F} = M^\sim$ for some flat A-module M. By [57, Ch.I, Th.1.2], M is the colimit of a filtered family of free A-modules of finite rank; in view of proposition 5.1.15(ii), we may then assume that $M = A^{\oplus n}$ for some $n \geq 0$, in which case the assertion is obvious. □

Corollary 5.1.19. *Consider a cartesian diagram of schemes*

$$\begin{tikzcd} Y' \arrow[r, "g'"] \arrow[d, "f'"'] & Y \arrow[d, "f"] \\ X' \arrow[r, "g"] & X \end{tikzcd}$$

such that f is quasi-compact and quasi-separated, and g is flat. Then the natural map

$$g^*Rf_*\mathscr{G} \to Rf'_*g'^*\mathscr{G} \tag{5.1.20}$$

is an isomorphism in $\mathsf{D}(\mathscr{O}_{X'}\text{-}\mathbf{Mod})$, for every $\mathscr{O}_Y$-module $\mathscr{G}$.

Proof. We easily reduce to the case where both X and X' are affine, hence g is an affine morphism. In this case, it suffices to show that g_*(5.1.20) is an isomorphism. However, we have an essentially commutative diagram

$$\begin{tikzcd} g_*(g^*Rf_*\mathscr{G}) \arrow[rr, "\alpha"] \arrow[d] & & Rf_*\mathscr{G} \otimes_{\mathscr{O}_X} g_*\mathscr{O}_{X'} \arrow[d] \\ g_*(Rf'_*g'^*\mathscr{G}) \arrow[r, "\sim"] & Rf_*g'_*(g'^*\mathscr{G}) \arrow[r, "\beta"] & Rf_*(\mathscr{G} \otimes_{\mathscr{O}_Y} g'_*\mathscr{O}_{Y'}) \end{tikzcd}$$

whose left vertical arrow is g_*(5.1.20) and whose right vertical arrow is the natural isomorphism provided by corollary 5.1.17 (applied to the flat $\mathscr{O}_Y$-module $\mathscr{F} := g_*\mathscr{O}_{Y'}$). Also, α and β are the natural maps obtained as in [26, Ch.0, (5.4.10)]; it is easily seen that these are isomorphisms, for any affine morphism g and g'. The assertion follows. $\square$

Remark 5.1.21. Notice that, for an affine morphism $f : Y \to X$, the map (5.1.18) is an isomorphism for any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ and any quasi-coherent $\mathscr{O}_Y$-module $\mathscr{G}$. The details shall be left to the reader.

5.1.22. Let $(X, \mathscr{A})$ be a ringed space; one defines as in example 4.1.8(v) a *total* Hom *cochain complex*, which is a functor :

$$\mathscr{H}om^\bullet_{\mathscr{A}} : \mathsf{C}(\mathscr{A}\text{-}\mathbf{Mod})^o \times \mathsf{C}(\mathscr{A}\text{-}\mathbf{Mod}) \to \mathsf{C}(\mathscr{A}\text{-}\mathbf{Mod})$$

on the category of complexes of $\mathscr{A}$-modules. Recall that, for any two complexes $M_\bullet$, $N^\bullet$, and any object U of X, the group of n-cocycles in $\mathscr{H}om^\bullet_{\mathscr{A}}(M_\bullet, N^\bullet)(U)$ is naturally isomorphic to $\mathrm{Hom}_{\mathsf{C}(\mathscr{A}_{|U}\text{-}\mathbf{Mod})}(M_{\bullet|U}, N^\bullet_{|U}[n])$, and $H^n\mathscr{H}om^\bullet_{\mathscr{A}}(M_\bullet, N^\bullet)(U)$ is naturally isomorphic to the group of homotopy classes of maps $M_{\bullet|U} \to N^\bullet_{|U}[n]$ (see example 4.1.7(i)). We also set :

$$\mathrm{Hom}^\bullet_{\mathscr{A}} := \Gamma \circ \mathscr{H}om^\bullet_{\mathscr{A}} : \mathsf{C}(\mathscr{A}\text{-}\mathbf{Mod})^o \times \mathsf{C}(\mathscr{A}\text{-}\mathbf{Mod}) \to \mathsf{C}(\Gamma(\mathscr{A})\text{-}\mathbf{Mod}). \tag{5.1.23}$$

The bifunctor $\mathscr{H}om^\bullet_{\mathscr{A}}$ admits a right derived functor :

$$R\mathscr{H}om^\bullet_{\mathscr{A}} : \mathsf{D}(\mathscr{A}\text{-}\mathbf{Mod})^o \times \mathsf{D}^+(\mathscr{A}\text{-}\mathbf{Mod}) \to \mathsf{D}(\mathscr{A}\text{-}\mathbf{Mod})$$

for whose construction we refer to [75, §10.7]. Likewise, one has a derived functor $R\mathrm{Hom}^\bullet_{\mathscr{A}}$ for (5.1.23), and there are natural isomorphisms of $\mathscr{A}$-modules :

$$H^iR\mathrm{Hom}^\bullet_{\mathscr{A}}(M^\bullet, N^\bullet) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{D}^+(\mathscr{A}\text{-}\mathbf{Mod})}(M^\bullet, N^\bullet[i]) \tag{5.1.24}$$

for every $i \in \mathbb{Z}$, and every bounded below complexes $M^\bullet$ and $N^\bullet$ of $\mathscr{A}$-modules.

Lemma 5.1.25. *Let $(X, \mathscr{A})$ be a ringed space, $\mathscr{B}$ an $\mathscr{A}$-algebra. Then :*

(i) *If $\mathscr{I}$ is an injective $\mathscr{A}$-module, $\mathscr{H}om_{\mathscr{A}}(\mathscr{F}, \mathscr{I})$ is flabby for every $\mathscr{A}$-module $\mathscr{F}$, and $\mathscr{H}om_{\mathscr{A}}(\mathscr{B}, \mathscr{I})$ is an injective $\mathscr{B}$-module.*

(ii) *There is a natural isomorphism of bifunctors :*

$$R\Gamma \circ R\mathscr{H}om^\bullet_{\mathscr{A}} \xrightarrow{\sim} R\mathrm{Hom}^\bullet_{\mathscr{A}}.$$

(iii) *The forgetful functor* $\mathsf{D}^+(\mathscr{B}\text{-}\mathbf{Mod}) \to \mathsf{D}^+(\mathscr{A}\text{-}\mathbf{Mod})$ *admits the right adjoint :*

$$\mathsf{D}^+(\mathscr{A}\text{-}\mathbf{Mod}) \to \mathsf{D}^+(\mathscr{B}\text{-}\mathbf{Mod}) \quad : \quad K^\bullet \mapsto R\mathscr{H}om^\bullet_{\mathscr{A}}(\mathscr{B}, K^\bullet).$$

Proof. (i) : Let $j : U \subset X$ be any open immersion; a section $s \in \mathscr{H}om_{\mathscr{A}}(\mathscr{F}, \mathscr{I})(U)$ is a map of $\mathscr{A}_{|U}$-modules $s : \mathscr{F}_{|U} \to \mathscr{I}_{|U}$. We deduce a map of $\mathscr{A}$-modules $j_!s : j_!\mathscr{F}_{|U} \to j_!\mathscr{I}_{|U} \to \mathscr{I}$; since $\mathscr{I}$ is injective, $j_!s$ extends to a map $\mathscr{F} \to \mathscr{I}$, as required. Next, let $\mathscr{B}$ be an $\mathscr{A}$-algebra; we recall the following :

Claim 5.1.26. The functor

$$\mathscr{A}\text{-}\mathbf{Mod} \to \mathscr{B}\text{-}\mathbf{Mod} \quad : \quad \mathscr{F} \mapsto \mathscr{H}om_{\mathscr{A}}(\mathscr{B}, \mathscr{F})$$

is right adjoint to the forgetful functor.

Proof of the claim. We have to exhibit a natural bijection

$$\mathrm{Hom}_{\mathscr{A}}(N, M) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{B}}(N, \mathrm{Hom}_{\mathscr{A}}(\mathscr{B}, M))$$

for every $\mathscr{A}$-module M and $\mathscr{B}$-module N. This is given by the following rule. To an $\mathscr{A}$-linear map $t : N \to M$ one assigns the $\mathscr{B}$-linear map $t' : N \to \mathrm{Hom}_{\mathscr{A}}(\mathscr{B}, M)$ such that $t'(x)(b) = t(x \cdot b)$ for every $x \in N(U)$ and $b \in \mathscr{B}(V)$, where $V \subset U$ are any two open subsets of X. To describe the inverse of this transformation, it suffices to remark that $t(x) = t'(x)(1)$ for every local section x of N. ◊

Since the forgetful functor is exact, the second assertion of (i) follows immediately from claim 5.1.26. Assertion (ii) follows from (i) and [75, Th.10.8.2].

(iii): Let $K^\bullet$ (resp. $L^\bullet$) be a bounded below complex of $\mathscr{B}$-modules (resp. of injective $\mathscr{A}$-modules); then :

$$\begin{aligned}\mathrm{Hom}_{\mathsf{D}^+(\mathscr{A}\text{-}\mathbf{Mod})}(K^\bullet, L^\bullet) &= H^0\mathrm{Hom}^\bullet_{\mathsf{C}(\mathscr{A}\text{-}\mathbf{Mod})}(K^\bullet, L^\bullet) \\ &= H^0\mathrm{Hom}^\bullet_{\mathsf{C}(\mathscr{B}\text{-}\mathbf{Mod})}(K^\bullet, \mathscr{H}om_{\mathscr{A}}(\mathscr{B}, L^\bullet)) \\ &= \mathrm{Hom}_{\mathsf{D}^+(\mathscr{B}\text{-}\mathbf{Mod})}(K^\bullet, \mathscr{H}om_{\mathscr{A}}(\mathscr{B}, L^\bullet))\end{aligned}$$

where the last identity follows from (i) and (5.1.24). □

Theorem 5.1.27. (Trivial duality) *Let* $f : (Y, \mathscr{B}) \to (X, \mathscr{A})$ *be a morphism of ringed spaces. For any two complexes* $M_\bullet$ *in* $\mathsf{D}^-(\mathscr{A}\text{-}\mathbf{Mod})$ *and* $N^\bullet$ *in* $\mathsf{D}^+(\mathscr{B}\text{-}\mathbf{Mod})$, *there are natural isomorphisms :*

(i) $R\mathrm{Hom}^\bullet_{\mathscr{A}}(M_\bullet, Rf_*N^\bullet) \xrightarrow{\sim} R\mathrm{Hom}^\bullet_{\mathscr{B}}(Lf^*M_\bullet, N^\bullet)$ *in* $\mathsf{D}^+(\mathscr{A}(X)\text{-}\mathbf{Mod})$.

(ii) $R\mathscr{H}om^\bullet_{\mathscr{A}}(M_\bullet, Rf_*N^\bullet) \xrightarrow{\sim} Rf_*R\mathscr{H}om^\bullet_{\mathscr{B}}(Lf^*M_\bullet, N^\bullet)$ *in* $\mathsf{D}^+(\mathscr{A}\text{-}\mathbf{Mod})$.

Proof. One applies lemma 5.1.25(ii) to deduce (i) from (ii).

(ii) : By standard adjunctions, for any two complexes $M_\bullet$ and $N^\bullet$ as in the proposition, we have a natural isomorphism of total Hom cochain complexes :

$$\mathscr{H}om^\bullet_{\mathscr{A}}(M_\bullet, f_*N^\bullet) \xrightarrow{\sim} f_*\mathscr{H}om^\bullet_{\mathscr{B}}(f^*M_\bullet, N^\bullet). \tag{5.1.28}$$

Now, let us choose a flat resolution $P_\bullet \xrightarrow{\sim} M_\bullet$ of $\mathscr{A}$-modules, and an injective resolution $N^\bullet \xrightarrow{\sim} I^\bullet$ of $\mathscr{B}$-modules. In view of (5.1.28) and lemma 5.1.25(i) we have natural isomorphisms :

$$Rf_*R\mathscr{H}om^\bullet_{\mathscr{B}}(Lf^*M_\bullet, N^\bullet) \xrightarrow{\sim} f_*\mathscr{H}om^\bullet_{\mathscr{B}}(f^*P_\bullet, I^\bullet) \xleftarrow{\sim} \mathscr{H}om^\bullet_{\mathscr{A}}(P_\bullet, f_*I^\bullet)$$

in $\mathsf{D}^+(\mathscr{A}\text{-}\mathbf{Mod})$. It remains to show that the natural map

$$\mathscr{H}om^\bullet_{\mathscr{A}}(P_\bullet, f_*I^\bullet) \to R\mathscr{H}om^\bullet_{\mathscr{A}}(P_\bullet, f_*I^\bullet) \tag{5.1.29}$$

is an isomorphism. However, we have two spectral sequences :

$$\begin{aligned}E^1_{pq} &:= H_p\mathscr{H}om^\bullet_{\mathscr{A}}(P_q, f_*I^\bullet) \Rightarrow H_{p+q}\mathscr{H}om^\bullet_{\mathscr{A}}(P_\bullet, f_*I^\bullet) \\ F^1_{pq} &:= H_pR\mathscr{H}om^\bullet_{\mathscr{A}}(P_q, f_*I^\bullet) \Rightarrow H_{p+q}R\mathscr{H}om^\bullet_{\mathscr{A}}(P_\bullet, f_*I^\bullet)\end{aligned}$$

and (5.1.29) induces a natural map of spectral sequences :

$$E^1_{pq} \to F^1_{pq} \tag{5.1.30}$$

Consequently, it suffices to show that (5.1.30) is an isomorphism for every $p, q \in \mathbb{N}$, and therefore we may assume that $P_\bullet$ consists of a single flat $\mathscr{A}$-module placed in degree zero. A similar argument reduces to the case where $I_\bullet$ consists of a single injective $\mathscr{B}$-module sitting in degree zero. To conclude, it suffices to show :

Claim 5.1.31. Let P be a flat $\mathscr{A}$-module, I an injective $\mathscr{B}$-module. Then the natural map :

$$\mathscr{H}\!om_{\mathscr{A}}(P, f_*I)[0] \to R\mathscr{H}\!om^\bullet_{\mathscr{A}}(P, f_*I)$$

is an isomorphism.

Proof of the claim. $R^i\mathscr{H}\!om^\bullet_{\mathscr{A}}(P, f_*I)$ is the sheaf associated to the presheaf on X :

$$U \mapsto R^i\mathrm{Hom}^\bullet_{\mathscr{A}_{|U}}(P_{|U}, f_*I_{|U}).$$

Since $I_{|U}$ is an injective $\mathscr{B}_{|U}$-module, it suffices therefore to show that $R^i\mathrm{Hom}^\bullet_{\mathscr{A}}(P, f_*I) = 0$ for $i > 0$. However, recall that there is a natural isomorphism :

$$R^i\mathrm{Hom}^\bullet_{\mathscr{A}}(P, f_*I) \simeq \mathrm{Hom}_{\mathsf{D}(\mathscr{A}\text{-}\mathbf{Mod})}(P, f_*I[-i]).$$

Since the homotopy category $\mathsf{Hot}(\mathscr{A}\text{-}\mathbf{Mod})$ admits a left calculus of fractions, we deduce a natural isomorphism :

$$R^i\mathrm{Hom}^\bullet_{\mathscr{A}}(P, f_*I) \simeq \operatorname*{colim}_{E_\bullet \to P} \mathrm{Hom}_{\mathsf{Hot}(\mathscr{A}\text{-}\mathbf{Mod})}(E_\bullet, f_*I[-i]) \tag{5.1.32}$$

where the colimit ranges over the family of quasi-isomorphisms $E_\bullet \to P$. We may furthermore restrict the colimit in (5.1.32) to the subfamily of all such $E_\bullet \to P$ where $E_\bullet$ is a bounded above complex of flat $\mathscr{A}$-modules, since this subfamily is cofinal. Every such map $E_\bullet \to P$ induces a commutative diagram :

$$\begin{array}{ccc} \mathrm{Hom}_{\mathscr{A}}(P, f_*I[-i]) & \longrightarrow & \mathrm{Hom}_{\mathsf{Hot}(\mathscr{A}\text{-}\mathbf{Mod})}(E_\bullet, f_*I[-i]) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathscr{B}}(f^*P, I[-i]) & \longrightarrow & \mathrm{Hom}_{\mathsf{Hot}(\mathscr{B}\text{-}\mathbf{Mod})}(f^*E_\bullet, I[-i]) \end{array}$$

whose vertical arrows are isomorphisms. Since $E_\bullet$ and P are complexes of flat $\mathscr{A}$-modules, the induced map $f^*E_\bullet \to f^*P$ is again a quasi-isomorphism; therefore, since I is injective, the bottom horizontal arrow is an isomorphism, hence the same holds for the top horizontal one, and the claim follows easily. $\square$

Corollary 5.1.33. *Let A be a ring, $M^\bullet$ (resp. $N_\bullet$) a bounded below (resp. above) complex of A-modules. Set $X := \operatorname{Spec} A$ and denote by $M^{\bullet\sim}$ (resp. $N^\sim_\bullet$) the associated complex of quasi-coherent $\mathscr{O}_X$-modules. Then the natural map :*

$$R\mathrm{Hom}^\bullet_A(N_\bullet, M^\bullet) \to R\mathrm{Hom}^\bullet_{\mathscr{O}_X}(N^\sim_\bullet, M^{\bullet\sim})$$

is an isomorphism in $\mathsf{D}^+(A\text{-}\mathbf{Mod})$.

Proof. We apply the trivial duality theorem 5.1.27 to the unique morphism

$$f : (X, \mathscr{O}_X) \to (\{\mathrm{pt}\}, A)$$

of ringed spaces, where $\{\mathrm{pt}\}$ denotes the one-point space. Since f is flat and all quasi-coherent $\mathscr{O}_X$-modules are f_*-acyclic, the assertion follows easily. $\square$

5.1.34. Suppose that $\mathscr{F}$ is an abelian group object in a given topos X, and let $(\mathrm{Fil}^n\mathscr{F} \mid n \in \mathbb{N})$ be a descending filtration by abelian subobjects of $\mathscr{F}$, such that $\mathrm{Fil}^0\mathscr{F} = \mathscr{F}$. Set $\mathscr{F}^n := \mathscr{F}/\mathrm{Fil}^{n+1}\mathscr{F}$ for every $n \in \mathbb{N}$, and denote by $\mathrm{gr}^\bullet\mathscr{F}$ the graded abelian object associated to the filtered object $\mathrm{Fil}^\bullet\mathscr{F}$; then we have :

Lemma 5.1.35. *In the situation of* (5.1.34), *suppose furthermore that the natural map :*

$$\mathscr{F} \to R\lim_{k\in\mathbb{N}} \mathscr{F}^k$$

is an isomorphism in $\mathsf{D}(\mathbb{Z}_X\text{-}\mathbf{Mod})$. *Then, for every* $q \in \mathbb{N}$ *the following conditions are equivalent :*

(a) $H^q(X, \mathrm{Fil}^n\mathscr{F}) = 0$ *for every* $n \in \mathbb{N}$.

(b) *The natural map* $H^q(X, \mathrm{Fil}^n\mathscr{F}) \to H^q(X, \mathrm{gr}^n\mathscr{F})$ *vanishes for every* $n \in \mathbb{N}$.

Proof. Obviously we have only to show that (b) implies (a). Moreover, let us set :

$$(\mathrm{Fil}^n\mathscr{F})^k := \mathrm{Fil}^n\mathscr{F}/\mathrm{Fil}^{k+1}\mathscr{F} \qquad \text{for every } n, k \in \mathbb{N} \text{ with } k \geq n.$$

There follows, for every $n \in \mathbb{N}$, a short exact sequence of inverse systems of sheaves :

$$0 \to ((\mathrm{Fil}^n\mathscr{F})^k \mid k \geq n) \to (\mathscr{F}^k \mid k \geq n) \to \mathscr{F}^{n-1} \to 0$$

(where the right-most term is the constant inverse system which equals $\mathscr{F}^{n-1}$ in all degrees, with transition maps given by the identity morphisms); whence a distinguished triangle :

$$R\lim_{k\in\mathbb{N}} (\mathrm{Fil}^n\mathscr{F})^k \to R\lim_{k\in\mathbb{N}} \mathscr{F}^k \to \mathscr{F}^{n-1}[0] \to R\lim_{k\in\mathbb{N}} (\mathrm{Fil}^n\mathscr{F})^k[1]$$

which, together with our assumption on $\mathscr{F}$, easily implies that the natural map $\mathrm{Fil}^n\mathscr{F} \to R\lim_{k\in\mathbb{N}}(\mathrm{Fil}^n\mathscr{F})^k$ is an isomorphism in $\mathsf{D}(\mathbb{Z}_X\text{-}\mathbf{Mod})$. Summing up, we may replace the datum $(\mathscr{F}, \mathrm{Fil}^\bullet\mathscr{F})$ by $(\mathrm{Fil}^n\mathscr{F}, \mathrm{Fil}^{\bullet+n}\mathscr{F})$, and reduce to the case where $n = 0$.

The inverse system $(\mathscr{F}^n \mid n \in \mathbb{N})$ defines an abelian group object of the topos $X^\mathbb{N}$ (see [36, §7.3.4]); whence a spectral sequence:

$$E_2^{pq} := \lim_{n\in\mathbb{N}}{}^p\, H^q(X, \mathscr{F}^n) \Rightarrow H^{p+q}(X^\mathbb{N}, \mathscr{F}^\bullet) \simeq H^{p+q}(X, R\lim_{n\in\mathbb{N}} \mathscr{F}^n). \tag{5.1.36}$$

By [75, Cor.3.5.4] we have $E_2^{pq} = 0$ whenever $p > 1$, and, in view of our assumptions, (5.1.36) decomposes into short exact sequences :

$$0 \to \lim_{n\in\mathbb{N}}{}^1\, H^{q-1}(X, \mathscr{F}^n) \xrightarrow{\alpha_q} H^q(X, \mathscr{F}) \xrightarrow{\beta_q} \lim_{n\in\mathbb{N}} H^q(X, \mathscr{F}^n) \to 0 \qquad \text{for every } q \in \mathbb{N}$$

where β_q is induced by the natural system of maps $(\beta_{q,n} : H^q(X, \mathscr{F}) \to H^q(X, \mathscr{F}^n) \mid n \in \mathbb{N})$.

Claim 5.1.37. If (b) holds, then $\beta_q = 0$.

Proof of the claim. Suppose that (b) holds; we consider the long exact cohomology sequence associated to the short exact sequence :

$$0 \to \mathrm{Fil}^{n+1}\mathscr{F} \to \mathrm{Fil}^n\mathscr{F} \to \mathrm{gr}^n\mathscr{F} \to 0$$

to deduce that the natural map $H^q(X, \mathrm{Fil}^{n+1}\mathscr{F}) \to H^q(X, \mathrm{Fil}^n\mathscr{F})$ is onto for every $n \in \mathbb{N}$; hence the same holds for the natural map $H^q(X, \mathrm{Fil}^n\mathscr{F}) \to H^q(X, \mathscr{F})$. By considering the long exact cohomology sequence attached to the short exact sequence :

$$0 \to \mathrm{Fil}^n\mathscr{F} \to \mathscr{F} \to \mathscr{F}^n \to 0 \tag{5.1.38}$$

we then deduce that $\beta_{q,n}$ vanishes for every $n \in \mathbb{N}$, which implies the claim. ◊

Claim 5.1.39. If (b) holds, then the inverse system $(H^{q-1}(X, \mathscr{F}^n) \mid n \in \mathbb{N})$ has surjective transition maps.

Proof of the claim. By considering the long exact cohomology sequence associated to the short exact sequence : $0 \to \mathrm{gr}^n\mathscr{F} \to \mathscr{F}^{n+1} \to \mathscr{F}^n \to 0$, we are reduced to showing that the boundary map $\partial_{q-1} : H^{q-1}(X, \mathscr{F}^n) \to H^q(X, \mathrm{gr}^n\mathscr{F})$ vanishes for every $n \in \mathbb{N}$. However, comparing with (5.1.38), we find that ∂_{q-1} factors through the natural map $H^q(X, \mathrm{Fil}^n\mathscr{F}) \to H^q(X, \mathrm{gr}^n\mathscr{F})$, which vanishes if (b) holds. $\Diamond$

The lemma follows from claims 5.1.37 and 5.1.39, and [75, Lemma 3.5.3]. $\square$

5.2. **Quasi-coherent modules.** Let X be any scheme, and $\mathscr{F}$ any abelian sheaf on X. The *support* of $\mathscr{F}$ is the subset :

$$\mathrm{Supp}\,\mathscr{F} := \{x \in X \mid \mathscr{F}_x \neq 0\} \subset X.$$

Recall that an $\mathscr{O}_X$-module $\mathscr{F}$ is said to be *f-flat at a point* $x \in X$ if $\mathscr{F}_x$ is a flat $\mathscr{O}_{Y,f(x)}$-module. $\mathscr{F}$ is said to be *f-flat over a point* $y \in Y$ if $\mathscr{F}$ is f-flat at all points of $f^{-1}(y)$. Finally, one says that $\mathscr{F}$ is *f-flat* if $\mathscr{F}$ is f-flat at all the points of X ([26, Ch.0, §6.7.1]).

5.2.1. We denote by $\mathscr{O}_X$-$\mathbf{Mod}$ (resp. $\mathscr{O}_X$-$\mathbf{Mod}_{\mathrm{qcoh}}$, $\mathscr{O}_X$-$\mathbf{Mod}_{\mathrm{coh}}$, resp. $\mathscr{O}_X$-$\mathbf{Mod}_{\mathrm{lfft}}$) the category of all (resp. of quasi-coherent, resp. of coherent, resp. of locally free of finite type) $\mathscr{O}_X$-modules. Also, we denote by $\mathsf{D}(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{qcoh}}$ (resp. $\mathsf{D}(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{coh}}$) the full triangulated subcategory of $\mathsf{D}(\mathscr{O}_X\text{-}\mathbf{Mod})$ consisting of the complexes $K^\bullet$ such that $H^iK^\bullet$ is a quasi-coherent (resp. coherent) $\mathscr{O}_X$-module for every $i \in \mathbb{Z}$. As usual, we shall use also the variants $\mathsf{D}^+(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{qcoh}}$ (resp. $\mathsf{D}^-(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{qcoh}}$, resp. $\mathsf{D}^b(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{qcoh}}$, resp. $\mathsf{D}^{[a,b]}(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{qcoh}}$) consisting of all objects of $\mathsf{D}(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{qcoh}}$ whose cohomology vanishes in sufficiently large negative degree (resp. sufficiently large positive degree, resp. outside a bounded interval, resp. outside the interval $[a, b]$), and likewise for the corresponding subcategories of $\mathsf{D}(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{coh}}$.

5.2.2. There are obvious forgetful functors:

$$\iota_X : \mathscr{O}_X\text{-}\mathbf{Mod}_{\mathrm{qcoh}} \to \mathscr{O}_X\text{-}\mathbf{Mod} \qquad R\iota_X : \mathsf{D}(\mathscr{O}_X\text{-}\mathbf{Mod}_{\mathrm{qcoh}}) \to \mathsf{D}(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{qcoh}}$$

and we wish to exhibit right adjoints to these functors. To this aim, suppose first that X is affine; then we may consider the functor:

$$\mathsf{qcoh}_X : \mathscr{O}_X\text{-}\mathbf{Mod} \to \mathscr{O}_X\text{-}\mathbf{Mod}_{\mathrm{qcoh}} \qquad \mathscr{F} \mapsto \mathscr{F}(X)^\sim$$

where, for an $\mathscr{O}_X(X)$-module M, we have denoted as usual by $M^\sim$ the quasi-coherent sheaf arising from M. If $\mathscr{G}$ is a quasi-coherent $\mathscr{O}_X$-module, then clearly $\mathsf{qcoh}_X\mathscr{G} \simeq \mathscr{G}$; moreover, for any other $\mathscr{O}_X$-module $\mathscr{F}$, there is a natural bijection:

$$\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{G}, \mathscr{F}) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{O}_X(X)}(\mathscr{G}(X), \mathscr{F}(X)).$$

It follows easily that qcoh_X is the sought right adjoint.

5.2.3. Slightly more generally, let U be *quasi-affine, i.e.* a quasi-compact open subset of an affine scheme, and choose a quasi-compact open immersion $j : U \to X$ into an affine scheme X. In this case, we may define

$$\mathsf{qcoh}_U : \mathscr{O}_U\text{-}\mathbf{Mod} \to \mathscr{O}_X\text{-}\mathbf{Mod} \qquad \mathscr{F} \mapsto (\mathsf{qcoh}_X j_*\mathscr{F})_{|U}.$$

Since $j_* : \mathscr{O}_U\text{-}\mathbf{Mod} \to \mathscr{O}_X\text{-}\mathbf{Mod}$ is right adjoint to $j^* : \mathscr{O}_X\text{-}\mathbf{Mod} \to \mathscr{O}_U\text{-}\mathbf{Mod}$, we have a natural isomorphism: $\mathrm{Hom}_{\mathscr{O}_U}(\mathscr{G}, \mathscr{F}) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{O}_X}(j_*\mathscr{G}, j_*\mathscr{F})$ for every $\mathscr{O}_U$-modules $\mathscr{G}$ and $\mathscr{F}$. Moreover, if $\mathscr{G}$ is quasi-coherent, the same holds for $j_*\mathscr{G}$ ([26, Ch.I, Cor.9.2.2]), whence a natural isomorphism $\mathscr{G} \simeq \mathsf{qcoh}_U\mathscr{G}$, by the foregoing discussion for the affine case. Summing up, this shows that qcoh_U is a right adjoint to ι_U, and especially it is independent, up to unique isomorphism, of the choice of j.

5.2.4. Next, suppose that X is quasi-compact and quasi-separated. We choose a finite covering $\mathfrak{U} := (U_i \mid i \in I)$ of X, consisting of affine open subsets, and set $U := \coprod_{i\in I} U_i$, the X-scheme which is the disjoint union (*i.e.* categorical coproduct) of the schemes U_i. We denote by $\mathfrak{U}_\bullet$ the simplicial covering such that $\mathfrak{U}_n := U \times_X \cdots \times_X U$, the $(n+1)$-th power of U, with the face and degeneracy maps defined as in (5.1.5); let also $\pi_n : \mathfrak{U}_n \to X$ be the natural morphism, for every $n \in \mathbb{N}$. Clearly we have $\pi_{n-1} \circ \partial_i = \pi_n$ for every face morphism $\partial_i : \mathfrak{U}_n \to \mathfrak{U}_{n-1}$. The simplicial scheme $\mathfrak{U}_\bullet$ (with the Zariski topology on each scheme $\mathfrak{U}_n$) can also be regarded as a fibred topos over the category Δ^o (notation of [36, §2.2]); then the datum $\mathscr{O}_{\mathfrak{U}_\bullet} := (\mathscr{O}_{\mathfrak{U}_n} \mid n \in \mathbb{N})$ consisting of the structure sheaves on each $\mathfrak{U}_n$ and the natural maps $\partial_i^* \mathscr{O}_{\mathfrak{U}_{n-1}} \to \mathscr{O}_{\mathfrak{U}_n}$ for every $n > 0$ and every $i = 0, \dots, n$ (and similarly for the degeneracy maps), defines a ring in the associated topos $\mathrm{Top}(\mathfrak{U}_\bullet)$ (see [36, §3.3.15]). We denote by $\mathscr{O}_{\mathfrak{U}_\bullet}$-$\mathbf{Mod}$ the category of $\mathscr{O}_{\mathfrak{U}_\bullet}$-modules in the topos $\mathrm{Top}(\mathfrak{U}_\bullet)$. The family $(\pi_n \mid n \in \mathbb{N})$ induces a morphism of topoi

$$\pi_\bullet : \mathrm{Top}(\mathfrak{U}_\bullet) \to s.X$$

where $s.X$ is the topos $\mathrm{Top}(X_\bullet)$ associated to the constant simplicial scheme $X_\bullet$ (with its Zariski topology) such that $X_n := X$ for every $n \in \mathbb{N}$ and such that all the face and degeneracy maps are $\mathbf{1}_X$. Clearly the objects of $s.X$ are nothing else than the cosimplicial Zariski sheaves on X. Especially, if we view a $\mathscr{O}_X$-module $\mathscr{F}$ as a constant cosimplicial $\mathscr{O}_X$-module, we may define the *augmented cosimplicial Čech $\mathscr{O}_X$-module*

$$\mathscr{F} \to \mathscr{C}^\bullet(\mathfrak{U}, \mathscr{F}) := \pi_{\bullet*} \circ \pi_\bullet^* \mathscr{F} \tag{5.2.5}$$

associated to $\mathscr{F}$ and the covering $\mathfrak{U}$; in every degree $n \in \mathbb{N}$ this is defined by the rule :

$$\mathscr{C}^n(\mathfrak{U}, \mathscr{F}) := \pi_{n*}\pi_n^* \mathscr{F}$$

and the coface operators ∂^i are induced by the face morphisms ∂_i in the obvious way.

Lemma 5.2.6. (i) *The augmented complex* (5.2.5) *is aspherical for every $\mathscr{O}_X$-module $\mathscr{F}$.*

(ii) *If $\mathscr{F}$ is an injective $\mathscr{O}_X$-module, then* (5.2.5) *is homotopically trivial.*

Proof. (See also *e.g.* [39, Th.5.2.1].) The proof relies on the following alternative description of the cosimplicial Čech $\mathscr{O}_X$-module. Consider the adjoint pair :

$$(\pi^*, \pi_*) : \mathscr{O}_{\mathfrak{U}}\text{-}\mathbf{Mod} \rightleftarrows \mathscr{O}_X\text{-}\mathbf{Mod}$$

arising from our covering $\pi : \mathfrak{U} \to X$. Let $(\top := \pi_* \circ \pi^*, \eta, \mu)$ be the associated triple (see (4.5.3)), $\mathscr{F}$ any $\mathscr{O}_X$-module; we leave to the reader the verification that the resulting augmented cosimplicial complex $\mathscr{F} \to \top^\bullet \mathscr{F}$ – as defined in (4.5.2) – is none else than the augmented Čech complex (5.2.5). Thus, for every $\mathscr{O}_X$-module $\mathscr{F}$, the augmented complex $\pi^*\mathscr{F} \to \pi^*\mathscr{C}^\bullet(\mathfrak{U}, \mathscr{F})$ is homotopically trivial (proposition 4.5.4); since π is a covering morphism, (i) follows. Furthermore, by the same token, the augmented complex $\pi_*\mathscr{G} \to \mathscr{C}^\bullet(\mathfrak{U}, \pi_*\mathscr{G})$ is homotopically trivial for every $\mathscr{O}_{\mathfrak{U}}$-module $\mathscr{G}$; especially we may take $\mathscr{G} := \pi^*\mathscr{I}$, where $\mathscr{I}$ is an injective $\mathscr{O}_X$-module. On the other hand, when $\mathscr{I}$ is injective, the unit of adjunction $\mathscr{I} \to \top\mathscr{I}$ is split injective; hence the augmented complex $\mathscr{I} \to \mathscr{C}^\bullet(\mathfrak{U}, \mathscr{I})$ is a direct summand of the homotopically trivial complex $\top\mathscr{I} \to \mathscr{C}^\bullet(\mathfrak{U}, \top\mathscr{I})$, and (ii) follows. □

Notice now that the schemes $\mathfrak{U}_n$ are quasi-affine for every $n \in \mathbb{N}$, hence the functors $\mathsf{qcoh}_{\mathfrak{U}_n}$ are well defined as in (5.2.3), and indeed, the rule : $(\mathscr{F}_n \mid n \in \mathbb{N}) \mapsto (\mathsf{qcoh}_{\mathfrak{U}_n}\mathscr{F}_n \mid n \in \mathbb{N})$ yields a functor :

$$\mathsf{qcoh}_{\mathfrak{U}_\bullet} : \mathscr{O}_{\mathfrak{U}_\bullet}\text{-}\mathbf{Mod} \to \mathscr{O}_{\mathfrak{U}_\bullet}\text{-}\mathbf{Mod}.$$

This suggests to introduce a *quasi-coherent cosimplicial Čech $\mathscr{O}_X$-module* :

$$\mathsf{q}\mathscr{C}^\bullet(\mathfrak{U}, \mathscr{F}) := \pi_{\bullet*} \circ \mathsf{qcoh}_{\mathfrak{U}_\bullet} \circ \pi_\bullet^* \mathscr{F}$$

for every $\mathscr{O}_X$-module $\mathscr{F}$ (regarded as a constant cosimplicial module in the usual way). More plainly, this is the cosimplicial $\mathscr{O}_X$-module such that :

$$\mathsf{q}\mathscr{C}^n(\mathfrak{U},\mathscr{F}) := \pi_{n*} \circ \mathsf{qcoh}_{\mathfrak{U}_n} \circ \pi_n^*\mathscr{F} \qquad \text{for every } n \in \mathbb{N}.$$

According to [26, Ch.I, Cor.9.2.2], the $\mathscr{O}_X$-modules $\mathsf{q}\mathscr{C}^n(\mathfrak{U},\mathscr{F})$ are quasi-coherent for all $n \in \mathbb{N}$. Finally, we define the functor :

$$\mathsf{qcoh}_X : \mathscr{O}_X\text{-}\mathbf{Mod} \to \mathscr{O}_X\text{-}\mathbf{Mod}_{\mathrm{qcoh}} \qquad \mathscr{F} \mapsto \mathrm{Equal}(\mathsf{q}\mathscr{C}^0(\mathfrak{U},\mathscr{F}) \overset{\partial^0}{\underset{\partial^1}{\rightrightarrows}} \mathsf{q}\mathscr{C}^1(\mathfrak{U},\mathscr{F}))$$

Proposition 5.2.7. (i) *In the situation of* (5.2.4)*, the functor* qcoh_X *is right adjoint to* ι_X.

(ii) *Let* Y *be any other quasi-compact and quasi-separated scheme,* $f : X \to Y$ *any morphism. Then the induced diagram of functors:*

$$\begin{tikzcd} \mathscr{O}_X\text{-}\mathbf{Mod} \arrow[r, "f_*"] \arrow[d, "\mathsf{qcoh}_X"'] & \mathscr{O}_Y\text{-}\mathbf{Mod} \arrow[d, "\mathsf{qcoh}_Y"] \\ \mathscr{O}_X\text{-}\mathbf{Mod}_{\mathrm{qcoh}} \arrow[r, "f_*"] & \mathscr{O}_Y\text{-}\mathbf{Mod}_{\mathrm{qcoh}} \end{tikzcd}$$

commutes up to a natural isomorphism of functors.

Proof. For every $n \in \mathbb{N}$ and every $\mathscr{O}_{\mathfrak{U}_n}$-module $\mathscr{H}$, the counit of the adjunction yields a natural map of $\mathscr{O}_{\mathfrak{U}_n}$-modules: $\mathsf{qcoh}_{\mathfrak{U}_n}\mathscr{H} \to \mathscr{H}$. Taking $\mathscr{H}$ to be $\pi_n^*\mathscr{F}$ on $\mathfrak{U}_n$ (for a given $\mathscr{O}_X$-module $\mathscr{F}$), these maps assemble to a morphism of cosimplicial $\mathscr{O}_X$-modules :

(5.2.8) $$\mathsf{q}\mathscr{C}^\bullet(\mathfrak{U},\mathscr{F}) \to \mathscr{C}^\bullet(\mathfrak{U},\mathscr{F})$$

and it is clear that (5.2.8) is an isomorphism whenever $\mathscr{F}$ is quasi-coherent. Let now $\varphi : \mathscr{G} \to \mathscr{F}$ be a map of $\mathscr{O}_X$-modules, with $\mathscr{G}$ quasi-coherent; after applying the natural transformation (5.2.8) and forming equalizers, we obtain a commutative diagram :

$$\begin{tikzcd} \mathsf{qcoh}_X\mathscr{G} \arrow[r, "\sim"] \arrow[d, "\mathsf{qcoh}_X\varphi"'] & \mathscr{G} \arrow[d, "\varphi"] \\ \mathsf{qcoh}_X\mathscr{F} \arrow[r] & \mathscr{F} \end{tikzcd}$$

from which we see that the rule: $\varphi \mapsto \mathsf{qcoh}_X\varphi$ establishes a natural injection:

(5.2.9) $$\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{G},\mathscr{F}) \to \mathrm{Hom}_{\mathscr{O}_X}(\mathscr{G},\mathsf{qcoh}_X\mathscr{F})$$

and since $\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{G},(5.2.8))$ is an isomorphism of cosimplicial $\mathscr{O}_X$-modules, (5.2.9) is actually a bijection, whence (i).

(ii) is obvious, since both $\mathsf{qcoh}_Y \circ f_*$ and $f_* \circ \mathsf{qcoh}_X$ are right adjoint to $f^* \circ \iota_Y = \iota_X \circ f^*$. $\square$

5.2.10. In the situation of (5.2.4), the functor qcoh_X is left exact, since it is a right adjoint, hence it gives rise to a left derived functor :

$$R\mathsf{qcoh}_X : \mathsf{D}^+(\mathscr{O}_X\text{-}\mathbf{Mod}) \to \mathsf{D}^+(\mathscr{O}_X\text{-}\mathbf{Mod}_{\mathrm{qcoh}}).$$

Proposition 5.2.11. *Let* X *be a quasi-compact and quasi-separated scheme. Then :*

(i) $R\mathsf{qcoh}_X$ *is right adjoint to* $R\iota_X$.

(ii) *Suppose moreover, that* X *is* semi-separated, i.e. *such that the intersection of any two affine open subsets of* X*, is still affine. Then the unit of the adjunction* $(R\iota_X, R\mathsf{qcoh}_X)$ *is an isomorphism of functors.*

Proof. (i): The exactness of the functor ι_X implies that qcoh_X preserves injectives; the assertion is a formal consequence : the details shall be left to the reader.

(ii): Let $\mathscr{F}^\bullet$ be any complex of quasi-coherent $\mathscr{O}_X$-modules; we have to show that the natural map $\mathscr{F}^\bullet \to R\mathsf{qcoh}_X\mathscr{F}^\bullet$ is an isomorphism. Using a Cartan-Eilenberg resolution $\mathscr{F}^\bullet \xrightarrow{\sim} \mathscr{I}^{\bullet\bullet}$ we deduce a spectral sequence ([75, §5.7])

$$E_1^{pq} := R^p\mathsf{qcoh}_X H^q\mathscr{F}^\bullet \Rightarrow R^{p+q}\mathsf{qcoh}_X\mathscr{F}^\bullet$$

which easily reduces to checking the assertion for the cohomology of $\mathscr{F}^\bullet$, so we may assume from start that $\mathscr{F}^\bullet$ is a single $\mathscr{O}_X$-module placed in degree zero. Let us then choose an injective resolution $\mathscr{F} \xrightarrow{\sim} \mathscr{I}^\bullet$ (that is, in the category of all $\mathscr{O}_X$-modules); we have to show that $H^p\mathsf{qcoh}_X\mathscr{I}^\bullet = 0$ for $p > 0$. We deal first with the following special case :

Claim 5.2.12. Assertion (i) holds if X is affine.

Proof of the claim. Indeed, in this case, the chosen injective resolution of $\mathscr{F}$ yields a long exact sequence $0 \to \mathscr{F}(X) \to \mathscr{I}^\bullet(X)$, and therefore a resolution $\mathsf{qcoh}_X\mathscr{F} := \mathscr{F}(X)^\sim \to \mathsf{qcoh}_X\mathscr{I}^\bullet := \mathscr{I}^\bullet(X)^\sim$. ◊

For the general case, we choose any affine covering $\mathfrak{U}$ of X and we consider the cochain complex of cosimplicial complexes $\mathsf{q}\mathscr{C}^\bullet(\mathfrak{U}, \mathscr{I}^\bullet)$.

Claim 5.2.13. For any injective $\mathscr{O}_X$-module $\mathscr{I}$, the augmented complex:

$$\mathsf{qcoh}_X\mathscr{I} \to \mathsf{q}\mathscr{C}^\bullet(\mathfrak{U}, \mathscr{I})$$

is homotopically trivial.

Proof of the claim. It follows easily from proposition 5.2.7(ii) that

$$\mathsf{q}\mathscr{C}^\bullet(\mathfrak{U}, \mathscr{F}) \simeq \mathsf{qcoh}_X(\mathscr{C}^\bullet(\mathfrak{U}, \mathscr{F}))$$

for any $\mathscr{O}_X$-module $\mathscr{F}$. Then the claim follows from lemma 5.2.6(ii). ◊

We have a spectral sequence :

$$E_1^{pq} := H^p\mathsf{q}\mathscr{C}^\bullet(\mathfrak{U}, \mathscr{I}^q) \Rightarrow \mathrm{Tot}^{p+q}(\mathsf{q}\mathscr{C}^\bullet(\mathfrak{U}, \mathscr{I}^\bullet))$$

and it follows from claim 5.2.13 that $E_1^{pq} = 0$ whenever $p > 0$, and $E^{0q} = \mathsf{qcoh}_X\mathscr{I}^q$ for every $q \in \mathbb{N}$, so the spectral sequence $E^{\bullet\bullet}$ degenerates, and we deduce a quasi-isomorphism

$$\mathsf{qcoh}_X\mathscr{I}^\bullet \xrightarrow{\sim} \mathrm{Tot}^\bullet(\mathsf{q}\mathscr{C}^\bullet(\mathfrak{U}, \mathscr{I}^\bullet)). \tag{5.2.14}$$

On the other hand, for fixed $q \in \mathbb{N}$, we have $\mathsf{q}\mathscr{C}^q(\mathfrak{U}, \mathscr{I}^\bullet) = \pi_{n*}\mathsf{qcoh}_{\mathfrak{U}_n}\pi_n^*\mathscr{I}^\bullet$; since X is semi-separated, $\mathfrak{U}_n$ is affine, so the complex $\mathsf{qcoh}_{\mathfrak{U}_n}\pi_n^*\mathscr{I}^\bullet$ is a resolution of $\mathsf{qcoh}_{\mathfrak{U}_n}\pi_n^*\mathscr{F} = \pi_n^*\mathscr{F}$, by claim 5.2.12. Furthermore, $\pi_n : \mathfrak{U}_n \to X$ is an affine morphism, so $\mathsf{q}\mathscr{C}^q(\mathfrak{U}, \mathscr{I}^\bullet)$ is a resolution of $\pi_{n*}\pi_n^*\mathscr{F}$. Summing up, we see that $\mathrm{Tot}^\bullet(\mathsf{q}\mathscr{C}^\bullet(\mathfrak{U}, \mathscr{I}^\bullet))$ is quasi-isomorphic to $\mathscr{C}^\bullet(\mathfrak{U}, \mathscr{F})$, which is a resolution of $\mathscr{F}$, by lemma 5.2.6(i). Combining with (5.2.14), we deduce (ii). □

Theorem 5.2.15. *Let X be a quasi-compact and semi-separated scheme. The forgetful functor:*

$$R\iota_X : \mathsf{D}^+(\mathscr{O}_X\text{-}\mathbf{Mod}_{\mathrm{qcoh}}) \to \mathsf{D}^+(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{qcoh}}$$

is an equivalence of categories, whose quasi-inverse is the restriction of $R\mathsf{qcoh}_X$.

Proof. By proposition 5.2.11(ii) we know already that the composition $R\mathsf{qcoh}_X \circ R\iota_X$ is a self-equivalence of $\mathsf{D}^+(\mathscr{O}_X\text{-}\mathbf{Mod}_{\mathrm{qcoh}})$. For every complex $\mathscr{F}^\bullet$ in $\mathsf{D}^+(\mathscr{O}_X\text{-}\mathbf{Mod})$, the counit of adjunction $\varepsilon_{\mathscr{F}^\bullet} : R\iota_X \circ R\mathsf{qcoh}_X\mathscr{F}^\bullet \to \mathscr{F}^\bullet$ can be described as follows. Pick an injective resolution $\alpha : \mathscr{F}^\bullet \to \mathscr{I}^\bullet$; then $\varepsilon_{\mathscr{F}^\bullet}$ is defined by the diagram :

$$\mathsf{qcoh}_X\mathscr{I}^\bullet \xrightarrow{\varepsilon^\bullet} \mathscr{I}^\bullet \xleftarrow{\alpha} \mathscr{F}^\bullet$$

where, for each $n \in \mathbb{N}$, the map $\varepsilon^n : \mathsf{qcoh}_X \mathscr{I}^n \to \mathscr{I}^n$ is the counit of the adjoint pair $(\iota_X, \mathsf{qcoh}_X)$. It suffices then to show that $\varepsilon^\bullet$ is a quasi-isomorphism, when $\mathscr{I}^\bullet$ is an object of $\mathsf{D}^+(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{qcoh}}$. To this aim, we may further choose $\mathscr{I}^\bullet$ of the form $\mathrm{Tot}^\bullet(\mathscr{I}^{\bullet\bullet})$, where $\mathscr{I}^{\bullet\bullet}$ is a Cartan-Eilenberg resolution of $\mathscr{F}^\bullet$ (see [75, §5.7]). We then deduce a spectral sequence :

$$E_2^{pq} := R^p \mathsf{qcoh}_X H^q \mathscr{F}^\bullet \Rightarrow R^{p+q} \mathsf{qcoh}_X \mathscr{F}^\bullet.$$

The double complex $\mathscr{I}^{\bullet\bullet}$ also gives rise to a similar spectral sequence F_2^{pq}, and clearly $F_2^{pq} = 0$ whenever $p > 0$, and $F^{0q} = H^q \mathscr{F}^\bullet$. Furthermore, the counit of adjunction $\varepsilon^{\bullet\bullet} : \mathsf{qcoh}_X \mathscr{I}^{\bullet\bullet} \to \mathscr{I}^{\bullet\bullet}$ induces a morphism of spectral sequences $\omega^{pq} : E_2^{pq} \to F_2^{pq}$. Consequently, in order to prove that the $\varepsilon_{\mathscr{F}^\bullet}$ is a quasi-isomorphism, it suffices to show that ω^{pq} is an isomorphism for every $p, q \in \mathbb{N}$. This comes down to the assertion that $R^p \mathsf{qcoh}_X \mathscr{G} = 0$ for every quasi-coherent $\mathscr{O}_X$-module $\mathscr{G}$, and every $p > 0$. However, we have $\mathscr{G} = R\iota_X \mathscr{G}$, so that $R\mathsf{qcoh}_X \mathscr{G} = R\mathsf{qcoh}_X \circ R\iota_X \mathscr{G}$, and then the contention follows from proposition 5.2.11(ii). □

Lemma 5.2.16. *Let X be a quasi-compact and quasi-separated scheme, U a quasi-compact open subset of X, $\mathscr{H}$ a quasi-coherent $\mathscr{O}_X$-module, $\mathscr{G}$ a finitely presented quasi-coherent $\mathscr{O}_U$-module, and $\varphi : \mathscr{G} \to \mathscr{H}_{|U}$ a $\mathscr{O}_U$-linear map. Then:*

(i) *There exists a finitely presented quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ on X, and a $\mathscr{O}_X$-linear map $\psi : \mathscr{F} \to \mathscr{H}$, such that $\mathscr{F}_{|U} = \mathscr{G}$ and $\psi_{|U} = \varphi$.*
(ii) *Especially, every finitely presented quasi-coherent $\mathscr{O}_U$-module extends to a finitely presented quasi-coherent $\mathscr{O}_X$-module.*

Proof. (i): Let $(V_i \mid i = 1, \dots, n)$ be a finite affine open covering of X. For every $i = 0, \dots, n$, let us set $U_i := U \cup V_1 \cup \cdots \cup V_i$; we construct, by induction on i, a family of finitely presented quasi-coherent $\mathscr{O}_{U_i}$-modules $\mathscr{F}_i$, and morphisms $\psi_i : \mathscr{F}_i \to \mathscr{H}_{|U_i}$ such that $\mathscr{F}_{i+1|U_i} = \mathscr{F}_i$ and $\psi_{i+1|U_{i+1}} = \psi_i$ for every $i < n$. For $i = 0$ we have $U_0 = U$, and we set $\mathscr{F}_0 := \mathscr{G}$, $\psi_0 := \varphi$. Suppose that $\mathscr{F}_i$ and ψ_i have already been given. Since X is quasi-separated, the same holds for U_{i+1}, and the immersion $j : U_i \to U_{i+1}$ is quasi-compact; it follows that $j_* \mathscr{F}_i$ and $j_* \mathscr{H}_{|U_i}$ are quasi-coherent $\mathscr{O}_{U_i}$-modules ([26, Ch.I, Prop.9.4.2(i)]). We let

$$\mathscr{M} := j_* \mathscr{F}_i \times_{j_* \mathscr{H}_{|U_i}} \mathscr{H}_{|U_{i+1}}.$$

Then $\mathscr{M}$ is a quasi-coherent $\mathscr{O}_{U_{i+1}}$-module admitting a map $\mathscr{M} \to \mathscr{H}_{|U_{i+1}}$, and such that $\mathscr{M}_{|U_i} = \mathscr{F}_i$. We can then find a filtered family of quasi-coherent $\mathscr{O}_{V_{i+1}}$-modules of finite presentation $(\mathscr{M}_\lambda \mid \lambda \in \Lambda)$, whose colimit is $\mathscr{M}_{|V_{i+1}}$. Since $\mathscr{F}_i$ is finitely presented and $U_i \cap V_{i+1}$ is quasi-compact, there exists $\lambda \in \Lambda$ such that the induced morphism $\beta : \mathscr{M}_{\lambda|U_i \cap V_{i+1}} \to \mathscr{F}_{i|U_i \cap V_{i+1}}$ is an isomorphism. We can thus define $\mathscr{F}_{i+1}$ by gluing $\mathscr{F}_i$ and $\mathscr{M}_\lambda$ along β; likewise, ψ_i and the induced map $\mathscr{M}_\lambda \to \mathscr{M}_{|V_{i+1}} \to \mathscr{H}_{|V_{i+1}}$ glue to a map ψ_{i+1} as required. Clearly the pair $(\mathscr{F} := \mathscr{F}_n, \psi := \psi_n)$ is the sought extension of $(\mathscr{G}, \varphi)$.

(ii): Let 0_X be the final object in the category of $\mathscr{O}_X$-modules; to extend a finitely presented quasi-coherent $\mathscr{O}_U$-module $\mathscr{G}$, it suffices to apply (i) to the unique map $\mathscr{G} \to 0_{X|U}$. □

For ease of reference, we point out the following simple consequence of lemma 5.2.16.

Corollary 5.2.17. *Let U be a quasi-affine scheme, $\mathscr{E}$ a locally free $\mathscr{O}_U$-module of finite type. Then we may find integers $n, m \in \mathbb{N}$ and a left exact sequence of $\mathscr{O}_U$-modules :*

$$0 \to \mathscr{E} \to \mathscr{O}_U^{\oplus m} \to \mathscr{O}_U^{\oplus n}.$$

Proof. Set $\mathscr{E}^\vee := \mathscr{H}om_{\mathscr{O}_U}(\mathscr{E}, \mathscr{O}_U)$, and notice that $\mathscr{E}^\vee$ is a locally free $\mathscr{O}_U$-module of finite type, especially it is quasi-coherent and of finite presentation. By assumption, U is a quasi-compact open subset of an affine scheme X, hence $\mathscr{E}^\vee$ extends to a quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ of finite presentation (lemma 5.2.16(ii)); we have $\mathscr{F} = F^\sim$, for some finitely presented A-module F; we choose a presentation of F as the cokernel of an A-linear map $A^{\oplus n} \to A^{\oplus m}$, for some

$m, n \in \mathbb{N}$, whence a presentation of $\mathscr{E}^\vee$ as the cokernel of a morphism $\mathscr{O}_U^{\oplus n} \to \mathscr{O}_U^{\oplus m}$, and after dualizing again, we get the sought left exact sequence. $\square$

5.2.18. Let X be a scheme; for any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, we consider the full subcategory $C/\mathscr{F}$ of the category $\mathscr{O}_X\text{-}\mathbf{Mod}_{\mathrm{qcoh}}/\mathscr{F}$ whose objects are all the maps $\varphi : \mathscr{G} \to \mathscr{F}$ such that $\mathscr{G}$ is a finitely presented $\mathscr{O}_X$-module (notation of (1.1.12)). One verifies easily that this category is filtered, and we have a source functor :

$$\iota_{\mathscr{F}} : C/\mathscr{F} \to \mathscr{O}_X\text{-}\mathbf{Mod}_{\mathrm{qcoh}} \qquad (\mathscr{G} \to \mathscr{F}) \mapsto \mathscr{G}$$

Proposition 5.2.19. *With the notation of* (5.2.18)*, suppose that X is quasi-compact and quasi-separated. Then, for any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, the induced map*

$$\operatorname*{colim}_{C/\mathscr{F}} \iota_{\mathscr{F}} \to \mathscr{F}$$

is an isomorphism.

Proof. Let $\mathscr{F}'$ be such colimit; clearly there is a natural map $\mathscr{F}' \to \mathscr{F}$, and we have to show that it is an isomorphism. To this aim, we can check on the stalk over the points $x \in X$, hence we come down to showing :

Claim 5.2.20. Let $\mathscr{F}$ be any quasi-coherent $\mathscr{O}_X$-module. Then :

(i) For every $s \in \mathscr{F}_x$ there exists a finitely presented quasi-coherent $\mathscr{O}_X$-module $\mathscr{G}$, a morphism $\varphi : \mathscr{G} \to \mathscr{F}$ and $t \in \mathscr{G}_x$ such that $\varphi_x(t) = s$.

(ii) For every map $\varphi : \mathscr{G} \to \mathscr{F}$ with $\mathscr{G}$ finitely presented and quasi-coherent, and every $s \in \operatorname{Ker} \varphi_x$, there exists a commutative diagram of quasi-coherent $\mathscr{O}_X$-modules :

$$\begin{tikzcd} \mathscr{G} \arrow[d, "\psi"'] \arrow[dr, "\varphi"] & \\ \mathscr{H} \arrow[r] & \mathscr{F} \end{tikzcd}$$

with $\mathscr{H}$ finitely presented and $\psi_x(s) = 0$.

Proof of the claim. (i): Let $U \subset X$ be an open subset such that s extends to a section $s_U \in \mathscr{F}(U)$; we deduce a map $\varphi_U : \mathscr{O}_U \to \mathscr{F}_{|U}$ by the rule $a \mapsto a \cdot s_U$ for every $a \in \mathscr{O}_U(U)$. In view of lemma 5.2.16(i), the pair $(\varphi_U, \mathscr{O}_U)$ extends to a pair $(\varphi, \mathscr{G})$ with the sought properties.

(ii): We apply (i) to the quasi-coherent $\mathscr{O}_X$-module $\operatorname{Ker} \varphi$, to find a finitely presented quasi-coherent $\mathscr{O}_X$-module $\mathscr{G}'$, a morphism $\beta : \mathscr{G}' \to \operatorname{Ker} \varphi$ and $t \in \mathscr{H}_x$ such that $\beta_x(t) = s$. Then we let $\psi : \mathscr{G} \to \mathscr{H} := \operatorname{Coker}(\mathscr{G}' \xrightarrow{\beta} \operatorname{Ker} \varphi \to \mathscr{G})$ be the natural map. By construction, $\mathscr{H}$ is finitely presented, $\psi_x(s) = 0$ and clearly φ factors through ψ. $\square$

Corollary 5.2.21. *Let X be a coherent, quasi-compact and quasi-separated scheme, $U \subset X$ a quasi-compact open subset. Then the induced functor*

$$\mathsf{D}^{[a,b]}(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{coh}} \to \mathsf{D}^{[a,b]}(\mathscr{O}_U\text{-}\mathbf{Mod})_{\mathrm{coh}}$$

is essentially surjective, for every $a, b \in \mathbb{N}$.

Proof. To start out, for any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, define the category $C/\mathscr{F}$ as in 5.2.18, and consider the full subcategory $C'/\mathscr{F}$ (resp. $C''/\mathscr{F}$) of $C/\mathscr{F}$ whose objects are all the maps $\varphi : \mathscr{G} \to \mathscr{F}$ such that $\varphi_{|U}$ is an isomorphism (resp. an epimorphism). It is easily seen that both $C'/\mathscr{F}$ and $C''/\mathscr{F}$ are filtered categories, and we have :

Claim 5.2.22. With the foregoing notation, the following holds :

(i) If $\mathscr{F}_{|U}$ is a coherent $\mathscr{O}_U$-module, $C'/\mathscr{F}$ is a cofinal subcategory of $C/\mathscr{F}$.

(ii) For every object $K^\bullet$ of $\mathsf{D}^b(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{coh}}$ and every $c \in \mathbb{Z}$, the natural map

$$\operatorname*{colim}_{C/\mathscr{F}} \mathrm{Hom}_{\mathsf{D}(\mathscr{O}_X\text{-}\mathbf{Mod})}(K^\bullet, \iota_{\mathscr{F}}[c]) \to \mathrm{Hom}_{\mathsf{D}(\mathscr{O}_X\text{-}\mathbf{Mod})}(K^\bullet, \mathscr{F}[c])$$

is an isomorphism (notation of (5.2.18)).

Proof of the claim. (i): Since U is quasi-compact, it is easily seen that $C''/\mathscr{F}$ is cofinal in $C/\mathscr{F}$, so we are reduced to checking that $C'/\mathscr{F}$ is cofinal in $C''/\mathscr{F}$. Hence, let $\psi : \mathscr{G}' \to \mathscr{F}$ be any object of $C''/\mathscr{F}$; since $\mathscr{O}_X$ is coherent, $\mathscr{K} := \mathrm{Ker}\,\psi_{|U}$ is a coherent $\mathscr{O}_U$-module, therefore we may extend $\mathscr{K}$ to a coherent $\mathscr{O}_X$-module $\mathscr{K}'$ and the identity map of $\mathscr{K}$ to a morphism $\psi' : \mathscr{K}' \to \mathrm{Ker}\,\psi$ of $\mathscr{O}_X$-modules (lemma 5.2.16(i)). Set $\mathscr{G} := \mathscr{G}'/\psi'(\mathscr{K}')$; the induced map $\mathscr{G} \to \mathscr{F}$ is an object of $C'/\mathscr{F}$, whence the contention.

(ii): We have two spectral sequences :

$$E_2^{pq} : \operatorname*{colim}_{C/\mathscr{F}} R^p\Gamma(X, R^q\mathscr{H}om^\bullet_{\mathscr{O}_X}(K^\bullet, \iota_{\mathscr{F}}[c])) \Rightarrow \operatorname*{colim}_{C/\mathscr{F}} \mathrm{Hom}_{\mathsf{D}(\mathscr{O}_X\text{-}\mathbf{Mod})}(K^\bullet, \iota_{\mathscr{F}}[c+p+q])$$
$$F_2^{pq} : \qquad R^p\Gamma(X, R^q\mathscr{H}om^\bullet_{\mathscr{O}_X}(K^\bullet, \mathscr{F}[c])) \Rightarrow \mathrm{Hom}_{\mathsf{D}(\mathscr{O}_X\text{-}\mathbf{Mod})}(K^\bullet, \mathscr{F}[c+p+q])$$

and a morphism of spectral sequences $E_2^{pq} \to F_2^{pq}$. Since the functors $R^p\Gamma$ commute with filtered colimits, we deduce that it suffices to show that the natural morphism

$$\operatorname*{colim}_{C/\mathscr{F}} R^q\mathscr{H}om^\bullet_{\mathscr{O}_X}(K^\bullet, \iota_{\mathscr{F}}[c])) \to R^q\mathscr{H}om^\bullet_{\mathscr{O}_X}(K^\bullet, \mathscr{F}[c])$$

is an isomorphism for every $q \in \mathbb{Z}$. Then, a standard *dévissage* argument further reduces to the case where $K^\bullet$ is concentrated in a single degree, so we come down to checking that the functor $\mathscr{F} \mapsto \mathscr{E}xt^q_{\mathscr{O}_X}(\mathscr{G}, \mathscr{F})$ commutes with filtered colimits of quasi-coherent $\mathscr{O}_X$-modules, for every coherent $\mathscr{O}_X$-module $\mathscr{G}$ and every $q \in \mathbb{Z}$. To this aim, we may assume that X is affine, in which case – since $\mathscr{O}_X$ is coherent – $\mathscr{G}$ admits a resolution $\mathscr{L}^\bullet \to \mathscr{G}$ consisting of free $\mathscr{O}_X$-modules of finite rank; the latter are acyclic for the functor $\mathscr{H}om_{\mathscr{O}_X}$, so we are left with the assertion that the functor $\mathscr{F} \mapsto H^q\mathscr{H}om_{\mathscr{O}_X}(\mathscr{L}^\bullet, \mathscr{F})$ commutes with filtered colimits, which is clear. $\diamondsuit$

Now, let $j : U \to X$ be the open immersion, $\mathscr{F}^\bullet$ an object of $\mathsf{D}^{[a,b]}(\mathscr{O}_U\text{-}\mathbf{Mod})_{\mathrm{coh}}$. By [28, Ch.III, Prop.1.4.10, Cor.1.4.12] and a standard spectral sequence argument, it is easily seen that

$$\mathscr{H}^\bullet := Rj_*\mathscr{F}^\bullet$$

lies in $\mathsf{D}^b(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{qcoh}}$. We may then replace $\mathscr{H}^a$ by $\mathrm{Coker}\,(d^{a-1} : \mathscr{H}^{a-1} \to \mathscr{H}^a)$, $\mathscr{H}^b$ by $\mathrm{Ker}\,(d^b : \mathscr{H}^b \to \mathscr{H}^b)$, $\mathscr{H}^i$ by 0 for $i \notin [a,b]$, and assume that $\mathscr{H}^\bullet$ lies in $\mathsf{D}^{[a,b]}(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{qcoh}}$ and $j^*\mathscr{H}^\bullet = \mathscr{F}^\bullet$. It then suffices to show the following :

Claim 5.2.23. Let $\mathscr{H}^\bullet$ be any object of $\mathsf{D}^{[a,b]}(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{qcoh}}$ such that $\mathscr{H}^\bullet_{|U}$ lies in the category $\mathsf{D}^{[a,b]}(\mathscr{O}_U\text{-}\mathbf{Mod})_{\mathrm{coh}}$. Then there exists an object $\mathscr{G}^\bullet$ of $\mathsf{D}^{[a,b]}(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{coh}}$, with a morphism $\varphi^\bullet : \mathscr{G}^\bullet \to \mathscr{H}^\bullet$ in $\mathsf{D}(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{qcoh}}$ such that $\varphi^\bullet_{|U}$ is an isomorphism.

Proof of the claim. We proceed by descending induction on a, and notice that in case $a = b$, the assertion follows immediately from claim 5.2.22(i) and proposition 5.2.19.

Thus, suppose that $a < b$, and that the assertion is already known for every object of $\mathsf{D}^{[a+1,b]}(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{qcoh}}$ fulfilling the stated condition. Define the *truncation* $\tau_{\geq a+1}\mathscr{H}^\bullet$ as the unique object (up to isomorphism) of $\mathsf{D}^{[a+1,b]}(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{qcoh}}$ fitting in a distinguished triangle of $\mathsf{D}(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{qcoh}}$

$$H^a\mathscr{H}^\bullet[-a] \to \mathscr{H}^\bullet \to \tau_{\geq a+1}\mathscr{H}^\bullet \to H^a\mathscr{H}^\bullet[1-a].$$

By inductive assumption, we may find an object $\mathscr{G}'^\bullet$ of $\mathsf{D}^{[a+1,b]}(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{coh}}$ and a morphism $\varphi'^\bullet : \mathscr{G}'^\bullet \to \tau_{\geq a+1}\mathscr{H}^\bullet$ in $\mathsf{D}(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{qcoh}}$ which restricts to an isomorphism on U. There follows a morphism of distinguished triangles :

$$\begin{tikzcd} H^a\mathscr{H}^\bullet[-a] \arrow[r] \arrow[d, equal] & \mathscr{H}'^\bullet \arrow[r] \arrow[d, "\beta^\bullet"] & \mathscr{G}'^\bullet \arrow[r] \arrow[d, "\varphi'^\bullet"] & H^a\mathscr{H}^\bullet[1-a] \arrow[d, equal] \\ H^a\mathscr{H}^\bullet[-a] \arrow[r] & \mathscr{H}^\bullet \arrow[r] & \tau_{\geq a+1}\mathscr{H}^\bullet \arrow[r, "\alpha^\bullet"] & H^a\mathscr{H}^\bullet[1-a] \end{tikzcd}$$

for some object $\mathscr{H}'^\bullet$ of $\mathsf{D}^{[a,b]}(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{qcoh}}$, and clearly $\beta^\bullet$ restricts to an isomorphism on U. We may then replace $\mathscr{H}^\bullet$ by $\mathscr{H}'^\bullet$, and assume that $\tau_{\geq a+1}\mathscr{H}^\bullet$ lies in $\mathsf{D}^{[a+1,b]}(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{coh}}$.

By claim 5.2.22, we may find an object $\psi : \mathscr{G} \to H^a\mathscr{H}^\bullet$ of $C'/\mathscr{F}$ such that $\alpha^\bullet$ factors through $\psi[1-a]$ and a morphism $\omega^\bullet : \tau_{\geq a+1}\mathscr{H}^\bullet \to \mathscr{G}[1-a]$ in $\mathsf{D}(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{coh}}$. Then $\mathrm{Cone}\,\omega^\bullet[-1]$ lies in $\mathsf{D}^{[a,b]}(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{coh}}$, and the induced morphism $\mathrm{Cone}\,\omega^\bullet[-1] \to \mathscr{H}^\bullet$ restricts to an isomorphism on U. $\square$

5.3. **Duality for quasi-coherent modules.** Let $f : X \to Y$ be a morphism of schemes; theorem 5.1.27 falls short of proving that Lf^* is a left adjoint to Rf_*, since the former functor is defined on bounded above complexes, while the latter is defined on complexes that are bounded below. This deficency has been overcome by N.Spaltenstein's paper [71], where he shows how to extend the usual constructions of derived functors to unbounded complexes. On the other hand, in many cases one can also construct a *right adjoint* to Rf_*. This is the subject of Grothendieck's duality theory. We shall collect the statements that we need from that theory, and refer to the original source [46] for the rather elaborate proofs.

5.3.1. For any morphism of schemes $f : X \to Y$, let $\overline{f} : (X, \mathscr{O}_X) \to (Y, f_*\mathscr{O}_X)$ be the corresponding morphism of ringed spaces. The functor

$$\overline{f}_* : \mathscr{O}_X\text{-}\mathbf{Mod} \to f_*\mathscr{O}_X\text{-}\mathbf{Mod}$$

admits a left adjoint $\overline{f}^*$ defined as usual ([26, Ch.0, §4.3.1]). In case f is affine, $\overline{f}$ is flat and the unit and counit of the adjunction restrict to isomorphisms of functors :

$$\mathbf{1} \to \overline{f}_* \circ \overline{f}^* : f_*\mathscr{O}_X\text{-}\mathbf{Mod}_{\mathrm{qcoh}} \to f_*\mathscr{O}_X\text{-}\mathbf{Mod}_{\mathrm{qcoh}}$$
$$\overline{f}^* \circ \overline{f}_* \to \mathbf{1} : \mathscr{O}_X\text{-}\mathbf{Mod}_{\mathrm{qcoh}} \to \mathscr{O}_X\text{-}\mathbf{Mod}_{\mathrm{qcoh}}.$$

on the corresponding subcategories of quasi-coherent modules ([27, Ch.II, Prop.1.4.3]). It follows easily that, on these subcategories, $\overline{f}^*$ is also a right adjoint to $\overline{f}_*$.

Lemma 5.3.2. *Keep the notation of* (5.3.1)*, and suppose that f is affine. Then the functor :*

$$\overline{f}^* : \mathsf{D}^+(f_*\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{qcoh}} \to \mathsf{D}^+(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{qcoh}}$$

is right adjoint to the functor $R\overline{f}_* : \mathsf{D}^+(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{qcoh}} \to \mathsf{D}^+(f_*\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{qcoh}}$. Moreover, the unit and counit of the resulting adjunction are isomorphisms of functors.

Proof. By trivial duality (theorem 5.1.27), $\overline{f}^*$ is left adjoint to $R\overline{f}_*$ on $\mathsf{D}^+(\mathscr{O}_X\text{-}\mathbf{Mod})$; it suffices to show that the unit (resp. and counit) of this latter adjunction are isomorphisms for every object $K^\bullet$ of $\mathsf{D}^+(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{qcoh}}$ (resp. $L^\bullet$ of $\mathsf{D}^+(f_*\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{qcoh}}$). Concerning the unit, we can use a Cartan-Eilenberg injective resolution of $\overline{f}^*K^\bullet$, and a standard spectral sequence, to reduce to the case where $K^\bullet = \mathscr{F}[0]$ for a quasi-coherent $f_*\mathscr{O}_X$-module $\mathscr{F}$; however, the natural map $\overline{f}_* \circ \overline{f}^*\mathscr{F} \to R\overline{f}_* \circ \overline{f}^*\mathscr{F}$ is an isomorphism, so the assertion follows from (5.3.1). $\square$

Remark 5.3.3. (i) Even when both X and Y are affine, neither the unit nor the counit of adjunction in (5.3.1) is an isomorphism on the categories of all modules. For a counterexample concerning the unit, consider the case of a finite injection of domains $A \to B$, where A is local and B is semi-local (but not local); let $f : X := \mathrm{Spec}\, B \to \mathrm{Spec}\, A$ be the corresponding morphism, and denote by $\mathscr{F}$ the $f_*\mathscr{O}_X$-module supported on the closed point, with stalk equal to B. Then one verifies easily that $\overline{f}_* \circ \overline{f}^* \mathscr{F}$ is a direct product of finitely many copies of $\mathscr{F}$, indexed by the closed points of X. Concerning the counit, keep the same morphism f, let $U := X \setminus \{x\}$, where x is a closed point and set $\mathscr{F} := j_! \mathscr{O}_U$; then it is clear that $\overline{f}^* \circ \overline{f}_* \mathscr{F}$ is supported on the complement of the closed fibre of f.

(ii) Similarly, one sees easily that $\overline{f}^*$ is not a right adjoint to $\overline{f}_*$ on the category of all $f_*\mathscr{O}_X$-modules.

(iii) Incidentally, the analogous functor $\overline{f}^*_{\text{ét}}$ defined on the étale (rather than Zariski) site is a right adjoint to $\overline{f}_{\text{ét}*} : \mathscr{O}_{X,\text{ét}}\text{-}\mathbf{Mod} \to f_*\mathscr{O}_{X,\text{ét}}\text{-}\mathbf{Mod}$, and on these sites the unit and counit of the adjunctions are isomorphisms for all modules.

5.3.4. For the construction of the right adjoint $f^!$ to Rf_*, one considers first the case where f factors as a composition of a finite morphism followed by a smooth one, in which case $f^!$ admits a corresponding decomposition. Namely :

- In case f is smooth, we shall consider the functor :
$$f^\sharp : \mathsf{D}(\mathscr{O}_Y\text{-}\mathbf{Mod}) \to \mathsf{D}(\mathscr{O}_X\text{-}\mathbf{Mod}) \qquad K^\bullet \mapsto f^*K^\bullet \otimes_{\mathscr{O}_X} \Lambda^n_{\mathscr{O}_X} \Omega^1_{X/Y}[n]$$
where n is the locally constant relative dimension function of f.
- In case f is finite, we shall consider the functor :
$$f^\flat : \mathsf{D}^+(\mathscr{O}_Y\text{-}\mathbf{Mod}) \to \mathsf{D}^+(\mathscr{O}_X\text{-}\mathbf{Mod}) \qquad K^\bullet \mapsto \overline{f}^* R\mathscr{H}om^\bullet_{\mathscr{O}_Y}(f_*\mathscr{O}_X, K^\bullet).$$

If f is any quasi-projective morphism, such factorization can always be found locally on X, and one is left with the problem of patching a family of locally defined functors $((f_{|U_i})^! \mid i \in I)$, corresponding to an open covering $X = \bigcup_{i \in I} U_i$. Since such patching must be carried out in the derived category, one has to take care of many cumbersome complications.

5.3.5. Recall that a finite morphism $f : X \to Y$ is said to be *pseudo-coherent* if $f_*\mathscr{O}_X$ is a pseudo-coherent $\mathscr{O}_Y$-module. This condition is equivalent to the pseudo-coherence of f in the sense of [8, Exp.III, Déf.1.2].

Lemma 5.3.6. *Let $f : X \to Y$ be a finite morphism of schemes. Then :*

(i) *If f is finitely presented, the functor*
$$\mathscr{F} \mapsto \overline{f}^* \mathscr{H}om_{\mathscr{O}_Y}(f_*\mathscr{O}_X, \mathscr{F})$$
is right adjoint to $f_* : \mathscr{O}_X\text{-}\mathbf{Mod}_{\text{qcoh}} \to \mathscr{O}_Y\text{-}\mathbf{Mod}_{\text{qcoh}}$.

(ii) *If f is pseudo-coherent, the functor*
$$K^\bullet \mapsto \overline{f}^* R\mathscr{H}om^\bullet_{\mathscr{O}_Y}(f_*\mathscr{O}_X, K^\bullet) \quad : \quad \mathsf{D}^+(\mathscr{O}_Y\text{-}\mathbf{Mod})_{\text{qcoh}} \to \mathsf{D}^+(\mathscr{O}_X\text{-}\mathbf{Mod})_{\text{qcoh}}$$
is right adjoint to
$$Rf_* : \mathsf{D}^+(\mathscr{O}_X\text{-}\mathbf{Mod})_{\text{qcoh}} \to \mathsf{D}^+(\mathscr{O}_Y\text{-}\mathbf{Mod})_{\text{qcoh}}$$
(notation of (5.2.1)*).*

Proof. (i): Under the stated assumptions, $f_*\mathscr{O}_X$ is a finitely presented $\mathscr{O}_Y$-module (by claim 5.7.8), hence the functor
$$\mathscr{F} \mapsto \mathscr{H}om_{\mathscr{O}_Y}(f_*\mathscr{O}_X, \mathscr{F}) \quad : \quad \mathscr{O}_Y\text{-}\mathbf{Mod} \to f_*\mathscr{O}_X\text{-}\mathbf{Mod}$$
preserves the subcategories of quasi-coherent modules, hence it restricts to a right adjoint for the forgetful functor $f_*\mathscr{O}_X\text{-}\mathbf{Mod}_{\text{qcoh}} \to \mathscr{O}_Y\text{-}\mathbf{Mod}_{\text{qcoh}}$ (claim 5.1.26). Then the assertion follows from (5.3.1).

(ii) is analogous : since $f_*\mathscr{O}_X$ is pseudo-coherent, the functor
$$K^\bullet \mapsto R\mathscr{H}om^\bullet_{\mathscr{O}_Y}(f_*\mathscr{O}_X, K^\bullet) \quad : \quad \mathsf{D}^+(\mathscr{O}_Y\text{-}\mathbf{Mod}) \to \mathsf{D}^+(f_*\mathscr{O}_X\text{-}\mathbf{Mod})$$

preserves the subcategories of complexes with quasi-coherent homology, hence it restricts to a right adjoint for the forgetful functor $\mathsf{D}^+(f_*\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{qcoh}} \to \mathsf{D}^+(\mathscr{O}_Y\text{-}\mathbf{Mod})_{\mathrm{qcoh}}$, by lemma 5.1.25(iii). To conclude, it then suffices to apply lemma 5.3.2. $\square$

Proposition 5.3.7. *Suppose that $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$ are morphisms of schemes. Then :*

(i) *If f and g are finite and f is pseudo-coherent, there is a natural isomorphism of functors on $\mathsf{D}^+(\mathscr{O}_Z\text{-}\mathbf{Mod})$:*
$$\xi_{f,g} : (g \circ f)^\flat \xrightarrow{\sim} f^\flat \circ g^\flat.$$

(ii) *If f and $g \circ f$ are finite and pseudo-coherent, and g is smooth of bounded fibre dimension, there is a natural isomorphism of functors on $\mathsf{D}^+(\mathscr{O}_Z\text{-}\mathbf{Mod})$:*
$$\zeta_{f,g} : (g \circ f)^\flat \xrightarrow{\sim} f^\flat \circ g^\sharp.$$

(iii) *If f, g and $h \circ g$ are finite and pseudo-coherent, and h is smooth of bounded fibre dimension, then the diagram of functors on $\mathsf{D}^+(\mathscr{O}_W\text{-}\mathbf{Mod})$:*
$$\begin{tikzcd} (h \circ g \circ f)^\flat \arrow[r, "\zeta_{g\circ f,h}"] \arrow[d, "\xi_{f,h\circ g}"'] & (g \circ f)^\flat \circ h^\sharp \arrow[d, "\xi_{f,g} \circ h^\sharp"] \\ f^\flat \circ (h \circ g)^\flat \arrow[r, "f^\flat(\zeta_{g,h})"] & f^\flat \circ g^\flat \circ h^\sharp \end{tikzcd}$$
commutes.

(iv) *If, moreover*
$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "h'"'] & Z \arrow[d, "h"] \\ W' \arrow[r, "j"] & W \end{tikzcd}$$
is a cartesian diagram of schemes, such that h is smooth of bounded fibre dimension and j is finite and pseudo-coherent, then there is a natural isomorphism of functors :
$$\vartheta_{j,h} : g^\flat \circ h^\sharp \xrightarrow{\sim} h'^\sharp \circ j^\flat.$$

Proof. This is [46, Ch.III, Prop.6.2, 8.2, 8.6]. We check only (i). There is a natural commutative diagram of ringed spaces :
$$\begin{tikzcd} (X, \mathscr{O}_X) \arrow[r, "f"] \arrow[dr, "\overline{f}"'] & (Y, \mathscr{O}_Y) \arrow[rr, "g"] \arrow[drr, "\overline{g}"'] & & (Z, \mathscr{O}_Z) \\ & (Y, f_*\mathscr{O}_X) \arrow[u, "\alpha"'] \arrow[drr, "\varphi"'] & & (Z, g_*\mathscr{O}_Y) \arrow[u, "\gamma"'] \\ & & & (Z, (g \circ f)_*\mathscr{O}_X) \arrow[u, "\beta"'] \end{tikzcd}$$
where :

(5.3.8) $$\varphi \circ \overline{f} = \overline{g \circ f}.$$

By claim 5.1.26, the (forgetful) functors α_*, β_* and γ_* admit right adjoints :
$$\begin{array}{lcl} \alpha^\flat : \mathscr{O}_Y\text{-}\mathbf{Mod} \to f_*\mathscr{O}_X\text{-}\mathbf{Mod} & : & \mathscr{F} \mapsto \mathscr{H}om_{\mathscr{O}_Y}(f_*\mathscr{O}_X, \mathscr{F}) \\ \beta^\flat : g_*\mathscr{O}_Y\text{-}\mathbf{Mod} \to (g \circ f)_*\mathscr{O}_X\text{-}\mathbf{Mod} & : & \mathscr{F} \mapsto \mathscr{H}om_{g_*\mathscr{O}_Y}((g \circ f)_*\mathscr{O}_X, \mathscr{F}) \\ \gamma^\flat : \mathscr{O}_Z\text{-}\mathbf{Mod} \to g_*\mathscr{O}_Y\text{-}\mathbf{Mod} & : & \mathscr{F} \mapsto \mathscr{H}om_{\mathscr{O}_Z}(g_*\mathscr{O}_Y, \mathscr{F}). \end{array}$$

Likewise, $(\gamma\circ\beta)_*$ admits a right adjoint $(\gamma\circ\beta)^\flat$ and the natural identification

$$\gamma_*\circ\beta_*\xrightarrow{\sim}(\gamma\circ\beta)_*$$

induces a natural isomorphisms of functors :

$$(5.3.9)\qquad (\gamma\circ\beta)^\flat:=\mathscr{H}om_{\mathscr{O}_Z}((g\circ f)_*\mathscr{O}_X,-)\xrightarrow{\sim}\beta^\flat\circ\gamma^\flat.$$

(Notice that, in general, there might be several choices of such natural transformations (5.3.9), but for the proof of (iii) – as well as for the construction of $f^!$ for more general morphisms f – it is essential to make an explicit and canonical choice.)

Claim 5.3.10. (i) $\alpha_*\circ\varphi^*=\overline{g}^*\circ\beta_*$.

(ii) The natural commutative diagram of sheaves :

$$\begin{tikzcd}
\overline{g}^{-1}g_*\mathscr{O}_Y \arrow[r] \arrow[d] & \mathscr{O}_Y \arrow[d] \\
\overline{g}^{-1}(g\circ f)_*\mathscr{O}_X \arrow[r] & f_*\mathscr{O}_X
\end{tikzcd}$$

is cocartesian.

Proof of the claim. (i) is an easy consequence of (ii). Assertion (ii) can be checked on the stalks, hence we may assume that $Z=\operatorname{Spec}A$, $Y=\operatorname{Spec}B$ and $X=\operatorname{Spec}C$ are affine schemes. Let $\mathfrak{p}\in Y$ be any prime ideal, and set $\mathfrak{q}:=g(\mathfrak{p})\in Z$; then $(\overline{g}^{-1}g_*\mathscr{O}_Y)_\mathfrak{p}=B_\mathfrak{q}$ (the $A_\mathfrak{q}$-module obtained by localizing the A-module B at the prime $\mathfrak{q}$). Likewise, $(\overline{g}^{-1}(g\circ f)_*\mathscr{O}_X)_\mathfrak{p}=C_\mathfrak{q}$ and $(f_*\mathscr{O}_X)_\mathfrak{p}=C_\mathfrak{p}$. Hence the claim boils down to the standard isomorphism : $C_\mathfrak{q}\otimes_{B_\mathfrak{q}}B_\mathfrak{p}\simeq C_\mathfrak{p}$. $\Diamond$

Let $I^\bullet$ be a bounded below complex of injective $\mathscr{O}_Z$-modules; by lemma 5.1.25(i), $\gamma^\flat I^\bullet$ is a complex of injective $g_*\mathscr{O}_Y$-modules; taking into account (5.3.9) we deduce a natural isomorphism of functors on $\mathsf{D}^+(\mathscr{O}_Z\text{-}\mathbf{Mod})$:

$$(5.3.11)\qquad R(\gamma\circ\beta)^\flat\xrightarrow{\sim}R(\beta^\flat\circ\gamma^\flat)=R\beta^\flat\circ R\gamma^\flat.$$

Furthermore, (5.3.8) implies that $\overline{g\circ f}^*=\overline{f}^*\circ\varphi^*$. Combining with (5.3.11), we see that the sought $\xi_{f,g}$ is a natural transformation :

$$\overline{f}^*\circ\varphi^*\circ R\beta^\flat\circ R\gamma^\flat\to\overline{f}^*\circ R\alpha^\flat\circ\overline{g}^*\circ R\gamma^\flat$$

of functors on $\mathsf{D}^+(\mathscr{O}_Z\text{-}\mathbf{Mod})$. Hence (i) will follow from :

Claim 5.3.12. (i) There exists a natural isomorphism of functors :

$$g_*\mathscr{O}_Y\text{-}\mathbf{Mod}\to f_*\mathscr{O}_X\text{-}\mathbf{Mod}\quad:\quad\varphi^*\circ\beta^\flat\xrightarrow{\sim}\alpha^\flat\circ\overline{g}^*.$$

(ii) The natural transformation :

$$R(\alpha^\flat\circ\overline{g}^*)\to R\alpha^\flat\circ\overline{g}^*$$

is an isomorphism of functors : $\mathsf{D}^+(g_*\mathscr{O}_Y\text{-}\mathbf{Mod})\to\mathsf{D}^+(f_*\mathscr{O}_X\text{-}\mathbf{Mod})$.

Proof of the claim. First we show how to construct a natural transformation as in (i); this is the same as exhibiting a map of functors : $\alpha_*\circ\varphi^*\circ\beta^\flat\to\overline{g}^*$. In view of claim 5.3.10(i), the latter can be defined as the composition of $\overline{g}^*$ and the counit of adjunction $\beta_*\circ\beta^\flat\to\mathbf{1}_{g_*\mathscr{O}_Y\text{-}\mathbf{Mod}}$.

Next, recall the natural transformation :

$$(5.3.13)\quad \overline{g}^*\mathscr{H}om_{g_*\mathscr{O}_Y}(\mathscr{F},\mathscr{G})\to\mathscr{H}om_{\mathscr{O}_Y}(\overline{g}^*\mathscr{F},\overline{g}^*\mathscr{G})\quad\text{for every }g_*\mathscr{O}_Y\text{-modules }\mathscr{F}\text{ and }\mathscr{G}$$

(defined so as to induce the pull-back map on global Hom functors). Notice that the natural map $\overline{g}^*\overline{g}_*\mathscr{F}\to\mathscr{F}$ (counit of adjunction) is an isomorphism for every quasi-coherent $\mathscr{O}_Y$-module $\mathscr{F}$. Especially, if $\mathscr{F}=g_*\mathscr{A}$ for some quasi-coherent $\mathscr{O}_Y$-module $\mathscr{A}$, then (5.3.13) takes the form :

$$(5.3.14)\qquad \overline{g}^*\mathscr{H}om_{g_*\mathscr{O}_Y}(\overline{g}_*\mathscr{A},\mathscr{G})\to\mathscr{H}om_{\mathscr{O}_Y}(\mathscr{A},\overline{g}^*\mathscr{G}).$$

Furthermore, by inspecting the definitions (and the proof of claim 5.1.26), one verifies easily that the map $\varphi^* \circ \beta^\flat(\mathscr{G}) \to \alpha^\flat \circ \overline{g}^*(\mathscr{G})$ constructed above is the same as the map (5.3.14), taken with $\mathscr{A} = f_*\mathscr{O}_X$. Notice that, since f is pseudo-coherent, $f_*\mathscr{O}_X$ is even a finitely presented $\mathscr{O}_Y$-module; thus, in order to conclude the proof of (i), it suffices to show that (5.3.14) is an isomorphism for every finitely presented $\mathscr{O}_Y$-module $\mathscr{A}$ and every $g_*\mathscr{O}_Y$-module $\mathscr{G}$. To this aim, we may assume that Z and Y are affine; then we may find a presentation $\underline{\mathscr{E}} := (\mathscr{O}_Y^{\oplus p} \to \mathscr{O}_Y^{\oplus q} \to \mathscr{A} \to 0)$. We apply the natural transformation (5.3.14) to $\underline{\mathscr{E}}$, thereby obtaining a commutative ladder with left exact rows; then the five-lemma reduces the assertion to the case where $\mathscr{A} = \mathscr{O}_Y$, which is obvious. To prove assertion (ii) we may again assume that Y and Z are affine, in which case we can find a resolution $L_\bullet \to f_*\mathscr{O}_X$ consisting of free $\mathscr{O}_Y$-modules of finite rank. For a given bounded below complex $I^\bullet$ of injective $g_*\mathscr{O}_Y$-modules, choose a resolution $\overline{g}^*I^\bullet \to J^\bullet$ consisting of injective $\mathscr{O}_Y$-modules. We deduce a commutative ladder :

$$\begin{tikzcd}
\overline{g}^*\mathscr{H}om^\bullet_{g_*\mathscr{O}_Y}(\overline{g}_* \circ f_*\mathscr{O}_X, I^\bullet) \arrow[r, "\mu_1"] \arrow[d, "\lambda_1"'] & \mathscr{H}om^\bullet_{\mathscr{O}_Y}(f_*\mathscr{O}_X, \overline{g}^*I^\bullet) \arrow[r, "\mu_3"] \arrow[d] & \mathscr{H}om^\bullet_{\mathscr{O}_Y}(f_*\mathscr{O}_X, J^\bullet) \arrow[d, "\lambda_2"] \\
\overline{g}^*\mathscr{H}om^\bullet_{g_*\mathscr{O}_Y}(\overline{g}_*L_\bullet, I^\bullet) \arrow[r, "\mu_2"] & \mathscr{H}om^\bullet_{\mathscr{O}_Y}(L_\bullet, \overline{g}^*I^\bullet) \arrow[r, "\mu_4"] & \mathscr{H}om^\bullet_{\mathscr{O}_Y}(L_\bullet, J^\bullet).
\end{tikzcd}$$

Since $\overline{g}$ is flat, λ_1 is a quasi-isomorphism, and the same holds for λ_2. The maps μ_1 and μ_2 are of the form (5.3.14), hence they are quasi-isomorphisms, by the foregoing proof of (i). Finally, it is clear that μ_4 is a quasi-isomorphism as well, hence the same holds for $\mu_3 \circ \mu_1$, and (ii) follows. □

5.3.15. Let now $f : X \to Y$ be an *embeddable* morphism, *i.e.* such that it can be factored as a composition $f = g \circ h$ where $h : X \to Z$ is a finite pseudo-coherent morphism, $g : Z \to Y$ is smooth and separated, and the fibres of g have bounded dimension. One defines :

$$f^! := h^\flat \circ g^\sharp.$$

Lemma 5.3.16. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$ be three embeddable morphisms of schemes, such that the compositions $g \circ f$, $h \circ g$ and $h \circ g \circ f$ are also embeddable. Then we have :*

(i) *There exists a natural isomorphism of functors*

$$\psi_{g,f} : (g \circ f)^! \xrightarrow{\sim} f^! \circ g^!.$$

Especially, $f^!$ is independent – up to a natural isomorphism of functors – of the choice of factorization as a finite morphism followed by a smooth morphism.

(ii) *The diagram of functors :*

$$\begin{tikzcd}
(h \circ g \circ f)^! \arrow[r, "\psi_{h,g\circ f}"] \arrow[d, "\psi_{h\circ g,f}"'] & (g \circ f)^! \circ h^! \arrow[d, "\psi_{g,f} \circ h^!"] \\
f^! \circ (h \circ g)^! \arrow[r, "f^!(\psi_{h,g})"] & f^! \circ g^! \circ h^!
\end{tikzcd}$$

commutes.

Proof. The proof is a complicated verification, starting from proposition 5.3.7. See [46, Ch.III, Th.8.7] for details. □

5.3.17. Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be two finite and pseudo-coherent morphisms of schemes. The map $f^\natural : \mathscr{O}_Y \to f_*\mathscr{O}_X$ induces a natural transformation :

(5.3.18) $$\mathscr{H}om^\bullet_{\mathscr{O}_Z}(g_*(f^\natural), -) : \beta_* \circ (\gamma \circ \beta)^\flat \to \gamma^\flat$$

(notation of the proof of proposition 5.3.7(i)) and we wish to conclude this section by exhibiting another compatibility involving (5.3.18) and the isomorphism $\xi_{f,g}$. To this aim, we consider the composition of isomorphism of functors on $\mathsf{D}^+(\mathscr{O}_Z\text{-}\mathbf{Mod})_{\mathrm{qcoh}}$:

$$(5.3.19)\qquad Rf_* \circ \overline{g\circ f}^* \xrightarrow{①} \alpha_* \circ R\overline{f}_* \circ \overline{f}^* \circ \varphi^* \xrightarrow{②} \alpha_* \circ \varphi^* \xrightarrow{③} \overline{g}^* \circ \beta_*$$

where :

① is deduced from (5.3.8) and the decomposition $f = \alpha \circ \overline{f}$.
② is induced from the counit of adjunction $\overline{\varepsilon} : R\overline{f}_* \circ \overline{f}^* \to \mathbf{1}$ provided by lemma 5.3.2.
③ is the identification of claim 5.3.10.

We define a morphism of functors as a composition :

$$(5.3.20)\qquad Rf_* \circ (g\circ f)^\flat = Rf_* \circ \overline{g\circ f}^* \circ R(\gamma\circ\beta)^\flat \xrightarrow{④} \overline{g}^* \circ \beta_* \circ R(\gamma\circ\beta)^\flat \xrightarrow{⑤} g^\flat$$

where :

④ is the isomorphism (5.3.19) $\circ\, R(\gamma\circ\beta)^\flat$.
⑤ is the composition of $\overline{g}^*$ and the morphism $\beta_*\circ R(\gamma\circ\beta)^\flat \to R\gamma^\flat$ deduced from (5.3.18).

We have a natural diagram of functors on $\mathsf{D}^+(\mathscr{O}_Z\text{-}\mathbf{Mod})_{\mathrm{qcoh}}$:

$$(5.3.21)\qquad \begin{tikzcd} R(g\circ f)_*\circ(g\circ f)^\flat \arrow[r, "⑥"] \arrow[d, "R(g\circ f)_*(\xi_{f,g})"'] & Rg_*\circ Rf_*\circ(g\circ f)^\flat \arrow[r, "⑨"] \arrow[d, "Rg_*\circ Rf_*(\xi_{f,g})"] & Rg_*\circ g^\flat \\ R(g\circ f)_*\circ(f^\flat\circ g^\flat) \arrow[r, "⑦"] & Rg_*\circ Rf_*\circ(f^\flat\circ g^\flat) \arrow[r, equal] & Rg_*\circ(Rf_*\circ f^\flat)\circ g^\flat \arrow[u, "⑧"'] \end{tikzcd}$$

where :

⑥ and ⑦ are induced by the natural isomorphism $R(g\circ f)_* \xrightarrow{\sim} Rg_*\circ Rf_*$.
⑧ is induced by the counit of the adjunction $\varepsilon : Rf_*\circ f^\flat \to \mathbf{1}$.
⑨ is $Rg_* \circ$ (5.3.20).

Lemma 5.3.22. *In the situation of* (5.3.17)*, diagram* (5.3.21) *commutes.*

Proof. The diagram splits into left and right subdiagrams, and clearly the left subdiagram commutes, hence it remains to show that the right one does too; to this aim, it suffices to consider the simpler diagram :

$$\begin{tikzcd} Rf_*\circ(g\circ f)^\flat \arrow[r] \arrow[d, "Rf_*(\xi_{f,g})"'] & g^\flat \\ Rf_*\circ f^\flat\circ g^\flat \arrow[ur] \end{tikzcd}$$

whose horizontal arrow is (5.3.20), and whose upward arrow is deduced from the counit ε. However, the counit ε, used in ⑧, can be expressed in terms of the counit $\overline{\varepsilon} : R\overline{f}_*\circ\overline{f}^* \to \mathbf{1}$, used in ②, therefore we are reduced to considering the diagram of functors on $\mathsf{D}^+(\mathscr{O}_Z\text{-}\mathbf{Mod})_{\mathrm{qcoh}}$:

$$\begin{tikzcd} \alpha_*\circ R\overline{f}_*\circ\overline{f}^*\circ\varphi^*\circ R(\gamma\circ\beta)^\flat \arrow[r, "ⓐ"] \arrow[dd, "Rf_*(\xi_{f,g})"'] & \alpha_*\circ\varphi^*\circ R(\gamma\circ\beta)^\flat \arrow[r, "ⓔ"] \arrow[d, "ⓒ"'] & \overline{g}^*\circ\beta_*\circ R(\gamma\circ\beta)^\flat \arrow[dd, "⑤"] \\ & \alpha_*\circ\varphi^*\circ R\beta^\flat\circ R\gamma^\flat \arrow[d, "ⓓ"'] & \\ \alpha_*\circ R\overline{f}_*\circ\overline{f}^*\circ R\alpha^\flat\circ g^\flat \arrow[r, "ⓑ"] & \alpha_*\circ R\alpha^\flat\circ\overline{g}^*\circ R\gamma^\flat \arrow[r, "ⓕ"] & g^\flat = \overline{g}^*\circ R\gamma^\flat \end{tikzcd}$$

where :

a and b are induced from $\overline{\varepsilon}$.
c is induced by the isomorphism (5.3.11).
d is induced by the isomorphism of claim 5.3.12(i).
e is induced by the identity of claim 5.3.10.
f is induced by the counit $\alpha_* \circ R\alpha^\flat \to \mathbf{1}$.

Now, by inspecting the construction of $\xi_{f,g}$ one checks that the left subdiagram of the latter diagram commutes; moreover, from claim 5.3.12(ii) (and from lemma 5.1.25(i)), one sees that the right subdiagram is obtained by evaluating on complexes of injective modules the analogous diagram of functors on $\mathscr{O}_Z$-$\mathbf{Mod}$:

$$\begin{tikzcd}
\alpha_* \circ \varphi^* \circ (\gamma \circ \beta)^\flat \arrow[r, "\boxed{e}"] \arrow[d, "\boxed{c}"'] & \overline{g}^* \circ \beta_* \circ (\gamma \circ \beta)^\flat \arrow[dd, "\boxed{5}"] \\
\alpha_* \circ \varphi^* \circ \beta^\flat \circ \gamma^\flat \arrow[d, "\boxed{d}"'] & \\
\alpha_* \circ \alpha^\flat \circ \overline{g}^* \circ \gamma^\flat \arrow[r, "\boxed{f}"] & \overline{g}^* \circ \gamma^\flat .
\end{tikzcd}$$

Furthermore, by inspecting the proof of claim 5.3.12(i), one verifies that $\boxed{f} \circ \boxed{d}$ is the same as the composition

$$\alpha_* \circ \varphi^* \circ \beta^\flat \circ \gamma^\flat \xrightarrow{i} \overline{g}^* \circ \beta_* \circ \beta^\flat \circ \gamma^\flat \xrightarrow{j} \overline{g}^* \circ \gamma^\flat$$

where j is deduced from the counit $\beta_* \circ \beta^\flat \to \mathbf{1}$, and i is deduced from the identity of claim 5.3.10. Hence we come down to showing that the diagram :

$$\begin{tikzcd}
\beta_* \circ (\gamma \circ \beta)^\flat \arrow[r, "(5.3.18)"] \arrow[d, "\beta_*(5.3.9)"'] & \gamma^\flat \\
\beta_* \circ \beta^\flat \circ \gamma^\flat \arrow[ur, "k"'] &
\end{tikzcd}$$

commutes in $g_*\mathscr{O}_Y$-$\mathbf{Mod}$, where k is deduced from the counit $\beta_* \circ \beta^\flat \to \mathbf{1}$. This is an easy verification, that shall be left to the reader. $\square$

5.4. **Depth and cohomology with supports.** The aim of this section is to clarify some general issues concerning local cohomology and the closely related notion of depth, in the context of arbitrary schemes (whereas the usual references restrict to the case of locally noetherian schemes).

5.4.1. To begin with, let X be any topological space, $i : Z \to X$ a closed immersion, and $j : U := X \setminus Z \to X$ the open immersion of the complement of Z. We let $\mathbb{Z}_X$ be the constant abelian sheaf on X with value $\mathbb{Z}$; hence $\mathbb{Z}_X$-$\mathbf{Mod}$ denotes the category of abelian sheaves on X. One defines the functor

$$\underline{\Gamma}_Z : \mathbb{Z}_X\text{-}\mathbf{Mod} \to \mathbb{Z}_X\text{-}\mathbf{Mod} \qquad \mathscr{F} \mapsto \mathrm{Ker}\,(\mathscr{F} \to j_*j^{-1}\mathscr{F})$$

as well as its composition with the global section functor :

$$\Gamma_Z : \mathbb{Z}_X\text{-}\mathbf{Mod} \to \mathbb{Z}\text{-}\mathbf{Mod} \qquad \mathscr{F} \mapsto \Gamma(X, \underline{\Gamma}_Z\mathscr{F}).$$

It is clear that $\underline{\Gamma}_Z$ and Γ_Z are left exact functors, hence they give rise to right derived functors

$$R\underline{\Gamma}_Z : \mathsf{D}^+(\mathbb{Z}_X\text{-}\mathbf{Mod}) \to \mathsf{D}^+(\mathbb{Z}_X\text{-}\mathbf{Mod}) \qquad R\Gamma_Z : \mathsf{D}^+(\mathbb{Z}_X\text{-}\mathbf{Mod}) \to \mathsf{D}^+(\mathbb{Z}\text{-}\mathbf{Mod}).$$

Moreover, suppose that $\mathscr{F} \to \mathscr{I}^\bullet$ is an injective resolution; since injective sheaves are flabby, we obtain a short exact sequence of complexes

$$0 \to \underline{\Gamma}_Z \mathscr{I}^\bullet \to \mathscr{I}^\bullet \to j_* j^{-1} \mathscr{I}^\bullet \to 0$$

whence a natural exact sequence of $\mathbb{Z}_X$-modules :

(5.4.2) $$0 \to \underline{\Gamma}_Z \mathscr{F} \to \mathscr{F} \to j_* j^{-1} \mathscr{F} \to R^1 \underline{\Gamma}_Z \mathscr{F} \to 0 \qquad \text{for every } \mathbb{Z}_X\text{-module } \mathscr{F}$$

and natural isomorphisms :

(5.4.3) $$R^{q-1} j_* j^{-1} \mathscr{F} \xrightarrow{\sim} R^q \underline{\Gamma}_Z \mathscr{F} \qquad \text{for all } q > 1.$$

Lemma 5.4.4. (i) *A flabby sheaf is $\underline{\Gamma}_Z$-acyclic.*

(ii) *Suppose that X fulfills conditions* (a) *and* (b) *of lemma* 5.1.2*, and that Z is a constructible closed subset of X. Then :*
  (a) *Every qc-flabby sheaf on X is $\underline{\Gamma}_Z$-acyclic.*
  (b) *For every $i \in \mathbb{N}$, the functor $R^i \underline{\Gamma}_Z$ commutes with filtered colimits of abelian sheaves.*

Proof. (i) follows by inspecting (5.4.2), (5.4.3), together with the fact that flabby sheaves are acyclic for direct image functors such as j_*. Assertion (ii) is checked in the same way, using lemmata 5.1.2, 5.1.3 and 5.1.10(ii). □

5.4.5. Next, if $(X, \mathscr{A})$ is a ringed space, one can consider the restriction of $\underline{\Gamma}_Z$ to the category $\mathscr{A}\text{-}\mathbf{Mod}$ of $\mathscr{A}$-modules, and the foregoing discussion carries over *verbatim* to such case. Moreover, the derived functor

$$R\underline{\Gamma}_Z : \mathsf{D}^+(\mathscr{A}\text{-}\mathbf{Mod}) \to \mathsf{D}^+(\mathscr{A}\text{-}\mathbf{Mod})$$

commutes with the forgetful functor $\mathsf{D}^+(\mathscr{A}\text{-}\mathbf{Mod}) \to \mathsf{D}^+(\mathbb{Z}_X\text{-}\mathbf{Mod})$. (Indeed, in view of lemma 5.4.4(i) one may compute $R\underline{\Gamma}_Z \mathscr{F}$ by a flabby resolution of $\mathscr{F}$.)

Furthermore, let $f : (Y, \mathscr{B}) \to (X, \mathscr{A})$ be a morphism of ringed spaces. For every $\mathscr{B}$-module $\mathscr{F}$, the direct image $f_* \mathscr{F}$ is an $\mathscr{A}$-module in a natural way, and the derived functor $Rf_* : \mathsf{D}^+(\mathscr{B}\text{-}\mathbf{Mod}) \to \mathsf{D}^+(\mathscr{A}\text{-}\mathbf{Mod})$ commutes with the forgetful functors $\mathsf{D}^+(\mathscr{A}\text{-}\mathbf{Mod}) \to \mathsf{D}^+(\mathbb{Z}_X\text{-}\mathbf{Mod})$ and $\mathsf{D}^+(\mathscr{B}\text{-}\mathbf{Mod}) \to \mathsf{D}^+(\mathbb{Z}_Y\text{-}\mathbf{Mod})$. (Again, one sees this easily when one remarks that flabby resolutions compute Rf_*.)

5.4.6. In the situation of (5.4.5), let $\mathfrak{U} := (U_t \mid t \in I)$ be a family of open subsets of X with

$$\bigcup_{t \in I} U_t = U.$$

We attach to $\mathfrak{U}$ an *alternating Čech resolution*, which is a complex $R_\bullet(\mathfrak{U})$ in $\mathsf{C}^{\leq 0}(\mathbb{Z}_X\text{-}\mathbf{Mod})$ constructed as follows. Fix an arbitrary total ordering on I, and for every integer $n > 0$, let I_n be the set of all sequences $(t_1, \dots, t_n)$ of elements of I, such that $t_1 < \cdots < t_n$. For every $\underline{t} \in I_n$, set $U_{\underline{t}} := U_{t_1} \cap \cdots \cap U_{t_n}$, and let $j_{\underline{t}} : U_{\underline{t}} \to X$ be the open immersion. We let

$$R_0(\mathfrak{U}) := \mathbb{Z}_X \qquad \text{and} \qquad R_n(\mathfrak{U}) := \bigoplus_{\underline{t} \in I_n} j_{\underline{t}!} \mathbb{Z}_{U_{\underline{t}}} \qquad \text{for every } n > 0$$

The differential $d_1 : R_1(\mathfrak{U}) \to R_0(\mathfrak{U})$ is just the sum of the natural morphisms $j_{t!} \mathbb{Z}_{U_t} \to \mathbb{Z}_X$, for every $t \in I$. For $n > 1$, the differential $d_n : R_n(\mathfrak{U}) \to R_{n-1}(\mathfrak{U})$ is the sum of the maps

$$d_{\underline{t}} : j_{\underline{t}!} \mathbb{Z}_{U_{\underline{t}}} \to R_{n-1}(\mathfrak{U}) \qquad \text{for every } \underline{t} \in I_n$$

defined as follows. Say that $\underline{t} = (t_1, \dots, t_n)$, and for every $k = 1, \dots, n$ denote by $\underline{t}^{(k)} \in I_{n-1}$ the sequence obtained by deleting the k-th term from $\underline{t}$; the inclusion $U_{\underline{t}} \subset U_{\underline{t}^{(k)}}$ induces a morphism of $\mathbb{Z}_X$-modules

$$d_{\underline{t}}^{(k)} : j_{\underline{t}!}\mathbb{Z}_{U_{\underline{t}}} \to j_{\underline{t}^{(k)}!}\mathbb{Z}_{U_{\underline{t}^{(k)}}} \subset R_{n-1}(\mathfrak{U})$$

and we set

$$d_{\underline{t}} := \sum_{k=1}^{n} (-1)^k \cdot d_{\underline{t}}^{(k)}.$$

A direct computation shows that $d_n \circ d_{n+1} = 0$ for every $n \in \mathbb{N}$. The name of $R_\bullet(\mathfrak{U})$ is justified by the following :

Lemma 5.4.7. *With the notation of* (5.4.6)*, the natural projection $\mathbb{Z}_X \to i_*\mathbb{Z}_Z$ induces a quasi-isomorphism of complexes of $\mathbb{Z}_X$-modules :*

$$R_\bullet(\mathfrak{U}) \xrightarrow{\sim} i_*\mathbb{Z}_Z[0]. \tag{5.4.8}$$

Proof. For every finite subset $J \subset I$, set $\mathfrak{U}_J := (U_t \mid t \in J)$, $Z_J := X \setminus \bigcup_{t \in J} U_t$, and denote by $i_J : Z_J \to X$ the closed immersion. Notice the natural isomorphisms

$$R_\bullet(\mathfrak{U}) \xrightarrow{\sim} \operatorname*{colim}_{J \subset I} R_\bullet(\mathfrak{U}_J) \qquad i_*\mathbb{Z}_Z \xrightarrow{\sim} \operatorname*{colim}_{J \subset I} i_{J*}\mathbb{Z}_{Z_J}$$

where J ranges over all the finite subsets of I. Since (5.4.8) is likewise the colimit of the corresponding system of maps $R_\bullet(\mathfrak{U}_J) \to i_{J*}\mathbb{Z}_{Z_J}[0]$, we may assume from start that I is a finite set. Now, the assertion can be checked at the stalks over the points of X, hence let $x \in X$ be any such point, and $R_\bullet(\mathfrak{U})_x$ the corresponding complex of $\mathbb{Z}$-modules (the stalk of $R_\bullet(\mathfrak{U})$ at x). If $x \in Z$, then obviously $R_\bullet(\mathfrak{U})_x$ is concentrated in degree 0, and the resulting map (5.4.8)$_x$ is clearly an isomorphism. If $x \notin Z$, set $J := \{t \in I \mid x \in U_t\}$; it is easily seen that $R_\bullet(\mathfrak{U})_x = R_\bullet(\mathfrak{U}_J)_x$, so we may replace $\mathfrak{U}$ by $\mathfrak{U}_J$, and assume that $x \in U_t$ for every $t \in I$. In this case, a simple inspection shows that $R_\bullet(\mathfrak{U})_x$ is naturally isomorphic to the Koszul complex $\mathbf{K}_\bullet(\mathbf{1}_I)$, where $\mathbf{1}_I = (1_t \mid t \in I)$ is a sequence of units $1 \in \mathbb{Z}$, indexed by I. The complex (of $\mathbb{Z}$-modules) $\mathbf{K}_\bullet(\mathbf{1}_I)$ is obviously acyclic, so the claim follows. $\square$

5.4.9. Keep the notation of (5.4.6), and notice the natural isomorphism of $\mathscr{A}$-modules :

$$\mathscr{H}om_{\mathbb{Z}}(i_*\mathbb{Z}_Z, \mathscr{F}) \xrightarrow{\sim} \underline{\Gamma}_Z(\mathscr{F}) \qquad \text{for every } \mathscr{A}\text{-module } \mathscr{F}$$

whence a natural isomorphism of $\mathscr{A}(X)$-modules :

$$\mathrm{Hom}_{\mathbb{Z}}(i_*\mathbb{Z}_Z, \mathscr{F}) \xrightarrow{\sim} \Gamma_Z(\mathscr{F}) \qquad \text{for every } \mathscr{A}\text{-module } \mathscr{F}.$$

By lemma 5.4.7, we deduce natural isomorphisms in $\mathsf{D}^+(\mathscr{A}\text{-}\mathbf{Mod})$ (resp. in $\mathsf{D}^+(\mathscr{A}(X)\text{-}\mathbf{Mod})$)

$$R\mathscr{H}om^\bullet_{\mathbb{Z}}(R_\bullet(\mathfrak{U}), K^\bullet) \xrightarrow{\sim} R\underline{\Gamma}_Z K^\bullet \qquad (\text{ resp. } R\mathrm{Hom}^\bullet_{\mathbb{Z}}(R_\bullet(\mathfrak{U}), K^\bullet) \xrightarrow{\sim} R\Gamma_Z K^\bullet \text{ })$$

for every bounded below complex $K^\bullet$ of $\mathscr{A}$-modules. These isomorphisms allow to compute $R\Gamma_Z K^\bullet$ as a *Čech cohomology functor*, as follows. For every $\mathscr{A}$-module $\mathscr{F}$, we set

$$C^\bullet_{\mathrm{alt}}(\mathfrak{U}_\bullet, \mathscr{F}) := \mathrm{Hom}^\bullet_{\mathbb{Z}}(R_\bullet(\mathfrak{U}), \mathscr{F}[0])$$

and we call it the *alternating Čech complex of $\mathscr{F}$ relative to the covering* $\mathfrak{U}$ of U. Unwinding the definitions, we see that the latter is the cochain complex :

$$0 \to \mathscr{F}(X) \to \prod_{t \in I} \mathscr{F}(U_t) \to \cdots \to \prod_{\underline{t} \in I_n} \mathscr{F}(U_{\underline{t}}) \to \cdots \tag{5.4.10}$$

The differential d^n of (5.4.10) is given by the following rule. In degree zero, d^0 assigns to each global section $s \in \mathscr{F}(X)$ the system of its restrictions $(s_{|U_t} \mid t \in I)$. In case $n > 0$, we have

$$d^n(s_\bullet)_{\underline{t}} := \sum_{k=1}^{n+1} (-1)^k \cdot (s_{\underline{t}(k)})_{|U_{\underline{t}}} \qquad \text{for every } \underline{t} \in I_{n+1} \text{ and every } s_\bullet \in C^n(\mathfrak{U}, \mathscr{F}).$$

Clearly, the rule : $\mathscr{F} \mapsto C^\bullet_{\mathrm{alt}}(\mathfrak{U}, \mathscr{F})$ defines a functor $\mathscr{A}\text{-}\mathbf{Mod} \to \mathsf{C}(\mathscr{A}(X)\text{-}\mathbf{Mod})$. Now, pick any resolution $K^\bullet \xrightarrow{\sim} \mathscr{I}^\bullet$ by a bounded below complex of injective $\mathscr{A}$-modules; the foregoing yields a natural isomorphism :

$$R\Gamma_Z K^\bullet \xrightarrow{\sim} \mathrm{Tot}\, C^\bullet_{\mathrm{alt}}(\mathfrak{U}, \mathscr{I}^\bullet) \qquad \text{in } \mathsf{D}^+(\mathscr{A}(X)\text{-}\mathbf{Mod}).$$

5.4.11. We define a bifunctor

$$\mathscr{H}om^\bullet_Z := \underline{\Gamma}_Z \circ \mathscr{H}om^\bullet_{\mathscr{A}} : \mathsf{C}(\mathscr{A}\text{-}\mathbf{Mod})^o \times \mathsf{C}(\mathscr{A}\text{-}\mathbf{Mod}) \to \mathsf{C}(\mathscr{A}\text{-}\mathbf{Mod}).$$

The bifunctor $\mathscr{H}om^\bullet_Z$ admits a right derived functor :

$$R\mathscr{H}om^\bullet_Z : \mathsf{D}(\mathscr{A}\text{-}\mathbf{Mod})^o \times \mathsf{D}^+(\mathscr{A}\text{-}\mathbf{Mod}) \to \mathsf{D}(\mathscr{A}\text{-}\mathbf{Mod}).$$

The construction can be outlined as follows. First, for a fixed complex $K_\bullet$ of $\mathscr{A}$-modules, one can consider the right derived functor of the functor $\mathscr{G} \mapsto \mathscr{H}om^\bullet_Z(K_\bullet, \mathscr{G})$, which is denoted $R\mathscr{H}om^\bullet_Z(K_\bullet, -) : \mathsf{D}^+(\mathscr{A}\text{-}\mathbf{Mod}) \to \mathsf{D}(\mathscr{A}\text{-}\mathbf{Mod})$. Next, one verifies that every quasi-isomorphism $K_\bullet \to K'_\bullet$ induces an isomorphism of functors : $R\mathscr{H}om^\bullet_Z(K'_\bullet, -) \xrightarrow{\sim} R\mathscr{H}om^\bullet_Z(K_\bullet, -)$, hence the natural transformation $K_\bullet \mapsto R\mathscr{H}om^\bullet_Z(K_\bullet, -)$ factors through $\mathsf{D}(\mathscr{A}\text{-}\mathbf{Mod})$, and this is the sought bifunctor $R\mathscr{H}om^\bullet_Z$. In case $Z = X$, one recovers the functor $R\mathscr{H}om^\bullet_{\mathscr{A}}$ of (5.1.22).

Lemma 5.4.12. *In the situation of* (5.4.5), *the following holds :*

(i) *The functor $\underline{\Gamma}_Z$ is right adjoint to i_*i^{-1}. More precisely, for any two $\mathscr{A}$-modules $\mathscr{F}$ and $\mathscr{G}$ we have natural isomorphisms :*
  (a) $\mathscr{H}om_{\mathscr{A}}(\mathscr{F}, \underline{\Gamma}_Z\mathscr{G}) \simeq \mathscr{H}om_{\mathscr{A}}(i_*i^{-1}\mathscr{F}, \mathscr{G})$.
  (b) $R\mathscr{H}om^\bullet_{\mathscr{A}}(\mathscr{F}, R\underline{\Gamma}_Z\mathscr{G}) \simeq R\mathscr{H}om^\bullet_{\mathscr{A}}(i_*i^{-1}\mathscr{F}, \mathscr{G})$.

(ii) *If $\mathscr{F}$ is an injective (resp. flabby) $\mathscr{A}$-module on X, then the same holds for $\underline{\Gamma}_Z\mathscr{F}$.*

(iii) *There are natural isomorphisms of bifunctors :*

$$R\mathscr{H}om^\bullet_Z \xrightarrow{\sim} R\underline{\Gamma}_Z \circ R\mathscr{H}om^\bullet_{\mathscr{A}} \xrightarrow{\sim} R\mathscr{H}om^\bullet_{\mathscr{A}}(-, R\underline{\Gamma}_Z -).$$

(iv) *If $W \subset Z$ is any closed subset, there is a natural isomorphism of functors :*

$$R\underline{\Gamma}_W \xrightarrow{\sim} R\underline{\Gamma}_W \circ R\underline{\Gamma}_Z.$$

(v) *There is a natural isomorphism of functors :*

$$Rf_* \circ R\underline{\Gamma}_{f^{-1}Z} \xrightarrow{\sim} R\underline{\Gamma}_Z \circ Rf_* : \mathsf{D}^+(\mathscr{B}\text{-}\mathbf{Mod}) \to \mathsf{D}^+(\mathscr{A}\text{-}\mathbf{Mod}).$$

Proof. (i): To establish the isomorphism (a), one uses the short exact sequence

$$0 \to j_!j^{-1}\mathscr{F} \to \mathscr{F} \to i_*i^{-1}\mathscr{F} \to 0$$

to show that any map $\mathscr{F} \to \underline{\Gamma}_Z\mathscr{G}$ factors through $\mathscr{H} := i_*i^{-1}\mathscr{F}$. Conversely, since $j_*j^{-1}\mathscr{H} = 0$, it is clear that every map $\mathscr{H} \to \mathscr{G}$ must factor through $\underline{\Gamma}_Z\mathscr{G}$. The isomorphism (b) is derived easily from (a) and (ii).

(iii): According to [75, Th.10.8.2], the first isomorphism in (iii) is deduced from lemmata 5.4.4(i) and 5.1.25(ii). Likewise, the second isomorphism in (iii) follows from [75, Th.10.8.2], and assertion (ii) concerning injective sheaves.

(ii): Suppose first that $\mathscr{F}$ is injective; let $\mathscr{G}_1 \to \mathscr{G}_2$ a monomorphism of $\mathscr{A}$-modules and $\varphi : \mathscr{G}_1 \to \underline{\Gamma}_Z\mathscr{F}$ any $\mathscr{A}$-linear map. By (i) we deduce a map $i_*i^*\mathscr{G}_1 \to \mathscr{F}$, which extends to a map $i_*i^*\mathscr{G}_2 \to \mathscr{F}$, by injectivity of $\mathscr{F}$. Finally, (i) again yields a map $\mathscr{G}_2 \to \underline{\Gamma}_Z\mathscr{F}$ extending φ.

Next, if $\mathscr{F}$ is flabby, let $V \subset X$ be any open subset, and $s \in \underline{\Gamma}_Z\mathscr{F}(V)$; since $s_{|V\setminus Z} = 0$, we can extend s to a section $s' \in \underline{\Gamma}_Z\mathscr{F}(V \cup (X \setminus Z))$. Since $\mathscr{F}$ is flabby, s' extends to a section $s'' \in \mathscr{F}(X)$; however, by construction s'' vanishes on the complement of Z.

(iv): Clearly $\underline{\Gamma}_W = \underline{\Gamma}_W \circ \underline{\Gamma}_Z$, so the claim follows easily from (ii) and [75, Th.10.8.2].

(v): Indeed, since flabby $\mathscr{B}$-modules are acyclic for both f_* and $\underline{\Gamma}_{f^{-1}Z}$, one can apply (ii) and [75, Th.10.8.2] to naturally identify both functors with the right derived functor of $\underline{\Gamma}_Z \circ f_*$. $\square$

Definition 5.4.13. Let $K^\bullet \in \mathsf{D}^+(\mathbb{Z}_X\text{-}\mathbf{Mod})$ be any bounded below complex. The *depth of* $K^\bullet$ *along* Z is by definition :

$$\mathrm{depth}_Z K^\bullet := \sup\{n \in \mathbb{N} \mid R^i\underline{\Gamma}_Z K^\bullet = 0 \text{ for all } i < n\} \in \mathbb{N} \cup \{+\infty\}.$$

Notice that

$$\mathrm{depth}_W K^\bullet \geq \mathrm{depth}_Z K^\bullet \qquad \text{whenever } W \subset Z. \tag{5.4.14}$$

More generally, suppose that $\Phi := (Z_\lambda \mid \lambda \in \Lambda)$ is a family of closed subsets of X, and suppose that Φ is cofiltered by inclusion. Then, due to (5.4.14) it is reasonable to define the *depth of* $K^\bullet$ *along* Φ by the rule :

$$\mathrm{depth}_\Phi K^\bullet := \sup\{\mathrm{depth}_{Z_\lambda} K^\bullet \mid \lambda \in \Lambda\}.$$

5.4.15. Suppose now that X is a scheme and $i : Z \to X$ a closed immersion of schemes, such that Z is constructible in X. It follows that the open immersion $j : X\backslash Z \to X$ is quasi-compact and separated, hence the functors $R^q j_*$ preserve the subcategories of quasi-coherent modules, for every $q \in \mathbb{N}$ ([28, Ch.III, Prop.1.4.10]). In light of (5.4.3) we derive that $R^q\underline{\Gamma}_Z$ restricts to a functor :

$$R^q\underline{\Gamma}_Z : \mathscr{O}_X\text{-}\mathbf{Mod}_{\mathrm{qcoh}} \to \mathscr{O}_X\text{-}\mathbf{Mod}_{\mathrm{qcoh}} \qquad \text{for every } q \in \mathbb{N}.$$

Lemma 5.4.16. *In the situation of* (5.4.15)*, the following holds :*

(i) *The functor* $R\underline{\Gamma}_Z$ *restricts to a triangulated functor :*

$$R\underline{\Gamma}_Z : \mathsf{D}^+(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{qcoh}} \to \mathsf{D}^+(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{qcoh}}$$

(notation of (5.2.2)*).*

(ii) *Let* $f : Y \to X$ *be an affine morphism of schemes,* $K^\bullet$ *any object of* $\mathsf{D}^+(\mathscr{O}_Y\text{-}\mathbf{Mod})_{\mathrm{qcoh}}$. *Then we have the identity :*

$$\mathrm{depth}_{f^{-1}Z} K^\bullet = \mathrm{depth}_Z Rf_* K^\bullet.$$

(iii) *Let* $f : Y \to X$ *be a flat morphism of schemes,* $K^\bullet$ *any object of* $\mathsf{D}^+(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{qcoh}}$. *Then the natural map*

$$f^* R\underline{\Gamma}_Z K^\bullet \to R\underline{\Gamma}_{f^{-1}Z} f^* K^\bullet$$

is an isomorphism in $\mathsf{D}^+(\mathscr{O}_Y\text{-}\mathbf{Mod})_{\mathrm{qcoh}}$. *Especially, we have*

$$\mathrm{depth}_{f^{-1}Z} f^* K^\bullet \geq \mathrm{depth}_Z K^\bullet$$

and if f *is faithfully flat, the inequality is actually an equality.*

Proof. (i): Indeed, if $K^\bullet$ is an object of $\mathsf{D}^+(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{qcoh}}$, we have a spectral sequence :

$$E_2^{pq} := R^p\underline{\Gamma}_Z H^q K^\bullet \Rightarrow R^{p+q}\underline{\Gamma}_Z K^\bullet.$$

(This spectral sequence is obtained from a Cartan-Eilenberg resolution of $K^\bullet$: see *e.g.* [75, §5.7].) Hence $R^\bullet\underline{\Gamma}_Z K^\bullet$ admits a finite filtration whose subquotients are quasi-coherent; then the lemma follows from [28, Ch.III, Prop.1.4.17].

(ii): To start with, a spectral sequence argument as in the foregoing shows that Rf_* restricts to a functor $\mathsf{D}^+(\mathscr{O}_Y\text{-}\mathbf{Mod})_{\mathrm{qcoh}} \to \mathsf{D}^+(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{qcoh}}$, and moreover we have natural identifications : $R^i f_* K^\bullet \xrightarrow{\sim} f_* H^i K^\bullet$ for every $i \in \mathbb{Z}$ and every object $K^\bullet$ of $\mathsf{D}^+(\mathscr{O}_Y\text{-}\mathbf{Mod})_{\mathrm{qcoh}}$. In view of lemma 5.4.12(v), we derive natural isomorphisms : $R^i\underline{\Gamma}_Z Rf_*K^\bullet \xrightarrow{\sim} Rf_*R^i\underline{\Gamma}_{f^{-1}Z}K^\bullet$ for every object $K^\bullet$ of $\mathsf{D}^+(\mathscr{O}_Y\text{-}\mathbf{Mod})_{\mathrm{qcoh}}$. The assertion follows easily.

(iii): A spectral sequence argument as in the proof of (i) allows to assume that $K^\bullet = \mathscr{F}[0]$ for some quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$. In this case, let $j_X : X\setminus Z \to X$ and $j_Y : Y\setminus f^{-1}Z \to Y$ be the open immersions; considering the exact sequences f^*(5.4.2) and f^*(5.4.3), we reduce to showing that the natural map $f^*Rj_{X*}j_X^{-1}\mathscr{F} \to Rj_{Y*}j_Y^*f^*\mathscr{F}$ is an isomorphism in $\mathsf{D}^+(\mathscr{O}_Y\text{-}\mathbf{Mod})_{\mathrm{qcoh}}$. The latter assertion follows from corollary 5.1.19. □

Remark 5.4.17. Let X be an affine scheme, $Z \subset X$ any closed subset, $\mathscr{F}^\bullet$ a bounded below complex of quasi-coherent $\mathscr{O}_X$-modules, and $\mathfrak{U} := (U_t \mid t \in I)$ a family of affine open subsets of X such that $\bigcup_{t\in I} U_t = X\setminus Z$. Then there exists a natural isomorphism

$$R\Gamma_Z\mathscr{F}^\bullet \xrightarrow{\sim} \mathrm{Tot}\, C^\bullet_{\mathrm{alt}}(\mathfrak{U}, \mathscr{F}^\bullet) \qquad \text{in } \mathsf{D}^+(\mathscr{O}_X\text{-}\mathbf{Mod})$$

(where $C^\bullet_{\mathrm{alt}}$ denotes the alternating Čech complex). Indeed, pick any resolution $\mathscr{F}^\bullet \xrightarrow{\sim} \mathscr{I}^\bullet$ by a bounded below complex of injective $\mathscr{O}_X$-modules; by virtue of (5.4.9) it suffices to show that the natural map of double complexes

$$C^\bullet_{\mathrm{alt}}(\mathfrak{U}, \mathscr{F}^\bullet) \to C^\bullet_{\mathrm{alt}}(\mathfrak{U}, \mathscr{I}^\bullet)$$

induces a quasi-isomorphism on total complexes. However, notice that the open subset $U_{\underline{t}} \subset X$ is affine, for every $n \in \mathbb{N}$ and every $\underline{t} \in I_n$ (notation of (5.4.9)); hence, it suffices to check that the induced map $\mathscr{F}^\bullet(V) \to \mathscr{I}^\bullet(V)$ is a quasi-isomorphism, for every affine open subset $V \subset X$. To this aim, consider the spectral sequence

$$E_1^{p,q} := H^p(V, \mathscr{F}^q) \Rightarrow H^{p+q}\mathscr{I}^\bullet(V).$$

Since V is affine, we have $E_1^{p,q} = 0$ for every $p > 0$; on the other hand, the differential $d^{0,q} : E_1^{0,q} \to E_1^{0,q+1}$ is nothing else than the differential $d^q(V) : \mathscr{F}^q(V) \to \mathscr{F}^{q+1}(V)$, for every $q \in \mathbb{Z}$, whence the claim.

5.4.18. Let X be a scheme, and $x \in X$ any point. We consider the cofiltered family $\Phi(x)$ of all non-empty constructible closed subsets of $X(x) := \mathrm{Spec}\,\mathscr{O}_{X,x}$; clearly $\bigcap_{Z\in\Phi(x)} Z = \{x\}$. Let $K^\bullet$ be any object of $\mathsf{D}^+(\mathbb{Z}_X\text{-}\mathbf{Mod})$; we are interested in the quantities :

$$\delta(x, K^\bullet) := \mathrm{depth}_{\{x\}} K^\bullet_{|X(x)} \qquad \text{and} \qquad \delta'(x, K^\bullet) := \mathrm{depth}_{\Phi(x)} K^\bullet_{|X(x)}.$$

Especially, we would like to be able to compute the depth of a complex $K^\bullet$ along a closed subset $Z \subset X$, in terms of the local invariants $\delta(x, K^\bullet)$ (resp. $\delta'(x, K^\bullet)$), evaluated at the points $x \in Z$. This shall be achieved by theorem 5.4.20. To begin with, we remark :

Lemma 5.4.19. *With the notation of* (5.4.18), *we have :*

(i) $\delta(x, K^\bullet) \geq \delta'(x, K^\bullet)$ *for every $x \in X$ and every complex $K^\bullet$ of $\mathbb{Z}_X$-modules.*

(ii) *If the topological space $|X|$ underlying X is locally noetherian, the inequality of* (i) *is actually an equality.*

(iii) *If $\mathscr{F}$ is a flat quasi-coherent $\mathscr{O}_X$-module, then $\delta'(x, \mathscr{F}) \geq \delta'(x, \mathscr{O}_X)$.*

Proof. (i) follows easily from lemma 5.4.12(iv). (ii) follows likewise from lemma 5.4.12(iv) and the fact that, if $|X|$ is locally noetherian, $\{x\}$ is the smallest element of the family $\Phi(x)$.

(iii): According to [57, Ch.I, Th.1.2], the $\mathscr{O}_{X(x)}$-module $\mathscr{F}_{|X(x)}$ is the colimit of a filtered system $(\mathscr{L}_\lambda \mid \lambda \in \Lambda)$ of free $\mathscr{O}_{X(x)}$-modules of finite rank. In view of lemma 5.4.4(iii), it is easily seen that

$$\mathrm{depth}_Z\mathscr{F} \geq \inf_{\lambda\in\Lambda} \mathrm{depth}_Z\mathscr{L}_\lambda = \mathrm{depth}_Z\mathscr{O}_X$$

for every closed onstructible subset $Z \subset X$. The assertion follows. □

Theorem 5.4.20. *With the notation of* (5.4.18), *let $Z \subset X$ be any closed constructible subset, $K^\bullet$ any object of* $\mathsf{D}^+(\mathbb{Z}_X\text{-}\mathbf{Mod})$. *Then the following holds :*

(i) $(R\underline{\Gamma}_Z K^\bullet)_{|X(x)} = R\underline{\Gamma}_{Z\cap X(x)}(K^\bullet_{|X(x)})$ *for every* $x \in Z$.
(ii) $\mathrm{depth}_Z K^\bullet = \inf\{\delta(x, K^\bullet) \mid x \in Z\} = \inf\{\delta'(x, K^\bullet) \mid x \in Z\}$.

Proof. We may assume that X is affine, hence separated; then, according to lemma 5.4.4(ii.a), cohomology with supports in Z can be computed by a qc-flabby resolution, hence we may assume that $K^\bullet$ is a complex of qc-flabby sheaves on X. In this case, arguing as in the proof of claim 5.1.13, one checks easily that the restriction $K^\bullet_{|X(x)}$ is a complex of qc-flabby sheaves on $X(x)$. Thus, for the proof of (i), it suffices to show that the natural map in $\mathbb{Z}_X$-$\mathbf{Mod}$:

$$(\underline{\Gamma}_Z \mathscr{F})_{|X(x)} \to \underline{\Gamma}_{Z\cap X(x)}(\mathscr{F}_{|X(x)})$$

is an isomorphism, for every object $\mathscr{F}$ of $\mathbb{Z}_X$-$\mathbf{Mod}$.

Now, for any point $x \in X$, we may view $X(x)$ as the limit of the cofiltered system $(V_\lambda \mid \lambda \in \Lambda)$ of all affine open neighborhoods of x in X. We denote by $(j_\lambda : V_\lambda \setminus Z \to V_\lambda \mid \lambda \in \Lambda)$ the induced cofiltered system of open immersions, whose limit is the morphism $j_\infty : X(x)\setminus Z \to X(x)$; also, let $\varphi_\lambda : X(x) \to V_\lambda$ be the natural morphisms. Then

$$\mathscr{F}_{|X(x)} = \operatorname*{colim}_{\lambda\in\Lambda} \varphi_\lambda^{-1}\mathscr{F}_{|U_\lambda}.$$

By proposition 5.1.15(ii), the natural map

$$\operatorname*{colim}_{\lambda\in\Lambda} \varphi_\lambda^{-1} j_{\lambda*}\mathscr{F}_{|V_\lambda\setminus Z} \to j_{\infty*}\mathscr{F}_{|X(x)\setminus Z}$$

is an isomorphism. We deduce natural isomorphisms :

$$\operatorname*{colim}_{\lambda\in\Lambda} \varphi_\lambda^{-1}\underline{\Gamma}_{Z\cap V_\lambda}\mathscr{F}_{|V_\lambda} \xrightarrow{\sim} \underline{\Gamma}_{Z\cap X(x)}\mathscr{F}_{|X(x)}.$$

However, we have as well natural isomorphisms :

$$\varphi_\lambda^{-1}\underline{\Gamma}_{Z\cap V_\lambda}\mathscr{F}_{|V_\lambda} \xrightarrow{\sim} (R\underline{\Gamma}_Z\mathscr{F})_{|X(x)} \qquad \text{for every } \lambda \in \Lambda.$$

whence (i). Next, let $d := \mathrm{depth}_Z K^\bullet$; in light of (i), it is clear that $\delta(x, K^\bullet) \geq d$ for all $x \in Z$, hence, in order to prove the first identity of (ii) it suffices to show :

Claim 5.4.21. Suppose $d < +\infty$. Then there exists $x \in Z$ such that $R^d\Gamma_{\{x\}}K^\bullet_{|X(x)} \neq 0$.

Proof of the claim. By definition of d, we can find an open subset $V \subset X$ and a non-zero section $s \in \Gamma(V, R^d\underline{\Gamma}_Z K^\bullet)$. The support of s is a closed subset $S \subset Z$. Let x be a maximal point of S. From lemma 5.4.12(iv) we deduce a spectral sequence

$$E_2^{pq} := R^p\underline{\Gamma}_{\{x\}}R^q\underline{\Gamma}_{Z\cap X(x)}K^\bullet_{|X(x)} \Rightarrow R^{p+q}\underline{\Gamma}_{\{x\}}K^\bullet_{|X(x)}$$

and (i) implies that $E_2^{pq} = 0$ whenever $q < d$, therefore :

$$R^d\underline{\Gamma}_{\{x\}}K^\bullet_{|X(x)} \simeq R^0\underline{\Gamma}_{\{x\}}R^d\underline{\Gamma}_{Z\cap X(x)}K^\bullet_{|X(x)}.$$

However, the image of s in $R^d\underline{\Gamma}_{Z\cap X(x)}K^\bullet_{|X(x)}$ is supported precisely at x, as required. ◊

Finally, since Z is constructible, $Z \cap X(x) \in \Phi(x)$ for every $x \in X$, hence $\delta'(x, K^\bullet) \geq d$. Then the second identity of (ii) follows from the first and lemma 5.4.19(i). □

Corollary 5.4.22. *Let $Z \subset X$ be a closed and constructible subset, $\mathscr{F}$ an abelian sheaf on X, $j : X\setminus Z \to X$ the induced open immersion. Then the natural map $\mathscr{F} \to j_*j^*\mathscr{F}$ is an isomorphism if and only if $\delta(x, \mathscr{F}) \geq 2$, if and only if $\delta'(x, \mathscr{F}) \geq 2$ for every $x \in Z$.* □

5.4.23. Let now X be an affine scheme, say $X := \mathrm{Spec}\, A$ for some ring A, and $Z \subset X$ a constructible closed subset. In this case, we wish to show that the depth of a complex of quasi-coherent $\mathscr{O}_X$-modules along Z can be computed in terms of Ext functors on the category of A-modules. This is the content of the following :

Proposition 5.4.24. *In the situation of* (5.4.23), *let N be any finitely presented A-module such that* $\mathrm{Supp}\, N = Z$, *and $M^\bullet$ any bounded below complex of A-modules. We denote by $M^{\bullet\sim}$ the complex of quasi-coherent $\mathscr{O}_X$-modules determined by $M^\bullet$. Then :*

$$\mathrm{depth}_Z M^{\bullet\sim} = \sup\{n \in \mathbb{N} \mid \mathrm{Ext}^j_A(N, M^\bullet) = 0 \textit{ for all } j < n\}.$$

Proof. Let $N^\sim$ be the quasi-coherent $\mathscr{O}_X$-module determined by N.

Claim 5.4.25. There is a natural isomorphism in $\mathsf{D}^+(A\text{-}\mathbf{Mod})$:

$$R\mathrm{Hom}^\bullet_A(N, M^\bullet) \xrightarrow{\sim} R\mathrm{Hom}^\bullet_{\mathscr{O}_X}(N^\sim, R\underline{\Gamma}_Z M^{\bullet\sim}).$$

Proof of the claim. Let $i : Z \to X$ be the closed immersion. The assumption on N implies that $N^\sim = i_* i^{-1} N^\sim$; hence lemma 5.4.12(i.b) yields a natural isomorphism

$$R\mathrm{Hom}^\bullet_{\mathscr{O}_X}(N^\sim, M^{\bullet\sim}) \xrightarrow{\sim} R\mathrm{Hom}^\bullet_{\mathscr{O}_X}(N^\sim, R\underline{\Gamma}_Z M^{\bullet\sim}). \tag{5.4.26}$$

The claim follows by combining corollary 5.1.33 and (5.4.26). $\diamondsuit$

From claim 5.4.25 we see already that $\mathrm{Ext}^j_A(N, M^\bullet) = 0$ whenever $j < \mathrm{depth}_Z M^{\bullet\sim}$. Suppose that $\mathrm{depth}_Z M^{\bullet\sim} = p < +\infty$; in this case, claim 5.4.25 gives an isomorphism :

$$\mathrm{Ext}^p_A(N, M^\bullet) \simeq \mathrm{Hom}_{\mathscr{O}_X}(N^\sim, R^p\underline{\Gamma}_Z M^{\bullet\sim}) \simeq \mathrm{Hom}_A(N, R^p\Gamma_Z M^{\bullet\sim})$$

where the last isomorphism holds by lemma 5.4.16. To conclude the proof, we have to exhibit a non-zero map from N to $Q := R^p\Gamma_Z M^{\bullet\sim}$. Let $F_0(N) \subset A$ denote the Fitting ideal of N; this is a finitely generated ideal, whose zero locus coincides with the support of N. More precisely, $\mathrm{Ann}_A(N)^r \subset F_0(N) \subset \mathrm{Ann}_A(N)$ for all sufficiently large integers $r > 0$. Let now $x \in Q$ be any non-zero element, and $f_1, \dots, f_k$ a finite system of generators for $F_0(N)$; since x vanishes on $X \setminus Z$, the image of x in Q_{f_i} is zero for every $i \leq k$, *i.e.* there exists $n_i \geq 0$ such that $f_i^{n_i} x = 0$ in Q. It follows easily that $F_0(N)^n \subset \mathrm{Ann}_A(x)$ for a sufficiently large integer n, and therefore x defines a map $\varphi : A/F_0(N)^n \to Q$ by the rule : $a \mapsto ax$. According to [36, Lemma 3.2.21], we can find a finite filtration $0 = J_m \subset \cdots \subset J_1 \subset J_0 := A/F_0(N)^n$ such that each J_i/J_{i+1} is quotient of a direct sum of copies of N. Let $s \leq m$ be the smallest integer such that $J_s \subset \mathrm{Ker}\,\varphi$. By restriction, φ induces a non-zero map $J_{s-1}/J_s \to Q$, whence a non-zero map $N^{(S)} \to Q$, for some set S, and finally a non-zero map $N \to Q$, as required. $\square$

Remark 5.4.27. Notice that the existence of a finitely presented A-module N with $\mathrm{Supp}\, N = Z$ is equivalent to the constructibility of Z.

5.4.28. In the situation of (5.4.23), let $I \subset A$ be a finitely generated ideal such that $V(I) = Z$, and $M^\bullet$ a bounded below complex of A-modules. Denote by $M^{\bullet\sim}$ the complex of quasi-coherent modules on $\mathrm{Spec}\, A$ associated to $M^\bullet$. Then it is customary to set :

$$\mathrm{depth}_I M^\bullet := \mathrm{depth}_Z M^{\bullet\sim}$$

and the depth along Z of $M^\bullet$ is also called the *I-depth* of $M^\bullet$. Moreover, if A is a local ring with maximal ideal $\mathfrak{m}_A$, we shall often use the standard notation :

$$\mathrm{depth}_A M^\bullet := \mathrm{depth}_{\{\mathfrak{m}_A\}} M^{\bullet\sim}$$

and this invariant is often called briefly the *depth* of $M^\bullet$. With this notation, theorem 5.4.20(ii) can be rephrased as the identity :

$$\mathrm{depth}_I M^\bullet = \inf_{\mathfrak{p} \in V(I)} \mathrm{depth}_{A_\mathfrak{p}} M^\bullet_\mathfrak{p}. \tag{5.4.29}$$

5.4.30. Suppose now that $\mathbf{f}$ is a system of generators of the ideal $I \subset A$, and set $Z := V(I)$. In this generality, the Koszul complex $\mathbf{K}^\bullet(\mathbf{f})$ of (4.1.16) is not a resolution of the A-module A/I, and consequently $H^\bullet(\mathbf{f}, M)$ (for an A-module M) does not necessarily agree with $\mathrm{Ext}^\bullet_A(A/I, M)$. Nevertheless, the following holds.

Proposition 5.4.31. *In the situation of* (5.4.30)*, let $M^\bullet$ be any object of* $\mathsf{D}^+(A\text{-}\mathbf{Mod})$*, and set*

$$d := \sup\{i \in \mathbb{Z} \mid H^j(\mathbf{f}, M^\bullet) = 0 \text{ for all } j < i\}.$$

Then we have :

(i) $\mathrm{depth}_I M^\bullet = d$.

(ii) *There are natural isomorphisms :*

(a) $H^d(\mathbf{f}, M^\bullet) \xrightarrow{\sim} \mathrm{Hom}_A(A/I, R^d\Gamma_Z M^{\bullet\sim}) \xrightarrow{\sim} \mathrm{Ext}^d_A(A/I, M^\bullet)$*, when $d < +\infty$.*

(b) $\operatorname*{colim}_{m\in\mathbb{N}} \mathbf{K}^\bullet(\mathbf{f}^m, M^\bullet) \xrightarrow{\sim} R\Gamma_Z M^{\bullet\sim}$ *in* $\mathsf{D}(A\text{-}\mathbf{Mod})$ *(where the transition maps in the colimit are the maps $\boldsymbol{\varphi}^\bullet_{\mathbf{f}^n}$ of* (4.1.23)*).*

Proof. (i): Let $B := \mathbb{Z}[t_1, \dots, t_r] \to A$ be the ring homomorphism defined by the rule : $t_i \mapsto f_i$ for $i = 1, \dots, r$; we denote $\psi : X \to Y := \mathrm{Spec}\, B$ the induced morphism, $J \subset B$ the ideal generated by the system $\mathbf{t} := (t_i \mid i = 1, \dots, r)$, and set $W := V(J) \subset Y$. From lemma 5.4.12(v) we deduce a natural isomorphism in $\mathsf{D}(\mathscr{O}_Y\text{-}\mathbf{Mod})$:

$$\psi_* R\underline{\Gamma}_Z M^{\bullet\sim} \xrightarrow{\sim} R\underline{\Gamma}_W \psi_* M^{\bullet\sim}. \tag{5.4.32}$$

Hence we are reduced to showing :

Claim 5.4.33. $d = \mathrm{depth}_J \psi_* M^\bullet$.

Proof of the claim. Notice that $\mathbf{t}$ is a regular sequence in B, hence $\mathrm{Ext}^\bullet_B(B/J, \psi_* M^\bullet) \simeq H^\bullet(\mathbf{t}, \psi_* M^\bullet)$; by proposition 5.4.24, there follows the identity :

$$\mathrm{depth}_J \psi_* M^\bullet = \sup\{n \in \mathbb{N} \mid H^i(\mathbf{t}, \psi_* M^\bullet) = 0 \text{ for all } i < n\}.$$

Clearly there is a natural isomorphism : $H^i(\mathbf{t}, \psi_* M^\bullet) \xrightarrow{\sim} H^i(\mathbf{f}, M^\bullet)$, whence the assertion. $\Diamond$

(ii.a): From lemma 5.4.12(i.b) and corollary 5.1.33 we derive a natural isomorphism :

$$R\mathrm{Hom}^\bullet_B(B/J, \psi_* M^\bullet) \xrightarrow{\sim} R\mathrm{Hom}^\bullet_{\mathscr{O}_Y}((B/J)^\sim, R\underline{\Gamma}_W \psi_* M^{\bullet\sim}).$$

However, due to (5.4.32) and claim 5.4.33 we may compute :

$$R^d\mathrm{Hom}^\bullet_{\mathscr{O}_Y}((B/J)^\sim, R\underline{\Gamma}_W \psi_* M^{\bullet\sim}) \xrightarrow{\sim} \mathrm{Hom}_B(B/J, R^d\Gamma_W \psi_* M^{\bullet\sim}) \xrightarrow{\sim} \mathrm{Hom}_A(A/I, R^d\Gamma_Z M^{\bullet\sim}).$$

The first claimed isomorphism easily follows. Similarly, one applies lemma 5.4.12(i.b) and corollary 5.1.33 to compute $\mathrm{Ext}^\bullet_A(A/I, M^\bullet)$, and deduces the second isomorphism of (ii.a) using (i).

(ii.b): For every $i \leq r$, let $U_i := \mathrm{Spec}\, A[f_i^{-1}] \subset X := \mathrm{Spec}\, A$, and set $\mathfrak{U} := (U_i \mid i = 1, \dots, r)$; then $\mathfrak{U}$ is an affine covering of $X \setminus Z$. By inspecting the definitions we deduce a natural isomorphism of complexes :

$$\operatorname*{colim}_{m\in\mathbb{N}} \mathbf{K}^\bullet(\mathbf{f}^m, M^\bullet) \xrightarrow{\sim} \mathrm{Tot}\, C^\bullet_{\mathrm{alt}}(\mathfrak{U}, M^{\bullet\sim}) \tag{5.4.34}$$

(notation of (5.4.9)) so the assertion follows from remark 5.4.17. $\Box$

Corollary 5.4.35. *In the situation of* (5.4.28)*, let B be a faithfully flat A-algebra. Then :*

$$\mathrm{depth}_{IB} B \otimes_A M^\bullet = \mathrm{depth}_I M^\bullet.$$

Proof. It is a special case of lemma 5.4.16(iii). Alternatively, one remarks that

$$\mathbf{K}^\bullet(\mathbf{g}, M^\bullet \otimes_A B) \simeq \mathbf{K}^\bullet(\mathbf{g}, M^\bullet) \otimes_A B \qquad \text{for every finite sequence of elements } \mathbf{g} \text{ in } A$$

which allows to apply proposition 5.4.31(ii.b). $\Box$

Theorem 5.4.36. *Let $A \to B$ be a local homomorphism of local rings, and suppose that the maximal ideals $\mathfrak{m}_A$ and $\mathfrak{m}_B$ of A and B are finitely generated. Let also M be an A-module and N a B-module which is flat over A. Then we have :*

$$\mathrm{depth}_B(M \otimes_A N) = \mathrm{depth}_A M + \mathrm{depth}_{B/\mathfrak{m}_A B}(N/\mathfrak{m}_A N).$$

Proof. Let $\mathbf{f} := (f_1, \dots, f_r)$ and $\overline{\mathbf{g}} := (\overline{g}_1, \dots, \overline{g}_s)$ be finite systems of generators for $\mathfrak{m}_A$ and respectively $\overline{\mathfrak{m}}_B$, the maximal ideal of $B/\mathfrak{m}_A B$. We choose an arbitrary lifting of $\overline{\mathbf{g}}$ to a finite system g of elements of $\mathfrak{m}_B$; then $(\mathbf{f}, \mathbf{g})$ is a system of generators for $\mathfrak{m}_B$. We have a natural identification of complexes of B-modules :

$$\mathbf{K}_\bullet(\mathbf{f}, \mathbf{g}) \xrightarrow{\sim} \mathrm{Tot}_\bullet(\mathbf{K}_\bullet(\mathbf{f}) \otimes_A \mathbf{K}_\bullet(\mathbf{g}))$$

whence natural isomorphisms :

$$\mathbf{K}^\bullet((\mathbf{f}, \mathbf{g}), M \otimes_A N) \xrightarrow{\sim} \mathrm{Tot}^\bullet(\mathbf{K}^\bullet(\mathbf{f}, M) \otimes_A \mathbf{K}^\bullet(\mathbf{g}, N)).$$

A standard spectral sequence, associated to the filtration by rows, converges to the cohomology of this total complex, and since N is a flat A-module, $\mathbf{K}^\bullet(\mathbf{g}, N)$ is a complex of flat A-modules, so that the E_1 term of this spectral sequence is found to be :

$$E_1^{pq} \simeq H^q(\mathbf{f}, M) \otimes_A \mathbf{K}^p(\mathbf{g}, N)$$

and the differentials $d_1^{pq} : E_1^{pq} \to E_1^{p+1,q}$ are induced by the differentials of the complex $\mathbf{K}^\bullet(\mathbf{g}, N)$. Set $\kappa_A := A/\mathfrak{m}_A$; notice that $H^q(\mathbf{f}, M)$ is a κ_A-vector space, hence

$$H^q(\mathbf{f}, M) \otimes_A H^\bullet(\mathbf{g}, N) \simeq H^\bullet(H^q(\mathbf{f}, M) \otimes_{\kappa_A} \mathbf{K}^\bullet(\overline{\mathbf{g}}, N/\mathfrak{m}_A N))$$

and consequently :

$$E_2^{pq} \simeq H^q(\mathbf{f}, M) \otimes_{\kappa_A} H^p(\overline{\mathbf{g}}, N/\mathfrak{m}_A N).$$

Especially :

(5.4.37) $\qquad E_2^{pq} = 0$ if either $p < d := \mathrm{depth}_A M$ or $q < d' := \mathrm{depth}_{B/\mathfrak{m}_A B}(N/\mathfrak{m}_A N)$.

Hence $H^i((\mathbf{f}, \mathbf{g}), M \otimes_A N) = 0$ for all $i < d + d'$ and moreover, if d and d' are finite, $H^{d+d'}((\mathbf{f}, \mathbf{g}), M \otimes_A N) \simeq E_2^{dd'} \neq 0$. Now the sought identity follows readily from proposition 5.4.31(i). $\square$

Remark 5.4.38. By inspection of the proof, we see that – under the assumptions of theorem 5.4.36 – the following identity has been proved :

$$\mathrm{Ext}_A^{d+d'}(B/\mathfrak{m}_B, M \otimes_A N) \simeq \mathrm{Ext}_A^d(A/\mathfrak{m}_A, M) \otimes_A \mathrm{Ext}_{B/\mathfrak{m}_A B}^{d'}(B/\mathfrak{m}_B, N/\mathfrak{m}_A N)$$

where d and d' are defined as in (5.4.37).

Corollary 5.4.39. *Let $f : X \to S$ be a locally finitely presented morphism of schemes, $\mathscr{F}$ a f-flat quasi-coherent $\mathscr{O}_X$-module of finite presentation, $\mathscr{G}$ any quasi-coherent $\mathscr{O}_S$-module, $x \in X$ any point, $i : f^{-1}f(x) \to X$ the natural morphism. Then :*

$$\delta'(x, \mathscr{F} \otimes_{\mathscr{O}_X} f^*\mathscr{G}) = \delta'(x, i^*\mathscr{F}) + \delta'(f(x), \mathscr{G}).$$

Proof. Set $s := f(x)$, and denote by

$$j_s : S(s) := \mathrm{Spec}\, \mathscr{O}_{S,s} \to S \qquad \text{and} \qquad f_s : X(s) := X \times_S S(s) \to S(s)$$

the induced morphisms. By inspecting the definitions, one checks easily that

$$\delta'(x, \mathscr{F} \otimes_{\mathscr{O}_X} f^*\mathscr{G}) = \delta'(x, \mathscr{F}_{|X(s)} \otimes_{\mathscr{O}_{X(s)}} f_s^*\mathscr{G}_{|S(s)}) \qquad \text{and} \qquad \delta'(s, \mathscr{G}) = \delta'(s, \mathscr{G}_{|S(s)}).$$

Thus, we may replace S by $S(s)$ and X by $X(s)$, and assume from start that S is a local scheme and s its closed point. Clearly we may also assume that X is affine and finitely presented over S. Then we can write S as the limit of a cofiltered family $(S_\lambda \mid \lambda \in \Lambda)$ of local noetherian schemes, and we may assume that $f : X \to S$ is a limit of a cofiltered family $(f_\lambda : X_\lambda \to S_\lambda \mid \lambda \in \Lambda)$

of morphisms of finite type, such that $X_\mu = X_\lambda \times_{S_\lambda} S_\mu$ whenever $\mu \geq \lambda$. Likewise, we may descend $\mathscr{F}$ to a family $(\mathscr{F}_\lambda \mid \lambda \in \Lambda)$ of finitely presented quasi-coherent $\mathscr{O}_{X_\lambda}$-modules, such that $\mathscr{F}_\mu = \varphi^*_{\mu\lambda}\mathscr{F}_\lambda$ for every $\mu \geq \lambda$ (where $\varphi_{\mu\lambda} : X_\mu \to X_\lambda$ is the natural morphism). Furthermore, up to replacing Λ by some cofinal subset, we may assume that $\mathscr{F}_\lambda$ is f_λ-flat for every $\lambda \in \Lambda$ ([32, Ch.IV, Cor.11.2.6.1(ii)]). For every $\lambda, \mu \in \Lambda$ with $\mu \geq \lambda$, denote by $\psi_\lambda : S \to S_\lambda$ and $\psi_{\mu\lambda} : S_\mu \to S_\lambda$ the natural morphisms, and let $s_\lambda := \psi_\lambda(s)$; given a constructible closed subscheme $Z_\lambda \subset S_\lambda$, set $Z := \psi_\lambda^{-1} Z_\lambda$ (resp. $Z_\mu := \psi_{\mu\lambda}^{-1} Z_\lambda$), which is constructible and closed in S (resp. in S_μ). According to lemma 5.4.16(ii) we have $\mathrm{depth}_Z \mathscr{G} = \mathrm{depth}_{Z_\mu} \psi_{\mu*}\mathscr{G} = \mathrm{depth}_{Z_\lambda} \psi_{\lambda*}\mathscr{G}$, whence the inequality :

$$\delta'(s_\lambda, \psi_{\lambda*}\mathscr{G}) \leq \delta'(s_\mu, \psi_{\mu*}\mathscr{G}) \leq \delta'(s, \mathscr{G}) \qquad \text{whenever } \mu \geq \lambda.$$

On the other hand, every closed constructible subscheme $Z \subset S$ is of the form $\psi_\lambda^{-1} Z_\lambda$ for some $\lambda \in \Lambda$ and Z_λ closed constructible in S_λ ([28, Ch.III, Cor.8.2.11]), so that :

$$\delta'(s, \mathscr{G}) = \sup\{\delta'(s_\lambda, \psi_{\lambda*}\mathscr{G}) \mid \lambda \in \Lambda\}. \tag{5.4.40}$$

Similarly, for every $\lambda \in \Lambda$ let x_λ be the image of x in X_λ and denote by $i_\lambda : f_\lambda^{-1}(s_\lambda) \to X_\lambda$ the natural morphism. Notice that $f_\lambda^{-1}(s_\lambda)$ is a scheme of finite type over $\mathrm{Spec}\, \kappa(s_\lambda)$, especially it is noetherian, hence $\delta'(x_\lambda, i^*_\lambda \mathscr{F}_\lambda) = \delta(x_\lambda, i^*_\lambda \mathscr{F}_\lambda)$, by lemma 5.4.19(ii); by the same token we have as well the identity : $\delta'(x, i^*\mathscr{F}) = \delta(x, i^*\mathscr{F})$. Then, since the natural morphism $f^{-1}(s) \to f_\lambda^{-1}(s_\lambda)$ is faithfully flat, corollary 5.4.35 yields the identity :

$$\delta'(x_\lambda, i^*_\lambda \mathscr{F}_\lambda) = \delta'(x, i^*\mathscr{F}) \qquad \text{for every } \lambda \in \Lambda. \tag{5.4.41}$$

Next, notice that the local scheme $X(x)$ is the limit of the cofiltered system of local schemes $(X_\lambda(x_\lambda) \mid \lambda \in \Lambda)$. Let $Z_\lambda \subset X_\lambda(x_\lambda)$ be a closed constructible subscheme and set $Z := \varphi_\lambda^{-1} Z_\lambda$, where $\varphi_\lambda : X(x) \to X_\lambda(x_\lambda)$ is the natural morphism. Let Y_λ be the fibre product in the cartesian diagram of schemes :

$$\begin{tikzcd} Y_\lambda \arrow[r, "q_\lambda"] \arrow[d, "p_\lambda"'] & X_\lambda(x_\lambda) \arrow[d] \\ S \arrow[r, "\psi_\lambda"] & S_\lambda. \end{tikzcd}$$

The morphism φ_λ factors through a unique morphism of S-schemes $\overline{\varphi}_\lambda : X(x) \to Y_\lambda$, and if $y_\lambda \in Y_\lambda$ is the image of the closed point x of $X(x)$, then $\overline{\varphi}_\lambda$ induces a natural identification $X(x) \stackrel{\sim}{\to} Y_\lambda(y_\lambda)$. To ease notation, let us set :

$$\mathscr{H} := \mathscr{F} \otimes_{\mathscr{O}_X} f^*\mathscr{G} \qquad \text{and} \qquad \mathscr{H}_\lambda := \mathscr{F}_\lambda \otimes_{\mathscr{O}_{X_\lambda}} f^*_\lambda \psi_{\lambda*}\mathscr{G}.$$

In view of theorem 5.4.20(i), there follows a natural isomorphism :

$$R\underline{\Gamma}_Z \mathscr{H}_{|X(x)} \stackrel{\sim}{\to} \overline{\varphi}^*_\lambda R\underline{\Gamma}_{q_\lambda^{-1}(Z_\lambda)}(q^*_\lambda \mathscr{F}_{\lambda|X_\lambda(x_\lambda)} \otimes_{\mathscr{O}_{Y_\lambda}} p^*_\lambda \mathscr{G}). \tag{5.4.42}$$

On the other hand, applying the projection formula (see remark 5.1.21) we get :

$$q_{\lambda*}(q^*_\lambda \mathscr{F}_{\lambda|X_\lambda(x_\lambda)} \otimes_{\mathscr{O}_{Y_\lambda}} p^*_\lambda \mathscr{G}) \simeq \mathscr{H}_{\lambda|X_\lambda(x_\lambda)}. \tag{5.4.43}$$

Combining (5.4.42), (5.4.43) and lemma 5.4.12(v) we deduce :

$$R\underline{\Gamma}_Z \mathscr{H}_{|X(x)} \simeq \varphi^*_\lambda R\underline{\Gamma}_{Z_\lambda} \mathscr{H}_{\lambda|X_\lambda(x_\lambda)}.$$

Notice that the foregoing argument applies also in case S is replaced by some S_μ for some $\mu \geq \lambda$ (and consequently X is replaced by X_μ); we arrive therefore at the inequality :

$$\delta'(x_\lambda, \mathscr{H}_\lambda) \leq \delta'(x_\mu, \mathscr{H}_\mu) \leq \delta'(x, \mathscr{H}) \qquad \text{whenever } \mu \geq \lambda. \tag{5.4.44}$$

Claim 5.4.45. $\delta'(x, \mathscr{H}) = \sup\{\delta'(x_\lambda, \mathscr{H}_\lambda) \mid \lambda \in \Lambda\}$.

Proof of the claim. Due to (5.4.44) we may assume that $d := \delta'(x_\lambda, \mathscr{H}_\lambda) < +\infty$ for every $\lambda \in \Lambda$. The condition means that $R^d\underline{\Gamma}_{Z_\lambda}\mathscr{H}_{\lambda|X_\lambda(x_\lambda)} \neq 0$ for every $\lambda \in \Lambda$ and every closed constructible subscheme $Z_\lambda \subset X_\lambda(x_\lambda)$. We have to show that $R^d\underline{\Gamma}_Z\mathscr{H}_{|X(x)} \neq 0$ for arbitrarily small constructible closed subschemes $Z \subset X(x)$. However, given such Z, there exists $\lambda \in \Lambda$ and Z_λ closed constructible in $X_\lambda(x_\lambda)$ such that $Z = \varphi^{-1}(Z_\lambda)$ ([28, Ch.III, Cor.8.2.11]). Say that:

$$S = \mathrm{Spec}\, A \qquad S_\lambda = \mathrm{Spec}\, A_\lambda \qquad X_\lambda(x_\lambda) = \mathrm{Spec}\, B_\lambda.$$

Hence $Y_\lambda \simeq \mathrm{Spec}\, A \otimes_{A_\lambda} B_\lambda$. Let also F_λ (resp. G) be a B_λ-module (resp. A-module) such that $F_\lambda^\sim \simeq \mathscr{F}_\lambda$ (resp. $G^\sim \simeq \mathscr{G}$). Then $\mathscr{H}_{|Y_\lambda} \simeq (F_\lambda \otimes_{A_\lambda} G)^\sim$. Let $\mathfrak{m}_\lambda \subset B_\lambda$ and $\mathfrak{n}_\lambda \subset A_\lambda$ be the maximal ideals. Up to replacing Z by a smaller subscheme, we may assume that $Z_\lambda = V(\mathfrak{m}_\lambda)$. To ease notation, set :

$$E_1 := \mathrm{Ext}^a_{A_\lambda}(\kappa(s_\lambda), G) \qquad E_2 := \mathrm{Ext}^b_{B_\lambda/\mathfrak{n}_\lambda B}(\kappa(x_\lambda), F_\lambda/\mathfrak{n}_\lambda B_\lambda)$$

with $a = \delta'(s_\lambda, \psi_{\lambda*}\mathscr{G})$ and $b = d - a$. Proposition 5.4.31(i),(ii) and remark 5.4.38 say that

$$0 \neq E_3 := \mathrm{Ext}^d_{B_\lambda}(\kappa(x_\lambda), F_\lambda \otimes_{A_\lambda} G) \simeq E_1 \otimes_{\kappa(s_\lambda)} E_2.$$

To conclude, it suffices to show that $y_\lambda \in \mathrm{Supp}\, E_3$. However :

$$\mathrm{Supp}\, E_3 = (p_\lambda^{-1}\mathrm{Supp}\, E_1) \cap (q_\lambda^{-1}\mathrm{Supp}\, E_2).$$

Now, $p_\lambda(y_\lambda) = s$, and clearly $s \in \mathrm{Supp}\, E_1$, since G is an A-module and s is the closed point of S; likewise, $q_\lambda(y_\lambda) = x_\lambda$ is the closed point of $X_\lambda(x_\lambda)$, therefore it is in the support of E_2. $\Diamond$

We can now conclude the proof : indeed, from theorem 5.4.36 we derive :

$$\delta'(x_\lambda, \mathscr{H}_\lambda) = \delta'(x, i_\lambda^*\mathscr{F}_\lambda) + \delta'(f(x), \psi_{\lambda*}\mathscr{G}) \qquad \text{for every } \lambda \in \Lambda$$

and then the corollary follows from (5.4.40), (5.4.41) and claim 5.4.45. $\square$

5.4.46. Consider now a finitely presented morphism of schemes $f : X \to Y$, and a sheaf of ideals $\mathscr{I} \subset \mathscr{O}_X$ of finite type. Let $x \in X$ be any point, and set $y := f(x)$; notice that $f^{-1}(y)$ is an algebraic $\kappa(y)$-scheme, so that the local ring $\mathscr{O}_{f^{-1}(y),x}$ is noetherian. We let $\mathscr{I}(y) := i_y^{-1}\mathscr{I} \cdot \mathscr{O}_{f^{-1}(y)}$, which is a coherent sheaf of ideals of the fibre $f^{-1}(y)$. The *fibrewise $\mathscr{I}$-depth of X over Y at the point x* is the integer:

$$\mathrm{depth}_{\mathscr{I},f}(x) := \mathrm{depth}_{\mathscr{I}(y)_x}\mathscr{O}_{f^{-1}(y),x}.$$

In case $\mathscr{I} = \mathscr{O}_X$, the fibrewise $\mathscr{I}$-depth shall be called the *fibrewise depth* at the point x, and shall be denoted by $\mathrm{depth}_f(x)$. In view of (5.4.29) we have the identity:

$$\mathrm{depth}_{\mathscr{I},f}(x) = \inf\{\mathrm{depth}_f(z) \mid z \in V(\mathscr{I}(y)) \text{ and } x \in \overline{\{z\}}\}. \tag{5.4.47}$$

5.4.48. The fibrewise $\mathscr{I}$-depth can be computed locally as follows. For a given $x \in X$, set $y := f(x)$ and let $U \subset X$ be any affine open neighborhood of x such that $\mathscr{I}_{|U}$ is generated by a finite family $\mathbf{f} := (f_i)_{1\leq i\leq r}$, where $f_i \in \mathscr{I}(U)$ for every $i \leq r$. Then proposition 5.4.31(i) implies that :

$$\mathrm{depth}_{\mathscr{I},f}(x) = \sup\{n \in \mathbb{N} \mid H^i(\mathbf{K}^\bullet(\mathbf{f}, \mathscr{O}_X(U))_x \otimes_{\mathscr{O}_{Y,y}} \kappa(y)) = 0 \text{ for all } i < n\}. \tag{5.4.49}$$

Proposition 5.4.50. *In the situation of* (5.4.46)*, for every integer $d \in \mathbb{N}$ we define the subset:*

$$L_{\mathscr{I}}(d) := \{x \in X \mid \mathrm{depth}_{\mathscr{I},f}(x) \geq d\}.$$

Then the following holds:

(i) *$L_{\mathscr{I}}(d)$ is a constructible subset of X.*

(ii) *Let $Y' \to Y$ be any morphism of schemes, set $X' := X \times_Y Y'$, and let $g : X' \to X$, $f' : X' \to Y'$ be the induced morphisms. Then:*

$$L_{g^*\mathscr{I}}(d) = g^{-1}L_{\mathscr{I}}(d) \qquad \textit{for every } d \in \mathbb{N}.$$

Proof. (ii): The identity can be checked on the fibres, hence we may assume that $Y = \mathrm{Spec}\, k$ and $Y' = \mathrm{Spec}\, k'$ for some fields $k \subset k'$. Let $x' \in X'$ be any point and $x \in X$ its image; since the map $\mathscr{O}_{X,x} \to \mathscr{O}_{X',x'}$ is faithfully flat, corollary 5.4.35 implies:

$$\mathrm{depth}_{g^*\mathscr{I},f'}(x') = \mathrm{depth}_{\mathscr{I},f}(x)$$

whence (ii).

(i): (We refer here to the notion of constructibility defined in [25, Ch.0, Déf.2.3.10], which is local on X; this corresponds to the local constructiblity as defined in [28, Ch.0, Déf.9.1.11].) We may assume that $Y = \mathrm{Spec}\, A$, $X = \mathrm{Spec}\, B$, $\mathscr{I} = I^\sim$, where A is a ring, B is a finitely presented A-algebra, and $I \subset B$ is a finitely generated ideal, for which we choose a finite system of generators $\mathbf{f} := (f_i)_{1\leq i\leq r}$. We may also assume that A is reduced. Suppose first that A is noetherian; then, for every $i \in \mathbb{N}$ and $y \in Y$ let us set:

$$N_{\mathscr{I}}(i) = \bigcup_{y\in Y} \mathrm{Supp}\, H^i(\mathbf{K}^\bullet(\mathbf{f}, B) \otimes_A \kappa(y)).$$

Taking into account (5.4.29) and (5.4.49), we deduce:

$$L_{\mathscr{I}}(d) = \bigcap_{i=0}^{d-1}(X\setminus N_{\mathscr{I}}(i)).$$

It suffices therefore to show that each subset $N_{\mathscr{I}}(i)$ is constructible. For every $j \in \mathbb{N}$ we set:

$$\mathbf{Z}^j := \mathrm{Ker}(d^j : \mathbf{K}^j(\mathbf{f}, B) \to \mathbf{K}^{j+1}(\mathbf{f}, B)) \qquad \mathbf{B}^j := \mathrm{Im}(d^{j-1} : \mathbf{K}^{j-1}(\mathbf{f}, B) \to \mathbf{K}^j(\mathbf{f}, B)).$$

Using [32, Ch.IV, Cor. 8.9.5], one deduces easily that there exists an affine open subscheme $U \subset Y$, say $U = \mathrm{Spec}\, A'$ for some flat A-algebra A', such that $A' \otimes_A \mathbf{Z}^\bullet$, $A' \otimes_A \mathbf{B}^\bullet$ and $A' \otimes_A H^\bullet(\mathbf{K}^\bullet(\mathbf{f}, B))$ are flat A'-modules. By noetherian induction, we can then replace Y by U and X by $X \times_Y U$, and assume from start that $\mathbf{Z}^\bullet$, $\mathbf{B}^\bullet$ and $H^\bullet(\mathbf{K}^\bullet(\mathbf{f}, B))$ are flat A-modules. In such case, taking homology of the complex $\mathbf{K}^\bullet(\mathbf{f}, B)$ commutes with any base change; therefore:

$$N_{\mathscr{I}}(i) = \bigcup_{y\in Y} \mathrm{Supp}\, H^i(\mathbf{f}, B) \otimes_A \kappa(y) = \mathrm{Supp}\, H^i(\mathbf{f}, B)$$

whence the claim, since the support of a B-module of finite type is closed in X.

Finally, for a general ring A, we can find a noetherian subalgebra $A' \subset A$, an A'-algebra B' of finite type and a finitely generated ideal $I' \subset B'$ such that $B = A \otimes_{A'} B$ and $I = I'B$. Let $\mathscr{I}'$ be the sheaf of ideals on $X' := \mathrm{Spec}\, B'$ determined by I'; by the foregoing, $L_{\mathscr{I}'}(d)$ is a constructible subset of X'. Thus, the assertion follows from (ii), and the fact that a morphism of schemes is continuous for the constructible topology ([30, Ch.IV, Prop.1.8.2]). □

5.5. **Depth and associated primes.**

Definition 5.5.1. Let X be a scheme, $\mathscr{F}$ a quasi-coherent $\mathscr{O}_X$-module.

(i) Let $x \in X$ be any point; we say that x is *associated to* $\mathscr{F}$ if there exists $f \in \mathscr{F}_x$ such that the radical of the annihilator of f in $\mathscr{O}_{X,x}$ is the maximal ideal of $\mathscr{O}_{X,x}$. If x is a point associated to $\mathscr{F}$ and x is not a maximal point of $\mathrm{Supp}\,\mathscr{F}$, we say that x is an *imbedded point* for $\mathscr{F}$. We shall denote :

$$\mathrm{Ass}\,\mathscr{F} := \{x \in X \mid x \text{ is associated to } \mathscr{F}\}.$$

(ii) We say that the $\mathscr{O}_X$-module $\mathscr{F}$ *satisfies condition* S_1 if every associated point of $\mathscr{F}$ is a maximal point of X.

(iii) Likewise, if X is affine, say $X = \operatorname{Spec} A$, and M is any A-module, we denote by $\operatorname{Ass}_A M \subset X$ (or just by $\operatorname{Ass} M$, if the notation is not ambiguous) the set of prime ideals associated to the $\mathscr{O}_X$-module $M^\sim$ arising from M. We say that M *satisfies condition* S_1 if the same holds for the $\mathscr{O}_X$-module $M^\sim$.

(iv) Let $x \in X$ be a point, $\mathscr{G}$ a quasi-coherent $\mathscr{O}_X$-submodule of $\mathscr{F}$. We say that $\mathscr{G}$ is a x*-primary submodule* of $\mathscr{F}$ if $\operatorname{Ass} \mathscr{F}/\mathscr{G} = \{x\}$. We say that $\mathscr{G}$ is a *primary submodule* of $\mathscr{F}$ if there exists a point $x \in X$ such that $\mathscr{G}$ is x-primary.

(v) We say that a submodule $\mathscr{G}$ of the $\mathscr{O}_X$-module $\mathscr{F}$ admits a *primary decomposition* if there exist primary submodules $\mathscr{G}_1, \dots, \mathscr{G}_n \subset \mathscr{F}$ such that $\mathscr{G} = \mathscr{G}_1 \cap \cdots \cap \mathscr{G}_n$.

Remark 5.5.2. (i) Our definition of associated point is borrowed from [45, Partie I, Déf.3.2.1]. It agrees with that of [44, Exp.III, Déf.1.1] in case X is a locally noetherian scheme (see lemma 5.5.18(ii)), but in general the two notions are distinct.

(ii) For a noetherian ring A and a finitely generated A-module M, our condition S_1 is the same as that of [48]. It also agrees with that of [31, Ch.IV, Déf.5.7.2], in case $\operatorname{Supp} M^\sim = \operatorname{Spec} A$.

Lemma 5.5.3. *Let X be a scheme, $\mathscr{F}$ a quasi-coherent $\mathscr{O}_X$-module. Then :*

(i) $\operatorname{Ass} \mathscr{F} \subset \operatorname{Supp} \mathscr{F}$.

(ii) $\operatorname{Ass} \mathscr{F}_{|U} = U \cap \operatorname{Ass} \mathscr{F}$ *for every open subset* $U \subset X$.

(iii) $\mathscr{F} = 0$ *if and only if* $\operatorname{Ass} \mathscr{F} = \varnothing$.

Proof. (i) and (ii) are obvious. To show (iii), we may assume that $\mathscr{F} \neq 0$, and we have to prove that $\operatorname{Ass} \mathscr{F}$ is not empty. Let $x \in X$ be any point such that $\mathscr{F}_x \neq 0$, and pick a non-zero section $f \in \mathscr{F}_x$; set $I := \operatorname{Ann}_{\mathscr{O}_{X,x}}(f)$, let $\overline{\mathfrak{q}}$ be a minimal prime ideal of the quotient ring $\mathscr{O}_{X,x}/I$, and $\mathfrak{q} \subset \mathscr{O}_{X,x}$ the preimage of $\overline{\mathfrak{q}}$. Then $\mathfrak{q}$ corresponds to a generization y of the point x in X; let $f_y \in \mathscr{F}_y$ be the image of f. Then $\operatorname{Ann}_{\mathscr{O}_{X,y}}(f_y) = I \cdot \mathscr{O}_{Y,y}$, whose radical is the maximal ideal $\mathfrak{q} \cdot \mathscr{O}_{X,y}$, *i.e.* $y \in \operatorname{Ass} \mathscr{F}$. ∎

Proposition 5.5.4. *Let X be a scheme, $\mathscr{F}$ a quasi-coherent $\mathscr{O}_X$-module. We have :*

(i) $\operatorname{Ass} \mathscr{F} = \{x \in X \mid \delta(x, \mathscr{F}) = 0\}$.

(ii) *If* $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}''$ *is an exact sequence of quasi-coherent $\mathscr{O}_X$-modules, then :*

$$\operatorname{Ass} \mathscr{F}' \subset \operatorname{Ass} \mathscr{F} \subset \operatorname{Ass} \mathscr{F}' \cup \operatorname{Ass} \mathscr{F}''.$$

(iii) *If $x \in X$ is any point, and $\mathscr{G}_1, \dots, \mathscr{G}_n$ (for some $n > 0$) are x-primary submodules of $\mathscr{F}$, then $\mathscr{G}_1 \cap \cdots \cap \mathscr{G}_n$ is also x-primary.*

(iv) *Let $f : X \to Y$ be a finite morphism of schemes. Then*

(a) *The natural map :*

$$\bigoplus_{x \in f^{-1}(y)} \Gamma_{\{x\}} \mathscr{F}_{|X(x)} \to \Gamma_{\{y\}} f_* \mathscr{F}_{|Y(y)}$$

is a bijection for all $y \in Y$.

(b) $\operatorname{Ass} f_* \mathscr{F} = f(\operatorname{Ass} \mathscr{F})$.

(v) *Suppose that $\mathscr{F}$ is the union of a filtered family $(\mathscr{F}_\lambda \mid \lambda \in \Lambda)$ of quasi-coherent $\mathscr{O}_X$-submodules. Then :*

$$\bigcup_{\lambda \in \Lambda} \operatorname{Ass} \mathscr{F}_\lambda = \operatorname{Ass} \mathscr{F}.$$

(vi) *Let $f : Y \to X$ be a flat morphism of schemes, and suppose that the topological space $|X|$ underlying X is locally noetherian. Then* $\operatorname{Ass} f^* \mathscr{F} \subset f^{-1} \operatorname{Ass} \mathscr{F}$.

Proof. (i) and (v) follow directly from the definitions.

(ii): Consider, for every point $x \in X$ the induced exact sequence of $\mathscr{O}_{X(x)}$-modules :

$$0 \to \underline{\Gamma}_{\{x\}} \mathscr{F}'_x \to \underline{\Gamma}_{\{x\}} \mathscr{F}_x \to \underline{\Gamma}_{\{x\}} \mathscr{F}''_x.$$

Then :

$$\mathrm{Supp}\,\underline{\Gamma}_{\{x\}}\mathscr{F}'_x \subset \mathrm{Supp}\,\underline{\Gamma}_{\{x\}}\mathscr{F}_x \subset \mathrm{Supp}\,\underline{\Gamma}_{\{x\}}\mathscr{F}'_x \cup \mathrm{Supp}\,\underline{\Gamma}_{\{x\}}\mathscr{F}''_x$$

which, in light of (i), is equivalent to the contention.

(iii): One applies (ii) to the natural injection : $\mathscr{F}/(\mathscr{G}_1 \cap \cdots \cap \mathscr{G}_n) \to \oplus_{i=1}^n \mathscr{F}/\mathscr{G}_i$.

(iv): We may assume that Y is a local scheme with closed point $y \in Y$. Let $x_1, \dots, x_n \in X$ be the finitely many points lying over y; for every $i, j \leq n$, we let

$$\pi_j : X(x_j) \to Y \qquad \text{and} \qquad \pi_{ij} : X(x_i) \times_X X(x_j) \to Y$$

be the natural morphisms. To ease notation, denote also by $\mathscr{F}_i$ (resp. $\mathscr{F}_{ij}$) the pull back of of $\mathscr{F}$ to $X(x_j)$ (resp. to $X(x_i) \times_X X(x_j)$). The induced morphism $U := X(x_1) \amalg \cdots \amalg X(x_n) \to X$ is faithfully flat, so descent theory yields an exact sequence of $\mathscr{O}_Y$-modules :

$$0 \longrightarrow f_*\mathscr{F} \longrightarrow \prod_{j=1}^n \pi_{j*}\mathscr{F}_j \underset{p_2^*}{\overset{p_1^*}{\rightrightarrows}} \prod_{i,j=1}^n \pi_{ij*}\mathscr{F}_{ij}. \tag{5.5.5}$$

where $p_1, p_2 : U \times_X U \to U$ are the natural morphisms.

Claim 5.5.6. The induced maps :

$$\Gamma_{\{y\}}p_1^*, \Gamma_{\{y\}}p_2^* : \prod_{j=1}^n \Gamma_{\{y\}}\pi_{j*}\mathscr{F}_j \to \prod_{i,j=1}^n \Gamma_{\{y\}}\pi_{ij*}\mathscr{F}_{ij}$$

coincide.

Proof of the claim. Indeed, it suffices to verify that they coincide after projecting onto each factor $\mathscr{G}_{ij} := \Gamma_{\{y\}}\pi_{ij*}\mathscr{F}_{ij}$. But this is clear from definitions if $i = j$. On the other hand, if $i \neq j$, the image of π_{ij} in Y does not contain y, so the corresponding factor $\mathscr{G}_{ij}$ vanishes. ◊

From (5.5.5) and claim 5.5.6 we deduce that the natural map

$$\Gamma_{\{y\}}f_*\mathscr{F} \to \prod_{j=1}^n \Gamma_{\{y\}}\pi_{j*}\mathscr{F}_j$$

is a bijection. Clearly $\Gamma_{\{y\}}\pi_{j*}\mathscr{F}_j = \Gamma_{\{x\}}\mathscr{F}_j$, so both assertions follow easily.

(vi): Let $x \in X$ be any point; since $|X|$ is locally noetherian, the subset $\{x\}$ is closed and constructible in $X(x)$, and then $f^{-1}(x)$ is closed and constructible in $Y \times_X X(x)$. Hence $\delta(y, f^*\mathscr{F}) \geq \delta(x, \mathscr{F})$ for every $y \in f^{-1}(x)$ (lemma 5.4.16(iii) and theorem 5.4.20(ii)). Then the assertion follows from (i). □

Corollary 5.5.7. *In the situation of corollary* 5.4.39*, suppose furthermore that $|S|$ is a locally noetherian topological space. Then we have :*

$$\mathrm{Ass}\,\mathscr{F} \otimes_{\mathscr{O}_X} f^*\mathscr{G} = \bigcup_{s \in \mathrm{Ass}\,\mathscr{G}} \mathrm{Ass}\,\mathscr{F} \otimes_{\mathscr{O}_{S,s}} \kappa(s).$$

Proof. Under the stated assumptions, it is easily seen that, for every $x \in X$ (resp. $s \in S$), the subset $\{x\}$ (resp. $\{s\}$) is constructible in X (resp. in S). Then the assertion follows immediately from proposition 5.5.4(i) and theorem 5.4.20(ii). □

Remark 5.5.8. Actually, it can be shown that the extra assumption on $|S|$ is superflous : see [45, Part I, Prop.3.4.3].

Corollary 5.5.9. *Let X be a quasi-compact and quasi-separated scheme, $\mathscr{F}$ a quasi-coherent $\mathscr{O}_X$-module and $\mathscr{G} \subset \mathscr{F}$ a quasi-coherent submodule. Then :*

(i) *For every quasi-compact open subset $U \subset X$ and every quasi-coherent primary $\mathscr{O}_U$-submodule $\mathscr{N} \subset \mathscr{F}_{|U}$, with $\mathscr{G}_{|U} \subset \mathscr{N}$, there exists a quasi-coherent primary $\mathscr{O}_X$-submodule $\mathscr{M} \subset \mathscr{F}$ such that $\mathscr{M}_{|U} = \mathscr{N}$ and $\mathscr{G} \subset \mathscr{N}$.*

(ii) *$\mathscr{G}$ admits a primary decomposition if and only if there exists a finite open covering $X = U_1 \cup \cdots \cup U_n$ consisting of quasi-compact open subsets, such that the submodules $\mathscr{G}_{|U_i} \subset \mathscr{F}_{|U_i}$ admit primary decompositions for every $i = 1, \dots, n$.*

Proof. Say that $\mathscr{N}$ is x-primary for some point $x \in U$, and set $Z := X \setminus U$. According to [26, Ch.I, Prop.9.4.2], we can extend $\mathscr{N}$ to a quasi-coherent $\mathscr{O}_X$-submodule $\mathscr{M}_1 \subset \mathscr{F}$; up to replacing $\mathscr{M}_1$ by $\mathscr{M}_1 + \mathscr{G}$, we may assume that $\mathscr{G} \subset \mathscr{M}_1$. Since $(\mathscr{F}/\mathscr{M}_1)_{|U} = \mathscr{F}_{|U}/\mathscr{N}$, it follows from lemma 5.5.3(ii) that $\mathrm{Ass}\,\mathscr{F}/\mathscr{M}_1 \subset \{x\} \cup Z$. Let $\overline{\mathscr{M}} := \underline{\Gamma}_Z(\mathscr{F}/\mathscr{M}_1)$, and denote by $\mathscr{M}$ the preimage of $\overline{\mathscr{M}}$ in $\mathscr{F}$. There follows a short exact sequence :

$$0 \to \overline{\mathscr{M}} \to \mathscr{F}/\mathscr{M}_1 \to \mathscr{F}/\mathscr{M} \to 0.$$

Clearly $R^1\underline{\Gamma}_Z\overline{\mathscr{M}} = 0$, whence a short exact sequence

$$0 \to \underline{\Gamma}_Z\overline{\mathscr{M}} \to \underline{\Gamma}_Z(\mathscr{F}/\mathscr{M}_1) \to \underline{\Gamma}_Z(\mathscr{F}/\mathscr{M}) \to 0.$$

We deduce that $\mathrm{depth}_Z(\mathscr{F}/\mathscr{M}) > 0$; then theorem 5.4.20(ii) and proposition 5.5.4(i) show that $Z \cap \mathrm{Ass}\,(\mathscr{F}/\mathscr{M}) = \varnothing$, so that $\mathscr{M}$ is x-primary, as required.

(ii): We may assume that a covering $X = U_1 \cup \cdots \cup U_n$ is given with the stated property. For every $i \leq n$, let $\mathscr{G}_{|U_i} = \mathscr{N}_{i1} \cap \cdots \cap \mathscr{N}_{ik_i}$ be a primary decomposition; by (i) we may extend every $\mathscr{N}_{ij}$ to a primary submodule $\mathscr{M}_{ij} \subset \mathscr{F}$ containing $\mathscr{G}$. Then $\mathscr{G} = \bigcap_{i=1}^n \bigcap_{j=1}^{k_i} \mathscr{M}_{ij}$ is a primary decomposition of $\mathscr{G}$. □

Lemma 5.5.10. *Let X be a quasi-separated and quasi-compact scheme, $\mathscr{F}$ a quasi-coherent $\mathscr{O}_X$-module of finite type. Then the submodule $0 \subset \mathscr{F}$ admits a primary decomposition if and only if the following conditions hold:*

(a) $\mathrm{Ass}\,\mathscr{F}$ *is a finite set.*
(b) *For every $x \in \mathrm{Ass}\,\mathscr{F}$ there is a x-primary ideal $I \subset \mathscr{O}_{X,x}$ such that the natural map*

$$\Gamma_{\{x\}}\mathscr{F}_x \to \mathscr{F}_x/I\mathscr{F}_x$$

is injective.

Proof. In view of corollary 5.5.9(i) we are reduced to the case where X is an affine scheme, say $X = \mathrm{Spec}\,A$, and $\mathscr{F} = M^\sim$ for an A-module M of finite type. Suppose first that 0 admits a primary decomposition :

$$0 = \bigcap_{i=1}^k N_i. \tag{5.5.11}$$

Then the natural map $M \to \oplus_{i=1}^k M/N_i$ is injective, hence

$$\mathrm{Ass}\,M \subset \mathrm{Ass}\bigoplus_{i=1}^k M/N_i \subset \bigcup_{i=1}^k \mathrm{Ass}\,M/N_i \tag{5.5.12}$$

by proposition 5.5.4(ii), and this shows that (a) holds. Next, if N_i and N_j are $\mathfrak{p}$-primary for the same prime ideal $\mathfrak{p} \subset A$, we may replace both of them by their intersection (proposition 5.5.4(iii)). Proceeding in this way, we achieve that the N_i appearing in (5.5.11) are primary submodules for pairwise distinct prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_k \subset A$. By (5.5.12) we have $\mathrm{Ass}\,M \subset \{\mathfrak{p}_1, \dots, \mathfrak{p}_k\}$. Suppose that $\mathfrak{p}_1 \notin \mathrm{Ass}\,M$, and set $Q := \mathrm{Ker}\,(M \to \oplus_{i=2}^k M/N_i)$. For every $j > 1$ we have $\Gamma_{\{\mathfrak{p}_j\}}(M/N_1)^\sim = 0$, therefore :

$$\Gamma_{\{\mathfrak{p}_j\}}Q^\sim_{\mathfrak{p}_j} = \mathrm{Ker}\,(\Gamma_{\{\mathfrak{p}_j\}}M^\sim_{\mathfrak{p}_j} \to \oplus_{i=1}^k (M/N_i)^\sim_{\mathfrak{p}_j}) = 0.$$

Hence $\mathrm{Ass}\, Q = \varnothing$, by proposition 5.5.4(i), and then $Q = 0$ by lemma 5.5.3(iii). In other words, we can omit N_1 from (5.5.11), and still obtain a primary decomposition of 0; iterating this argument, we may achieve that $\mathrm{Ass}\, M = \{\mathfrak{p}_1, \dots, \mathfrak{p}_k\}$. After these reductions, we see that

$$\Gamma_{\{\mathfrak{p}_j\}}(M/N_i)^\sim_{\mathfrak{p}_j} = 0 \qquad \text{whenever } i \neq j$$

and consequently :

$$\mathrm{Ker}\,(\varphi_j : \Gamma_{\{\mathfrak{p}_j\}} M^\sim_{\mathfrak{p}_j} \to (M/N_j)_{\mathfrak{p}_j}) = 0 \qquad \text{for every } j = 1, \dots, k.$$

Now, for given $j \leq k$, let $\overline{f}_1, \dots, \overline{f}_n$ be a system of non-zero generators for M/N_j; by assumption $I_i := \mathrm{Ann}_A(\overline{f}_i)$ is a $\mathfrak{p}_j$-primary ideal for every $i \leq n$. Hence $\mathfrak{q}_j := \mathrm{Ann}_A(M/N_j) = \bigcap_{i=1}^n I_i$ is $\mathfrak{p}_j$-primary as well; since φ_j factors through $(M/\mathfrak{q}_j M)_{\mathfrak{p}_j}$, we see that (b) holds.

Conversely, suppose that (a) and (b) hold. Say that $\mathrm{Ass}\, M = \{\mathfrak{p}_1, \dots, \mathfrak{p}_k\}$; for every $j \leq k$ we choose a $\mathfrak{p}_j$-primary ideal $\mathfrak{q}_j$ such that the map $\Gamma_{\{\mathfrak{p}_j\}} M^\sim_{\mathfrak{p}_j} \to M_{\mathfrak{p}_j}/\mathfrak{q}_j M_{\mathfrak{p}_j}$ is injective. Clearly $N_j := \mathrm{Ker}\,(M \to M_{\mathfrak{p}_j}/\mathfrak{q}_j M_{\mathfrak{p}_j})$ is a $\mathfrak{p}_j$-primary submodule of M; moreover, the induced map $\varphi : M \to \oplus_{j=1}^k M/N_j$ is injective, since $\mathrm{Ass}\,\mathrm{Ker}\,\varphi = \varnothing$ (proposition 5.5.4(ii) and lemma 5.5.3(iii)). In other words, $0 = \bigcap_{j=1}^k N_j$ is a primary decomposition of 0. ☐

Proposition 5.5.13. *Let X be a quasi-compact and quasi-separated scheme, $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ a short exact sequence of quasi-coherent $\mathscr{O}_X$-modules of finite type, and suppose that*

(a) $\mathrm{Ass}\,\mathscr{F}' \cap \mathrm{Supp}\,\mathscr{F}'' = \varnothing$.
(b) *The submodules $0 \subset \mathscr{F}'$ and $0 \subset \mathscr{F}''$ admit primary decompositions.*

Then the submodule $0 \subset \mathscr{F}$ admits a primary decomposition.

Proof. The assumptions imply that $\mathrm{Ass}\,\mathscr{F}'$ and $\mathrm{Ass}\,\mathscr{F}''$ are finite sets, (lemma 5.5.10), hence the same holds for $\mathrm{Ass}\,\mathscr{F}$ (proposition 5.5.4(ii)). Given any point $x \in X$ and any x-primary ideal $I \subset \mathscr{O}_{X,x}$, we may consider the commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \Gamma_{\{x\}}\mathscr{F}'_x \arrow[r] \arrow[d, "\alpha"'] & \Gamma_{\{x\}}\mathscr{F}_x \arrow[r] \arrow[d, "\beta"'] & \Gamma_{\{x\}}\mathscr{F}''_x \arrow[d, "\gamma"'] \\
\mathrm{Tor}_1^{\mathscr{O}_{X,x}}(I, \mathscr{F}''_x) \arrow[r] & \mathscr{F}'_x/I\mathscr{F}'_x \arrow[r] & \mathscr{F}_x/I\mathscr{F}_x \arrow[r] & \mathscr{F}''_x/I\mathscr{F}''_x
\end{tikzcd}$$

Now, if $x \in \mathrm{Ass}\,\mathscr{F}'$, assumption (a) implies that $\Gamma_{\{x\}}\mathscr{F}''_x = 0 = \mathrm{Tor}_1^{\mathscr{O}_{X,x}}(I_x, \mathscr{F}''_x)$, and by (b) and lemma 5.5.10 we can choose I such that α is injective; a little diagram chase then shows that β is injective as well. Similarly, if $x \in \mathrm{Ass}\,\mathscr{F}''$, we have $\Gamma_{\{x\}}\mathscr{F}'_x = 0$ and we may choose I such that γ is injective, which implies again that β is injective. To conclude the proof it suffices to apply again lemma 5.5.10. ☐

Proposition 5.5.14. *Let Y be a quasi-compact and quasi-separated scheme, $f : X \to Y$ a finite morphism and $\mathscr{F}$ a quasi-coherent $\mathscr{O}_X$-module of finite type. Then the $\mathscr{O}_X$-submodule $0 \subset \mathscr{F}$ admits a primary decomposition if and only if the same holds for the $\mathscr{O}_Y$-submodule $0 \subset f_*\mathscr{F}$.*

Proof. Under the stated assumptions, we can apply the criterion of lemma 5.5.10. To start out, it is clear from proposition 5.5.4(iv.b) that $\mathrm{Ass}\,\mathscr{F}$ is finite if and only if $\mathrm{Ass}\, f_*\mathscr{F}$ is finite. Next, suppose that $0 \subset \mathscr{F}$ admits a primary decomposition, let $y \in Y$ be any point and set $f^{-1}(y) := \{x_1, \dots, x_n\}$; we may also assume that Y and X are affine, and then for every $j \leq n$ we can find an x_j-primary ideal $I_j \subset \mathscr{O}_X$ such that the map $\Gamma_{\{x_j\}}\mathscr{F}_{x_j} \to \mathscr{F}_{x_j}/I_j\mathscr{F}_{x_j}$ is injective. Let I be the kernel of the natural map $\mathscr{O}_Y \to f_*(\mathscr{O}_X/(I_1 \cap \dots \cap I_n)$. Then I is a quasi-coherent

y-primary ideal of $\mathscr{O}_Y$ and we deduce a commutative diagram :

$$\begin{tikzcd} \bigoplus_{j=1}^n \Gamma_{\{x_j\}}(\mathscr{F}_x) \arrow[r] \arrow[d, "\alpha"'] & \Gamma_{\{y\}}(f_*\mathscr{F}_y) \arrow[d, "\beta"] \\ \bigoplus_{j=1}^n \mathscr{F}_{x_j}/I\mathscr{F}_{x_j} \arrow[r] & f_*\mathscr{F}_y/If_*\mathscr{F}_y \end{tikzcd} \tag{5.5.15}$$

whose horizontal arrows are isomorphisms, in view of proposition 5.5.4(iv.a), and where α is injective by construction. It follows that β is injective, so condition (b) of lemma 5.5.10 holds for the stalk $f_*\mathscr{F}_y$, and since $y \in Y$ is arbitrary, we see that $0 \subset f_*\mathscr{F}$ admits a primary decomposition. Conversely, suppose that $0 \subset f_*\mathscr{F}$ admits a primary decomposition; then for every $y \in Y$ we can find a quasi-coherent y-primary ideal $I \subset \mathscr{O}_Y$ such that the map β of (5.5.15) is injective; hence α is injective as well, and again we deduce easily that $0 \subset \mathscr{F}$ admits a primary decomposition. $\square$

Proposition 5.5.16. *Let X and $\mathscr{F}$ be as in lemma* 5.5.10, *$i : Z \to X$ a closed constructible immersion, and $U := X\setminus Z$. Suppose that :*

(a) *The $\mathscr{O}_U$-submodule $0 \subset \mathscr{F}_{|U}$ and the $\mathscr{O}_Z$-submodule $0 \subset i^*\mathscr{F}$ admit primary decompositions.*

(b) *The natural map $\underline{\Gamma}_Z\mathscr{F} \to i_*i^*\mathscr{F}$ is injective.*

Then the $\mathscr{O}_X$-submodule $0 \subset \mathscr{F}$ admits a primary decomposition.

Proof. We shall verify that conditions (a) and (b) of lemma 5.5.10 hold for $\mathscr{F}$. To check condition (a), it suffices to remark that $U \cap \mathrm{Ass}\,\mathscr{F} = \mathrm{Ass}\,\mathscr{F}_{|U}$ (which is obvious) and that $Z \cap \mathrm{Ass}\,\mathscr{F} \subset \mathrm{Ass}\,i^*\mathscr{F}$, which follows easily from our current assumption (b).

Next we check that condition (b) of *loc.cit.* holds. This is no problem for the points $x \in U \cap \mathrm{Ass}\,\mathscr{F}$, so suppose that $x \in Z \cap \mathrm{Ass}\,\mathscr{F}$. Moreover, we may also assume that X is affine, say $X = \mathrm{Spec}\,A$, so that $Z = V(J)$ for some ideal $J \subset A$. Due to proposition 5.5.14 we know that $0 \subset \mathscr{G} := i_*i^*\mathscr{F}$ admits a primary decomposition, hence we can find an x-primary ideal $I \subset A$ such that the natural map $\Gamma_{\{x\}}\mathscr{G} \to \mathscr{G}_x/I\mathscr{G}_x \simeq \mathscr{F}_x/(I+J)\mathscr{F}_x$ is injective. Clearly $I+J$ is again a x-primary ideal; since Z is closed and constructible, theorem 5.4.20(i) and our assumption (b) imply that the natural map $\Gamma_{\{x\}}\mathscr{F} \to \Gamma_{\{x\}}\mathscr{G}$ is injective, hence the same holds for the map $\Gamma_{\{x\}}\mathscr{F} \to \mathscr{F}_x/(I+J)\mathscr{F}_x$, as required. $\square$

5.5.17. Let now A be a ring, and set $X := \mathrm{Spec}\,A$. An associated (resp. imbedded) point for the quasi-coherent $\mathscr{O}_X$-module $\mathscr{O}_X$ is also called an *associated* (resp. *imbedded*) *prime ideal* of A. We notice :

Lemma 5.5.18. *Let M be any A-module. The following holds :*

(i) $\mathrm{Ass}\,M$ *is the set of all $\mathfrak{p} \in \mathrm{Spec}\,A$ with the following property. There exists $m \in M$, such that $\mathfrak{p}$ is a maximal point of the closed subset* $\mathrm{Supp}(m)$ *(*i.e. *$\mathfrak{p}$ is the preimage of a minimal prime ideal of the ring $A/\mathrm{Ann}_A(m)$).*

(ii) *If $\mathfrak{p} \in \mathrm{Ass}\,M$, and $\mathfrak{p}$ is finitely generated, there exists $m \in M$ with $\mathfrak{p} = \mathrm{Ann}_A(m)$.*

Proof. (i): Indeed, suppose that $m \in M\setminus\{0\}$ is any element, and $\mathfrak{p}$ is a maximal point of $V(\mathrm{Ann}_A(m))$; denote by $m_\mathfrak{p} \in M_\mathfrak{p} := M \otimes_A A_\mathfrak{p}$ the image of m. Then :

$$\mathrm{Ann}_{A_\mathfrak{p}}(m_\mathfrak{p}) = \mathrm{Ann}_A(m) \otimes_A A_\mathfrak{p}. \tag{5.5.19}$$

Hence $A_\mathfrak{p}/\mathrm{Ann}_{A_\mathfrak{p}}(m_\mathfrak{p}) = (A/\mathrm{Ann}_A(m)) \otimes_A A_\mathfrak{p}$, and the latter is by assumption a local ring of Krull dimension zero; it follows easily that the radical of $\mathrm{Ann}_{A_\mathfrak{p}}(m_\mathfrak{p})$ is $\mathfrak{p}A_\mathfrak{p}$, *i.e.* $\mathfrak{p} \in \mathrm{Ass}_A M$, as stated.

Conversely, suppose that $\mathfrak{p} \in \mathrm{Ass}_A M$; then there exists $m_\mathfrak{p} \in M_\mathfrak{p}$ such that the radical of $\mathrm{Ann}_{A_\mathfrak{p}}(m_\mathfrak{p})$ equals $\mathfrak{p}A_\mathfrak{p}$. We may assume that $m_\mathfrak{p}$ is the image of some $m \in M$; then (5.5.19) implies that $V(\mathrm{Ann}_A(m)) \cap \mathrm{Spec}\, A_{\mathfrak p} = V(\mathrm{Ann}_{A_{\mathfrak p}}(m_{\mathfrak p})) = \{\mathfrak p\}$, so $\mathfrak p$ is a maximal point of $V(\mathrm{Ann}_A(m))$.

(ii): Suppose $\mathfrak p \in \mathrm{Ass}\, M$ is finitely generated; by definition, there exists $y \in M_{\mathfrak p}$ such that the radical of $I := \mathrm{Ann}_{A_{\mathfrak p}}(y)$ equals $\mathfrak p A_{\mathfrak p}$. Since $\mathfrak p$ is finitely generated, some power of $\mathfrak p A_{\mathfrak p}$ is contained in I. Let $t \in \mathbb N$ be the smallest integer such that $\mathfrak p^t A_{\mathfrak p} \subset I$, and pick any non-zero element x in $\mathfrak p^{t-1} A_{\mathfrak p} y$; then $\mathrm{Ann}_{A_{\mathfrak p}}(x) = \mathfrak p A_{\mathfrak p}$. After clearing some denominators, we may assume that $x \in M$. Let $a_1, \dots, a_r$ be a finite set of generators for $\mathfrak p$; we may then find $t_1, \dots, t_r \in A \setminus \mathfrak p$ such that $t_i a_i x = 0$ in M, for every $i \le r$. Set $x' := t_1 \cdot \ldots \cdot t_r x$; then $\mathfrak p \subset \mathrm{Ann}_A(x')$; however $\mathrm{Ann}_{A_{\mathfrak p}}(x') = \mathfrak p A_{\mathfrak p}$, therefore $\mathfrak p = \mathrm{Ann}_A(x')$, whence the contention. □

5.5.20. Let now $\mathfrak p \subset A$ be any prime ideal, and $n \ge 1$; if $A/\mathfrak p^n$ does not admit imbedded primes, then the ideal $\mathfrak p^n$ is $\mathfrak p$-primary. In the presence of imbedded primes, this fails. For instance, we have the following :

Example 5.5.21. Let k be a field, $k[x,y]$ the free polynomial algebra in indeterminates x and y; consider the ideal $I := (xy, y^2)$, and set $A := k[x,y]/I$. Let $\overline{x}$ and $\overline{y}$ be the images of x and y in A; then $\mathfrak p := (\overline{y}) \subset A$ is a prime ideal, and $\mathfrak p^2 = 0$. However, $\mathfrak m := \mathrm{Ann}_A(\overline{y})$ is the maximal ideal generated by $\overline{x}$ and $\overline{y}$, so the ideal $0 \subset A$ is not $\mathfrak p$-primary.

There is however a natural sequence of $\mathfrak p$-primary ideals naturally attached to $\mathfrak p$. To explain this, let us remark, more generally, the following :

Lemma 5.5.22. *Let A be a ring, $\mathfrak p \subset A$ a prime ideal. Denote by $\varphi_{\mathfrak p} : A \to A_{\mathfrak p}$ the localization map. The rule :*

$$I \mapsto \varphi_{\mathfrak p}^{-1} I$$

induces a bijection from the set of $\mathfrak p A_{\mathfrak p}$-primary ideals of $A_{\mathfrak p}$, to the set of $\mathfrak p$-primary ideals of A.

Proof. Suppose that $I \subset A_{\mathfrak p}$ is $\mathfrak p A_{\mathfrak p}$-primary; since the natural map $A/\varphi_{\mathfrak p}^{-1} I \to A_{\mathfrak p}/I$ is injective, it is clear that $\varphi_{\mathfrak p}^{-1} I$ is $\mathfrak p$-primary. Conversely, suppose that $J \subset A$ is $\mathfrak p$-primary; we claim that $J = \varphi_{\mathfrak p}^{-1}(J_{\mathfrak p})$. Indeed, by assumption (and by lemma 5.5.3(iii)) we have $(A/J)_{\mathfrak q} = 0$ whenever $\mathfrak q \neq \mathfrak p$; it follows easily that the localization map $A/J \to A_{\mathfrak p}/J_{\mathfrak p}$ is an isomorphism, whence the contention. □

Definition 5.5.23. Keep the notation of lemma 5.5.22; for every $n \ge 0$ one defines the *n-th symbolic power* of $\mathfrak p$, as the ideal :

$$\mathfrak p^{(n)} := \varphi_{\mathfrak p}^{-1}(\mathfrak p^n A_{\mathfrak p}).$$

By lemma 5.5.22, the ideal $\mathfrak p^{(n)}$ is $\mathfrak p$-primary for every $n \ge 1$. More generally, for every A-module M, let $\varphi_{M,\mathfrak p} : M \to M_{\mathfrak p}$ be the localization map; then one defines the $\mathfrak p$-*symbolic filtration* on M, by the rule :

$$\mathrm{Fil}^{(n)}_{\mathfrak p} M := \varphi_{M,\mathfrak p}^{-1}(\mathfrak p^n M_{\mathfrak p}) \qquad \text{for every } n \ge 0.$$

The filtration $\mathrm{Fil}^{(\bullet)}_{\mathfrak p} M$ induces a (linear) topology on M, called the $\mathfrak p$-*symbolic topology*.

More generally, if $\Sigma \subset \mathrm{Spec}\, A$ is any subset, we define the Σ-*symbolic topology* on M, as the coarsest linear topology $\mathscr T_\Sigma$ on M such that $\mathrm{Fil}^{(n)}_{\mathfrak p} M$ is an open subset of $\mathscr T_\Sigma$, for every $\mathfrak p \in \Sigma$ and every $n \ge 0$. If Σ is finite, it is induced by the Σ-*symbolic filtration*, defined by the submodules :

$$\mathrm{Fil}^{(n)}_{\Sigma} M := \bigcap_{\mathfrak p \in \Sigma} \mathrm{Fil}^{(n)}_{\mathfrak p} M \qquad \text{for every } n \ge 0.$$

5.5.24. Let A be a ring, M an A-module, $I \subset A$ an ideal. We shall show how to characterize the finite subsets $\Sigma \subset \mathrm{Spec}\, A$ such that the Σ-symbolic topology on M agrees with the I-preadic topology (see theorem 5.5.33). Hereafter, we begin with a few preliminary observations. First, suppose that A is noetherian; then it is easily seen that, for every prime ideal $\mathfrak{p} \subset A$, and every $\mathfrak{p}A_\mathfrak{p}$-primary ideal $I \subset A_\mathfrak{p}$, there exists $n \in \mathbb{N}$ such that $\mathfrak{p}^n A_\mathfrak{p} \subset I$. From lemma 5.5.22 we deduce that every $\mathfrak{p}$-primary ideal of A contains a symbolic power of $\mathfrak{p}$; *i.e.*, every $\mathfrak{p}$-primary ideal is open in the $\mathfrak{p}$-symbolic topology of A. More generally, let $\Sigma \subset \mathrm{Spec}\, A$ be any subset, M a finitely generated A-module, and $N \subset M$ a submodule; from the existence of a primary decomposition for N ([61, Th.6.8]), we see that N is open in the Σ-symbolic topology of M, whenever $\mathrm{Ass}\, M/N \subset \Sigma$. Especially, if $\mathrm{Ass}\, M/I^nM \subset \Sigma$ for every $n \in \mathbb{N}$, then the Σ-symbolic topology is finer than the I-preadic topology on M. On the other hand, clearly $I^nM \subset \mathrm{Fil}^{(n)}_\mathfrak{p} M$ for every $\mathfrak{p} \in \mathrm{Supp}\, M/IM$ and every $n \in \mathbb{N}$, so the I-preadic topology on M is finer than the $\mathrm{Supp}\, M/IM$-symbolic topology. Summing up, if we have :

$$\Sigma_0(M) := \bigcup_{n\in\mathbb{N}} \mathrm{Ass}\, M/I^nM \subset \Sigma \subset \mathrm{Supp}\, M/IM$$

then the Σ-symbolic topology on M agrees with the I-preadic topology. Notice that the above expression for $\Sigma_0(M)$ is a union of finite subsets ([61, Th.6.5(i)]), hence $\Sigma_0(M)$ is – *a priori* – at most countable; in fact, we shall show that $\Sigma_0(M)$ is finite. Indeed, for every $n \in \mathbb{N}$, set :

$$\mathrm{gr}^n_I A := I^n/I^{n+1} \qquad \mathrm{gr}^n_I M := I^nM/I^{n+1}M.$$

Then $\mathrm{gr}^\bullet_I A := \oplus_{n\in\mathbb{N}} \mathrm{gr}^n_I A$ is naturally a graded A/I-algebra, and $\mathrm{gr}^\bullet_I M := \oplus_{n\in\mathbb{N}} \mathrm{gr}^n_I M$ is a graded $\mathrm{gr}^\bullet_I A$-module. Let $\psi : \mathrm{Spec}\, \mathrm{gr}^\bullet_I A \to \mathrm{Spec}\, A/I$ be the natural morphism, and set :

$$\Sigma(M) := \psi(\mathrm{Ass}_{\mathrm{gr}^\bullet_I A}(\mathrm{gr}^\bullet_I M)).$$

Lemma 5.5.25. *With the notation of* (5.5.24)*, we have :*

$$\mathrm{Ass}_{A/I}(\mathrm{gr}^n_I M) \subset \Sigma(M) \qquad \textit{for every } n \in \mathbb{N}.$$

Proof. To ease notation, set $A_0 := A/I$ and $B := \mathrm{gr}^\bullet_I A$. Suppose that $\mathfrak{p} \in \mathrm{Ass}_{A_0}(\mathrm{gr}^n_I M)$; by lemma 5.5.18(i), there exists $m \in \mathrm{gr}^n_I M$ such that $\mathfrak{p}$ is the preimage of a minimal prime ideal of $A_0/\mathrm{Ann}_{A_0}(m)$. However, if we regard m as a homogeneous element of the B-module $\mathrm{gr}^\bullet_I M$, we have the obvious identity :

$$\mathrm{Ann}_{A_0}(m) = A_0 \cap \mathrm{Ann}_B(m) \subset B$$

from which we see that the induced map

$$A_0/\mathrm{Ann}_{A_0}(m) \to B/\mathrm{Ann}_B(m)$$

is injective, hence ψ restricts to a dominant morphism $V(\mathrm{Ann}_B(m)) \to V(\mathrm{Ann}_{A_0}(m))$. Especially, there exists $\mathfrak{q} \in V(\mathrm{Ann}_B(m))$ with $\psi(\mathfrak{q}) = \mathfrak{p}$; up to replacing $\mathfrak{q}$ by a generization, we may assume that $\mathfrak{q}$ is a maximal point of $V(\mathrm{Ann}_B(m))$, hence $\mathfrak{q}$ is an associated prime for $\mathrm{gr}^\bullet_I M$, again by lemma 5.5.18(i). $\square$

5.5.26. An easy induction, starting from lemma 5.5.25, shows that $\mathrm{Ass}_A M/I^nM \subset \Sigma(M)$, for every $n \in \mathbb{N}$, therefore $\Sigma_0(M) \subset \Sigma(M)$. However, if A is noetherian, the same holds for $\mathrm{gr}^\bullet_I A$ (since the latter is a quotient of an A/I-algebra of finite type), hence $\Sigma(M)$ is finite, provided M is finitely generated ([61, Th.6.5(i)]), and *a fortiori*, the same holds for $\Sigma_0(M)$.

Remark 5.5.27. Another proof of the finiteness of $\Sigma_0(M)$ can be found in [21].

5.5.28. Next, we wish to show that actually there exists a *smallest* subset $\Sigma_{\min}(M) \subset \mathrm{Spec}\, A$ such that the $\Sigma_{\min}(M)$-symbolic topology on M agrees with I-preadic topology; after some simple reductions, this boils down to the following assertion. Let $\Sigma \subset \mathrm{Spec}\, A$ be a finite subset, and $\mathfrak{p}, \mathfrak{p}' \in \Sigma$ two elements, such that the Σ-symbolic topology on M agrees with both the $\Sigma \setminus \{\mathfrak{p}\}$-symbolic topology and the $\Sigma\setminus\{\mathfrak{p}'\}$-symbolic topology; then these topologies agree as well with the $\Sigma \setminus \{\mathfrak{p}, \mathfrak{p}'\}$-symbolic topology. Indeed, given any subset $\Sigma' \subset \mathrm{Spec}\, A$ with $\mathfrak{p} \in \Sigma'$, for the Σ'-symbolic and the $\Sigma' \setminus \{\mathfrak{p}\}$-symbolic topologies to agree, it is necessary and sufficient that, for every $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that :

$$\mathrm{Fil}^{(m)}_{\Sigma'\setminus\{\mathfrak{p}\}} M \subset \mathrm{Fil}^{(n)}_{\mathfrak{p}} M \qquad \text{or, what is the same :} \qquad \mathrm{Fil}^{(m)}_{\Sigma'\setminus\{\mathfrak{p}\}} M_{\mathfrak{p}} \subset \mathfrak{p}^n M_{\mathfrak{p}}.$$

In the latter inclusion, we may then replace $\Sigma' \setminus \{\mathfrak{p}\}$ by the smaller subset $\Sigma' \cap \mathrm{Spec}\, A_{\mathfrak{p}} \setminus \{\mathfrak{p}\}$, without changing the two terms. Suppose now that $\mathfrak{p}' \notin \mathrm{Spec}\, A_{\mathfrak{p}}$; then if we apply the above, first with $\Sigma' := \Sigma$, and then with $\Sigma' := \Sigma \setminus \{\mathfrak{p}'\}$, we see that the Σ-symbolic topology agrees with the $\Sigma\setminus\{\mathfrak{p}\}$-symbolic topology, if and only if the $\Sigma\setminus\{\mathfrak{p}'\}$-symbolic topology agrees with the $\Sigma\setminus\{\mathfrak{p}, \mathfrak{p}'\}$-symbolic topology, whence the contention. In case $\mathfrak{p}' \in \mathrm{Spec}\, A_{\mathfrak{p}}$, we may assume that $\mathfrak{p} \neq \mathfrak{p}'$, otherwise there is nothing to prove; then we shall have $\mathfrak{p} \notin \mathrm{Spec}\, A_{\mathfrak{p}'}$, so the foregoing argument still goes through, after reversing the roles of $\mathfrak{p}$ and $\mathfrak{p}'$.

5.5.29. Finally, theorem 5.5.33 will characterize the subset $\Sigma_{\min}(M)$ as in (5.5.26). To this aim, for every prime ideal $\mathfrak{p} \subset A$, let $A^\wedge_{\mathfrak{p}}$ denote the $\mathfrak{p}$-adic completion of A; we set :

$$\mathrm{Ass}_A(I, M) := \{\mathfrak{p} \in \mathrm{Spec}\, A \mid \text{there exists } \mathfrak{q} \in \mathrm{Ass}_{A^\wedge_{\mathfrak{p}}} M \otimes_A A^\wedge_{\mathfrak{p}} \text{ such that } \sqrt{\mathfrak{q} + IA^\wedge_{\mathfrak{p}}} = \mathfrak{p}A^\wedge_{\mathfrak{p}}\}$$

(where, for any ideal $J \subset A^\wedge_{\mathfrak{p}}$ we denote by $\sqrt{J} \subset A^\wedge_{\mathfrak{p}}$ the radical of J, so the above condition selects the points $\mathfrak{q} \in \mathrm{Spec}\, A^\wedge_{\mathfrak{p}}$, such that $\overline{\{\mathfrak{q}\}} \cap V(IA^\wedge_{\mathfrak{p}}) = \{\mathfrak{p}A^\wedge_{\mathfrak{p}}\}$).

Lemma 5.5.30. *Let A be a noetherian ring, M an A-module. Then we have :*

(i) $\mathrm{depth}_{A_{\mathfrak{p}}} M \otimes_A A_{\mathfrak{p}} = \mathrm{depth}_{A^\wedge_{\mathfrak{p}}} M \otimes_A A^\wedge_{\mathfrak{p}}$ *for every* $\mathfrak{p} \in \mathrm{Spec}\, A$.

(ii) $\mathrm{Ass}_A(0, M) = \mathrm{Ass}_A M$.

(iii) $\mathrm{Ass}_A(I, M) \subset V(I) \cap \mathrm{Supp}\, M$.

(iv) *Suppose that M is the union of a filtered family $(M_\lambda \mid \lambda \in \Lambda)$ of A-submodules. Then :*

$$\mathrm{Ass}_A(I, M) = \bigcup_{\lambda\in\Lambda} \mathrm{Ass}_A(I, M_\lambda).$$

(v) $\mathrm{Ass}_A(I, M)$ *contains the maximal points of* $V(I) \cap \mathrm{Supp}\, M$.

Proof. (iii) is immediate, and in view of [61, Th.8.8], (i) is a special case of corollary 5.4.35.

(ii): By definition, $\mathrm{Ass}_A(0, M)$ consists of all the prime ideals $\mathfrak{p} \subset A$ such that $\mathfrak{p}A^\wedge_{\mathfrak{p}}$ is an associated prime ideal of $M \otimes_A A^\wedge_{\mathfrak{p}}$; in light of proposition 5.5.4(i), the latter condition holds if and only if $\mathrm{depth}_{A^\wedge_{\mathfrak{p}}} M \otimes_A A^\wedge_{\mathfrak{p}} = 0$, hence if and only if $\mathrm{depth}_{A_{\mathfrak{p}}} M \otimes_A A_{\mathfrak{p}} = 0$, by (i); to conclude, one appeals again to proposition 5.5.4(i).

(iv): In view of proposition 5.5.4(v), we have $\mathrm{Ass}_{A^\wedge_{\mathfrak{p}}} M \otimes_A A^\wedge_{\mathfrak{p}} = \bigcup_{\lambda\in\Lambda} \mathrm{Ass}_{A^\wedge_{\mathfrak{p}}} M_\lambda \otimes_A A^\wedge_{\mathfrak{p}}$; the contention is an immediate consequence.

(v): In view of (iv), we are easily reduced to the case where M is a finitely generated A-module, in which case $\mathrm{Supp}\, M = V(J)$ for some ideal $J \subset A$. Hence, suppose that $\mathfrak{p} \subset A$ is a maximal point of $\mathrm{Spec}\, A/(I + J)$, in other words, the preimage of a minimal prime ideal of $A/(I + J)$; notice that $\mathrm{Supp}\, M \otimes_A A^\wedge_{\mathfrak{p}} = V(JA^\wedge_{\mathfrak{p}})$. Hence we have $(J + I)A^\wedge_{\mathfrak{p}} \subset \mathfrak{q} + IA^\wedge_{\mathfrak{p}}$ for every $\mathfrak{q} \in \mathrm{Supp}\, M \otimes_A A^\wedge_{\mathfrak{p}}$, so the radical of $\mathfrak{q} + IA^\wedge_{\mathfrak{p}}$ equals $\mathfrak{p}A^\wedge_{\mathfrak{p}}$, as required. $\square$

Lemma 5.5.31. *Let A be a local noetherian ring, $\mathfrak{m}$ its maximal ideal, and suppose that A is $\mathfrak{m}$-adically complete. Let M be a finitely generated A-module, and $(\mathrm{Fil}^n M \mid n \in \mathbb{N})$ a descending filtration consisting of A-submodules of M. Then the following conditions are equivalent :*

(a) $\bigcap_{n\in\mathbb{N}} \mathrm{Fil}^n M = 0$.
(b) *For every* $i \in \mathbb{N}$ *there exists* $n \in \mathbb{N}$ *such that* $\mathrm{Fil}^n M \subset \mathfrak{m}^i M$.

Proof. Clearly (b)⇒(a), hence suppose that (a) holds. For every $i, n \in \mathbb{N}$, set

$$J_{i,n} := \mathrm{Im}(\mathrm{Fil}^n M \to M/\mathfrak{m}^i M).$$

For given $i \in \mathbb{N}$, the A-module $M/\mathfrak{m}^i M$ is artinian, hence there exists $n \in \mathbb{N}$ such that $J_i := J_{i,n} = J_{i,n'}$ for every $n' \geq n$. Assertion (b) then follows from the following :

Claim 5.5.32. If (a) holds, $J_i = 0$ for every $i \in \mathbb{N}$.

Proof of the claim. By inspecting the definition, it is easily seen that the natural $A/\mathfrak{m}^{i+1}$-linear map $J_{i+1} \to J_i$ is surjective for every $i \in \mathbb{N}$, hence we are reduced to showing that $J := \lim\limits_{i\in\mathbb{N}} J_i$ vanishes. However, J is naturally a submodule of $\lim\limits_{i\in\mathbb{N}} M/\mathfrak{m}^i M \simeq M$, and if $x \in M$ lies in J, then we have $x \in \mathrm{Fil}^n M + \mathfrak{m}^i M$ for every $i, n \in \mathbb{N}$. Since $\mathrm{Fil}^n M$ is a closed subset for the $\mathfrak{m}$-adic topology of M ([61, Th.8.10(i)]), we have $\bigcap_{i\in\mathbb{N}}(\mathrm{Fil}^n M + \mathfrak{m}^i M) = \mathrm{Fil}^n M$, for every $n \in \mathbb{N}$, hence $x \in \bigcap_{n\in\mathbb{N}} \mathrm{Fil}^n M$, which vanishes, if (a) holds. $\square$

The following theorem generalizes [70, Ch.VIII, §5, Cor.5] and [47, Prop.7.1].

Theorem 5.5.33. *Let A be a noetherian ring, $I \subset A$ an ideal, M a finitely generated A-module, and $\Sigma \subset \mathrm{Spec}\, A/I$ a finite subset. Then the Σ-symbolic topology on M agrees with the I-preadic topology if and only if* $\mathrm{Ass}_A(I, M) \subset \Sigma$.

Proof. Let $\mathfrak{p} \in \mathrm{Ass}_A(I, M) \setminus \Sigma$, and suppose, by way of contradiction, that the Σ-symbolic topology agrees with the I-adic topology, *i.e.* for every $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that :

$$\mathrm{Fil}^{(m)}_\Sigma M \subset I^n M.$$

Let $X := \mathrm{Spec}\, A_\mathfrak{p}$ and $U := X \setminus \{\mathfrak{p}\}$; after localizing at the prime $\mathfrak{p}$, we deduce the inclusion :

$$\mathrm{Fil}^{(m)}_{\Sigma\cap U} M_\mathfrak{p} \subset I^n M_\mathfrak{p} \tag{5.5.34}$$

(cp. the discussion in (5.5.26)). Let also $\mathscr{M} := M^\sim$, the quasi-coherent $\mathscr{O}_X$-module associated to M; clearly we have $I^m M \subset \mathrm{Fil}^{(m)}_\mathfrak{q} M$ for every $\mathfrak{q} \in \mathrm{Spec}\, A/I$, hence (5.5.34) implies the inclusion :

$$\{x \in M_\mathfrak{p} \mid x_{|U} \in I^m \mathscr{M}(U)\} \subset I^n M_\mathfrak{p}.$$

Let $A^\wedge_\mathfrak{p}$ (resp. $M^\wedge_\mathfrak{p}$) be the $\mathfrak{p}$-adic completion of A (resp. of M), $f : X^\wedge := \mathrm{Spec}\, A^\wedge_\mathfrak{p} \to X$ the natural morphism, and set $U^\wedge := f^{-1}U = X^\wedge \setminus \{\mathfrak{p}A^\wedge_\mathfrak{p}\}$. Since f is faithfully flat, and U is quasi-compact, we deduce that :

$$\{x \in M^\wedge_\mathfrak{p} \mid x_{|U^\wedge} \in I^m f^*\mathscr{M}(U^\wedge)\} \subset I^n M^\wedge_\mathfrak{p}. \tag{5.5.35}$$

On the other hand, by assumption there exists $\mathfrak{q} \in \mathrm{Ass}_{A^\wedge_\mathfrak{p}} M^\wedge_\mathfrak{p}$ such that $\overline{\{\mathfrak{q}\}} \cap V(IA^\wedge_\mathfrak{p}) = \{\mathfrak{p}A^\wedge_\mathfrak{p}\}$, hence we may find $x \in M^\wedge_\mathfrak{p}$ whose support is $\overline{\{\mathfrak{q}\}}$ (lemma 5.5.18(ii)). It follows easily that the image of x vanishes in $f^*\mathscr{M}/I^m f^*\mathscr{M}(U)$ for all $m \in \mathbb{N}$, *i.e.* $x_{|U} \in I^m f^*\mathscr{M}(U)$ for every $m \in \mathbb{N}$, hence $x \in I^n M^\wedge_\mathfrak{p}$ for every $n \in \mathbb{N}$, in view of (5.5.35). However, $M^\wedge_\mathfrak{p}$ is separated for the $\mathfrak{p}$-adic topology, *a fortiori* also for the I-preadic topology, so $x = 0$, a contradiction.

Next, let $\mathfrak{p} \in \Sigma \setminus \mathrm{Ass}_A(I, M)$, set $\Sigma' := \Sigma \setminus \{\mathfrak{p}\}$, and suppose that the Σ-symbolic topology agrees with the I-adic topology; we have to prove that the latter agrees as well with the Σ'-symbolic topology. This amounts to showing that, for every $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that :

$$\mathrm{Fil}^{(m)}_{\Sigma'} M \subset \mathrm{Fil}^{(n)}_\mathfrak{p} M \qquad \text{or, what is the same :} \qquad \mathrm{Fil}^{(m)}_{\Sigma'} M_\mathfrak{p} \subset \mathfrak{p}^n M_\mathfrak{p}.$$

We may write :

$$\mathrm{Fil}^{(m)}_{\Sigma'} M_{\mathfrak{p}} = \{x \in M_{\mathfrak{p}} \mid x_{|U} \in (\mathrm{Fil}^{(m)}_{\Sigma'} M_{\mathfrak{p}})^{\sim}(U)\} = \{x \in M_{\mathfrak{p}} \mid x_{|U} \in (\mathrm{Fil}^{(m)}_{\Sigma} M_{\mathfrak{p}})^{\sim}(U)\}$$

from which it is clear that the contention holds if and only if, for every $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that :

$$\{x \in M_{\mathfrak{p}} \mid x_{|U} \in I^m \mathscr{M}(U)\} \subset \mathfrak{p}^n M_{\mathfrak{p}}.$$

Arguing as in the foregoing, we see that the latter condition holds if and only if :

$$\mathrm{Fil}^m M^{\wedge}_{\mathfrak{p}} := \{x \in M^{\wedge}_{\mathfrak{p}} \mid x_{|U^{\wedge}} \in I^m f^* \mathscr{M}(U^{\wedge})\} \subset \mathfrak{p}^n M^{\wedge}_{\mathfrak{p}}.$$

In view of lemma 5.5.31, we are then reduced to showing the following :

Claim 5.5.36. $\bigcap_{m \in \mathbb{N}} \mathrm{Fil}^m M^{\wedge}_{\mathfrak{p}} = 0.$

Proof of the claim. Let x be an element in this intersection; for any $\mathfrak{q} \in U^{\wedge} \cap V(IA^{\wedge}_{\mathfrak{p}})$, we have $x_{\mathfrak{q}} \in \bigcap_{m \in \mathbb{N}} \mathfrak{q}^m (M^{\wedge}_{\mathfrak{p}})_{\mathfrak{q}}$, hence $x_{\mathfrak{q}} = 0$, by [61, Th.8.10(i)]. In other words, $\mathrm{Supp}(x) \cap U \cap V(I) = \varnothing$. Suppose that $x \neq 0$, and let $\mathfrak{q}$ be any maximal point of $\mathrm{Supp}(x)$; by lemma 5.5.18(i), $\mathfrak{q}$ is an associated prime, and the foregoing implies that $\{\mathfrak{q}\} \cap V(IA^{\wedge}_{\mathfrak{p}}) = \{\mathfrak{p}A^{\wedge}_{\mathfrak{p}}\}$, which contradicts the assumption that $\mathfrak{p} \notin \mathrm{Ass}_A(I, M)$. □

Corollary 5.5.37. *In the situation of theorem* 5.5.33, $\mathrm{Ass}_A(I, M)$ *is a finite set.*

Proof. We have already found a finite subset $\Sigma \subset \mathrm{Spec}\, A/I$ such that the Σ-symbolic topology on M agrees with the I-preadic topology (see (5.5.26)). The contention then follows straightforwardly from theorem 5.5.33. □

Example 5.5.38. Let k be a field with $\mathrm{char}\, k \neq 2$, and let $C \subset \mathbb{A}^2_k := \mathrm{Spec}\, k[X, Y]$ be the nodal curve cut by the equation $Y^2 = X^2 + X^3$, so that the only singularity of C is the node at the origin $p := (0, 0) \in C$. Let $R := k[X, Y, Z]$, $A := R/(Y^2 - X^2 - X^3)$; denote by $\pi : \mathbb{A}^3_k := \mathrm{Spec}\, R \to \mathbb{A}^2_k$ the linear projection which is dual to the inclusion $k[X, Y] \to R$, so that $D := \pi^{-1}C = \mathrm{Spec}\, A$. We define a morphism $\varphi : \mathbb{A}^1_k := \mathrm{Spec}\, k[T] \to D$ by the rule : $T \mapsto (T^2 - 1, T(T^2 - 1), T)$ (*i.e.* φ is dual to the homomorphism of k-algebras such that $X \mapsto T^2 - 1$, $Y \mapsto T(T^2 - 1)$ and $Z \mapsto T$). Let $C' \subset D$ be the image of φ, with its reduced subscheme structure. It is easy to check that the restriction of π maps C' birationally onto C, so there are precisely two points $p'_0, p'_1 \in C'$ lying over p. Let $\mathfrak{n} := I(C') \subset A$, the prime ideal which is the generic point of the (irreducible) curve C'. We claim that the $\mathfrak{n}$-preadic topology on A does not agree with the $\mathfrak{n}$-symbolic topology. To this aim – in view of theorem 5.5.33 – it suffices to show that $\{p'_0, p'_1\} \subset \mathrm{Ass}_A(\mathfrak{n}, A)$. However, for any closed point $\mathfrak{p} \in \pi^{-1}(p)$, the $\mathfrak{p}$-adic completion $A^{\wedge}_{\mathfrak{p}}$ admits two distinct minimal primes, corresponding to the two branches of the nodal conic C at the node p, and the corresponding irreducible components of $B := \mathrm{Spec}\, A^{\wedge}_{\mathfrak{p}}$ meet along the affine line $V(Z)$. To see this, we may suppose that $\mathfrak{p} = (X, Y, Z)$, hence $A^{\wedge}_{\mathfrak{p}} \simeq k[[X, Y, Z]]/(Y^2 - X^2(1 + X))$, and notice that the latter is isomorphic to $k[[S, Y, Z]]/(Y^2 - S^2)$, via the isomorphism that sends $Y \mapsto Y$, $Z \mapsto Z$ and $S \mapsto X(1+X)^{1/2}$ (the assumption on the characteristic of k ensures that $1 + X$ admits a square root in $k[[X]]$). Now, say that $\mathfrak{p} = p'_0$; then $C'_{\mathfrak{p}} := C' \cap B$ is contained in only one of the two irreducible components of B. Let $\mathfrak{q} \in B$ be the minimal prime ideal whose closure does not contain $C'_{\mathfrak{p}}$; then $\mathfrak{q} \in \mathrm{Ass} A^{\wedge}_{\mathfrak{p}}$ and $\overline{\{\mathfrak{q}\}} \cap C'_{\mathfrak{p}} = \{\mathfrak{p}A^{\wedge}_{\mathfrak{p}}\}$, therefore $p'_0 \in \mathrm{Ass}_A(\mathfrak{n}, A)$, as stated.

Example 5.5.39. Let A be an excellent normal ring, $I \subset A$ any ideal, and set $Z := V(I)$. Then $\mathrm{Ass}_A(I, A)$ is the set $\mathrm{Max}(Z)$ of all maximal points of Z. Indeed, $\mathrm{Max}(Z) \subset \mathrm{Ass}_A(I, A)$ by lemma 5.5.30(v). Conversely, suppose $\mathfrak{p} \in \mathrm{Ass}_A(I, A)$; the completion $A^{\wedge}_{\mathfrak{p}}$ is still normal ([61, Th.32.2(i)]), and therefore its only associated prime is 0, so the assumption means that the radical of $IA^{\wedge}_{\mathfrak{p}}$ is $\mathfrak{p}A^{\wedge}_{\mathfrak{p}}$. Equivalently, $\dim A^{\wedge}_{\mathfrak{p}}/IA^{\wedge}_{\mathfrak{p}} = 0$, so $\dim A_{\mathfrak{p}}/IA_{\mathfrak{p}} = 0$, which is the contention.

Definition 5.5.40. Let X be a noetherian scheme, $Y \subset X$ a closed subset, $\mathfrak{X}$ the formal completion of X along Y ([26, Ch.I, §10.8]), and $f : \mathfrak{X} \to X$ the natural morphism of locally ringed spaces. We say that the pair (X, Y) *satisfies the Lefschetz condition*, if for every open subset $U \subset X$ such that $Y \subset U$, and every locally free $\mathscr{O}_U$-module $\mathscr{E}$ of finite type, the natural map :

$$\Gamma(U, \mathscr{E}) \to \Gamma(\mathfrak{X}, f^*\mathscr{E})$$

is an isomorphism. In this case, we also say that $\mathrm{Lef}(X, Y)$ holds. (Cp. [44, Exp.X, §2].)

Lemma 5.5.41. *In the situation of definition* 5.5.40*, suppose that* $\mathrm{Lef}(X, Y)$ *holds, and let* $U \subset X$ *be any open subset such that* $Y \subset U$*. Then :*

(i) *The functor :*

$$\mathscr{O}_U\text{-}\mathbf{Mod}_{\mathrm{lfft}} \to \mathscr{O}_{\mathfrak{X}}\text{-}\mathbf{Mod}_{\mathrm{lfft}} \quad : \quad \mathscr{E} \mapsto f^*\mathscr{E}$$

is fully faithful (notation of (5.2.1)*).*

(ii) *Denote by* $\mathscr{O}_U\text{-}\mathbf{Alg}_{\mathrm{lfft}}$ *the category of* $\mathscr{O}_U$*-algebras, whose underlying* $\mathscr{O}_U$*-module is free of finite type, and define likewise* $\mathscr{O}_{\mathfrak{X}}\text{-}\mathbf{Alg}_{\mathrm{lfft}}$*. Then the functor :*

$$\mathscr{O}_U\text{-}\mathbf{Alg}_{\mathrm{lfft}} \to \mathscr{O}_{\mathfrak{X}}\text{-}\mathbf{Alg}_{\mathrm{lfft}} \quad : \quad \mathscr{A} \mapsto f^*\mathscr{A}$$

is fully faithful.

Proof. (i): Let $\mathscr{E}$ and $\mathscr{F}$ be any two locally free $\mathscr{O}_U$-modules of finite type. We have :

$$\mathrm{Hom}_{\mathscr{O}_U}(\mathscr{E}, \mathscr{F}) = \Gamma(U, \mathscr{H}\!om_{\mathscr{O}_U}(\mathscr{E}, \mathscr{F}))$$

and likewise we may compute $\mathrm{Hom}_{\mathscr{O}_{\mathfrak{X}}}(f^*\mathscr{E}, f^*\mathscr{F})$. However, the natural map :

$$f^*\mathscr{H}\!om_{\mathscr{O}_U}(\mathscr{E}, \mathscr{F}) \to \mathscr{H}\!om_{\mathscr{O}_{\mathfrak{X}}}(f^*\mathscr{E}, f^*\mathscr{F})$$

is an isomorphism of $\mathscr{O}_{\mathfrak{X}}$-modules. The assertion follows.

(ii): An object of $\mathscr{O}_U\text{-}\mathbf{Alg}_{\mathrm{lfft}}$ is a locally free $\mathscr{O}_U$-module $\mathscr{A}$ of finite type, together with morphisms $\mathscr{A} \otimes_{\mathscr{O}_U} \mathscr{A} \to \mathscr{A}$ and $1_{\mathscr{A}} : \mathscr{O}_U \to \mathscr{A}$ of $\mathscr{O}_U$-modules, fulfilling the usual unitarity, commutativity and associativity conditions. An analogous description holds for the objects of $\mathscr{O}_{\mathfrak{X}}\text{-}\mathbf{Alg}_{\mathrm{lfft}}$, and for the morphisms of either category. Since $\mathscr{A} \otimes_{\mathscr{O}_U} \mathscr{A}$ is again locally free of finite type, the assertion follows easily from (i) : the details are left to the reader. □

Lemma 5.5.42. *Let* A *be a noetherian ring,* $I \subset A$ *an ideal,* $U \subset \mathrm{Spec}\, A$ *an open subset,* $\mathfrak{U}$ *the formal completion of* U *along* $U \cap V(I)$*. Consider the following conditions :*

(a) $\mathrm{Lef}(U, U \cap V(I))$ *holds.*

(b) *The natural map* $\rho_U : \Gamma(U, \mathscr{O}_U) \to \Gamma(\mathfrak{U}, \mathscr{O}_{\mathfrak{U}})$ *is an isomorphism.*

(c) *The natural map* $\rho : A \to \Gamma(\mathfrak{U}, \mathscr{O}_{\mathfrak{U}})$ *is an isomorphism.*

Then (c)⇒(b)⇔(a)*, and* (c) *implies that* A *is* I*-adically complete.*

Proof. Clearly (a)⇒(b), hence we assume that (b) holds, and we show (a). Let $V \subset U$ be an open subset with $U \cap V(I) \subset V$ and $\mathscr{E}$ a coherent locally free $\mathscr{O}_V$-module. As A is noetherian, V is quasi-compact, so we may find a left exact sequence $P_\bullet := (0 \to \mathscr{E} \to \mathscr{O}_V^{\oplus m} \to \mathscr{O}_V^{\oplus n})$ of $\mathscr{O}_V$-modules (corollary 5.2.17). Since the natural map of locally ringed spaces $f : \mathfrak{U} \to U$ is flat, the sequence f^*P is still left exact. Since the global section functors are left exact, there follows a ladder of left exact sequences :

$$\Gamma(V, P_\bullet) \to \Gamma(\mathfrak{U}, f^*P_\bullet)$$

which reduces the assertion to the case where $\mathscr{E} = \mathscr{O}_V$; the latter is covered by the following :

Claim 5.5.43. The natural map $\rho_V : \Gamma(V, \mathscr{O}_V) \to \Gamma(\mathfrak{U}, \mathscr{O}_{\mathfrak{U}})$ is an isomorphism.

Proof of the claim. The isomorphism ρ_U factors through ρ_V, hence the latter is a surjection. Suppose that $s \in \mathrm{Ker}\,\rho_V$, and $s \neq 0$; then we may find $x \in V$ such that the image s_x of s in $\mathscr{O}_{V,x}$ does not vanish. Moreover, we may find $a \in A$ whose image $a(x)$ in $\kappa(x)$ does not vanish, and such that as is the restriction of an element of A; especially, $as \in \Gamma(U, \mathscr{O}_U)$, and clearly the image of as in $\Gamma(V, \mathscr{O}_V)$ lies in $\mathrm{Ker}\,\rho_V$. Therefore, $as = 0$ in $\Gamma(U, \mathscr{O}_U)$; however the image as_x of as in $\mathscr{O}_{U,x}$ is non-zero by construction, a contradiction. This shows that ρ_V is injective, whence the claim. ◊

Finally, suppose that (c) holds; arguing as in the proof of claim 5.5.43, one sees that ρ_U is an isomorphism. Moreover, since $\Gamma(\mathfrak{U}, \mathscr{O}_\mathfrak{U}) \simeq \lim\limits_{n\in\mathbb{N}} \Gamma(\mathfrak{U}, \mathscr{O}_\mathfrak{U}/I^n\mathscr{O}_\mathfrak{U})$, the morphism ρ factors through the natural A-linear map $i : A \to A^\wedge$ to the I-adic completion of A. The composition with ρ^{-1} yields an A-linear left inverse $s : A^\wedge \to A$ to i. Set $N := \mathrm{Ker}\,s$; clearly s is surjective, hence $A^\wedge \simeq A \oplus N$. It follows easily that $N/I^nN = 0$ for every $n \in \mathbb{N}$, especially $N \subset \bigcap_{n\in\mathbb{N}} I^nA^\wedge$. Therefore $N = 0$, since $A^\wedge$ is separated for the I-adic topology. □

Proposition 5.5.44. *Let $\varphi : A \to B$ be a flat homomorphism of noetherian rings, $I \subset A$ an ideal, $U \subset \mathrm{Spec}\,B$ an open subset. Set $f := \mathrm{Spec}\,\varphi : \mathrm{Spec}\,B \to \mathrm{Spec}\,A$, and assume that :*

(a) *B is complete for the IB-adic topology.*
(b) *For every $x \in V(I)$, we have : $\{y \in f^{-1}(x) \mid \delta(y, \mathscr{O}_{f^{-1}(x)}) = 0\} \subset U$.*
(c) *For every $x \in \mathrm{Ass}_A(I, A)$, we have : $\{y \in f^{-1}(x) \mid \delta(y, \mathscr{O}_{f^{-1}(x)}) \leq 1\} \subset U$.*

Then $\mathrm{Lef}(U, U \cap V(IB))$ holds.

Proof. Set $\Sigma := \mathrm{Ass}_A(I, A)$, and let $\mathfrak{U}$ be the formal completion of U along $V(IB)$; by theorem 5.5.33, the I-preadic topology on A agrees with the Σ-symbolic topology. Let also $\mathscr{J}$ be the family consisting of all ideals $J \subset A$ such that $\mathrm{Ass}\,A/J \subset \Sigma$; it follows that the natural maps :

$$B \to \lim_{J\in\mathscr{J}} B/JB \qquad \Gamma(\mathfrak{U}, \mathscr{O}_\mathfrak{U}) \to \lim_{J\in\mathscr{J}} \Gamma(U, \mathscr{O}_U/J\mathscr{O}_U)$$

are isomorphisms (see the discussion in (5.5.24)). In view of lemma 5.5.42, we are then reduced to showing :

Claim 5.5.45. The natural map $B/JB \to \Gamma(U, \mathscr{O}_U/J\mathscr{O}_U)$ is an isomorphism for every $J \in \mathscr{J}$.

Proof of the claim. Let $f : Y := \mathrm{Spec}\,B/JB \to X := \mathrm{Spec}\,A/J$ be the induced morphism; in view of corollary 5.4.22, it suffices to prove that $\delta(y, \mathscr{O}_Y) \geq 2$ whenever $y \in Y \setminus U$. Thus, set $x := f(y)$; by corollary 5.4.39 we have :

$$\delta(y, \mathscr{O}_Y) = \delta(y, \mathscr{O}_{f^{-1}(x)}) + \delta(x, \mathscr{O}_X).$$

Now, if $\delta(x, \mathscr{O}_X) = 1$, notice that $f(Y) \subset V(J) \subset V(I)$, by lemmata 5.5.18(i) and 5.5.30(iii); hence (b) implies the contention in this case. Lastly, if $\delta(x, \mathscr{O}_X) = 0$, then $x \in \mathrm{Ass}\,A/J$ by proposition 5.5.4, hence we use assumption (c) to conclude. □

5.6. **Duality over coherent schemes.** We say that a scheme X is *coherent* if $\mathscr{O}_X$ is coherent. If X is coherent and $\mathscr{F}$ is an $\mathscr{O}_X$-module, we define the *dual $\mathscr{O}_X$-module*

$$\mathscr{F}^\vee := \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{O}_X).$$

Notice that $\mathscr{F}^\vee$ is coherent whenever $\mathscr{F}$ is. Moreover, for every morphism of $\mathscr{O}_X$-modules $\varphi : \mathscr{F} \to \mathscr{G}$, we denote by $\varphi^\vee : \mathscr{G}^\vee \to \mathscr{F}^\vee$ the induced (transpose) morphism. As usual, there is a natural morphism of $\mathscr{O}_X$-modules:

$$\beta_\mathscr{F} : \mathscr{F} \to \mathscr{F}^{\vee\vee}.$$

One says that $\mathscr{F}$ is *reflexive at a point* $x \in X$ if there exists an open neighborhood $U \subset X$ of x such that $\mathscr{F}_{|U}$ is a coherent $\mathscr{O}_U$-module, and $\beta_{\mathscr{F}|U}$ is an isomorphism. One says that $\mathscr{F}$ is *reflexive* if it is reflexive at all points of X. We denote by $\mathscr{O}_X$-**Rflx** the full subcategory of the category $\mathscr{O}_X$-$\mathbf{Mod}$, consisting of all the reflexive $\mathscr{O}_X$-modules. It contains $\mathscr{O}_X$-$\mathbf{Mod}_{\mathrm{lfft}}$ as a full subcategory (see (5.2.1)).

Lemma 5.6.1. *Let X be a coherent scheme, $x \in X$ any point, and $\mathscr{F}$ a coherent $\mathscr{O}_X$-module. Then the following conditions are equivalent:*

(a) *$\mathscr{F}$ is reflexive at the point x.*
(b) *$\mathscr{F}_x$ is a reflexive $\mathscr{O}_{X,x}$-module, by which we mean that the coherent sheaf induced by $\mathscr{F}_x$ on $\mathrm{Spec}\,\mathscr{O}_{X,x}$ is reflexive.*
(c) *The map $\beta_{\mathscr{F},x} : \mathscr{F}_x \to (\mathscr{F}^{\vee\vee})_x$ is an isomorphism.*

Proof. Left to the reader. □

Lemma 5.6.2. *Suppose that X is a reduced coherent scheme, and $\mathscr{F}$ a coherent $\mathscr{O}_X$-module. Then $\beta_{\mathscr{F}^\vee}$ is an isomorphism of $\mathscr{O}_X$-modules and its inverse is $\beta^\vee_{\mathscr{F}}$. Especially, $\mathscr{F}^\vee$ is reflexive.*

Proof. The assumption on X implies that the only associated points of $\mathscr{O}_X$ are the maximal points of X (*i.e.* $\mathscr{O}_X$ has no imbedded points). It follows easily that, for every coherent $\mathscr{O}_X$-module $\mathscr{G}$, the dual $\mathscr{G}^\vee$ satisfies condition S_1 (see definition 5.5.1(ii)). Now, rather generally, let $\mathscr{M}$ be any $\mathscr{O}_X$-module; directly from the definitions one derives the identity:

$$\beta^\vee_{\mathscr{M}} \circ \beta_{\mathscr{M}^\vee} = \mathbf{1}_{\mathscr{M}^\vee}. \tag{5.6.3}$$

It remains therefore only to show that $\beta^\vee_{\mathscr{F}}$ is a right inverse for $\beta_{\mathscr{F}^*}$ when $\mathscr{F}$ is coherent. Since $\mathscr{F}^{\vee\vee\vee}$ satisfies condition S_1, it suffices to check that, for every maximal point ξ, the induced map on stalks

$$\beta^\vee_{\mathscr{F},\xi} : \mathscr{F}^{\vee\vee\vee}_\xi \to \mathscr{F}^\vee_\xi$$

is a right inverse for $\beta_{\mathscr{F}^\vee,\xi}$. However, since $\mathscr{O}_{X,\xi}$ is a field, $\beta^\vee_{\mathscr{F},\xi}$ is a linear map of $\mathscr{O}_{X,\xi}$-vector spaces of the same dimension, hence it is an isomorphism, in view of (5.6.3). □

Remark 5.6.4. The following observation is often useful. Suppose that X is a coherent scheme, $\mathscr{F}$ an $\mathscr{O}_X$-module, reflexive at a given point $x \in X$. We can then choose a presentation $\mathscr{O}^{\oplus n}_{X,x} \to \mathscr{O}^{\oplus m}_{X,x} \to \mathscr{F}^\vee_x \to 0$, and after dualizing we deduce a left exact sequence

$$0 \to \mathscr{F}_x \to \mathscr{O}^{\oplus m}_{X,x} \xrightarrow{u} \mathscr{O}^{\oplus n}_{X,x}. \tag{5.6.5}$$

Especially, if $\mathscr{O}_{X,x}$ satisfies condition S_1 (in the sense of definition 5.5.1(iii)), then the same holds for $\mathscr{F}_x$. For the converse, suppose additionally that X is reduced, and let $x \in X$ be a point for which there exists a left exact sequence such as (5.6.5); then lemma 5.6.2 says that $\mathscr{F}$ is reflexive at x : indeed, $\mathscr{F}_x \simeq (\mathrm{Coker}\,u^\vee)^\vee$.

Lemma 5.6.6. (i) *Let $f : X \to Y$ be a flat morphism of coherent schemes. The induced functor $\mathscr{O}_Y$-$\mathbf{Mod} \to \mathscr{O}_X$-$\mathbf{Mod}$ restricts to a functor*

$$f^* : \mathscr{O}_Y\text{-}\mathbf{Rflx} \to \mathscr{O}_X\text{-}\mathbf{Rflx}.$$

(ii) *Let X_0 be a quasi-compact and quasi-separated scheme, $(X_\lambda \mid \lambda \in \Lambda)$ a cofiltered family of coherent X_0-schemes with flat transition morphisms $\psi_{\lambda\mu} : X_\lambda \to X_\mu$ such that the structure morphisms $X_\lambda \to X_0$ are affine, and set $X := \lim\limits_{\lambda\in\Lambda} X_\lambda$. Then:*

(a) *X is coherent.*
(b) *the natural functor: $2\text{-}\operatorname*{colim}\limits_{\lambda\in\Lambda^o} \mathscr{O}_{X_\lambda}\text{-}\mathbf{Rflx} \to \mathscr{O}_X\text{-}\mathbf{Rflx}$ is an equivalence.*

(iii) *Suppose that f is surjective, and let $\mathscr{F}$ be any coherent $\mathscr{O}_Y$-module. Then $\mathscr{F}$ is reflexive if and only if $f^*\mathscr{F}$ is a reflexive $\mathscr{O}_X$-module.*

Proof. (i) follows easily from [36, Lemma 2.4.29(i.a)].

(ii): For every $\lambda \in \Lambda$, denote by $\psi_\lambda : X \to X_\lambda$ the natural morphism. Let $U \subset X$ be a quasi-compact open subset, $u : \mathscr{O}^{\oplus n}_U \to \mathscr{O}_U$ any morphism of $\mathscr{O}_U$-modules; by [32, Ch.IV, Cor.8.2.11] there exists $\lambda \in \Lambda$ and a quasi-compact open subset $U_\lambda \subset X_\lambda$ such that $U = \psi^{-1}(U_\lambda)$. By [32, Ch.IV, Th.8.5.2](i) we may then suppose that u descends to a homomorphism $u_\lambda : \mathscr{O}_{U_\lambda}^{\oplus n} \to \mathscr{O}_{U_\lambda}$, whose kernel is of finite type, since X_λ is coherent. Since the transition morphisms are flat, we have $\mathrm{Ker}\, u = \psi_\lambda^*(\mathrm{Ker}\, u_\lambda)$, whence (ii.a). Next, using [32, Ch.IV, Th.8.5.2] one sees easily that the functor of (ii.b) is fully faithful and moreover, every reflexive $\mathscr{O}_X$-module $\mathscr{F}$ descends to a coherent $\mathscr{O}_{X_\lambda}$-module $\mathscr{F}_\lambda$ for some $\lambda \in \Lambda$. For every $\mu \geq \lambda$ let $\mathscr{F}_\mu := \psi_{\mu\lambda}^* \mathscr{F}_\lambda$; since $\beta_\mathscr{F}$ is an isomorphism, *loc.cit.* shows that $\beta_{\mathscr{F}_\mu}$ is already an isomorphism for some $\mu \geq \lambda$, whence (ii.b).

(iii): By virtue of (i), we may assume that $f^*\mathscr{F}$ is reflexive, and we need to show that the same holds for $\mathscr{F}$. However, the natural map $f^*(\mathscr{F}^{\vee\vee}) \to (f^*\mathscr{F})^{\vee\vee}$ is an isomorphism ([28, Ch.0, Prop.12.3.5]), hence $f^*\beta_\mathscr{F} = \beta_{f^*\mathscr{F}}$ is an isomorphism. Since f is faithfully flat, we deduce that $\beta_\mathscr{F}$ is an isomorphism, as stated. $\square$

Proposition 5.6.7. *Let X be a coherent, reduced, quasi-compact and quasi-separated scheme, $U \subset X$ a quasi-compact open subset. We have :*

(i) *The restriction functor $\mathscr{O}_X$-$\mathbf{Rflx} \to \mathscr{O}_U$-$\mathbf{Rflx}$ is essentially surjective.*
(ii) *Suppose furthermore, that $\delta'(x, \mathscr{O}_X) \geq 2$ for every $x \in X \setminus U$. Then the natural map*
$$\mathscr{F} \to j_* j^* \mathscr{F}$$
is an isomorphism, for every reflexive $\mathscr{O}_X$-module $\mathscr{F}$.

Proof. (i): Given a reflexive $\mathscr{O}_U$-module $\mathscr{F}$, lemma 5.2.16(ii) says that we can find a finitely presented quasi-coherent $\mathscr{O}_X$-module $\mathscr{G}$ extending $\mathscr{F}^\vee$; since X is coherent, $\mathscr{G}$ is a coherent $\mathscr{O}_X$-module, hence the same holds for $\mathscr{G}^\vee$, which extends $\mathscr{F}$ and is reflexive in light of lemma 5.6.2.

(ii): Since the assertion is local on X, we can suppose that there exists a left exact sequence $0 \to \mathscr{F} \to \mathscr{O}_X^{\oplus m} \to \mathscr{O}_X^{\oplus n}$ (see remark 5.6.4). Since the functor j_* is left exact, it then suffices to prove the contention for the sheaves $\mathscr{O}_X^{\oplus m}$ and $\mathscr{O}_X^{\oplus n}$, and thus we may assume from start that $\mathscr{F} = \mathscr{O}_X$. Then, since $X \setminus U$ is constructible, corollary 5.4.22 applies and yields the assertion. $\square$

Corollary 5.6.8. *Let X be a coherent scheme, $f : X \to S$ be a flat, locally finitely presented morphism, $j : U \to X$ a quasi-compact open immersion, $\mathscr{F}$ a reflexive $\mathscr{O}_X$-module. Suppose that*

(a) $\mathrm{depth}_f(x) \geq 1$ *for every point $x \in X \setminus U$, and*
(b) $\mathrm{depth}_f(x) \geq 2$ *for every maximal point η of S and every $x \in (X \setminus U) \cap f^{-1}(\eta)$.*
(c) *$\mathscr{O}_S$ has no imbedded points.*

Then the natural morphism $\mathscr{F} \to j_* j^* \mathscr{F}$ is an isomorphism.

Proof. Since f is flat and $\mathscr{O}_S$ has no imbedded points, corollary 5.4.39 and our assumptions (a) and (b) imply that $\delta'(x, \mathscr{O}_X) \geq 2$ for every $x \in X \setminus U$, so the assertion follows from proposition 5.6.7(ii). $\square$

5.6.9. Let X be any scheme. Recall that the *rank* of an $\mathscr{O}_X$-module $\mathscr{F}$ of finite type, is the upper semicontinuous function:
$$\mathrm{rk}\, \mathscr{F} : X \to \mathbb{N} \qquad x \mapsto \dim_{\kappa(x)} \mathscr{F}_x \otimes_{\mathscr{O}_{X,x}} \kappa(x).$$
Clearly, if $\mathscr{F}$ is a is locally free $\mathscr{O}_X$-module of finite type, $\mathrm{rk}\, \mathscr{F}$ is a continuous function on X. The converse holds, provided X is a reduced scheme. Moreover, if $\mathscr{F}$ is of finite presentation, $\mathrm{rk}\, \mathscr{F}$ is a constructible function and there exists a dense open subset $U \subset X$ such that $\mathrm{rk}\, \mathscr{F}$ restricts to a continuous function on U. We denote by $\mathbf{Pic}\, X$ the full subcategory of $\mathscr{O}_X$-$\mathbf{Mod}_{\mathrm{lfft}}$ consisting of all the objects whose rank is constant equal to one (*i.e.* the *invertible* $\mathscr{O}_X$-modules).

In case X is coherent, we shall also consider the category $\mathbf{Div}\,X$ of *generically invertible* $\mathscr{O}_X$-modules, defined as the full subcategory of $\mathscr{O}_X$-$\mathbf{Rflx}$ consisting of all objects which are locally free of rank one on a dense open subset of X. If X is coherent, $\mathbf{Pic}\,X$ is a full subcategory of $\mathbf{Div}X$.

Remark 5.6.10. (i) Let A be any integral domain, and set $X = \mathrm{Spec}\,A$. Classically, one has a notion of reflexive fractional ideal of A (see [61, p.80] or example 3.4.21). Suppose now that X is also coherent, in which case we have the notion of reflexive $\mathscr{O}_X$-module of (5.6). We claim that these two notions overlap on the subclass of reflexive fractional ideals of finite type: more precisely, let $\mathbf{Div}(A)$ be the full subcategory of A-$\mathbf{Mod}$ whose objects are the finitely generated reflexive fractional ideals of A. Then the essential image of the natural functor

$$(5.6.11) \qquad \mathbf{Div}(A) \to \mathscr{O}_X\text{-}\mathbf{Mod} \qquad M \mapsto M^\sim$$

is the category $\mathbf{Div}\,X$, and (5.6.11) yields an equivalence of $\mathbf{Div}(A)$ with the latter category. Indeed, let $\mathscr{F}$ be any generically invertible $\mathscr{O}_X$-module, and set $I := \mathscr{F}(X)$; if K denotes the field of fractions of A, then $\dim_K I \otimes_A K = 1$. Let us then fix a K-linear isomorphism $I \otimes_A K \xrightarrow{\sim} K$, and notice that the induced A-linear map $I \to K$ is injective, since $\mathscr{F}$ is S_1 (remark 5.6.4). We may then view I as a finitely generated A-submodule of K, and then it is clear that I is a fractional ideal of A. Moreover, on the one hand $\mathscr{F}^\vee$ is the coherent $\mathscr{O}_X$-module $\mathrm{Hom}_A(I, A)^\sim$; on the other hand, the natural map $\mathrm{Hom}_A(I, A) \to \mathrm{Hom}_K(I \otimes_A K, K) = K$ is injective, and its image is the fractional ideal I^{-1} (see (3.4.18)). We easily deduce that I is a reflexive fractional ideal, and conversely it is easily seen that every $\mathscr{O}_X$-module in the essential image of (5.6.11) is generically invertible; since this functor is also obviously fully faithful, the assertion follows.

(ii) Let A be a coherent integral domain, and denote by $\mathrm{coh.Div}(A)$ the set of all coherent reflexive fractional ideals of A. It is easily seen that $\mathrm{coh.Div}(A)$ is a submonoid of $\mathrm{Div}(A)$, for the natural monoid structure introduced in example 3.4.21. More generally, if X is a coherent integral scheme, we may define a sheaf of monoids $\mathscr{D}iv_X$ on X, as follows. First, to any affine open subset $U \subset X$, we assign the monoid $\mathscr{D}iv_X(U) := \mathrm{coh.Div}(\mathscr{O}_X(U))$. For each inclusion $j : U' \subset U$ of affine open subset, notice that the restriction map $\mathscr{O}_X(U) \to \mathscr{O}_X(U')$ is a flat ring homomorphism, hence it gives a flat morphism of monoids $\mathscr{O}_X(U) \setminus \{0\} \to \mathscr{O}_X(U') \setminus \{0\}$. We then have an induced map of monoids $\mathscr{D}iv_X(j) : \mathscr{D}iv_X(U) \to \mathscr{D}iv_X(U')$, by virtue of lemma 3.4.27(iv). It is easily seen that the resulting presheaf $\mathscr{D}iv_X$ is a sheaf on the site of all affine open subsets of X. By [26, Ch.0, §3.2.2], the latter extends uniquely to a sheaf of monoids on X, which we denote again $\mathscr{D}iv_X$. We set

$$\mathrm{Div}(X) := \mathscr{D}iv_X(X).$$

If X is normal and locally noetherian (or more generally, if X is a *Krull scheme*, *i.e.* $\mathscr{O}_X(U)$ is a Krull ring, for every affine open subset $U \subset X$) then $\mathscr{D}iv_X$ is an abelian sheaf, and $\mathrm{Div}(X)$ is an abelian group (proposition 3.4.25(i,iii)).

5.6.12. For future reference, it is useful to recall some preliminaries concerning the determinant functors defined in [55]. Let X be a scheme. We denote by $\mathbf{gr.Pic}\,X$ the category of graded invertible $\mathscr{O}_X$-modules. An object of $\mathbf{gr.Pic}\,X$ is a pair (L, α), where L is an invertible $\mathscr{O}_X$-module and $\alpha : X \to \mathbb{Z}$ is a continuous function. A homomorphism $h : (L, \alpha) \to (M, \beta)$ is a homomorphism of $\mathscr{O}_X$-modules $h : L \to M$ such that $h_x = 0$ for every $x \in X$ with $\alpha(x) \neq \beta(x)$. We denote by $\mathbf{gr.Pic}^*X$ the subcategory of $\mathbf{gr.Pic}\,X$ with the same objects, and whose morphisms are the isomorphisms in $\mathbf{gr.Pic}\,X$. Notice that $\mathbf{gr.Pic}\,X$ is a tensor category : the tensor product of two objects (L, α) and (M, β) is the pair $(L, \alpha) \otimes (M, \beta) := (L \otimes_{\mathscr{O}_X} M, \alpha + \beta)$. We denote by $\mathscr{O}_X$-$\mathbf{Mod}^*_{\mathrm{lfft}}$ the category whose objects are the locally free $\mathscr{O}_X$-modules of finite type, and whose morphisms are the $\mathscr{O}_X$-linear isomorphisms. The *determinant* is the functor:

$$\det : \mathscr{O}_X\text{-}\mathbf{Mod}^*_{\mathrm{lfft}} \to \mathbf{gr}.\mathbf{Pic}^* X \qquad F \mapsto (\Lambda^{\mathrm{rk}\, F}_{\mathscr{O}_X} F, \mathrm{rk}\, F).$$

Let $\mathsf{D}(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{perf}}$ be the category of perfect complexes of $\mathscr{O}_X$-modules; recall that, by definition, every perfect complex is locally isomorphic to a bounded complex of locally free $\mathscr{O}_X$-modules of finite type. The category $\mathsf{D}(\mathscr{O}_X\text{-}\mathbf{Mod})^*_{\mathrm{perf}}$ is the subcategory of $\mathsf{D}(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{perf}}$ with the same objects, and whose morphisms are the isomorphisms (*i.e.* the quasi-isomorphisms of complexes). The main theorem of chapter 1 of [55] can be stated as follows.

Lemma 5.6.13. *(*[55, Th.1]*) With the notation of* (5.6.12) *there exists, for every scheme X, an extension of the determinant functor to a functor:*

$$\mathsf{det} : \mathsf{D}(\mathscr{O}_X\text{-}\mathbf{Mod})^*_{\mathrm{perf}} \to \mathbf{gr}.\mathbf{Pic}\, X.$$

These determinant functors commute with every base change. □

Proposition 5.6.14. *Let X be a regular scheme. Then every reflexive generically invertible $\mathscr{O}_X$-module is invertible.*

Proof. The question is local on X, hence we may assume that X is affine. Let $\mathscr{F}$ be a generically invertible $\mathscr{O}_X$-module, and $U \subset X$ a dense open subset such that $\mathscr{F}_{|U}$ is invertible. Denote by $Z_1, \dots, Z_t$ the irreducible components of $Z := X \setminus U$ whose codimension in X equals one, and for every $i = 1, \dots, t$, let η_i be the maximal point of Z_i, and set $A_i := \mathscr{O}_{X,\eta_i}$. Since $\mathscr{F}$ is S_1 (remark 5.6.4), the stalk $\mathscr{F}_{\eta_i}$ is a torsion-free A_i-module of finite type for every $i \leq t$; however, A_i is a discrete valuation ring, hence $\mathscr{F}_{\eta_i}$ is a free A_i-module, necessarily of rank one, for $i = 1, \dots, t$. Since $\mathscr{F}$ is coherent, it follows that there exists an open neighborhood U_i of η_i in X, such that $\mathscr{F}_{|U_i}$ is a free $\mathscr{O}_{|U_i}$-module. Hence, we may replace U by $U \cup U_1 \cup \dots \cup U_t$ and assume that every irreducible component of Z has codimension > 1, therefore $\delta'(x, \mathscr{O}_X) \geq 2$ for every $x \in Z$. Now, $\mathscr{F}[0]$ is a perfect complex by Serre's theorem ([75, Th.4.4.16]), so the invertible $\mathscr{O}_X$-module $\mathsf{det}\,\mathscr{F}$ is well defined (lemma 5.6.13). Let $j : U \to X$ be the open immersion; in view of proposition 5.6.7, we deduce natural isomorphisms

$$\mathscr{F} \xrightarrow{\sim} j_* j^* \mathscr{F} \xrightarrow{\sim} j_* \mathsf{det}(j^* \mathscr{F}[0]) \xrightarrow{\sim} j_* j^* \mathsf{det}\, \mathscr{F}[0] \xleftarrow{\sim} \mathsf{det}\, \mathscr{F}[0]$$

and the assertion follows. □

We wish now to introduce a notion of duality better suited to derived categories of $\mathscr{O}_X$-modules (over a scheme X). Hereafter we only carry out a preliminary investigation of such derived duality – the full development of which, will be the task of section 5.8.

Definition 5.6.15. Let X be a coherent scheme. A complex $\omega^\bullet$ in $\mathsf{D}^b(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{coh}}$ (notation of (5.2.1)) is called *dualizing* if it fulfills the following two conditions :

(a) The functor :

$$\mathscr{D} : \mathsf{D}(\mathscr{O}_X\text{-}\mathbf{Mod})^o \to \mathsf{D}(\mathscr{O}_X\text{-}\mathbf{Mod}) \qquad C^\bullet \mapsto R\mathscr{H}om^\bullet_{\mathscr{O}_X}(C^\bullet, \omega^\bullet)$$

restricts to a *duality functor* : $\mathscr{D} : \mathsf{D}^b(\mathscr{O}_X\text{-}\mathbf{Mod})^o_{\mathrm{coh}} \to \mathsf{D}^b(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{coh}}$.

(b) The natural transformation : $\eta_{C^\bullet} : C^\bullet \to \mathscr{D} \circ \mathscr{D}(C^\bullet)$ restricts to a *biduality isomorphism* of functors on the category $\mathsf{D}^b(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{coh}}$.

Remark 5.6.16. Let X be a coherent scheme, $\omega^\bullet$ an object of $\mathsf{D}^b(\mathscr{O}\text{-}\mathbf{Mod})_{\mathrm{coh}}$, and define the functor $\mathscr{D}$ as in definition 5.6.15. A standard *dévissage* argument shows that $\omega^\bullet$ is dualizing on X if and only if $\mathscr{D}(\mathscr{F}[0])$ lies in $\mathsf{D}^b(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{coh}}$, and the biduality map $\mathscr{F}[0] \to \mathscr{D} \circ \mathscr{D}(\mathscr{F}[0])$ is an isomorphism for every $\mathscr{O}_X$-module $\mathscr{F}$.

Example 5.6.17. Suppose that X is a noetherian regular scheme (*i.e.* all the stalks $\mathscr{O}_{X,x}$ are regular rings). In light of Serre's theorem [75, Th.4.4.16], every object of $\mathsf{D}^b(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{coh}}$ is a perfect complex. It follows easily that the complex $\mathscr{O}_X[0]$ is dualizing. For more general schemes, the structure sheaf does not necessarily work, and the existence of a dualizing complex is a delicate issue. On the other hand, one may ask to what extent a complex is determined by the properties (a) and (b) of definition 5.6.15. Clearly, if $\omega^\bullet$ is dualizing on X, then so is any other complex of the form $\omega^\bullet \otimes_{\mathscr{O}_X} \mathscr{L}$, where $\mathscr{L}$ is an invertible $\mathscr{O}_X$-module. Also, any shift of $\omega^\bullet$ is again dualizing. Conversely, the following proposition 5.6.24 says that any two dualizing complexes are related in such manner, up to quasi-isomorphism.

Lemma 5.6.18. *Let $(X, \mathscr{O}_X)$ be any locally ringed space, and $P^\bullet$ and $Q^\bullet$ two objects of $\mathsf{D}^-(\mathscr{O}_X\text{-}\mathbf{Mod})$ with a quasi-isomorphism :*

$$P^\bullet \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_X} Q^\bullet \xrightarrow{\sim} \mathscr{O}_X[0].$$

Then there exists an invertible $\mathscr{O}_X$-module $\mathscr{L}$, a continuous function $\sigma : |X| \to \mathbb{Z}$ and quasi-isomorphisms:

$$P^\bullet \xrightarrow{\sim} \mathscr{L}[\sigma] \qquad \textit{and} \qquad Q^\bullet \xrightarrow{\sim} \mathscr{L}^{-1}[-\sigma].$$

Proof. It suffices to verify that, locally on X, the complexes $P^\bullet$ and $Q^\bullet$ are of the required form; indeed in this case X will be a disjoint union of open sets U_n on which $H^\bullet P^\bullet$ is concentrated in degree n and $H^\bullet Q^\bullet$ is concentrated in degree $-n$. Note that $\mathscr{O}_X\text{-}\mathbf{Mod}$ is equivalent to the product of the categories $\mathscr{O}_{U_n}\text{-}\mathbf{Mod}$ and that the derived category of a product of abelian categories is the product of the derived categories of the factors.

Claim 5.6.19. Let A be a (commutative) local ring, $K^\bullet$ and $L^\bullet$ two objects of $\mathsf{D}^-(A\text{-}\mathbf{Mod})$ with a quasi-isomorphism

$$K^\bullet \overset{\mathbf{L}}{\otimes}_A L^\bullet \xrightarrow{\sim} A[0]. \tag{5.6.20}$$

Then there exists $s \in \mathbb{Z}$ and quasi-isomorphisms :

$$K^\bullet \xrightarrow{\sim} A[s] \qquad L^\bullet \xrightarrow{\sim} A[-s].$$

Proof of the claim. Set :

$$i_0 := \max(i \in \mathbb{Z} \mid (H^i K^\bullet) \neq 0) \quad \text{and} \quad j_0 := \max(i \in \mathbb{Z} \mid (H^i L^\bullet) \neq 0).$$

We may assume that $K^i = 0$ for every $i > i_0$, and that $L^\bullet$ is a bounded above complex of free A-modules. Then we may find a filtered system $(K^\bullet_\lambda \mid \lambda \in \Lambda)$ of complexes of A-modules bounded from above, such that

- $H^{i_0}(K^\bullet_\lambda)$ is a finitely generated A-module, for every $\lambda \in \Lambda$;
- the colimit of the system $(K^\bullet_\lambda \mid \lambda \in \Lambda)$ (in the category of complexes of A-modules) is isomorphic to $K^\bullet$.

From (5.6.20) we get an isomorphism $H^0(K^\bullet \otimes_A L^\bullet) \xrightarrow{\sim} A$, and it follows easily that the natural map $H^0(K^\bullet_\lambda \otimes_A L^\bullet) \to H^0(K^\bullet \otimes_A L^\bullet)$ is surjective for some $\lambda \in \Lambda$. For such λ, we may then find a morphism in $\mathsf{D}^-(A\text{-}\mathbf{Mod})$:

$$A[0] \to K^\bullet_\lambda \overset{\mathbf{L}}{\otimes}_A L^\bullet \tag{5.6.21}$$

whose composition with the natural map $K^\bullet_\lambda \overset{\mathbf{L}}{\otimes}_A L^\bullet \to K^\bullet \overset{\mathbf{L}}{\otimes}_A L^\bullet$ is the inverse of (5.6.20). Hence

$$(5.6.21) \overset{\mathbf{L}}{\otimes}_A K^\bullet : K^\bullet \to (K^\bullet_\lambda \overset{\mathbf{L}}{\otimes}_A L^\bullet) \overset{\mathbf{L}}{\otimes}_A K^\bullet \xrightarrow{\sim} K^\bullet_\lambda \overset{\mathbf{L}}{\otimes}_A (L^\bullet \overset{\mathbf{L}}{\otimes}_A K^\bullet) \xrightarrow{\sim} K^\bullet_\lambda$$

is a right inverse of the natural morphism $K^\bullet_\lambda \to K^\bullet$ in $\mathsf{D}^-(A\text{-}\mathbf{Mod})$. Especially, the induced map $H^{i_0}K^\bullet_\lambda \to H^{i_0}K^\bullet$ is surjective, *i.e.* $H^{i_0}K^\bullet$ is a finitely generated A-module. Likewise, we see that $H^{j_0}L^\bullet$ is a finitely generated A-module. Now, notice that

$$H^k(K^\bullet \overset{\mathbf{L}}{\otimes}_A L^\bullet) \simeq \begin{cases} H^{i_0}K^\bullet \otimes_A H^{j_0}L^\bullet & \text{for } k = i_0 + j_0 \\ 0 & \text{for } k > i_0 + j_0. \end{cases}$$

From Nakayama's lemma it follows easily that $H^{i_0+j_0}(K^\bullet \overset{\mathbf{L}}{\otimes}_A L^\bullet) \neq 0$, and then our assumptions imply that $i_0 + j_0 = 0$ and $H^{i_0}K^\bullet \otimes_A H^{j_0}L^\bullet \simeq A$. One deduces easily that $H^{i_0}K^\bullet \simeq A \simeq H^{j_0}L^\bullet$ (see *e.g.* [36, Lemma 4.1.5]). Furthermore, we can find a complex $K^\bullet_1$ in $\mathsf{D}^{<i_0}(A\text{-}\mathbf{Mod})$ (resp. $L^\bullet_1$ in $\mathsf{D}^{<j_0}(A\text{-}\mathbf{Mod})$) such that :

$$K^\bullet \simeq A[-i_0] \oplus K^\bullet_1 \qquad \text{(resp. } L^\bullet \simeq A[-j_0] \oplus L^\bullet_1\text{)}$$

whence a quasi-isomorphism :

$$\varphi : A[0] \xrightarrow{\sim} K^\bullet \overset{\mathbf{L}}{\otimes}_A L^\bullet \xrightarrow{\sim} A[0] \oplus K^\bullet_1[-j_0] \oplus L^\bullet_1[-i_0] \oplus (K^\bullet_1 \overset{\mathbf{L}}{\otimes}_A L^\bullet_1).$$

However, by construction φ^{-1} restricts to an isomorphism on the direct summand $A[0]$, therefore $K^\bullet_1 \simeq 0 \simeq L^\bullet_1$ in $\mathsf{D}(A\text{-}\mathbf{Mod})$, and the claim follows. ◊

Now, for any point $x \in X$, let $i_x : \{x\} \to X$ be the inclusion map, and set $K^\bullet_x := i^*_x K^\bullet$ for every complex $K^\bullet$ of $\mathscr{O}_X$-modules. Notice that, if $K^\bullet$ is a complex of flat $\mathscr{O}_X$-modules, then $K^\bullet_x$ is a complex of flat $\mathscr{O}_{X,x}$-modules ([4, Exp.V, Prop.1.6(1)]), therefore the rule $K^\bullet \mapsto K^\bullet_x$ yields a well-defined functor $\mathsf{D}^-(\mathscr{O}_X\text{-}\mathbf{Mod}) \to \mathsf{D}^-(\mathscr{O}_{X,x}\text{-}\mathbf{Mod})$, and moreover we have a natural isomorphism

$$K^\bullet_x \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_{X,x}} L^\bullet_x \xrightarrow{\sim} (K^\bullet \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_X} L^\bullet)_x \tag{5.6.22}$$

for every objects $K^\bullet, L^\bullet$ of $\mathsf{D}^-(\mathscr{O}_X\text{-}\mathbf{Mod})$. Especially, under the current assumptions, and in view of claim 5.6.19, we may find $s \in \mathbb{Z}$, and an isomorphism of $\mathscr{O}_{X,x}$-modules

$$H^s P^\bullet_x \otimes_{\mathscr{O}_{X,x}} H^{-s} Q^\bullet_x \xrightarrow{\sim} H^0(P^\bullet_x \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_{X,x}} Q^\bullet_x) \xrightarrow{\sim} \mathscr{O}_{X,x}. \tag{5.6.23}$$

Since $\mathscr{O}_{X,x}$ is local, we may thus find $a_x \in H^s P^\bullet_x$ and $b_x \in H^{-s}Q^\bullet_x$ such that (5.6.23) maps $a_x \otimes b_x$ to 1. Then a_x and b_x extend to local sections

$$a \in \Gamma(U, \mathrm{Ker}(d : P^s \to P^{s-1})) \qquad b \in \Gamma(U, \mathrm{Ker}(d : Q^{-s} \to P^{-s-1}))$$

on some neighborhood $U \subset X$ of x, and after shrinking U, we may assume that $a \otimes b$ gets mapped to 1, under the induced morphism of $\mathscr{O}_U$-modules

$$H^s P^\bullet_{|U} \otimes_{\mathscr{O}_U} H^{-s} Q^\bullet_{|U} \to H^0(P^\bullet \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_X} Q^\bullet)_{|U} \xrightarrow{\sim} \mathscr{O}_U.$$

Then we obtain a well defined morphism in $\mathsf{D}^-(\mathscr{O}_U\text{-}\mathbf{Mod})$

$$\varphi : \mathscr{O}_U[s] \to P^\bullet_{|U} \qquad \text{(resp.} \quad \psi : \mathscr{O}_U[-s] \to Q^\bullet_{|U}\text{)}$$

by the rule : $t \mapsto t \cdot a$ (resp. $t \mapsto t \cdot b$) for every local section t of $\mathscr{O}_U$. Again by claim 5.6.19 and (5.6.22) we deduce that $\varphi_y : \mathscr{O}_{U,y}[s] \to P^\bullet_y$ is a quasi-isomorphism for every $y \in U$ (and likewise for ψ_y); . *i.e.* φ and ψ are the sought isomorphisms in $\mathsf{D}^-(\mathscr{O}_U\text{-}\mathbf{Mod})$. □

Proposition 5.6.24. *Suppose that $\omega^\bullet_1$ and $\omega^\bullet_2$ are two dualizing complexes for the coherent scheme X. Then there exists an invertible $\mathscr{O}_X$-module $\mathscr{L}$ and a continuous function $\sigma : |X| \to \mathbb{Z}$ such that*

$$\omega^\bullet_2 \simeq \omega^\bullet_1 \otimes_{\mathscr{O}_X} \mathscr{L}[\sigma] \qquad \textit{in } \mathsf{D}^b(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{coh}}.$$

Proof. Denote by $\mathscr{D}_1$ and $\mathscr{D}_2$ the duality functors associated to ω_1 and respectively ω_2. By assumption, we can find complexes $P^\bullet, Q^\bullet$ in $\mathsf{D}^b(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{coh}}$ such that $\omega_2 \simeq \mathscr{D}_1(P^\bullet)$ and $\omega_1 \simeq \mathscr{D}_2(Q^\bullet)$, and therefore

$$\mathscr{D}_2(\mathscr{F}^\bullet) \simeq R\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}^\bullet, \mathscr{D}_1(P^\bullet)) \simeq \mathscr{D}_1(\mathscr{F}^\bullet \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_X} P^\bullet)$$

for every object $\mathscr{F}^\bullet$ of $\mathsf{D}^b(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{coh}}$ ([75, Th.10.8.7]).

Claim 5.6.25. Let $C^\bullet$ be an object of $\mathsf{D}^-(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{coh}}$, such that $\mathscr{D}_1(C^\bullet)$ is in $\mathsf{D}^b(\mathscr{O}_X\text{-}\mathbf{Mod})$. Then $C^\bullet$ is in $\mathsf{D}^b(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{coh}}$.

Proof of the claim. For given $m, n \in \mathbb{N}$, the natural maps: $\tau_{-n]}C^\bullet \xrightarrow{\alpha} C^\bullet \xrightarrow{\beta} \tau_{[-m}C^\bullet$ induce morphisms

$$\mathscr{D}_1(\tau_{[-m}C^\bullet) \xrightarrow{\mathscr{D}_1(\beta)} \mathscr{D}_1(C^\bullet) \xrightarrow{\mathscr{D}_1(\alpha)} \mathscr{D}_1(\tau_{-n]}C^\bullet)$$

Say that $\omega^\bullet \simeq \tau_{[a}\omega^\bullet$ for some integer $a \in \mathbb{N}$. Then $\mathscr{D}_1(\tau_{-n]}C^\bullet)$ lies in $\mathsf{D}^{\geq n+a}(\mathscr{O}_X\text{-}\mathbf{Mod})$. Since $\mathscr{D}_1(C^\bullet)$ is bounded, it follows that $\mathscr{D}_1(\beta) = 0$ for n large enough. Consider now the commutative diagram :

$$\begin{tikzcd}
\tau_{[-n}C^\bullet \arrow[r, "\beta\circ\alpha"] \arrow[d] & \tau_{[-m}C^\bullet \arrow[d, "\eta"] \\
\mathscr{D}_1 \circ \mathscr{D}_1(\tau_{[-n}C^\bullet) \arrow[r, "\mathscr{D}_1\circ\mathscr{D}_1(\beta\circ\alpha)"] & \mathscr{D}_1 \circ \mathscr{D}_1(\tau_{[-m}C^\bullet)
\end{tikzcd}$$

Since $\tau_{[-m}C^\bullet$ is a bounded complex, η is an isomorphism in $\mathsf{D}(\mathscr{O}_X\text{-}\mathbf{Mod})$, so $\beta \circ \alpha = 0$ whenever n is large enough. Clearly this means that $C^\bullet$ is bounded, as claimed. ◊

Applying claim 5.6.25 to $C^\bullet := \mathscr{F}^\bullet \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_X} P^\bullet$ (which is in $\mathsf{D}^-(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{coh}}$, since X is coherent) we see that the latter is a bounded complex, and by reversing the roles of ω_1 and ω_2 it follows that the same holds for $\mathscr{F}^\bullet \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_X} Q^\bullet$. We then deduce isomorphisms in $\mathsf{D}^b(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{coh}}$

$$\mathscr{D}_1 \circ \mathscr{D}_2(\mathscr{F}^\bullet) \simeq \mathscr{F}^\bullet \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_X} P^\bullet \qquad \text{and} \qquad \mathscr{D}_2 \circ \mathscr{D}_1(\mathscr{F}^\bullet) \simeq \mathscr{F}^\bullet \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_X} Q^\bullet.$$

Letting $\mathscr{F}^\bullet := \mathscr{O}_X[0]$ we derive :

$$\mathscr{O}_X[0] \simeq \mathscr{D}_2 \circ \mathscr{D}_1 \circ \mathscr{D}_1 \circ \mathscr{D}_2(\mathscr{O}_X[0]) \simeq \mathscr{D}_2 \circ \mathscr{D}_1(P^\bullet) \simeq P^\bullet \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_X} Q^\bullet.$$

Then lemma 5.6.18 says that $P^\bullet \simeq \mathscr{E}[\tau]$ for an invertible $\mathscr{O}_X$-module $\mathscr{E}$ and a continuous function $\tau : |X| \to \mathbb{Z}$. Consequently : $\omega_2 \simeq \mathscr{E}^\vee[-\tau] \otimes_{\mathscr{O}_X} \omega_1$, so the proposition holds with $\mathscr{L} := \mathscr{E}^\vee$ and $\sigma := -\tau$. □

Proposition 5.6.26. *Let A be a noetherian local ring, κ its residue field, $M^\bullet$ a bounded complex of finitely generated A-modules. Set $X := \operatorname{Spec} A$ and suppose that there exists $c \in \mathbb{Z}$ such that*

$$R\mathrm{Hom}^\bullet_A(\kappa[0], M^\bullet) \simeq \kappa[c].$$

Then the complex of $\mathscr{O}_X$-modules $M^{\bullet\sim}$ arising from $M^\bullet$ is dualizing on X.

Proof. In view of remark 5.6.16 and corollary 5.1.33, it suffices to check :

Claim 5.6.27. For every finitely generated A-module N, the following holds :

(i) $D^\bullet(N) := R\mathrm{Hom}^\bullet_A(N[0], M^\bullet)$ is a bounded complex.

(ii) The natural map

$$N[0] \to DD^\bullet(N) := R\mathrm{Hom}^\bullet_A(D^\bullet(N), M^\bullet)$$

is an isomorphism.

Proof of the claim. We first show the claim for $N = \kappa$, in which case (i) holds by assumption. To check (ii), let $M^\bullet \xrightarrow{\sim} I^\bullet$ be a resolution consisting of a bounded below complex of injective A-modules, so that

$$D^\bullet(\kappa[0], M^\bullet) \xrightarrow{\sim} \mathrm{Hom}^\bullet_A(\kappa[0], I^\bullet) \xrightarrow{\sim} I^\bullet[\mathfrak{m}]$$

where $\mathfrak{m} \subset A$ is the maximal ideal, and $I^k[\mathfrak{m}]$ denotes the submodule of $\mathfrak{m}$-torsion elements in I^k, for every $k \in \mathbb{Z}$. Under these isomorphisms, it is easily seen that the biduality map of (ii) is identified with the unique one

$$\kappa \to H := \mathrm{Hom}_{\mathsf{Hot}(A\text{-}\mathbf{Mod})}(I^\bullet[\mathfrak{m}], I^\bullet)$$

that sends $1 \in \kappa$ to the inclusion map $j^\bullet : I^\bullet[\mathfrak{m}] \to I^\bullet$ (details left to the reader). Now, it is clear that any morphism $I^\bullet[\mathfrak{m}] \to I^\bullet$ in $\mathsf{Hot}(A\text{-}\mathbf{Mod})$ factors through $j^\bullet$, and on the other hand, our assumption implies that $H \simeq \kappa$. Hence, pick a morphism $f^\bullet : I^\bullet[\mathfrak{m}] \to I^\bullet$ representing a generator for the A-module H, and write $f^\bullet = j^\bullet \circ g^\bullet$ for some endomorphism $g^\bullet$ of $I^\bullet[\mathfrak{m}]$; in other words, $f^\bullet$ is the image of $j^\bullet$ under the A-linear map

$$\mathrm{Hom}_{\mathsf{Hot}(A\text{-}\mathbf{Mod})}(g^\bullet, I^\bullet) : H \to H$$

so the class of $j^\bullet$ cannot vanish in H, and (ii) follows in this case.

Next, we shall argue by induction on $d := \dim \mathrm{Supp}\, N$. If $d = 0$, then N is an A-module of finite length, in which case we argue by induction on the length l of N. If $l = 1$, we have $N \simeq \kappa$, so the assertions are already known. Suppose $l > 1$, and that both (i) and (ii) are already known for all A-modules of length $< d$; we may find an A-submodule $N' \subset N$ such that both N' and $N'' := N/N'$ have length $< d$. From the inductive assumption for N' and N'', and the induced distiguished triangle

$$D^\bullet(N') \to D^\bullet(N) \to D(N'') \to D^\bullet(N')[1]$$

we deduce that (i) holds for N. Likewise, since (ii) is known for both N' and N'', using the 5-lemma we deduce easily that the same holds also for N.

Lastly, suppose that $d > 0$, and both (i) and (ii) are already known for all A-modules of finite type whose support has dimension $< d$. Let $N' := \Gamma_{\{\mathfrak{m}\}} N$; both (i) and (ii) are already known for N', so the same *dévissage* argument as in the foregoing reduces to showing the claim for N/N', *i.e.* we may assume that $\mathfrak{m} \notin \mathrm{Ass}\, N$. Thus, let $t \in \mathfrak{m}$ be any element such that the scalar multiplication map $t \cdot \mathbf{1}_N$ is injective, so we have a short exact sequence

$$0 \to N \xrightarrow{t} N \to N_k := N/t^k N \to 0 \qquad \text{for every } k > 0.$$

Notice that $\dim \mathrm{Supp}\, N_k < d$: indeed, if $\mathfrak{p}$ is a minimal element of $\mathrm{Supp}\, N$, then $\mathfrak{p} \in \mathrm{Ass}\, N$ ([61, Th.6.5(iii)]), hence $t \notin \mathfrak{p}$, and therefore $\mathfrak{p} \notin \mathrm{Supp}\, N_k$. By inductive assumption, both (i) and (ii) hold for N_k (for every $k > 0$), and by considering the induced distinguished triangle

$$D^\bullet(N) \xrightarrow{t^k} D^\bullet(N) \to D^\bullet(N_k) \to D^\bullet(N)[1]$$

we deduce that scalar multiplication by t is an isomorphism on $H^k D^\bullet(N)$ whenever $|k|$ is large enough; but the latter is an A-module of finite type, so it must vanish, by Nakayama's lemma, and we conclude that (i) holds for N. Furthermore, by the same token we get an exact sequence

$$H^i DD^\bullet(N) \xrightarrow{t^k} H^i DD^\bullet(N) \to 0 \qquad \text{for every } i \neq 0$$

whence $H^iDD(N) = 0$ for $i \neq 0$, again by Nakayama's lemma. Lastly, for every $k > 0$ consider the ladder with exact rows :

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r, "t^k"] \arrow[d, "\alpha"'] & N \arrow[r] \arrow[d, "\alpha"] & N_k \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & H^0DD^\bullet(N) \arrow[r, "t^k"] & H^0DD^\bullet(N) \arrow[r] & H^0DD^\bullet(N_k) \arrow[r] & 0 \end{tikzcd}$$

whose right-most vertical arrow is an isomorphism, by inductive assumption. A simple diagram chase then yields

$$H^0DD^\bullet(N) = t^k \cdot H^0DD^\bullet(N) + \alpha(N)$$

so α is surjective, by Nakayama's lemma. To show the injectivity of α, let $x \in N$ be any non-zero element, and choose $k > 0$ such that $x \notin t^kN$, so the image of x does not vanish in N_k, whence necessarily $\alpha(x) \neq 0$, and the claim follows. $\square$

Lemma 5.6.28. *Let $f : X \to Y$ be a morphism of coherent schemes, and $\omega_Y^\bullet$ a dualizing complex on Y. We have :*

(i) *If f is finite and finitely presented, then $f^!\omega_Y^\bullet$ is dualizing on X.*
(ii) *If Y is quasi-compact and quasi-separated, and f is an open immersion, then $f^*\omega_Y^\bullet$ is dualizing on X.*

Proof. (i): Denote by $\overline{f} : (X, \mathscr{O}_X) \to (Y, f_*\mathscr{O}_X)$ the morphism of ringed spaces deduced from f. For any object $C^\bullet$ of $\mathsf{D}^-(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{coh}}$ we have natural isomorphisms :

$$\begin{aligned} \mathscr{D}(C^\bullet) := R\mathscr{H}om_{\mathscr{O}_X}(C^\bullet, f^!\omega_Y^\bullet) &\xrightarrow{\sim} R\mathscr{H}om_{\mathscr{O}_X}(C^\bullet, \overline{f}^*R\mathscr{H}om_{\mathscr{O}_Y}(f_*\mathscr{O}_X, \omega_Y^\bullet)) \\ &\xrightarrow{\sim} \overline{f}^*R\mathscr{H}om_{f_*\mathscr{O}_X}(f_*C^\bullet, R\mathscr{H}om_{\mathscr{O}_Y}(f_*\mathscr{O}_X, \omega_Y^\bullet)) \\ &\xrightarrow{\sim} \overline{f}^*R\mathscr{H}om_{\mathscr{O}_Y}(f_*C^\bullet, \omega_Y^\bullet). \end{aligned}$$

Hence, if $C^\bullet$ is in $\mathsf{D}^b(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{coh}}$, the same holds for $\mathscr{D}(C^\bullet)$, and we can compute :

$$\begin{aligned} \mathscr{D} \circ \mathscr{D}(C^\bullet) &\xrightarrow{\sim} \overline{f}^*R\mathscr{H}om_{\mathscr{O}_Y}(R\mathscr{H}om_{\mathscr{O}_Y}(f_*C^\bullet, \omega_Y^\bullet), \omega_Y^\bullet) \\ &\xrightarrow{\sim} \overline{f}^*f_*C^\bullet \xrightarrow{\sim} C^\bullet \end{aligned}$$

and by inspecting the definitions, one verifies that the resulting natural transformation $C^\bullet \to \mathscr{D} \circ \mathscr{D}(C^\bullet)$ is the biduality isomorphism. The claim follows.

(ii): Since $\mathscr{O}_Y$ is coherent, the natural map

$$f^*R\mathscr{H}om_{\mathscr{O}_Y}(\mathscr{G}[0], \omega_Y^\bullet) \to R\mathscr{H}om_{\mathscr{O}_X}(f^*\mathscr{G}[0], f^*\omega_Y^\bullet)$$

is an isomorphism in $\mathsf{D}^b(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{coh}}$, for every $\mathscr{O}_Y$-module $\mathscr{G}$. Then the assertion follows easily from lemma 5.2.16(ii) and remark 5.6.16. $\square$

5.6.29. For every coherent, quasi-compact and quasi-separated scheme X, consider the category Dual_X whose objects are the pairs $(U, \omega_U^\bullet)$, where $U \subset X$ is an open subset, and $\omega_U^\bullet$ is a dualizing complex on X. The morphisms $(U, \omega_U^\bullet) \to (U', \omega_{U'}^\bullet)$ are the pairs (j, β), where $j : U \to U'$ is an inclusion map of open subsets of X, and $\beta : j^*\omega_{U'}^\bullet \xrightarrow{\sim} \omega_U^\bullet$ is an isomorphism in $\mathsf{D}^b(\mathscr{O}_U\text{-}\mathbf{Mod})$. Let X_{Zar} denote the full subcategory of $\mathbf{Sch}/X$ whose objects are the (Zariski) open subsets of X; it follows easily from lemma 5.6.28(ii), that the forgetful functor

$$\mathsf{Dual}_X \to X_{\mathrm{Zar}} \qquad (U, \omega_U^\bullet) \mapsto U \tag{5.6.30}$$

is a fibration (see definition 1.4.1(ii)) With this notation, we have :

Proposition 5.6.31. *Every descent datum for the fibration* (5.6.30) *is effective.*

Proof. Let $((U_i, \omega_i^\bullet); \beta_{ij} \mid i, j = 1, \dots, n)$ be a descent datum for the fibration (5.6.30); this means that each $(U_i, \omega_i^\bullet)$ is an object of Dual_X, and if let $U_{ij} := U_i \cap U_j$ for every $i, j \leq n$, then

$$\beta_{ij} : \omega^\bullet_{i|U_{ij}} \xrightarrow{\sim} \omega^\bullet_{j|U_{ij}}$$

are isomorphisms in $\mathsf{D}^b(\mathscr{O}_{U_{ij}}\text{-}\mathbf{Mod})$, fulfilling a suitable cocycle condition (see (1.5.27)).

We shall show, by induction on $k = 1, \dots, n$, that there exists a dualizing complex $\omega^\bullet_{X_k}$ on $X_k := U_1 \cup \cdots \cup U_k$, such that the descent datum $((U_i, \omega_i^\bullet); \beta_{ij} \mid i, j = 1, \dots, k)$ is isomorphic to the descent datum relative to the family $(U_i \mid i = 1, \dots, k)$ determined by $(X_k, \omega^\bullet_{X_k})$.

For $k = 1$, there is nothing to prove. Next, suppose that $k > 1$, and $\omega^\bullet_{X_{k-1}}$ with the sought properties has already been constructed. If $k - 1 = n$, we are done; otherwise, let $V_k := X_k \cap U_{k+1}$, set $\mathfrak{U}_k := (U_{i,k+1} \mid i = 1, \dots, k)$, and $C^\bullet := R\mathscr{H}om^\bullet_{\mathscr{O}_{V_k}}(\omega^\bullet_{X_k|V_k}, \omega^\bullet_{k+1|V_k})$. The system of morphisms $(\beta_{i,k+1} \mid i = 1, \dots, k)$ determines a class c_k in the Čech cohomology group $H^0(\mathfrak{U}_k, C^\bullet)$. However, we have a spectral sequence (see lemma 5.2.6) :

$$E_2^{pq} := H^p(\mathfrak{U}_k, H^q C^\bullet) \Rightarrow H^{p+q} R\mathrm{Hom}^\bullet_{\mathscr{O}_{V_k}}(\omega^\bullet_{X_k|V_k}, \omega^\bullet_{k+1|V_k}).$$

Claim 5.6.32. $E_2^{pq} = 0$ for every $q < 0$.

Proof of the claim. More precisely, we check that $H^q C^\bullet = 0$ for every $q < 0$. Indeed, by lemma 5.6.28(ii), that both $\omega^\bullet_{X_k|V_k}$ and $\omega^\bullet_{k+1|V_k}$ are dualizing on V_k. Moreover, on $U_{i,k+1}$ they restrict to isomorphic objects of $\mathsf{D}^b(\mathscr{O}_{U_{i,k+1}}\text{-}\mathbf{Mod})$, for every $i = 1, \dots, k$. It follows easily from proposition 5.6.24 that there exists an invertible $\mathscr{O}_{V_k}$-module $\mathscr{L}$ and an isomorphism

$$\omega^\bullet_{X_k|V_k} \xrightarrow{\sim} \omega^\bullet_{k+1|V_k} \otimes_{\mathscr{O}_{V_k}} \mathscr{L} \qquad \text{in } \mathsf{D}^b(\mathscr{O}_{V_k}\text{-}\mathbf{Mod}).$$

Therefore, the biduality map induces an isomorphism

$$R\mathscr{H}om^\bullet_{\mathscr{O}_{V_k}}(\omega^\bullet_{X_k|V_k}, \omega^\bullet_{k+1|V_k}) \xrightarrow{\sim} \mathscr{L}[0]$$

whence the claim. ◊

From claim 5.6.32 we see that c_k corresponds to an element of $H^0 R\mathrm{Hom}^\bullet_{\mathscr{O}_{V_k}}(\omega^\bullet_{X_k|V_k}, \omega^\bullet_{k+1|V_k})$, and by construction, it is clear that this global section is an isomorphism $c_k : \omega^\bullet_{X_k|V_k} \xrightarrow{\sim} \omega^\bullet_{k+1|V_k}$ in $\mathsf{D}^b(\mathscr{O}_{V_k}\text{-}\mathbf{Mod})$. By definition, c_k is represented by a diagram of quasi-isomorphisms :

$$\omega^\bullet_{X_k|V_k} \leftarrow T^\bullet \to \omega^\bullet_{k+1|V_k} \qquad \text{in } \mathsf{C}(\mathscr{O}_{V_k}\text{-}\mathbf{Mod}) \tag{5.6.33}$$

for some complex $T^\bullet$. Let $T_!$ be the $\mathscr{O}_{X_{k+1}}$-module obtained as extension by zero of T, and define likewise $(\omega^\bullet_{X_k})_!$ and $(\omega^\bullet_{k+1})_!$. We let $\omega^\bullet_{X_{k+1}}$ be the cone of the map of complexes

$$T_! \to (\omega^\bullet_{X_k})_! \oplus (\omega^\bullet_{k+1})_!$$

deduced from (5.6.33). An easy inspection shows that this complex is dualizing on X_{k+1}, and it fulfills the stated condition. □

Remark 5.6.34. The proof of proposition 5.6.31 is a special case of a general technique for "glueing perverse sheaves" developed in [7].

In the study of duality for derived categories of modules, the role of reflexive modules is taken by the more general class of Cohen-Macaulay modules. There is a version of this theory for noetherian regular schemes, and a relative variant, for smooth morphisms. Let us begin by recalling the following :

Definition 5.6.35. Let $(A, \mathfrak{m}_A)$ be a local ring, M an A-module.

(i) The *dimension* of M, denoted $\dim_A M$, is the (Krull) dimension of $\mathrm{Supp}\, M \subset \mathrm{Spec}\, A$. (By convention, the empty set has dimension $-\infty$.)

(ii) If $\dim_A M = \mathrm{depth}_A M$, we say that M is a *Cohen-Macaulay* A-module. The category

$$A\text{-}\mathsf{CM}$$

is the full subcategory of A-$\mathbf{Mod}$ consisting of all finitely presented Cohen-Macaulay A-modules. For every $n \in \mathbb{N}$, we let A-CM_n be the full subcategory of A-CM whose objects are the Cohen-Macaulay A-modules of dimension n.

(iii) If A is a Cohen-Macaulay A-module, we say that A is a *Cohen-Macaulay* local ring.

(iv) Let X be a scheme, $\mathscr{F}$ a quasi-coherent $\mathscr{O}_X$-module. We say that $\mathscr{F}$ is a *Cohen-Macaulay* $\mathscr{O}_X$-module, if $\mathscr{F}_x$ is a Cohen-Macaulay $\mathscr{O}_{X,x}$-module, for every $x \in X$. We say that X is a *Cohen-Macaulay* scheme, if $\mathscr{O}_X$ is a Cohen-Macaulay $\mathscr{O}_X$-module.

(v) Let B be another local ring, $\varphi : A \to B$ a local ring homomorphism. The category

$$\varphi\text{-}\mathsf{CM}$$

of *φ-Cohen-Macaulay modules* is the full subcategory of B-$\mathbf{Mod}$ whose objects are all the finitely presented B-modules N that are φ-flat, and such that $N/\mathfrak{m}_A N$ is a Cohen-Macaulay B-module. For every $n \in \mathbb{N}$, we let φ-CM_n be the full subcategory of φ-CM whose objects are the φ-Cohen-Macaulay modules N with $\dim_B N/\mathfrak{m}_A N = n$.

(vi) Let $f : X \to Y$ be a locally finitely presented morphism of schemes, $\mathscr{F}$ a quasi-coherent $\mathscr{O}_X$-module, $x \in X$ any point, and set $y := f(x)$. We say that $\mathscr{F}$ is *f-Cohen-Macaulay* at the point x, if $\mathscr{F}_x$ is a $f^\natural_x$-Cohen-Macaulay module. We say that f is *Cohen-Macaulay* at the point x, if $\mathscr{O}_X$ is f-Cohen-Macaulay at the point x (cp. [31, Ch.IV, Déf.6.8.1]).

(vii) Let f and $\mathscr{F}$ be as in (vi). We say that $\mathscr{F}$ is f-Cohen-Macaulay (resp. that f is Cohen-Macaulay) if $\mathscr{F}$ (resp. $\mathscr{O}_X$) is f-Cohen-Macaulay at every point $x \in X$.

(viii) Let f and $\mathscr{F}$ be as in (vi). The *f-Cohen-Macaulay locus* of $\mathscr{F}$ is the subset

$$CM(f, \mathscr{F}) \subset X$$

consisting of all $x \in X$ such that f is Cohen-Macaulay at x. The subset $CM(f, \mathscr{O}_X)$ is also called the *Cohen-Macaulay locus of f* and is denoted briefly $CM(f)$.

Lemma 5.6.36. *Let $f : X \to Y$ be a locally finitely presented morphism of schemes, and $\mathscr{F}$ a finitely presented $\mathscr{O}_X$-module such that $\mathrm{Supp}\,\mathscr{F} = X$. We have :*

(i) *$CM(f, \mathscr{F})$ is an open subset of X,*

(ii) *The restriction $CM(f) \to Y$ of f is locally equidimensional.*

(iii) *Let $g : X' \to X$ be another morphism of schemes, $x' \in X'$ a point such that g is étale at x'. Then $g(x') \in CM(f, \mathscr{F})$ if and only if $x' \in CM(f \circ g, g^*\mathscr{F})$.*

(iv) *If $x \in CM(f)$, then $\delta'(x, \mathscr{O}_X) = \delta'(f(x), \mathscr{O}_Y) + \dim \mathscr{O}_{f^{-1}(fx),x}$ (notation of* (5.4.18)*).*

Proof. (i) and (ii) follow easily from [32, Ch.IV, Prop.15.4.3], and (iii) follows from corollary 5.4.35. Lastly, (iv) is an immediate consequence of corollary 5.4.39. □

Proposition 5.6.37. *Let A be a regular local ring of dimension d, and M a finitely generated Cohen-Macaulay A-module. Then :*

(i) *$\mathrm{Ext}^i_A(M, A) = 0$ for every $i \neq c := d - \dim M$.*

(ii) *The A-module $\mathscr{D}(M) := \mathrm{Ext}^c_A(M, A)$ is Cohen-Macaulay.*

(iii) *The natural map $M \to \mathrm{Ext}^c(\mathscr{D}(M), A)$ is an isomorphism, and $\mathrm{Supp}\,\mathscr{D}(M) = \mathrm{Supp}\,M$.*

(iv) *For every $n \in \mathbb{N}$, the functor*

$$\mathscr{D} : A\text{-}\mathsf{CM}_n \to A\text{-}\mathsf{CM}_n^o \qquad N \mapsto \mathrm{Ext}^{d-n}_A(N, A)$$

is an equivalence of categories.

Proof. (i): According to [30, Ch.0, Prop.17.3.4], the projective dimension of M equals c, so we may find a minimal free resolution $0 \to L_c \to \cdots \to L_1 \to L_0 \to M$ for M of length c. Hence, we need only prove the sought vanishing for every $i < c$. Set $X := \operatorname{Spec} A$, $Z := \operatorname{Supp} M \subset X$. By virtue of proposition 5.4.24, it suffices to show that $\operatorname{depth}_Z \mathscr{O}_X \geq c$. In light of (5.4.29), this comes down to showing that $\mathscr{O}_{X,z}$ is a local ring of depth $\geq c$, for every $z \in Z$. The latter holds, since $\mathscr{O}_{X,z}$ is a regular local ring of dimension $\geq c$ ([30, Ch.0, Cor.16.5.12]).

(ii): From (i) we deduce that $L_\bullet^\vee := (L_0^\vee \to \cdots \to L_n^\vee)$, together with its natural augmentation $L_n^\vee \to \mathscr{D}(M)$, is a free resolution of $\mathscr{D}(M)$; especially, the projective dimension of $\mathscr{D}(M)$ is $\leq c$, and therefore

$$\operatorname{depth}_A \mathscr{D}(M) \geq \dim M$$

again by [30, Ch.0, Prop.17.3.4]. On the other hand, it follows easily from [75, Prop.3.3.10] that $\operatorname{Supp} \mathscr{D}(M) \subset \operatorname{Supp} M$, so $\mathscr{D}(M)$ is Cohen-Macaulay.

(iii): From the proof of (ii) we see that $R\mathrm{Hom}_A(\mathscr{D}(M), A)$ is computed by the complex $L_\bullet^{\vee\vee} = L_\bullet$, whence the first assertion. Invoking again [75, Prop.3.3.10], we deduce that $\operatorname{Supp} M \subset \operatorname{Supp} \mathscr{D}(M)$; since the converse inclusion is already known, these two supports coincide.

(iv) follows straightforwardly from (ii) and (iii). $\square$

Corollary 5.6.38. *Let X be a regular noetherian scheme, and $j : Y \to X$ a closed immersion. We have :*

(i) *Y admits a dualizing complex $\omega_Y^\bullet$.*

(ii) *If Y is a Cohen-Macaulay scheme, then we may find a dualizing complex $\omega_Y^\bullet$ that is concentrated in degree 0. Moreover, $H^0(\omega_Y^\bullet)$ is a Cohen-Macaulay $\mathscr{O}_Y$-module.*

Proof. Indeed, lemma 5.6.28(i) shows that the complex $\omega_Y^\bullet := j^* R\mathscr{H}om_{\mathscr{O}_X}(j_*\mathscr{O}_Y, \mathscr{O}_X)$ will do. According to proposition 5.6.37(i,ii), this complex fulfills the condition of (ii), up to a suitable shift. $\square$

5.6.39. Let $f : X \to S$ be a smooth quasi-compact morphism of schemes, $\mathscr{F}$ a finitely presented quasi-coherent $\mathscr{O}_X$-module, $x \in X$ any point, and $s := f(x)$. Set

$$A := \mathscr{O}_{S,s} \qquad B := \mathscr{O}_{X,x} \qquad F := \mathscr{F}_x \qquad B_0 := B \otimes_A \kappa(s) \qquad F_0 := F \otimes_A \kappa(s).$$

Theorem 5.6.40. *In the situation of* (5.6.39)*, suppose that F is a $f_x^\natural$-Cohen-Macaulay module. Then we may find an open neighborhood $U \subset X$ of x in X such that the following holds:*

(i) *There exists a finite resolution of the $\mathscr{O}_U$-module $\mathscr{F}_{|U}$, of length $n := \dim B_0 - \dim_B F_0$*

$$\Sigma_\bullet \quad : \quad 0 \to \mathscr{L}_n \to \mathscr{L}_{n-1} \to \cdots \to \mathscr{L}_0 \to \mathscr{F}_{|U} \to 0$$

consisting of free $\mathscr{O}_U$-modules of finite rank. Moreover, $\Sigma_\bullet$ is universally $\mathscr{O}_S$-exact, i.e. *for every coherent $\mathscr{O}_S$-module $\mathscr{G}$, the complex $\Sigma_\bullet \otimes_{\mathscr{O}_X} f^*\mathscr{G}$ is still exact.*

(ii) *The complex $K^\bullet := R\mathscr{H}om^\bullet_{\mathscr{O}_U}(\mathscr{F}_{|U}, \mathscr{O}_U)$ is concentrated in degree n.*

(iii) *The natural map $\mathscr{F}_{|U} \to R\mathscr{H}om_{\mathscr{O}_U}(K^\bullet, \mathscr{O}_U)$ is an isomorphism in $\mathsf{D}(\mathscr{O}_U\text{-}\mathbf{Mod})$.*

(iv) *The B-module $G := H^n K_x^\bullet$ is $f_x^\natural$-Cohen-Macaulay, and* $\operatorname{Supp} G = \operatorname{Supp} F$.

Proof. (i): According to proposition 4.1.37, the B-module F admits a minimal free resolution

$$\Sigma_{x,\bullet} \quad : \quad \cdots \to L_2 \xrightarrow{d_2} L_1 \xrightarrow{d_1} L_0 \xrightarrow{\varepsilon} F \to 0$$

that is universally A-exact; especially, $d_i(L_i)$ is a flat A-module, for every $i \in \mathbb{N}$. It also follows that $\Sigma_{x,\bullet} \otimes_A \kappa(s)$ is a minimal free resolution of the B_0-module F_0. Since the latter is Cohen-Macaulay, and B_0 is a regular local ring ([33, Ch.IV, Th.17.5.1]), the projective dimension of the B_0-module F_0 equals n ([30, Ch.0, Prop.17.3.4]), therefore $d_n(L_n) \otimes_A \kappa(s)$ is a free B_0-module, so $d_n(L_n)$ is a flat B-module (lemma 4.3.35); then we deduce that it is actually a free

B-module, as it is finitely presented. Since $\Sigma_{x,\bullet}$ is minimal, we conclude that $L_i = 0$ for every $i > n$. We may now extend $\Sigma_{x,\bullet}$ to a finite resolution $\Sigma_\bullet$ of $\mathscr{F}_{|U}$ by free $\mathscr{O}_U$-modules on some open neighborhood U of x, and after replacing U by a smaller neighborhood of U, we may assume that $\mathscr{F}_{|U}$ is $f_{|U}$-flat ([32, Ch.IV, Th.11.3.1]), hence $\Sigma_\bullet$ is universally $\mathscr{O}_S$-exact, as stated.

(ii): Let $L_\bullet := (L_n \to \cdots \to L_0)$ be the complex obtained after omitting F from the resolution $\Sigma_{x,\bullet}$ (where L_0 is placed in degree 0). Then $K^\bullet$ is isomorphic to $L_\bullet^\vee := \mathrm{Hom}_B(L_\bullet, B)$. On the other hand, the universal exactness property of $\Sigma_{x,\bullet}$ implies that

$$\mathrm{Hom}_{B_0}(L_\bullet \otimes_A \kappa(s), B_0) = L_\bullet^\vee \otimes_A \kappa(s)$$

computes $R\mathrm{Hom}^\bullet_{B_0}(F_0, B_0)$. In view of proposition 5.6.37(i), we have $\mathrm{Ext}^i_{B_0}(F_0, B_0) = 0$ for every $i \neq n$. From this, a repeated application of [32, Ch.IV, Prop.11.3.7] shows that $L_\bullet^\vee$ is concentrated in degree n as well, and after shrinking U, we may assume that (ii) holds (details left to the reader).

(iii): Let $\mathscr{L}_\bullet := (\mathscr{L}_n \to \cdots \to \mathscr{L}_0)$ be the complex obtained by omitting $\mathscr{F}_{|U}$ form the resolution $\Sigma_\bullet$; the proof of (iii) shows that $R\mathscr{H}om_{\mathscr{O}_U}(K^\bullet, \mathscr{O}_U)$ is computed by $\mathscr{L}_\bullet^{\vee\vee} = \mathscr{L}_\bullet$, whence the contention.

(iv): By the same token, [32, Ch.IV, Prop.11.3.7] implies that $\mathrm{Coker}\, d_i^\vee$ is a flat A-module, for every $i = 1, \dots, n$. Especially, G is a flat A-module, and the complex $L_\bullet^\vee[-n]$, with its natural augmentation $L_n^\vee \to G$, is a universally A-exact and free resolution of the B-module G. Especially, the projective dimension of G is $\leq n$, therefore

$$\mathrm{depth}_B G \geq \dim_B F_0 \qquad (5.6.41)$$

by [30, Ch.0, Prop.17.3.4]. From (iii) we also see that the induced map

$$F_\mathfrak{p} \to \mathrm{Ext}^n_{B_\mathfrak{p}}(G_\mathfrak{p}, B_\mathfrak{p})$$

is an isomorphism, for every prime ideal $\mathfrak{p} \subset B$; therefore $\mathrm{Supp}\, F \subset \mathrm{Supp}\, G$. Symmetrically, the same argument yields $\mathrm{Supp}\, G \subset \mathrm{Supp}\, F$. Hence the supports of F and G agree. Lastly, combining with (5.6.41) we see that G is Cohen-Macaulay. □

5.6.42. Keep the notation of (5.6.39), and let $\varphi := f_x^\natural : A \to B$. From theorem 5.6.40(iii,iv) we deduce that, for every $n \in \mathbb{N}$, the functor

$$\mathscr{D}_\varphi : \varphi\text{-}\mathsf{CM}_n \to \varphi\text{-}\mathsf{CM}_n^o \qquad M \mapsto \mathrm{Ext}_B^{\dim B_0 - n}(M, B)$$

is an equivalence, and the natural map $M \to \mathscr{D}_\varphi \circ \mathscr{D}_\varphi(M)$ is an isomorphism, for every φ-Cohen-Macaulay module M. We shall see later also a relative variant of corollary 5.6.38, in a more special situation (see proposition 5.8.5).

5.7. **Schemes over a valuation ring.**

Throughout this section, $(K, |\cdot|)$ denotes a valued field, whose valuation ring (resp. maximal ideal, resp. residue field, resp. value group) shall be denoted K^+ (resp. $\mathfrak{m}_K$, resp. κ, resp. Γ). Also, we let

$$S := \mathrm{Spec}\, K^+ \qquad \text{and} \qquad S_{/b} := \mathrm{Spec}\, K^+/bK^+ \qquad \text{for every } b \in \mathfrak{m}_K$$

(so $S_{/0} = S$) and we denote by $s := \mathrm{Spec}\, \kappa$ (resp. by $\eta := \mathrm{Spec}\, K$) the closed (resp. generic) point of S. More generally, for every S-scheme X we let as well

$$X_{/b} := X \times_S S_{/b} \qquad \text{for every } b \in \mathfrak{m}_K.$$

A basic fact, which can be deduced easily from [45, Part I, Th.3.4.6], is that every finitely presented S-scheme is coherent; we present here a direct proof, based on the following

Proposition 5.7.1. *Let A be an essentially finitely presented K^+-algebra, M a finitely generated K^+-flat A-module. Then M is a finitely presented A-module.*

Proof. Let us write $A = S^{-1}B$ for some finitely presented K^+-algebra B, and some multiplicative subset $S \subset B$; then we may find a finitely generated B-module M_B with an isomorphism $\varphi : S^{-1}M_B \xrightarrow{\sim} M$ of A-modules. Let M'_B be the image of M_B in $M_B \otimes_{K^+} K$; since M is K^+-flat, φ induces an isomorphism $S^{-1}M'_B \to M$. It then suffices to show that M'_B is a finitely presented B-module; hence we may replace A by B and M by M'_B, and assume that A is a finitely presented K^+-algebra.

We are further easily reduced to the case where $A = K^+[T_1, \dots, T_n]$ is a free polynomial K^+-algebra. Pick a finite system of generators Σ for M; define increasing filtrations $\mathrm{Fil}_\bullet A$ and $\mathrm{Fil}_\bullet M$ on A and M, by letting $\mathrm{Fil}_k A$ be the K^+-submodule of all polynomials $P(T_1, \dots, T_n) \in A$ of total degree $\leq k$, and setting $\mathrm{Fil}_k M := \mathrm{Fil}_k A \cdot \Sigma \subset M$ for every $k \in \mathbb{N}$. We consider the Rees algebra $\mathsf{R}(\underline{A})_\bullet$ and the Rees module $\mathsf{R}(\underline{M})_\bullet$ associated to these filtrations as in definition 4.4.25(iii,iv). Say that Σ is a subset of cardinality N; we obtain an A-linear surjection $\varphi : A^{\oplus N} \to M$, and we set $M'_k := (\mathrm{Fil}_k A)^{\oplus N} \cap \mathrm{Ker}\,\varphi$ for every $k \in \mathbb{N}$. Notice that the resulting graded K^+-module $M'_\bullet := \bigoplus_{k \in \mathbb{N}} M'_k$ is actually a $\mathsf{R}(\underline{A})_\bullet$-module and we get a short exact sequence of graded $\mathsf{R}(\underline{A})_\bullet$-modules

$$C_\bullet \quad : \quad 0 \to M'_\bullet \to \mathsf{R}(\underline{A})^{\oplus N}_\bullet \to \mathsf{R}(\underline{M})_\bullet \to 0.$$

Notice also that the K^+-modules $\mathsf{R}(\underline{A})^{\oplus N}_k$ and $\mathsf{R}(\underline{M})_k$ are torsion-free and finitely generated, hence they are free of finite rank (see [36, Rem.6.1.12(ii)]); then the same holds for M'_k, for every $k \in \mathbb{N}$. It follows that the complex $C_\bullet \otimes_{K^+} \kappa$ is still short exact. On the other hand, $\mathsf{R}(\underline{A})_\bullet$ is a K^+-algebra of finite type (see example 4.4.28), hence $R := \mathsf{R}(\underline{A})_\bullet \otimes_{K^+} \kappa$ is a noetherian ring; therefore, $M'_\bullet \otimes_{K^+} \kappa$ is a graded R-module of finite type, say generated by the system of homogenenous elements $\{\overline{x}_1, \dots, \overline{x}_t\}$. Lift these elements to a system $\Sigma' := \{x_1, \dots, x_t\}$ of homogeneous elements of $M'_\bullet$, and denote by $M''_\bullet \subset M'_\bullet$ the $\mathsf{R}(\underline{A})_\bullet$-submodule generated by Σ'. By construction, $M''_k \otimes_{K^+} \kappa = M'_k \otimes_{K^+} \kappa$, hence $M''_k = M'_k$ for every $k \in \mathbb{N}$, by Nakayama's lemma. In other words, Σ' is a system of generators for $M'_\bullet$; it follows easily that the image of Σ' in $\mathrm{Ker}\,\varphi$ is also a system of generator for the latter A-module, and the proposition follows. $\square$

Corollary 5.7.2. *Every essentially finitely presented K^+-algebra is a coherent ring.*

Proof. One reduces easily to the case of a free polynomial K^+-algebra $K^+[T_1, \dots, T_n]$, in which case the assertion follows immediately from proposition 5.7.1 : the details shall be left to the reader. $\square$

Lemma 5.7.3. *If A if a finitely presented K^+-algebra, the subset $\mathrm{Min}\,A \subset \mathrm{Spec}\,A$ of minimal prime ideals is finite.*

Proof. We begin with the following :

Claim 5.7.4. The assertion holds when A is a flat K^+-algebra.

Proof of the claim. Indeed, in this case, by the going-down theorem ([61, Th.9.5]) the minimal primes lie in $\mathrm{Spec}\,A \otimes_{K^+} K$, which is a noetherian ring. $\Diamond$

In general, A is defined over a $\mathbb{Z}$-subalgebra $R \subset K^+$ of finite type. Let $L \subset K$ be the field of fractions of R, and set $L^+ := L \cap K^+$; then L^+ is a valuation ring of finite rank, and there exists a finitely presented L^+-algebra A' such that $A \simeq A' \otimes_{L^+} K^+$. For every prime ideal $\mathfrak{p} \subset L^+$, let $\kappa(\mathfrak{p})$ be the residue field of $L^+_\mathfrak{p}$, and set :

$$A'(\mathfrak{p}) := A' \otimes_{L^+} \kappa(\mathfrak{p}) \qquad K^+(\mathfrak{p}) := K^+ \otimes_{L^+} \kappa(\mathfrak{p}).$$

Since $\mathrm{Spec}\,L^+$ is a finite set, it suffices to show that the subset $\mathrm{Min}\,A \otimes_{L^+} \kappa(\mathfrak{p})$ is finite for every $\mathfrak{p} \in \mathrm{Spec}\,L^+$. However, $A \otimes_{L^+} \kappa(\mathfrak{p}) \simeq A'(\mathfrak{p}) \otimes_{\kappa(\mathfrak{p})} K^+(\mathfrak{p})$, so this ring is a flat $K^+(\mathfrak{p})$-algebra of finite presentation. Let I be the nilradical of $K^+(\mathfrak{p})$; since $K^+(\mathfrak{p})/I$ is a valuation ring, claim 5.7.4 applies with A replaced by $A \otimes_{K^+} K^+(\mathfrak{p})/I$, and concludes the proof. $\square$

Lemma 5.7.5. *Let $(A, \mathfrak{m}_A)$ be a local ring, $K^+ \to A$ a local and essentially finitely presented ring homomorphism. Then there exists a local morphism $\varphi : V \to A$ of essentially finitely presented K^+-algebras, such that the following holds :*

(i) *V is a valuation ring, its maximal ideal $\mathfrak{m}_V$ is generated by the image of $\mathfrak{m}_K$, and the map $K^+ \to V$ induces an isomorphism $\Gamma \xrightarrow{\sim} \Gamma_V$ of value groups.*
(ii) *φ induces a finite field extension $V/\mathfrak{m}_V \to A/\mathfrak{m}_A$.*

Proof. Set $X := \operatorname{Spec} A$ and let $x \in X$ be the closed point. Let us pick $a_1, \dots, a_d \in A$ whose classes in $\kappa(x)$ form a transcendence basis over $\kappa(s)$. The system $(a_i \mid i \leq d)$ defines a factorization of the morphism $\operatorname{Spec} \varphi : X \to S$ as a composition $X \xrightarrow{g} Y := \mathbb{A}^d_S \xrightarrow{h} S$, such that $\xi := g(x)$ is the generic point of $h^{-1}(s) \subset Y$. The morphism g is essentially finitely presented ([30, Ch.IV, Prop.1.4.3(v)]) and moreover :

Claim 5.7.6. The stalk $\mathscr{O}_{Y,\xi}$ is a valuation ring with value group Γ.

Proof of the claim. Indeed, set $B := K^+[T_1, \dots, T_d]$; one sees easily that $V := \mathscr{O}_{X,\xi}$ is the valuation ring of the Gauss valuation $|\cdot|_B : \operatorname{Frac}(B) \to \Gamma \cup \{0\}$ such that

$$\left| \sum_{\alpha \in \mathbb{N}^d} a_\alpha T^\alpha \right|_B = \max\{|a_\alpha| \mid \alpha \in \mathbb{N}^d\}$$

(where $a_\alpha \in K^+$ and $T^\alpha := T_1^{\alpha_1} \cdots T_r^{\alpha_r}$ for all $\alpha := (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$, and $a_\alpha = 0$ except for finitely many $\alpha \in \mathbb{N}^d$). ◊

In view of claim 5.7.6, to conclude the proof it suffices to notice that $\kappa(x)$ is a finite extension of $\kappa(\xi)$. □

Proposition 5.7.7. *Let $\varphi : K^+ \to A$ and $\psi : A \to B$ be two essentially finitely presented ring homomorphisms. Then :*

(i) *If ψ is integral, B is a finitely presented A-module.*
(ii) *If A is local, φ is local and flat, and $A/\mathfrak{m}_K A$ is a field, then A is a valuation ring and φ induces an isomorphism $\Gamma \xrightarrow{\sim} \Gamma_A$ of value groups.*

Proof. (i): The assumption means that there exists a finitely presented A-algebra C and a multiplicative system $T \subset C$ such that $B = T^{-1}C$. Let $I := \bigcup_{t \in T} \operatorname{Ann}_C(t)$. Then I is the kernel of the localization map $C \to B$; the latter is integral by hypothesis, hence for every $t \in T$ there is a monic polynomial $P(X) \in C[X]$, say of degree n, such that $P(t^{-1}) = 0$ in B, hence $t't^n \cdot P(t^{-1}) = 0$ in C, for some $t' \in T$, *i.e.* $t'(1 - ct) = 0$ holds in C for some $c \in C$, in other words, the image of t is already a unit in C/I, so that $B = C/I$. Then lemma 4.3.41 says that $V(I) \subset \operatorname{Spec} C$ is closed under generizations, hence $V(I)$ is open, by corollary 4.3.44 and lemma 5.7.3, and finally I is finitely generated, by lemma 4.3.38(ii). So B is a finitely presented A-algebra. Then (i) follows from the well known :

Claim 5.7.8. Let R be any ring, S a finitely presented and integral R-algebra. Then S is a finitely presented R-module.

Proof of the claim. Let us pick a presentation : $S = R[x_1, \dots, x_n]/(f_1, \dots, f_m)$. By assumption, for every $i \leq n$ we can find a monic polynomial $P_i(T) \in R[T]$ such that $P_i(x_i) = 0$ in S. Let us set $S' := R[x_1, \dots, x_n]/(P_1(x_1), \dots, P_n(x_n))$; then S' is a free R-module of finite rank, and there is an obvious surjection $S' \to S$ of R-algebras, the kernel of which is generated by the images of the polynomials $f_1, \dots, f_m$. The claim follows. ◊

(ii): By lemma 5.7.5, we may assume from start that the residue field of A is a finite extension of κ. We can write $A = C_\mathfrak{p}$ for a K^+-algebra C of finite presentation, and a prime ideal $\mathfrak{p} \subset C$ containing $\mathfrak{m}_K$; under the stated assumptions, $A/\mathfrak{m}_K A$ is a finite field extension of κ, hence $\mathfrak{p}$ is a maximal ideal of C, and moreover $\mathfrak{p}$ is isolated in the fibre $g^{-1}(s)$ of the structure morphism $g : Z := \mathrm{Spec}\, C \to S$. It then follows from [32, Ch.IV, Cor.13.1.4] that, up to shrinking Z, we may assume that g is a quasi-finite morphism ([27, Ch.II, §6.2]). Let $K^{\mathrm{h}+}$ be the henselization of K^+, and set $C^{\mathrm{h}} := C \otimes_{K^+} K^{\mathrm{h}+}$. There exists precisely one prime ideal $\mathfrak{q} \subset C^{\mathrm{h}}$ lying over $\mathfrak{p}$, and $C^{\mathrm{h}}_{\mathfrak{q}}$ is a flat, finite and finitely presented $K^{\mathrm{h}+}$-algebra by [33, Ch.IV, Th.18.5.11], so $C^{\mathrm{h}}_{\mathfrak{q}}$ is henselian, by [33, Ch.IV, Prop.18.5.6(i)], and then $C^{\mathrm{h}}_{\mathfrak{q}}$ is the henselization of A, since the natural map $A \to C^{\mathrm{h}}_{\mathfrak{q}}$ is ind-étale and induces an isomorphism $C^{\mathrm{h}}_{\mathfrak{q}} \otimes_{K^+} \kappa \simeq \kappa(x)$. Finally, $C^{\mathrm{h}}_{\mathfrak{q}}$ is also a valuation ring with value group Γ, by virtue of [36, Lemma 6.1.13 and Rem.6.1.12(vi)]. To conclude the proof of (ii), it now suffices to remark :

Claim 5.7.9. Let V be a valuation ring, $\varphi : R \to V$ a faithfully flat morphism. Then R is a valuation ring and φ induces an injection $\varphi_\Gamma : \Gamma_R \to \Gamma_V$ of the respective value groups.

Proof of the claim. Clearly R is a domain, since it is a subring of V. To show that R is a valuation ring, it then suffices to prove that, for any two ideals $I, J \subset R$, we have either $I \subset J$ or $J \subset I$. However, since V is a valuation ring, we know already that either $I \cdot V \subset J \cdot V$ or $J \cdot V \subset I \cdot V$; since φ is faithfully flat, the assertion follows. By the same token, we deduce that an ideal $I \subset R$ is equal to R if and only if $I \cdot V = V$, which implies that φ_Γ is injective (see [36, §6.1.11]). □

Proposition 5.7.10. *Let $f : X \to S$ be a flat and finitely presented morphism, $\mathscr{F}$ a coherent $\mathscr{O}_X$-module, $x \in X$ any point, $y := f(x)$, and suppose that $f^{-1}(y)$ is a regular scheme (this holds* e.g. *if f is a smooth morphism). Let also $n := \dim \mathscr{O}_{f^{-1}(y),x}$. Then:*

(i) $\mathrm{hom.dim}_{\mathscr{O}_{X,x}} \mathscr{F}_x \leq n + 1$.

(ii) *If $\mathscr{F}$ is f-flat at the point x, then* $\mathrm{hom.dim}_{\mathscr{O}_{X,x}} \mathscr{F}_x \leq n$.

(iii) *If $\mathscr{F}$ is reflexive at the point x, then* $\mathrm{hom.dim}_{\mathscr{O}_{X,x}} \mathscr{F}_x \leq n - 1$.

(iv) *If f has regular fibres, and $\mathscr{F}$ is reflexive and generically invertible (see* (5.6.9)*), then $\mathscr{F}$ is invertible.*

Proof. To start out, we may assume that X is affine, and then it follows easily from lemma 5.6.6(ii.a) that $\mathscr{O}_X$ is coherent.

(ii): Since $\mathscr{O}_X$ is coherent, we can find a possibly infinite resolution

$$\cdots \to E_2 \to E_1 \to E_0 \to \mathscr{F}_x \to 0$$

by free $\mathscr{O}_{X,x}$-modules E_i ($i \in \mathbb{N}$) of finite rank; set $E_{-1} := \mathscr{F}_x$ and $L := \mathrm{Im}(E_n \to E_{n-1})$. It suffices to show that the $\mathscr{O}_{X,x}$-module L is free. For every $\mathscr{O}_{X,x}$-module M we shall let $M(y) := M \otimes_{\mathscr{O}_{X,x}} \mathscr{O}_{f^{-1}(y),x}$. Since L and $\mathscr{F}_x$ are torsion-free, hence flat K^+-modules, the induced sequence of $\mathscr{O}_{f^{-1}(y),x}$-modules:

$$0 \to L(y) \to E_{n-1}(y) \to \cdots E_1(y) \to E_0(y) \to \mathscr{F}_x(y) \to 0$$

is exact; since $f^{-1}(y)$ is a regular scheme, $L(y)$ is a free $\mathscr{O}_{f^{-1}(y),x}$-module of finite rank. Since L is also flat as a K^+-module, [30, Prop.11.3.7] and Nakayama's lemma show that any set of elements of L lifting a basis of $L(y)$, is a free basis of L.

(i): Locally on X we can find an epimorphism $\mathscr{O}_X^{\oplus n} \to \mathscr{F}$, whose kernel $\mathscr{G}$ is again a coherent $\mathscr{O}_X$-module, since $\mathscr{O}_X$ is coherent. Clearly it suffices to show that $\mathscr{G}$ admits, locally on X, a finite free resolution of length $\leq n$, which holds by (ii), since $\mathscr{G}$ is f-flat.

(iii): Suppose $\mathscr{F}$ is reflexive at x; by remark 5.6.4 we can find a left exact sequence

$$0 \to \mathscr{F}_x \to \mathscr{O}_{X,x}^{\oplus m} \xrightarrow{\alpha} \mathscr{O}_{X,x}^{\oplus n}.$$

It follows that $\mathrm{hom.dim}_{\mathscr{O}_{X,x}} \mathscr{F}_x = \max(0, \mathrm{hom.dim}_{\mathscr{O}_{X,x}}(\mathrm{Coker}\,\alpha) - 2) \leq n - 1$, by (i).

(iv): Suppose that $\mathscr{F}$ is generically invertible, let $j : U \to X$ be the maximal open immersion such that $j^*\mathscr{F}$ is an invertible $\mathscr{O}_U$-module, and set $Z := X \setminus U$. Under the current assumptions,

X is reduced; it follows easily that U is the set where $\mathrm{rk}\,\mathscr{F} = 1$, and that j is quasi-compact (since the rank function is constructible : see (5.6.9)). It follows from (iii) that $Z \cap f^{-1}(y)$ has codimension ≥ 2 in $f^{-1}(y)$, for every $y \in S$. Hence the conditions of corollary 5.6.8 are fulfilled, and furthermore $\mathscr{F}[0]$ is a perfect complex by (ii), so the invertible $\mathscr{O}_X$-module $\det \mathscr{F}$ is well defined (lemma 5.6.13). We deduce natural isomorphisms:

$$\mathscr{F} \xrightarrow{\sim} j_*j^*\mathscr{F} \xrightarrow{\sim} j_*\mathsf{det}(j^*\mathscr{F}[0]) \xrightarrow{\sim} j_*j^*\mathsf{det}\,\mathscr{F}[0] \xleftarrow{\sim} \mathsf{det}\,\mathscr{F}[0]$$

whence (iv). □

Proposition 5.7.11. *Let X be a quasi-compact and quasi-separated scheme, $n \in \mathbb{N}$ an integer, and denote by $\pi : \mathbb{P}^n_X \to X$ the natural projection. Then the map*

$$H^0(X, \mathbb{Z}) \oplus \mathrm{Pic}\, X \to \mathrm{Pic}\, \mathbb{P}^n_X \qquad (r, \mathscr{L}) \mapsto \mathscr{O}_{\mathbb{P}^n_X}(r) \otimes \pi^*\mathscr{L}$$

is an isomorphism of abelian groups.

Proof. To begin with, let us recall the following well known :

Claim 5.7.12. Let $f : Y \to T$ be a proper morphism of noetherian schemes, $\mathscr{F}$ a coherent f-flat $\mathscr{O}_Y$-module, $t \in T$ a point, $p \in \mathbb{N}$ an integer, and denote $i_t : f^{-1}(t) \to Y$ the natural immersion. Suppose that $H^{p+1}(f^{-1}(t), i_t^*\mathscr{F}) = 0$. Then the natural map

$$(R^p f_*\mathscr{F})_t \to H^p(f^{-1}(t), i_t^*\mathscr{F})$$

is surjective.

Proof of the claim. In light of [28, Ch.III, Prop.1.4.15], we easily reduce to the case where T is a local scheme, say $T = \mathrm{Spec}\, A$ for some noetherian local ring A, and t is the closed point of T. Set $k := \kappa(t)$, and for every $q \in \mathbb{N}$, consider the functor

$$F_{-q} : A\text{-}\mathbf{Mod} \to A\text{-}\mathbf{Mod} \qquad M \mapsto H^0(T, R^p f_*(f^*M^\sim \otimes_{\mathscr{O}_Y} \mathscr{F}))$$

(where, as usual, $M^\sim$ denotes the quasi-coherent $\mathscr{O}_T$-module arising from an A-module M). Since $\mathscr{F}$ is f-flat, the system $(F_{-q} \mid q \in \mathbb{N})$ defines a homological functor. The assertion is that $F_{-p-1}(k) = 0$, in which case [29, Ch.III, Cor.7.5.3] (together with [28, Ch.III, Th.3.2.1 and Th.4.1.5]) says that $F_{-p-1}(M) = 0$ for every A-module M. Hence, F_{-p-1} is trivially an exact functor, and it follows that the natural map $F_{-p}(A) \to F_{-p}(k)$ is surjective ([29, Ch.III, Prop.7.5.4]), which is the claim. ◊

Now, the injectivity of the stated map is clear (details left to the reader). For the surjectivity, let $\mathscr{G}$ be an invertible $\mathscr{O}_{\mathbb{P}^n_X}$-module, write X as the limit of a filtered system $(X_\lambda \mid \lambda \in \Lambda)$ of noetherian schemes, and for every $\lambda \in \Lambda$, let $p_\lambda : \mathbb{P}^n_X \to \mathbb{P}^n_{X_\lambda}$ be the induced morphism; for some $\lambda \in \Lambda$, we may find an invertible $\mathscr{O}_{\mathbb{P}^n_{X_\lambda}}$-module $\mathscr{F}_\lambda$ with an isomorphism $\mathscr{G} \xrightarrow{\sim} p^*_\lambda\mathscr{G}_\lambda$ of $\mathscr{O}_{\mathbb{P}^n_X}$-modules. Clearly, it then suffices to check the sought surjectivity for the scheme X_λ; we may therefore replace X by X_λ, and assume from start that X is noetherian.

Next, let $x \in X$ be any point, and $i_x : \mathbb{P}^n_{\kappa(x)} \to \mathbb{P}^n_X$ the natural immersion; we have $i^*_x\mathscr{G} \simeq \mathscr{O}_{\mathbb{P}^n_{\kappa(x)}}(r_x)$ for some $r_x \in \mathbb{Z}$. Set $\mathscr{F} := \mathscr{G}(-r_x)$, and notice that $H^1(\mathbb{P}^n_{\kappa(x)}, i^*_x\mathscr{F}) = 0$; moreover, $H_x := H^0(\mathbb{P}^n_{\kappa(x)}, i^*_x\mathscr{F})$ is a one-dimensional $\kappa(x)$-vector space. From claim 5.7.12, it follows that there exists an open neighborhood U of x in X, and a section $s \in H^0(\pi^{-1}U, \mathscr{F})$ mapping to a generator of H_x. The section s defines a map of $\mathscr{O}_{\mathbb{P}^n_U}$-modules $\varphi : \mathscr{O}_{\mathbb{P}^n_U}(r_x) \to \mathscr{G}_{|\pi^{-1}U}$. Furthermore, since $i^*_x\mathscr{F} \simeq \mathscr{O}_{\mathbb{P}^n_{\kappa(x)}}$, it is easily seen that φ restricts to an isomorphism on some open subset of the form $\pi^{-1}U'$, where $U' \subset U$ is an open neighborhood of x in X.

Summing up, since $x \in X$ is arbitrary, and since X is quasi-compact, we may find a finite covering $X = U_1 \cup \cdots \cup U_d$ consisting of open subsets, and for every $i = 1, \dots, d$ an integer r_i and an isomorphism $\varphi_i : \mathscr{O}_{\mathbb{P}^n_{U_i}}(r_i) \xrightarrow{\sim} \mathscr{G}_{|\pi^{-1}U_i}$ of $\mathscr{O}_{\mathbb{P}^n_{U_i}}$-modules. Then, for every $i, j \leq d$ such that $U_{ij} := U_i \cap U_j \neq \varnothing$, the composition of the restriction of φ_i followed by the restriction of φ_j^{-1}, is an isomorphism

$$\varphi_{ij} : \mathscr{O}_{\mathbb{P}^n_{U_{ij}}}(r_i) \xrightarrow{\sim} \mathscr{O}_{\mathbb{P}^n_{U_{ij}}}(r_j)$$

of $\mathscr{O}_{\mathbb{P}^n_{U_{ij}}}$-modules. This already implies that $r_i = r_j$ whenever $U_{ij} \neq \varnothing$, hence the rule : $x \mapsto r_i$ if $x \in U_i$ yields a well defined continuous map $r : X \to \mathbb{Z}$. Lastly, φ_{ij} is the scalar multiplication by a section

$$u_{ij} \in H^0(\mathbb{P}^n_{U_{ij}}, \mathscr{O}^\times_{\mathbb{P}^n_{U_{ij}}}) = H^0(U_{ij}, \mathscr{O}^\times_{U_{ij}}) \qquad \text{for every } i, j = 1, \dots, d.$$

The system $(u_{ij} \mid i, j = 1, \dots, d)$ is a 1-cocycle whose class in Čech cohomology $H^1(U_\bullet, \mathscr{O}^\times_X)$ corresponds to an invertible $\mathscr{O}_X$-module $\mathscr{L}$, with a system of trivializations $\mathscr{O}_{U_i} \xrightarrow{\sim} \mathscr{L}_{|U_i}$, for $i = 1, \dots, d$; especially, φ_i can be regarded as an isomorphism $(\pi^{-1}\mathscr{L})_{|\pi^{-1}U_i}(r_i) \xrightarrow{\sim} \mathscr{G}_{|\pi^{-1}U_i}$ for every $i = 1, \dots, d$. By inspecting the construction, it is easily seen that these latter maps patch to a single isomorphism of $\mathscr{O}_{\mathbb{P}^n_X}$-modules $\pi^*\mathscr{L}(r) \xrightarrow{\sim} \mathscr{G}$, as required. ☐

Corollary 5.7.13. *Let $f : X \to S$ be a smooth morphism of finite presentation, set $\mathbb{G}_{m,X} := X \times_S \operatorname{Spec} K^+[T, T^{-1}]$, and denote by $\pi : \mathbb{G}_{m,X} \to X$ the natural morphism. Then the map*

$$\operatorname{Pic} X \to \operatorname{Pic} \mathbb{G}_{m,X} \qquad \mathscr{L} \mapsto \pi^*\mathscr{L}$$

is an isomorphism.

Proof. To start out, the map is injective, since π admits a section. Next, let $j : \mathbb{G}_{m,X} \to \mathbb{P}^1_X$ be the natural open immersion, and denote again by $\pi : \mathbb{P}^1_X \to X$ the natural projection. According to theorem 5.2.15 and proposition 5.7.10(i), every invertible $\mathscr{O}_{\mathbb{G}_{m,X}}$-module extends to an invertible $\mathscr{O}_{\mathbb{P}^1_X}$-module (namely the determinant of a coherent extension). Since $j^*\mathscr{O}_{\mathbb{P}^1_X}(n) \simeq \mathscr{O}_{\mathbb{G}_{m,X}}$, the claim follows from proposition 5.7.11. ☐

Remark 5.7.14. The proof of proposition 5.7.11 is based on the argument given in [62, Lecture 13, Prop.3]. Of course, statements related to corollary 5.7.13 abound in the literature, and more general results are available: see *e.g.* [6] and [73].

Proposition 5.7.15. *Let $f : X \to S$ be a smooth morphism of finite type, $j : U \to X$ an open immersion, and set $Z := X \setminus U$. Then:*

(i) *Every invertible $\mathscr{O}_U$-module extends to an invertible $\mathscr{O}_X$-module.*
(ii) *Suppose furthermore, that:*
  (a) *For every point $y \in S$, the codimension of $Z \cap f^{-1}(y)$ in $f^{-1}(y)$ is ≥ 1, and*
  (b) *The codimension of $Z \cap f^{-1}(\eta)$ in $f^{-1}(\eta)$ is ≥ 2.*
  Then the restriction functors:

(5.7.16) $$\mathscr{O}_X\text{-}\mathbf{Rflx} \to \mathscr{O}_U\text{-}\mathbf{Rflx} \qquad \mathbf{Pic}\, X \to \mathbf{Pic}\, U$$

are equivalences.

Proof. (i): Notice first that the underling space $|X|$ is noetherian, since it is the union $|f^{-1}(s)| \cup |f^{-1}(\eta)|$ of two noetherian spaces. Especially, every open immersion is quasi-compact, and X is quasi-separated. We may then reduce easily to the case where U is dense in X, in which case the assertion follows from propositions 5.6.7(i) and 5.7.10(iv).

(ii): We begin with the following:

Claim 5.7.17. Under the assumptions of (ii), the functors (5.7.16) are faithful.

Proof of the claim. Since f is smooth, all the stalks of $\mathscr{O}_X$ are reduced ([33, Ch.IV, Prop.17.5.7]). Since every point of Z is specialization of a point of U, the claim follows easily from remark 5.6.4. ♢

Under assumptions (a) and (b), the conditions of corollary 5.6.8 are fulfilled, and one deduces that (5.7.16) are full functors, hence fully faithful, in view of claim 5.7.17. The essential surjectivity of the restriction functor for reflexive $\mathscr{O}_X$-modules is already known, by proposition 5.6.7(i). □

Lemma 5.7.18. *Let $f : X \to S$ be a flat, finitely presented morphism, and denote by $U \subset X$ the maximal open subset such that the restriction $f_{|U} : U \to S$ is smooth. Suppose that the following two conditions hold:*

(a) *For every point $y \in S$, the fibre $f^{-1}(y)$ is geometrically reduced.*
(b) *The generic fibre $f^{-1}(\eta)$ is geometrically normal.*

Then the restriction functor induces an equivalence of categories (notation of (5.6.9)*) :*

$$\mathbf{Div}\, X \xrightarrow{\sim} \mathbf{Pic}\, U. \tag{5.7.19}$$

Proof. Let $\mathscr{F}$ be any generically invertible reflexive $\mathscr{O}_X$-module; it follows from proposition 5.7.10(iv) that $\mathscr{F}_{|U}$ is an invertible $\mathscr{O}_U$-module, so the functor (5.7.19) is well defined. Let $y \in S$ be any point; condition (a) says in particular that the fibre $f^{-1}(y)$ is generically smooth over $\kappa(y)$ ([61, Th.28.7]). Then it follows from [33, Ch.IV, Th.17.5.1] that $U \cap f^{-1}(y)$ is a dense open subset of $f^{-1}(y)$, and if $x \in f^{-1}(y)\setminus U$, then the depth of $\mathscr{O}_{f^{-1}(y),x}$ is ≥ 1, since the latter is a reduced local ring of dimension ≥ 1. Similarly, condition (b) and Serre's criterion [61, Ch.8, Th.23.8] say that for every $x \in f^{-1}(\eta)\setminus U$, the local ring $\mathscr{O}_{f^{-1}(\eta),x}$ has depth ≥ 2. Furthermore, [32, Ch.IV, Prop.9.9.4] implies that the open immersion $j : U \to X$ is quasi-compact; summing up, we see that all the conditions of corollary 5.6.8 are fulfilled, so that $\mathscr{F} = j_*j^*\mathscr{F}$ for every reflexive $\mathscr{O}_X$-module $\mathscr{F}$. This already means that (5.7.19) is fully faithful. To conclude, it suffices to invoke proposition 5.6.7(i). □

The rest of this section shall be concerned with some results that hold in the special case where the valuation of K has rank one.

Theorem 5.7.20. *Suppose that K is a valued field of rank one. Let $f : X \to S$ be a finitely presented morphism, $\mathscr{F}$ a coherent $\mathscr{O}_X$-module. Then :*

(i) *Every coherent proper submodule $\mathscr{G} \subset \mathscr{F}$ admits a primary decomposition.*
(ii) $\mathrm{Ass}\,\mathscr{F}$ *is a finite set.*

Proof. By lemma 5.5.10, assertion (ii) follows from (i). Using corollary 5.5.9(ii) we reduce easily to the case where X is affine, say $X = \mathrm{Spec}\, A$ for a finitely presented K^+-algebra A, and $\mathscr{F} = M^\sim$, $\mathscr{G} = N^\sim$ for some finitely presented A-modules $N \subset M \neq 0$. By considering the quotient M/N, we further reduce the proof to the case where $N = 0$. Let us choose a closed imbedding $i : X \to \mathrm{Spec}\, B$, where $B := K^+[T_1, \dots, T_r]$ is a free polynomial K^+-algebra; by proposition 5.5.14, the submodule $0 \subset \mathscr{F}$ admits a primary decomposition if and only if the submodule $0 \subset i_*\mathscr{F}$ does. Thus, we may replace A by B, and assume that $A = K^+[T_1, \dots, T_r]$, in which case we shall argue by induction on r. Let ξ denote the maximal point of $f^{-1}(s)$; by claim 5.7.6, $V := \mathscr{O}_{X,\xi}$ is a valuation ring with value group Γ. Then [36, Lemma 6.1.14] says that

$$M_\xi \simeq V^{\oplus m} \oplus (V/b_1V) \oplus \cdots \oplus (V/b_kV)$$

for some $m \in \mathbb{N}$ and elements $b_1, \dots, b_k \in \mathfrak{m}_K \setminus \{0\}$. After clearing some denominators, we may then find a map $\varphi : M' := A^{\oplus m} \oplus (A/b_1A) \oplus \cdots \oplus (A/b_kA) \to M$ whose localization φ_ξ is an isomorphism.

Claim 5.7.21. $\mathrm{Ass}\, A/bA = \{\xi\}$ whenever $b \in \mathfrak{m}_K \setminus \{0\}$.

Proof of the claim. Clearly $\mathrm{Ass}\, A/bA \subset f^{-1}(s)$. Let $x \in f^{-1}(s)$ be a non-maximal point. Thus, the prime ideal $\mathfrak{p} \subset A$ corresponding to x is not contained in $\mathfrak{m}_K A$, *i.e.* there exists $a \in \mathfrak{p}$

such that $|a|_A = 1$. Suppose by way of contradiction, that $x \in \mathrm{Ass}\, A/bA$; then we may find $c \in A$ such that $c \notin bA$ but $a^n c \in bA$ for some $n \in \mathbb{N}$. The conditions translate respectively as the inequalities :

$$|c|_A > |b|_A \qquad \text{and} \qquad |a^n|_A \cdot |c|_A = |a^n c|_A \leq |b|_A$$

which are incompatible, since $|a^n|_A = 1$. $\Diamond$

Since $(\mathrm{Ker}\, \varphi)_\xi = 0$, we deduce from claim 5.7.21 and proposition 5.5.4(ii) that $\mathrm{Ass}\,\mathrm{Ker}\, \varphi = \varnothing$; therefore $\mathrm{Ker}\, \varphi = 0$, by lemma 5.5.3(iii). Moreover, the submodule $N_1 := A^{\oplus m}$ (resp. $N_2 := (A/b_1 A) \oplus \cdots \oplus (A/b_k A)$) of M' is either ξ-primary (resp. $\{0\}$-primary), or else equal to M'; in the first case, $0 \subset M'$ admits the primary decomposition $0 = N_1 \cap N_2$, and in the second case, either 0 is primary, or else $M' = 0$ has the empty primary decomposition. Now, if $r = 0$ then $M = M'$ and we are done. Suppose therefore that $r > 0$ and that the theorem is known for all integers $< r$. Set $M'' := \mathrm{Coker}\, \varphi$; clearly $\xi \notin \mathrm{Supp}\, M''$, so by proposition 5.5.13 we are reduced to showing that the submodule $0 \subset M''$ admits a primary decomposition. We may then replace M by M'' and assume from start that $\xi \notin \mathrm{Supp}\, \mathscr{F}$. Let $\mathscr{F}^t \subset \mathscr{F}$ be the K^+-torsion submodule; clearly $\mathscr{F}/\mathscr{F}^t$ is a flat K^+-module, hence it is a finitely presented $\mathscr{O}_X$-module (proposition 5.7.1), so $\mathscr{F}^t$ is a finitely generated $\mathscr{O}_X$-module, by [17, Ch.10, §.1, n.4, Prop.6]. Hence we may find a non-zero $a \in \mathfrak{m}_K$ such that $a\mathscr{F}^t = 0$, *i.e.* the natural map $\mathscr{F}^t \to \mathscr{F}/a\mathscr{F}$ is injective. Denote by $i : Z := V(\mathrm{Ann}\, \mathscr{F}/a\mathscr{F}) \to X$ the closed immersion.

Claim 5.7.22. There exists a finite morphism $g : Z \to \mathbb{A}^{r-1}_S$.

Proof of the claim. Since $\xi \notin \mathrm{Supp}\, \mathscr{F}$, we have $I \nsubseteq \mathfrak{m}_K A$, *i.e.* we can find $b \in I$ such that $|b|_A = 1$. Set $W := V(a, b) \subset X$; it suffices to exhibit a finite morphism $W \to \mathbb{A}^{r-1}_S$. By [13, §5.2.4, Prop.2], we can find an automorphism $\sigma : A/aA \to A/aA$ such that $\sigma(b) = u \cdot b'$, where u is a unit, and $b' \in (K^+/aK^+)[T_1, \dots, T_r]$ is of the form $b' = T_r^k + \sum_{j=0}^{k-1} a_j T_r^j$, for some $a_0, \dots, a_{k-1} \in (K^+/aK^+)[T_1, \dots, T_{r-1}]$. Hence the ring $A/(aA + bA) = A/(aA + b'A)$ is finite over $K^+[T_1, \dots, T_{r-1}]$, and the claim follows. $\Diamond$

Claim 5.7.23. The $\mathscr{O}_X$-submodule $0 \subset \mathscr{F}/a\mathscr{F}$ admits a primary decomposition.

Proof of the claim. Let $\mathscr{G} := (\mathscr{F}/a\mathscr{F})_{|Z}$. By our inductive assumption the submodule $0 \subset g_*\mathscr{G}$ on $\mathbb{A}^{r-1}_S$ admits a primary decomposition; a first application of proposition 5.5.14 then shows that the submodule $0 \subset \mathscr{G}$ on Z admits a primary decomposition, and a second application proves the same for the submodule $0 \subset i_*\mathscr{G} = \mathscr{F}/a\mathscr{F}$. $\Diamond$

Finally, let $j : Y := V(a) \subset X$ be the closed immersion, and set $U := X \setminus Y$. By construction, the natural map $\underline{\Gamma}_Y \mathscr{F} \to j_* j^* \mathscr{F}$ is injective. Furthermore, U is an affine noetherian scheme, so [61, Th.6.8] ensures that the $\mathscr{O}_U$-submodule $0 \subset \mathscr{F}_{|U}$ admits a primary decomposition. The same holds for $j^*\mathscr{F}$, in view of claim 5.7.23. To conclude the proof, it remains only to invoke proposition 5.5.16. $\Box$

Corollary 5.7.24. *Let K be a valued field of rank one, $f : X \to S$ a finitely presented morphism, $\mathscr{F}$ a coherent $\mathscr{O}_X$-module, and suppose there is given a cofiltered family $\underline{\mathscr{I}} := (\mathscr{I}_\lambda \mid \lambda \in \Lambda)$ such that :*

(a) *$\mathscr{I}_\lambda \subset \mathscr{O}_X$ is a coherent ideal for every $\lambda \in \Lambda$.*
(b) *$\mathscr{I}_\lambda \cdot \mathscr{I}_\mu \in \underline{\mathscr{I}}$ whenever $\mathscr{I}_\lambda, \mathscr{I}_\mu \in \underline{\mathscr{I}}$.*

Then the following holds :

(i) *$\mathscr{F}_\lambda := \mathrm{Ann}_{\mathscr{F}}(\mathscr{I}_\lambda) \subset \mathscr{F}$ is a submodule of finite type for every $\lambda \in \Lambda$.*
(ii) *There exists $\lambda \in \Lambda$ such that $\mathscr{F}_\mu \subset \mathscr{F}_\lambda$ for every $\mu \in \Lambda$.*

Proof. We easily reduce to the case where X is affine, say $X = \mathrm{Spec}\, A$ with A finitely presented over K^+, and for each $\lambda \in \Lambda$ the ideal $I_\lambda := \Gamma(X, \mathscr{I}_\lambda) \subset A$ is finitely generated.

(i): For given $\lambda \in \Lambda$, let $f_1, \dots, f_k$ be a finite system of generators of I_λ; then $\mathscr{F}_\lambda$ is the kernel of the map $\varphi : \mathscr{F} \to \mathscr{F}^{\oplus k}$ defined by the rule : $m \mapsto (f_1 m, \dots, f_k m)$, which is finitely generated because A is coherent.

(ii): By theorem 5.7.20 we can find primary submodules $\mathscr{G}_1, \dots, \mathscr{G}_n \subset \mathscr{F}$ such that $\mathscr{G}_1 \cap \cdots \cap \mathscr{G}_n = 0$; for every $i \leq n$ and $\lambda \in \Lambda$ set $\mathscr{H}_i := \mathscr{F}/\mathscr{G}_i$ and $\mathscr{H}_{i,\lambda} := \mathrm{Ann}_{\mathscr{H}_i}(\mathscr{I}_\lambda)$. Since the natural map $\mathscr{F} \to \oplus_{i=1}^n \mathscr{H}_i$ is injective, we have :

$$\mathscr{F}_\lambda = \mathscr{F} \cap (\mathscr{H}_{1,\lambda} \oplus \cdots \oplus \mathscr{H}_{n,\lambda}) \qquad \text{for every } \lambda \in \Lambda.$$

It suffices therefore to prove that, for every $i \leq n$, there exists $\lambda \in \Lambda$ such that $\mathscr{H}_{i,\mu} = \mathscr{H}_{i,\lambda}$ for every $\mu \in \Lambda$. Say that $\mathscr{H}_i$ is $\mathfrak{p}$-primary, for some prime ideal $\mathfrak{p} \subset A$; suppose now that there exists λ such that $I_\lambda \subset \mathfrak{p}$; since I_λ is finitely generated, we deduce that $I_\lambda^n \mathscr{H}_i = 0$ for $n \in \mathbb{N}$ large enough; from (b) we see that $\mathscr{I}_\lambda^n \in \underline{\mathscr{I}}$, so (ii) holds in this case. In case $I_\lambda \not\subseteq \mathfrak{p}$ for every $\lambda \in \Lambda$, we have $\mathrm{Ann}_{\mathscr{H}_i}(\mathscr{I}_\lambda) = 0$ for every $\lambda \in \Lambda$, so (ii) holds in this case as well. $\square$

Corollary 5.7.25. *Suppose A is a valuation ring with value group Γ_A, and $\varphi : K^+ \to A$ is an essentially finitely presented local homomorphism from a valuation ring K^+ of rank one. Then:*

(i) *If the valuation of K is not discrete, φ induces an isomorphism $\Gamma \xrightarrow{\sim} \Gamma_A$.*
(ii) *If $\Gamma \simeq \mathbb{Z}$ and φ is flat, φ induces an inclusion $\Gamma \subset \Gamma_A$, and $(\Gamma_A : \Gamma)$ is finite.*

Proof. Suppose first that $\Gamma \simeq \mathbb{Z}$; then A is noetherian, hence Γ_A is discrete of rank one as well, and the assertion follows easily. In case Γ is not discrete, we claim that A has rank ≤ 1. Indeed, suppose by way of contradiction, that the rank of A is higher than one, and let $\mathfrak{m}_A \subset A$ be the maximal ideal; then we can find $a, b \in \mathfrak{m}_A \setminus \{0\}$ such that $a^{-i}b \in A$ for every $i \in \mathbb{N}$. Let us consider the A-module $M := A/bA$; we notice that $\mathrm{Ann}_M(a^i) = a^{-i}bA$ form a strictly increasing sequence of ideals, contradicting corollary 5.7.24(ii). Next we claim that φ is flat. Indeed, suppose this is not the case; then A is a κ-algebra. Let $\mathfrak{p} \subset A$ be the maximal ideal; by lemma 5.7.5, we may assume that $A/\mathfrak{p}$ is a finite extension of κ. Now, choose any finitely presented quotient $\overline{A}$ of A supported at $\mathfrak{p}$; it follows easily that $\overline{A}$ is integral over K^+, hence it is a finitely presented K^+-module by proposition 5.7.7(i), and its annihilator contains $\mathfrak{m}_K$, which is absurd in view of [36, Lemma 6.1.14]. Next, since Γ is not discrete, one sees easily that $\mathfrak{m}_K A$ is a prime ideal, and in light of the foregoing, it must then be the maximal ideal. Now the assertion follows from proposition 5.7.7(ii). $\square$

5.7.26. Let B be a finitely presented K^+-algebra, $\underline{A} := (A, \mathrm{Fil}_\bullet A)$ be a pair consisting of a B-algebra and a B-algebra filtration on A; let also M be a finitely generated A-module, and $\mathrm{Fil}_\bullet M$ a good $\underline{A}$-filtration on M (see definition 4.4.27). By definition, this means that the Rees module $\mathsf{R}(\underline{M})_\bullet$ of the filtered $\underline{A}$-module $\underline{M} := (M, \mathrm{Fil}_\bullet M)$ is finitely generated over the graded Rees B-algebra $\mathsf{R}(\underline{A})_\bullet$. We have :

Proposition 5.7.27. *In the situation of* (5.7.26)*, suppose that A is a finitely presented B-algebra, M is a finitely presented A-module, and* $\mathrm{Fil}_\bullet A$ *is a good filtration. Then :*

(i) $\mathsf{R}(\underline{A})_\bullet$ *is a finitely presented B-algebra, and* $\mathsf{R}(\underline{M})_\bullet$ *is a finitely presented* $\mathsf{R}(\underline{A})_\bullet$*-module.*
(ii) *If furthermore,* $\mathrm{Fil}_\bullet A$ *is a positive filtration (see definiion* 4.4.27(ii)*), then* $\mathrm{Fil}_i M$ *is a finitely presented B-module, for every $i \in \mathbb{Z}$.*

Proof. By lemma 4.4.31, there exists a finite system of generators $\mathbf{m} := (m_1, \dots, m_n)$ of M, and a sequence of integers $\mathbf{k} := (k_1, \dots, k_n)$ such that $\mathrm{Fil}_\bullet M$ is of the form (4.4.30).

(i): Consider first the case where A is a free B-algebra of finite type, say $A = B[t_1, \dots, t_p]$, such that $\mathrm{Fil}_\bullet A$ is the good filtration associated to the system of generators $\mathbf{t} := (t_1, \dots, t_p)$ and the sequence of integers $\mathbf{r} := (r_1, \dots, r_p)$, and moreover M is a free A-module with basis $\mathbf{m}$. Then $\mathsf{R}(\underline{A})_\bullet$ is also a free B-algebra of finite type (see example 4.4.28). Moreover, for every $j \leq n$, let $M_j \subset M$ be the A-submodule generated by m_j, and denote by $\mathrm{Fil}_\bullet M_j$ the good $\underline{A}$-filtration associated to the pair $(\{m_j\}, \{k_j\})$; clearly $\mathrm{Fil}_\bullet M = \mathrm{Fil}_\bullet M_1 \oplus \cdots \oplus \mathrm{Fil}_\bullet M_n$, therefore $\mathsf{R}(\underline{M})_\bullet = \mathsf{R}(\underline{M}_1)_\bullet \oplus \cdots \oplus \mathsf{R}(\underline{M}_n)_\bullet$, where $\underline{M}_j := (M_j, \mathrm{Fil}_\bullet M_j)$ for every $j \leq n$. Obviously each $\mathsf{R}(\underline{A})_\bullet$-module $\mathsf{R}(\underline{M}_j)_\bullet$ is free of rank one, so the proposition follows in this case.

Next, suppose that A is a free B-algebra of finite type, and M is arbitrary. Let F be a free A-module of rank n, $\mathbf{e} := (e_1, \dots, e_n)$ a basis of F, and define an A-linear surjection $\varphi : F \to M$ by the rule $e_j \mapsto m_j$ for every $j \leq n$. Then φ is even a map of filtered $\underline{A}$-modules, provided we endow F with the good $\underline{A}$-filtration $\mathrm{Fil}_\bullet F$ associated to the pair $(\mathbf{e}, \mathbf{k})$. More precisely, let $N := \mathrm{Ker}\,\varphi$; then the filtration $\mathrm{Fil}_\bullet M$ is induced by $\mathrm{Fil}_\bullet F$, meaning that $\mathrm{Fil}_i M := (N + \mathrm{Fil}_i F)/N$ for every $i \in \mathbb{Z}$. Obviously the natural map $\mathsf{R}(\underline{F})_\bullet \to \mathsf{R}(\underline{M})_\bullet$ is surjective, and its kernel is the Rees module $\mathsf{R}(\underline{N})_\bullet$ corresponding to the filtered $\underline{A}$-module $\underline{N} := (N, \mathrm{Fil}_\bullet N)$ with $\mathrm{Fil}_i N := N \cap \mathrm{Fil}_i F$ for every $i \in \mathbb{Z}$. Let $\mathbf{x} := (x_1, \dots, x_s)$ be a finite system of generators of N, and choose a sequence of integers $\mathbf{j} := (j_1, \dots, j_s)$ such that $x_i \in \mathrm{Fil}_{j_i} N$ for every $i \leq s$; denote by $\underline{L}$ the filtered $\underline{A}$-module associated as in (4.4.29), to the A-module N and the pair $(\mathbf{x}, \mathbf{j})$. Thus, $\mathsf{R}(\underline{L})_\bullet$ is a finitely generated graded $\mathsf{R}(\underline{A})_\bullet$-submodule of $\mathsf{R}(\underline{N})_\bullet$. To ease notation, set $\overline{N} := \mathsf{R}(\underline{N})_\bullet/\mathsf{R}(\underline{L})_\bullet$ and $\overline{F} := \mathsf{R}(\underline{F})_\bullet/\mathsf{R}(\underline{L})_\bullet$. Recall (definition 4.4.25(iii)) that $\mathsf{R}(\underline{A})_\bullet$ is a graded B-subalgebra of $A[U, U^{-1}]$; then we have :

Claim 5.7.28. $\overline{N} = \bigcup_{n \in \mathbb{N}} \mathrm{Ann}_{\overline{F}}(U^n)$.

Proof of the claim. It suffices to consider a homogeneous element $y := U^i z \in \mathsf{R}(\underline{F})_i$, for some some $z \in \mathrm{Fil}_i F$. Suppose that $U^n y \in \mathsf{R}(\underline{L})_{i+n}$; especially, $U^{i+n} z \in \mathsf{R}(\underline{N})_{i+n}$, so $z \in N$, hence $y \in \mathsf{R}(\underline{N})_i$. Conversely, suppose that $y \in \mathsf{R}(\underline{N})_i$; write $z = x_1 a_1 + \cdots + x_s a_s$ for some $a_1, \dots, a_s \in A$. Say that $a_r \in \mathrm{Fil}_{b_r} A$ for every $r \leq s$, and set $l := \max(i, b_1 + j_1, \dots, b_s + j_s)$. Then $U^{l-i} y = U^l z \in \mathsf{R}(\underline{L})_l$. ◊

By the foregoing, $\mathsf{R}(\underline{F})_\bullet$ is finitely presented over $\mathsf{R}(\underline{A})_\bullet$. It follows that $\overline{N}$ is finitely generated (corollary 5.7.24), hence $\mathsf{R}(\underline{N})_\bullet$ is a finitely generated $\mathsf{R}(\underline{A})_\bullet$-module, so finally $\mathsf{R}(\underline{M})_\bullet$ is finitely presented over $\mathsf{R}(\underline{A})_\bullet$.

Lastly, consider the case of an arbitrary finitely presented B-algebra A, endowed with the good B-algebra filtration associated to a system of generators $\mathbf{x} := (x_1, \dots, x_p)$ and the sequence of integers $\mathbf{r}$. We map the free B-algebra $P := B[t_1, \dots, t_p]$ onto A, by the rule: $t_j \mapsto x_j$ for every $j \leq p$. This is even a map of filtered algebras, provided we endow P with the good filtration $\mathrm{Fil}_\bullet P$ associated to the pair $(\mathbf{t}, \mathbf{r})$; more precisely, $\mathrm{Fil}_\bullet P$ induces the filtration $\mathrm{Fil}_\bullet A$ on A, hence $\mathrm{Fil}_\bullet A$ is a good $\underline{P}$-filtration. By the foregoing case, we then deduce that $\mathsf{R}(\underline{A})_\bullet$ is a finitely presented $\mathsf{R}(\underline{P})_\bullet$-module (where $\underline{P} := (P, \mathrm{Fil}_\bullet P)$), hence also a finitely presented K^+-algebra. Moreover, M is a finitely presented P-module, and clearly $\mathrm{Fil}_\bullet M$ is good when regarded as a $\underline{P}$-filtration, hence $\mathsf{R}(\underline{M})_\bullet$ is a finitely presented $\mathsf{R}(\underline{P})_\bullet$-module, so also a finitely presented $\mathsf{R}(\underline{A})_\bullet$-module.

(ii): The positivity condition implies that $\mathsf{R}(\underline{A})_0 = B$; then, taking into account (i), the assertion is just a special case of proposition 4.4.16(iii). □

Theorem 5.7.29 (Artin-Rees lemma). *Let A be an essentially finitely presented K^+-algebra, $I \subset A$ an ideal of finite type, M a finitely presented A-module, $N \subset M$ a finitely generated submodule. Then there exists an integer $c \in \mathbb{N}$ such that :*

$$I^n M \cap N = I^{n-c}(I^c M \cap N) \qquad \textit{for every } n \geq c.$$

Proof. We reduce easily to the case where A is a finitely presented K^+-algebra; then, with all the work done so far, we only have to repeat the argument familiar from the classical noetherian case. Indeed, let us define a filtered K^+-algebra $\underline{A} := (A, \mathrm{Fil}_\bullet A)$ by the rule : $\mathrm{Fil}_n A := I^{-n}$ if $n \leq 0$, and $\mathrm{Fil}_n A := A$ otherwise; also let $\underline{M} := (M, \mathrm{Fil}_\bullet M)$ be the filtered $\underline{A}$-module such that $\mathrm{Fil}_i M := M \cdot \mathrm{Fil}_i A$ for every $i \in \mathbb{Z}$. We endow $Q := M/N$ with the filtration $\mathrm{Fil}_\bullet Q$ induced from $\mathrm{Fil}_\bullet M$, so that the natural projection $M \to M/N$ yields a map of filtered $\underline{A}$-modules $\underline{M} \to \underline{Q} := (Q, \mathrm{Fil}_\bullet Q)$. By proposition 5.7.27, there follows a surjective map of finitely presented graded $\mathsf{R}(\underline{A})_\bullet$-modules : $\pi_\bullet : \mathsf{R}(\underline{M})_\bullet \to \mathsf{R}(\underline{Q})_\bullet$, whose kernel in degree $k \leq 0$ is the A-module $I^{-k}M \cap N$. Hence we can find a finite (non-empty) system $f_1, \dots, f_r$ of generators for the $\mathsf{R}(\underline{A})_\bullet$-module $\mathrm{Ker}\,\pi_\bullet$; clearly we may assume that each f_i is homogeneous, say of degree d_i; we may also suppose that $d_i \leq 0$ for every $i \leq r$, since $\mathrm{Ker}\,\pi_0$ generates $\mathrm{Ker}\,\pi_n$, for every $n \geq 0$. Let $d := \min(d_i \mid i = 1, \dots, r)$; by inspecting the definitions, one verifies easily that

$$\mathsf{R}(\underline{A})_i \cdot \mathrm{Ker}\,\pi_j \subset \mathsf{R}(\underline{A})_{i+j-d} \cdot \mathrm{Ker}\,\pi_d \qquad \text{whenever } i + j \leq d \text{ and } 0 \geq j \geq d.$$

Therefore $\mathrm{Ker}\,\pi_{d+k} = \mathsf{R}(\underline{A})_k \cdot \mathrm{Ker}\,\pi_d$ for every $k \leq 0$, so the assertion holds with $c := -d$. $\square$

5.7.30. In the situation of theorem 5.7.29, let M be any finitely presented A-module; we denote by $M^\wedge$ the I-adic completion of M, which is an $A^\wedge$-module.

Corollary 5.7.31. *With the notation of* (5.7.30), *the following holds :*

(i) *The functor $M \mapsto M^\wedge$ is exact on the category of finitely presented A-modules.*
(ii) *For every finitely presented A-module M, the natural map $M \otimes_A A^\wedge \to M^\wedge$ is an isomorphism.*
(iii) *$A^\wedge$ is flat over A. If additionally $I \subset \mathrm{rad}\,A$ (the Jacobson radical of A), then $A^\wedge$ is faithfully flat over A.*

Proof. (i): Let $N \subset M$ be any injection of finitely presented A-modules; by theorem 5.7.29, the I-adic topology on M induces the I-adic topology on N. Then the assertion follows from [61, Th.8.1].

(ii): Choose a presentation $A^{\oplus p} \to A^{\oplus q} \to M \to 0$. By (i) we deduce a commutative diagram with exact rows :

$$\begin{array}{ccccccc}
A^{\oplus p} \otimes_A A^\wedge & \longrightarrow & A^{\oplus q} \otimes_A A^\wedge & \longrightarrow & M \otimes_A A^\wedge & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
(A^{\oplus p})^\wedge & \longrightarrow & (A^{\oplus q})^\wedge & \longrightarrow & M^\wedge & \longrightarrow & 0
\end{array}$$

and clearly the two left-most vertical arrows are isomorphisms. The claim follows.

(iii): The first assertion means that the functor $M \mapsto M \otimes_A A^\wedge$ is exact; this follows from (i) and (ii), via a standard reduction to the case where M is finitely presented. Suppose next that $I \subset \mathrm{rad}\,A$; to conclude, it suffices to show that the image of the natural map $\mathrm{Spec}\,A^\wedge \to \mathrm{Spec}\,A$ contains the maximal spectrum $\mathrm{Max}\,A$ ([61, Th.7.3]). This is clear, since $A^\wedge/IA^\wedge \simeq A/IA$, and the natural map $\mathrm{Max}\,A/IA \to \mathrm{Max}\,A$ is a bijection. $\square$

Theorem 5.7.32. *Let $B \to A$ be a map of finitely presented K^+-algebras. Then :*

(i) *If M is an ω-coherent A-module, M is ω-coherent as a B-module.*
(ii) *If J is a coh-injective B-module, and $I \subset B$ is a finitely generated ideal, we have :*
  (a) *$\mathrm{Hom}_B(A, J)$ is a coh-injective A-module.*
  (b) *$\bigcup_{n\in\mathbb{N}} \mathrm{Ann}_J(I^n)$ is a coh-injective B-module.*
(iii) *If $(J_n, \varphi_n \mid n \in \mathbb{N})$ is a direct system consisting of coh-injective B/I^n-modules J_n and B-linear maps $\varphi_n : J_n \to J_{n+1}$ (for every $n \in \mathbb{N}$), then $\mathop{\mathrm{colim}}\limits_{n\in\mathbb{N}} J_n$ is a coh-injective B-module.*

Proof. (See (4.3.25) for the generalities on coh-injective and ω-coherent modules.)

(i): We reduce easily to the case where M is finitely presented over A. Let $x_1,\dots,x_p$ be a system of generators for the B-algebra A, and $m_1,\dots,m_n$ a system of generators for the A-module M. We let $\underline{A} := (A, \mathrm{Fil}_\bullet A)$, where $\mathrm{Fil}_\bullet A$ is the good B-algebra filtration associated to the pair $\mathbf{x} := (x_1,\dots,x_p)$ and $\mathbf{r} := (1,\dots,1)$; likewise, let $\underline{M}$ be the filtered $\underline{A}$-module defined by the good $\underline{A}$-filtration on M associated to $\mathbf{m} := (m_1,\dots,m_n)$ and $\mathbf{k} := (0,\dots,0)$ (see definition 4.4.27). Then claim follows easily after applying proposition 5.7.27(ii) to the filtered B-algebra $\underline{A}$ and the filtered $\underline{A}$-module $\underline{M}$.

(ii.a): Let $N \subset M$ be two coherent A-modules, and $\varphi : N \to \mathrm{Hom}_B(A, J)$ an A-linear map. According to claim 5.1.26, φ corresponds by adjunction to a unique B-linear map $\overline{\varphi} : N \to J$; on the other hand by (i), M, and M/N are ω-coherent B-modules, hence $\overline{\varphi}$ extends to a B-linear map $\overline{\psi} : M \to J$ (lemma 4.3.27). Under the adjunction, $\overline{\psi}$ corresponds to an A-linear extension $\psi : M \to \mathrm{Hom}_B(A, J)$ of φ.

(iii): Let $M \subset N$ be two finitely presented B-modules, $f : M \to J := \operatorname*{colim}_{n\in\mathbb{N}} J_n$ a B-linear map. Since M is finitely generated, f factors through a map $f_n : M \to J_n$ and the natural map $J_n \to J$, provided n is large enough ([36, Prop.2.3.16(ii)]). By theorem 5.7.29 there exists $c \in \mathbb{N}$ such that $I^{n+c}N \cap M \subset I^n M$. Hence

$$f_{n+c} := \varphi_{n+c-1} \circ \cdots \circ \varphi_n \circ f_n : M \to J_{n+c}$$

factors through a unique B/I^{n+c}-linear map $\overline{f}_{n+c} : \overline{M} := M/(I^{n+c}N \cap M) \to J_{n+c}$; since I^{n+c} is finitely generated, $\overline{M}$ is a coherent submodule of the coherent B-module $\overline{N} := N/I^{n+c}N$, therefore $\overline{f}_{n+c}$ extends to a map $\overline{g} : \overline{N} \to J_{n+c}$. The composition of $\overline{g} : \overline{N} \to J_{n+c}$ with the projection $N \to \overline{N}$ and the natural map $J_{n+c} \to J$, is the sought extension of f.

(ii.b): Letting $A := B/I^n$ in (ii.a), we deduce that $J_n := \mathrm{Ann}_J(I^n)$ is a coh-injective B/I^n-module, for every $n \in \mathbb{N}$. Then the assertion follows from (iii). $\square$

5.8. **Local duality.** Throughout this section we let $(K, |\cdot|)$ be a valued field of rank one. We shall continue to use the general notation of (5.7).

5.8.1. Let A be finitely presented K^+-algebra, $I \subset A$ an ideal generated by a finite system $\mathbf{f} := (f_i \mid i = 1,\dots,r)$, and denote by $i : Z := V(I) \to X := \operatorname{Spec} A$ the natural closed immersion. Let also $\mathscr{I} \subset \mathscr{O}_X$ be the ideal arising from I. For every $n \geq 0$ there is a natural epimorphism

$$i_* i^{-1}\mathscr{O}_X \to \mathscr{O}_X/\mathscr{I}^n$$

whence natural morphisms in $\mathsf{D}(A\text{-}\mathbf{Mod})$:

$$R\mathrm{Hom}^\bullet_{\mathscr{O}_X}(\mathscr{O}_X/\mathscr{I}^n, \mathscr{F}^\bullet) \to R\mathrm{Hom}^\bullet_{\mathscr{O}_X}(i_* i^{-1}\mathscr{O}_X, \mathscr{F}^\bullet) \xrightarrow{\sim} R\Gamma_Z \mathscr{F}^\bullet \tag{5.8.2}$$

for any bounded below complex $\mathscr{F}^\bullet$ of $\mathscr{O}_X$-modules.

Theorem 5.8.3. *In the situation of* (5.8.1), *let $M^\bullet$ be any object of* $\mathsf{D}^+(A\text{-}\mathbf{Mod})$, *and $M^{\bullet\sim}$ the complex of $\mathscr{O}_X$-modules determined by M. Then* (5.8.2) *induces natural isomorphisms :*

$$\operatorname*{colim}_{n\in\mathbb{N}} \mathrm{Ext}^i_A(A/I^n, M^\bullet) \xrightarrow{\sim} R^i\Gamma_Z M^{\bullet\sim} \qquad \textit{for every } i \in \mathbb{N}.$$

Proof. For $\mathscr{F}^\bullet := M^{\bullet\sim}$, trivial duality (theorem 5.1.27) identifies the source of (5.8.2) with $R\mathrm{Hom}^\bullet_A(A/I^n, M^\bullet)$; then one takes cohomology in degree i and forms the colimit over n to define the sought map. Next, by usual spectral sequence arguments, we may reduce to the case where $M^\bullet$ is a single A-module M sitting in degree zero (see *e.g.* the proof of proposition 5.2.11(i)). By inspecting the definitions, one verifies easily that the morphism thus defined is the composition of the isomorphism of proposition 5.4.31(ii.b) and the map (4.1.27) (see the proof of proposition 5.4.31(ii.b)); then it suffices to show that the inverse system $(H_i\mathbf{K}_\bullet(\mathbf{f}^n) \mid n \in \mathbb{N})$ is essentially zero when $i > 0$ (lemma 4.1.28). By lemma 4.1.30, this will in turn follow, provided the following holds. For every finitely presented quotient B of A, and every $b \in B$, there exists $p \in \mathbb{N}$ such that $\mathrm{Ann}_B(b^q) = \mathrm{Ann}_B(b^p)$ for every $q \geq p$. This latter assertion is a special case of corollary 5.7.24. $\square$

Corollary 5.8.4. *In the situation of theorem* 5.8.3*, we have natural isomorphisms :*

$$\operatorname*{colim}_{n\in\mathbb{N}} \mathrm{Ext}^i_A(I^n, M^\bullet) \xrightarrow{\sim} H^i(X\setminus Z, M^{\bullet\sim}) \qquad \textit{for every } i \in \mathbb{N}.$$

Proof. Set $U := X \setminus Z$. We may assume that $M^\bullet$ is a complex of injective A-modules, in which case the sought map is obtained by taking colimits over the direct system of composed morphisms :

$$\mathrm{Hom}_A(I^n, M^\bullet) \xrightarrow{\beta_n} \Gamma(U, M^\bullet) \to R\Gamma(U, M^{\bullet\sim})$$

where β_n is induced by the identification $(I^n)^\sim_{|U} = \mathscr{O}_U$ and the natural isomorphism :

$$\mathrm{Hom}_{\mathscr{O}_U}(\mathscr{O}_U, M^{\bullet\sim}) \simeq \Gamma(U, M^{\bullet\sim}).$$

The complex $R\Gamma(U, M^{\bullet\sim})$ is computed by a Cartan-Eilenberg injective resolution $M^{\bullet\sim} \xrightarrow{\sim} \mathscr{M}^\bullet$ of $\mathscr{O}_X$-modules, and then the usual arguments allow to reduce to the case where $M^\bullet$ consists of a single injective A-module M placed in degree zero. Finally, from the short exact sequence $0 \to I^n \to A \to A/I^n \to 0$ we deduce a commutative ladder with exact rows :

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_A(A/I^n, M) \arrow[r] \arrow[d, "\alpha_n"'] & M \arrow[r] \arrow[d, equal] & \mathrm{Hom}_A(I^n, M) \arrow[r] \arrow[d, "\beta_n"] & 0 \arrow[d] \\
0 \arrow[r] & \Gamma_Z M^\sim \arrow[r] & M \arrow[r] & \Gamma(U, M^\sim) \arrow[r] & R^1\Gamma_Z M^\sim \arrow[r] & 0
\end{tikzcd}$$

where α_n is induced from (5.8.2). From theorem 5.8.3 it follows that $\operatorname*{colim}_{n\in\mathbb{N}} \alpha_n$ is an isomorphism, and $R^i\Gamma_Z M^\sim$ vanishes for all $i > 0$; hence $\operatorname*{colim}_{n\in\mathbb{N}} \beta_n$ is an isomorphism, and the contention follows. $\square$

Proposition 5.8.5. *Let $X \xrightarrow{f} Y \xrightarrow{g} S$ be two finitely presented morphisms of schemes.*

(i) *If g is smooth, $\mathscr{O}_Y[0]$ is a dualizing complex on Y.*
(ii) *If g is smooth and f is a closed immersion, then X admits a dualizing complex $\omega^\bullet_X$.*
(iii) *If X and Y are affine and $\omega^\bullet_Y$ is a dualizing complex on Y, then $f^!\omega^\bullet_Y$ is dualizing on X (notation of* (5.3.15)*).*
(iv) *Every finitely presented quasi-separated S-scheme admits a dualizing complex.*

Proof. (i) is an immediate consequence of proposition 5.7.10(i).

(ii): As in the proof of corollary 5.6.38, this follows directly from (i) and lemma 5.6.28(i).

(iii): Let us choose a factorization $f = p_Y \circ i$ where $i : X \to \mathbb{A}^n_Y$ is a finitely presented closed immersion, and $p_Y : \mathbb{A}^n_Y \to Y$ the smooth projection onto Y. In view of lemma 5.6.28(i) and (5.3.15), it suffices to prove the assertion for the morphism p_Y. To this aim, we pick a closed finitely presented immersion $h : Y \to \mathbb{A}^m_S$ and consider the fibre diagram :

$$\begin{tikzcd}
\mathbb{A}^n_Y \arrow[r, "h'"] \arrow[d, "p_Y"'] & \mathbb{A}^{n+m}_S \arrow[d, "p_S"] \\
Y \arrow[r, "h"] & \mathbb{A}^m_S.
\end{tikzcd}$$

By (i) we know that the scheme $\mathbb{A}^m_S$ admits a dualizing complex $\omega^\bullet$, and then lemma 5.6.28(i) says that $h^!\omega^\bullet$ is dualizing as well. By proposition 5.6.24 we deduce that $\omega^\bullet_Y \simeq \mathscr{L}[\sigma] \otimes_{\mathscr{O}_Y} h^!\omega^\bullet$ for some invertible $\mathscr{O}_Y$-module $\mathscr{L}$ and some continuous function $\sigma : |Y| \to \mathbb{Z}$. Since p_Y is smooth, we can compute: $p^!_Y\omega^\bullet_Y \simeq (p^!_Y \circ h^!\omega^\bullet) \otimes_{\mathscr{O}_{\mathbb{A}^n_Y}} p^*_Y\mathscr{L}[\sigma]$, hence $p^!_Y\omega^\bullet_Y$ is dualizing if and only if the same holds for $p^!_Y \circ h^!\omega^\bullet$. By proposition 5.3.7(iv), the latter complex is isomorphic to $h'^{\flat} \circ p^!_S\omega^\bullet$ and again using lemma 5.6.28(i) we reduce to checking that $p^!_S\omega^\bullet$ is dualizing, which is clear from (i).

(iv): Let $f : X \to S$ be a finitely presented morphism, with X quasi-separated. If X is affine, (iii) says that $f^!\mathscr{O}_S[0]$ is dualizing on X. In the general case, let $(U_i \mid i = 1, \dots, n)$ be a finite covering of X consisting of affine open subsets; for each $i, j, k = 1, \dots, n$, denote by $f_i : U_i \to S$ the restriction of f, set $U_{ij} := U_i \cap U_j$, $U_{ijk} := U_{ij} \cap U_k$, and let $g_{ij} : U_{ij} \to U_i$ be the inclusion map. We know that $f_i^!\mathscr{O}_S[0]$ is dualizing on U_i, for every $i = 1, \dots, n$; moreover, for every $i, j = 1, \dots, n$ there exists a natural isomorphism

$$\psi_{ij} : g^*_{ij}f^!_i\mathscr{O}_S[0] \xrightarrow{\sim} g^*_{ji}f^!_j\mathscr{O}_S[0]$$

fulfilling the cocycle condition

$$\psi_{jk|U_{ijk}} \circ \psi_{ij|U_{ijk}} = \psi_{ik|U_{ijk}} \qquad \text{for every } i, j, k = 1, \dots, n$$

(lemma 5.3.16). In other words, $((U_i, f^!_i\mathscr{O}_S[0]); \psi_{ij} \mid i, j = 1, \dots, n)$ is a descent datum for the fibration (5.6.30). Then the assertion follows from proposition 5.6.31. ☐

Example 5.8.6. (i) For given $b \in \mathfrak{m}_K$, let $i_b : S_{/b} \to S$ be the closed immersion. If $b \neq 0$, a simple computation yields a natural isomorphism in $\mathsf{D}(\mathscr{O}_{S_{/b}}\text{-}\mathbf{Mod})$:

$$i^!_b\mathscr{O}_S \xrightarrow{\sim} \mathscr{O}_{S_{/b}}[-1].$$

By proposition 5.8.5(i,iii), we deduce that $\mathscr{O}_{S_{/b}}[0]$ is a dualizing complex on $S_{/b}$, for every $b \in \mathfrak{m}_K$.

(ii) Next, let $f : X \to S_{/b}$ be an affine finitely presented Cohen-Macaulay morphism, of constant fibre dimension n. Then $\omega^\bullet_X := f^!\mathscr{O}_{S_{/b}}[0]$ is a dualizing complex on X, by (i) and proposition 5.8.5(iii). Moreover, $\omega^\bullet_X$ is concentrated in degree $-n$, and $H^{-n}\omega^\bullet_X$ is a finitely presented f-Cohen-Macaulay $\mathscr{O}_X$-module. Indeed, let $i : X \to Y := \mathbb{A}^m_{S_{/b}}$ be a closed immersion of $S_{/b}$-schemes, and denote by $g : Y \to S_{/b}$ the projection. Fix any $x \in X$, let $y := i(x)$ and set

$$d_x := \dim \mathscr{O}_{f^{-1}(fx),x} \qquad d_y := \dim \mathscr{O}_{g^{-1}(gy),y}.$$

We have a natural isomorphism in $\mathsf{D}(\mathscr{O}_X\text{-}\mathbf{Mod})$

$$\omega^\bullet_X \xrightarrow{\sim} i^!\mathscr{O}_Y[m] \xrightarrow{\sim} i^*R\mathscr{H}om^\bullet_{\mathscr{O}_Y}(i_*\mathscr{O}_X, \mathscr{O}_Y[m])$$

(lemma 5.3.16(i)), and by assumption $i_*\mathscr{O}_X$ is a g-Cohen-Macaulay $\mathscr{O}_Y$-module; by theorem 5.6.40(ii), it follows that $\omega^\bullet_{X,x}$ is concentrated in degree $d_y - d_x - m$, and its cohomology in that degree is f-Cohen-Macaulay, as asserted. Lastly, since the fibres of f and g are equidimensional ([31, Ch.IV, Prop.5.2.1] and lemma 5.6.36(ii)), it is easily seen that $d_y - d_x = m - n$ (details left to the reader), whence the claim.

5.8.7. For any finitely presented morphism $f : X \to S$ we consider the map :

$$d : |X| \to \mathbb{Z} \qquad x \mapsto \mathrm{tr.deg}(\kappa(x)/\kappa(f(x))) + \dim \overline{\{f(x)\}}.$$

Lemma 5.8.8. *With the notation of* (5.8.7)*, let $x, y \in X$, and suppose that x is a specialization of y. We have :*

(i) *x is an immediate specialization of y if and only if $d(y) = d(x) + 1$.*
(ii) *If X is irreducible, $d(x) - d(y) = \dim X(y) - \dim X(x)$.*
(iii) *X is catenarian and of finite Krull dimension.*
(iv) *If X is irreducible and f is flat, then $\mathscr{O}_{f^{-1}(fx),x}$ is equidimensional.*
(v) *If f is flat, then $\dim \mathscr{O}_{X,x} = \dim \mathscr{O}_{f^{-1}(fx),x} + \dim \mathscr{O}_{S,f(x)}$.*
(vi) *If f is Cohen-Macaulay at the point x, then $\mathscr{O}_{X,x}$ is equidimensional.*

Proof. (i): In case $f(x) = f(y)$, the assertion follows from [31, Ch.IV, Prop.5.2.1]. Hence, suppose that $f(x) = s$, $f(y) = \eta$; let Z be the topological closure of $\{y\}$ in X, and endow Z with its reduced subscheme structure; notice that Z is an S-scheme of finite type. By assumption, X is quasi-compact and quasi-separated, hence $\{y\}$ is a pro-constructible subset of X, and therefore Z is the set of all specializations of y in X ([30, Ch.IV, Th.1.10.1]). Especially, y is the unique maximal point of $Z_\eta := Z \cap f^{-1}(\eta)$; also, if x is an immediate specialization of y, then x is a maximal point of $Z_s := Z \cap f^{-1}(s)$, and by [32, Ch.IV, Lemme 14.3.10], the latter implies that $d(y) = d(x) + 1$.

Conversely, suppose that $d(y) = d(x) + 1$; from [32, Ch.IV, Lemme 14.3.10] and [31, Ch.IV, Prop.5.2.1] we deduce that x is a maximal point of Z_s; then x is an immediate specialization of y in X, since otherwise x would be a specialization in X of a proper specialization y' of y in Z_η, and in this case [32, Ch.IV, Lemme 14.3.10] would say that $d(y') = d(x) + 1$, *i.e.* $d(y) = d(y')$, contradicting [31, Ch.IV, Prop.5.2.1].

(iii) It is easily seen that X has finite Krull dimension. Now, consider any sequence $y_0, \dots, y_n$ of points of X, with $y_0 := y$, $y_n := x$, and such that y_{i+1} is an immediate specialization of y_i, for $i = 0, \dots, n-1$; from (i) we deduce that $n = d(y) - d(x)$, especially n is independent of the chosen chain of specializations, so X is catenarian.

(ii): We reduce easily to the case where y is the maximal point of X, in which case we may argue as in the proof of (iii) (details left to the reader).

(iv): This is trivial, if $f(x) = \eta$; hence let us assume that $f(x) = s$, and let y_1, y_2 be two maximal generizations of x in $f^{-1}(s)$. We may find an affine open subset $U \subset X$ such that $y_1 \in U$ and $y_2 \notin U$; arguing as in the proof of (i), we see that the maximal point z of U (which is also the maximal point of X) is an immediate generization of y_1 in U, hence $d(z) = d(y_1) + 1$. Likewise, $d(z) = d(y_2) + 1$, hence $d(y_1) = d(y_2)$. In view of (i), the assertion follows easily.

(v): If $f(x) = \eta$, the identity is [31, Ch.IV, Cor.6.1.2]. Hence, suppose that $f(x) = s$. In this case, the proof of (iv) shows that the identity holds, when X is irreducible. In the general case, notice that – by the flatness assumption – every irreducible component of X intersect $f^{-1}(\eta)$, and it is therefore itself a flat S-scheme (with its reduced subscheme structure). Since $f^{-1}(\eta)$ is a noetherian scheme, it also follows that the set of irreducible components of X is finite; thus, let $X_1, \dots, X_n$ be the reduced irreducible components of X containing x, and $f_i : X_i \to S$ ($i = 1, \dots, n$) the corresponding restrictions of f. Then

$$\dim \mathscr{O}_{X,x} = \max_{1 \le i \le n} \dim \mathscr{O}_{X_i,x} \qquad \text{and} \qquad \dim \mathscr{O}_{f^{-1}(s),x} = \max_{1 \le i \le n} \dim \mathscr{O}_{f_i^{-1}(s),x}$$

whence the contention.

(vi): By assumption, $\mathscr{O}_{f^{-1}(fx),x}$ is a Cohen-Macaulay noetherian local ring, especially it is equidimensional ([61, Th.17.3(i)]). Thus, the assertion already follows, in case $f(x) = \eta$. If $f(x) = s$, let y be any maximal generization of x in $f^{-1}(s)$; arguing as in the proof of (i), we see that every immediate generization of y in X is a maximal point of X, whence the contention. $\square$

Lemma 5.8.9. (i) *The K^+-module K/K^+ is coh-injective.*

(ii) *Let A be a finitely presented K^+-algebra, $I \subset A$ a finitely generated ideal, J a coh-injective K^+-module. Then the A-module*

$$J_A := \operatorname*{colim}_{n \in \mathbb{N}} \mathrm{Hom}_{K^+}(A/I^n, J)$$

is coh-injective.

Proof. (i): It suffices to show that $\mathrm{Ext}^1_{K^+}(M, K/K^+) = 0$ whenever M is a finitely presented K^+-module. This is clear when $M = K^+$, and then [36, Ch.6, Lemma 6.1.14] reduces to the case where $M = K^+/aK^+$ for some $a \in \mathfrak{m}_K \setminus \{0\}$; in that case we can compute using the free resolution $K^+ \xrightarrow{a} K^+ \to M$, and the claim follows easily.

(ii): According to theorem 5.7.32(ii.a), the A-module $J'_A := \mathrm{Hom}_{K^+}(A, J)$ is coh-injective. However, $J_A = \bigcup_{n\in\mathbb{N}} \mathrm{Ann}_{J'_A}(I^n)$, so the assertion follows from theorem 5.7.32(ii.b). □

Theorem 5.8.10. *Let $f : X \to S$ be a finitely presented affine morphism. Then for every point $x \in X$, the $\mathscr{O}_{X,x}$-module*

$$J(x) := R^{1-d(x)}\Gamma_{\{x\}}(f^!\mathscr{O}_S[0])_{|X(x)}$$

is coh-injective, and we have a natural isomorphism in $\mathsf{D}(\mathscr{O}_{X,x}\text{-}\mathbf{Mod})$:

$$R\Gamma_{\{x\}}(f^!\mathscr{O}_S[0])_{|X(x)} \simeq J(x)[d(x) - 1].$$

Proof. Fix $x \in X$ and set $d := d(x)$; in case $f(x) = \eta$, the assertion follows from [46, Ch.V, Cor.8.4 and following remark]. Hence, suppose that $f(x) = s$, the closed point of S, and say that $X = \mathrm{Spec}\, A$.

Claim 5.8.11. We may assume that x is closed in X, hence that $\kappa(x)$ is finite over $\kappa(s)$.

Proof of the claim. Arguing as in the proof of lemma 5.7.5, we can find a factorization of the morphism f as a composition $X \xrightarrow{g} Y := \mathbb{A}^d_S \xrightarrow{h} S$, such that $\xi := g(x)$ is the generic point of $h^{-1}(s) \subset Y$, the morphism g is finitely presented, and the stalk $\mathscr{O}_{Y,\xi}$ is a valuation ring. Moreover, let $g_y := g \times_Y \mathbf{1}_{Y(y)} : X(y) := X \times_Y Y(y) \to Y(y)$; we have a natural isomorphism

$$(f^!\mathscr{O}_S[0])_{|X(y)} \simeq g_y^!\mathscr{O}_{Y(y)}[d].$$

Then we may replace f by g_y, and K^+ by $\mathscr{O}_{Y,\xi}$, whence the claim. ◇

Hence, suppose now that x is closed in X, let $\mathfrak{p} \subset A$ be the maximal ideal corresponding to x, and $\overline{\mathfrak{p}}$ the image of $\mathfrak{p}$ in $\overline{A} := A \otimes_{K^+} \kappa$; we choose a finite system of elements $b_1, \dots, b_r \in \mathscr{O}_X(X)$ whose images in $\overline{A}$ generate $\overline{\mathfrak{p}}$. Pick any non-zero $a \in \mathfrak{m}_K$, and let $I \subset A$ be the ideal generated by the system $(a, b_1, \dots, b_r)$, and $\mathscr{I} \subset \mathscr{O}_X$ the corresponding coherent ideal; clearly $V(I) = \{x\}$, hence theorem 5.8.3 yields a natural isomorphism :

$$\text{(5.8.12)} \qquad \operatorname*{colim}_{n\in\mathbb{N}} R^i\mathrm{Hom}^\bullet_{\mathscr{O}_X}(\mathscr{O}_X/\mathscr{I}^n, f^!\mathscr{O}_S[0]) \xrightarrow{\sim} R^i\Gamma_{\{x\}}f^!\mathscr{O}_S[0] \qquad \text{for every } i \in \mathbb{Z}$$

where the transition maps in the colimit are induced by the natural maps $\mathscr{O}_X/\mathscr{I}^n \to \mathscr{O}_X/\mathscr{I}^m$, for every $n \geq m$. However, from lemmata 5.3.6(ii), 5.3.16(i) and corollary 5.1.33 we deduce as well natural isomorphisms :

$$R\mathrm{Hom}^\bullet_{\mathscr{O}_X}(\mathscr{O}_X/\mathscr{I}^n, f^!\mathscr{O}_S[0]) \xrightarrow{\sim} R\mathrm{Hom}^\bullet_{\mathscr{O}_S}(f_*\mathscr{O}_X/\mathscr{I}^n, \mathscr{O}_S) \xrightarrow{\sim} R\mathrm{Hom}^\bullet_{K^+}(A/I^n, K^+).$$

We wish to compute these Ext groups by means of lemma 4.3.27(iii); to this aim, let us remark first that A/I^n is an integral K^+-algebra for every $n \in \mathbb{N}$, hence it is a finitely presented torsion K^+-module, according to proposition 5.7.7(i). We may then use the coh-injective resolution $K^+[0] \to (0 \to K \to K/K^+ \to 0)$ (lemmata 5.8.9(i), 4.3.27(iii)) to compute :

$$\mathrm{Ext}^i_{K^+}(A/I^n, K^+) = \begin{cases} J_n := \mathrm{Hom}_{K^+}(A/I^n, K/K^+) & \text{if } i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Now the theorem follows from lemma 5.8.9(i) and theorem 5.7.32(ii.a),(iii).

We could also appeal directly to lemma 5.8.9(ii), provided we already knew that the foregoing natural identifications transform the direct system whose colimit appears in (5.8.12), into the direct system $(J_n \mid n \in \mathbb{N})$ whose transition maps are induced by the natural maps $A/I^n \to A/I^m$, for every $n \geq m$. For the sake of completeness, we check this latter assertion.

For every $n \geq m$, let $j_n : X_n := \mathrm{Spec}\, A/I^n \to X$ and $j_{mn} : X_m \to X_n$ be the natural closed immersions. We have a diagram of functors :

$$\begin{tikzcd}
j_{m*} \circ j^!_{mn} \circ (f \circ j_n)^! \arrow[r, "\zeta_1"] \arrow[dr, "\varepsilon_1"] & j_{n*} \circ (j_{mn*} \circ j^!_{mn}) \circ j^!_n \circ f^! \arrow[dr, "\varepsilon_2"] & j_{m*} \circ j^!_m \circ f^! \arrow[l, "\xi_2"'] \arrow[d, "\alpha_{mn}"] \\
j_{m*} \circ (f \circ j_m)^! \arrow[u, "\xi_1"] \arrow[r, "\beta_{mn}"] & j_{n*} \circ (f \circ j_n)^! \arrow[r, "\zeta_2"] & j_{n*} \circ j^!_n \circ f^!
\end{tikzcd}$$

where :

- ζ_1 and ζ_2 are induced by the natural isomorphism $\psi_{f,j_n} : (f \circ j_n)^! \xrightarrow{\sim} j^!_n \circ f^!$ of lemma 5.3.16(i).
- ξ_1 (resp. ξ_2) is induced by the isomorphism $\psi_{f\circ j_n, j_{mn}}$ (resp. $\psi_{j_n, j_{mn}}$).
- ε_1 and ε_2 are induced by the counit of adjunction $j_{mn*} \circ j^!_{mn} \to \mathbf{1}_{\mathrm{D}^+(\mathscr{O}_{X_n}\text{-}\mathbf{Mod})}$.
- β_{mn} (resp. α_{mn}) is induced by the natural map $(f \circ j_n)_*\mathscr{O}_{X_n} \to (f \circ j_m)_*\mathscr{O}_{X_m}$ (resp. by the map $j_{n*}\mathscr{O}_{X_n} \to j_{m*}\mathscr{O}_{X_m}$).

It follows from lemma 5.3.22 that the two triangular subdiagrams commute, and it is also clear that the same holds for the inner quadrangular subdiagram. Moreover, lemma 5.3.16(ii) yields a commutative diagram

$$(5.8.13)\qquad \begin{tikzcd}
(f \circ j_m)^! \arrow[r, "\psi_{f,j_m}"] \arrow[d, "\psi_{f\circ j_n, j_{mn}}"'] & j^!_m \circ f^! \arrow[d, "\psi_{j_n, j_{mn}} \circ f^!"] \\
j^!_{mn} \circ (f \circ j_n)^! \arrow[r, "j^!_{mn}(\psi_{f,j_n})"] & j^!_{mn} \circ j^!_n \circ f^!
\end{tikzcd}$$

such that j_{m*}(5.8.13) is the diagram :

$$\begin{tikzcd}
j_{m*} \circ (f \circ j_m)^! \arrow[r, "j_{m*}(\psi_{f,j_m})"] \arrow[d, "\xi_1"'] & j_{m*} \circ j^!_m \circ f^! \arrow[d, "\xi_2"] \\
j_{m*} \circ j^!_{mn} \circ (f \circ j_n)^! \arrow[r, "\zeta_1"] & j_{m*} \circ j^!_{mn} \circ j^!_n \circ f^!.
\end{tikzcd}$$

We then arrive at the following commutative diagram :

$$\begin{tikzcd}
R\mathrm{Hom}^\bullet_{\mathscr{O}_S}(\mathscr{O}_X/\mathscr{I}^m, \mathscr{O}_S) \arrow[r, "R\Gamma(\psi_{f,j_m})"] \arrow[d, "R\Gamma(\beta_{mn})"'] & R\mathrm{Hom}^\bullet_{\mathscr{O}_X}(\mathscr{O}_X/\mathscr{I}^m, f^!\mathscr{O}_S[0]) \arrow[d, "R\Gamma(\alpha_{mn})"] \\
R\mathrm{Hom}^\bullet_{\mathscr{O}_S}(\mathscr{O}_X/\mathscr{I}^n, \mathscr{O}_S) \arrow[r, "R\Gamma(\psi_{f,j_n})"] & R\mathrm{Hom}^\bullet_{\mathscr{O}_X}(\mathscr{O}_X/\mathscr{I}^n, f^!\mathscr{O}_S[0]).
\end{tikzcd}$$

Now, the maps $R\Gamma(\alpha_{mn})$ are the transition morphisms of the inductive system whose colimit appears in (5.8.12), hence we may replace the latter by the inductive system formed by the maps $R\Gamma(\beta_{mn})$. Combining with corollary 5.1.33, we finally deduce natural A-linear isomorphisms

$$\operatorname*{colim}_{n\in\mathbb{N}} \mathrm{Ext}^i_{K^+}(A/I^n, K^+) \xrightarrow{\sim} R^i\Gamma_{\{x\}} f^!\mathscr{O}_S[0]$$

where the transition maps in the colimit are induced by the natural maps $A/I^n \to A/I^m$, for every $n \geq m$. Our assertion is an immediate consequence. □

5.8.14. Let A be a local ring, essentially of finite presentation over K^+. Set $X := \mathrm{Spec}\, A$ and let $x \in X$ be the closed point. Notice that X admits a dualizing complex. Indeed, one can find a finitely presented affine S-scheme Y and a point $y \in Y$ such that $X \simeq \mathrm{Spec}\, \mathscr{O}_{Y,y}$; by proposition 5.8.5(iv), Y admits a dualizing complex $\omega^\bullet_Y$, and since every coherent $\mathscr{O}_X$-module extends to a coherent $\mathscr{O}_Y$-module, one verifies easily that the restriction of $\omega^\bullet_Y$ is dualizing for

X. Hence, let $\omega^\bullet$ be a dualizing complex for X. It follows easily from theorem 5.8.10 and propositions 5.8.5(iii) and 5.6.24, that there exists a unique $c \in \mathbb{Z}$ such that

$$J(x) := R^c\Gamma_{\{x\}}\omega^\bullet \neq 0.$$

Moreover, $J(x)$ is a coh-injective A-module, hence by lemma 4.3.27(iii), we obtain a well defined functor

$$\mathsf{D}^b(A\text{-}\mathbf{Mod}_{\mathrm{coh}}) \to \mathsf{D}^b(A\text{-}\mathbf{Mod})^o \quad : \quad C^\bullet \mapsto \mathrm{Hom}^\bullet_A(C^\bullet, J(x)). \tag{5.8.15}$$

Furthermore, let $\mathsf{D}^b_{\{x\}}(A\text{-}\mathbf{Mod}_{\mathrm{coh}})$ be the full subcategory of $\mathsf{D}^b(A\text{-}\mathbf{Mod}_{\mathrm{coh}})$ consisting of all complexes $C^\bullet$ such that $\mathrm{Supp}\, H^\bullet C^\bullet \subset \{x\}$. We have the following :

Corollary 5.8.16. (Local duality) *In the situation of* (5.8.14)*, the following holds :*

(i) *The functor* (5.8.15) *restricts to an equivalence of categories :*

$$D : \mathsf{D}^b_{\{x\}}(A\text{-}\mathbf{Mod}_{\mathrm{coh}}) \to \mathsf{D}^b_{\{x\}}(A\text{-}\mathbf{Mod}_{\mathrm{coh}})^o$$

and the natural transformation $C^\bullet \to D \circ D(C^\bullet)$ is an isomorphism of functors.

(ii) *For every $i \in \mathbb{Z}$ there exists a natural isomorphism of functors :*

$$R^i\Gamma_{\{x\}} \circ \mathscr{D} \xrightarrow{\sim} D \circ R^{c-i}\Gamma \quad : \quad \mathsf{D}^b(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{coh}} \to A\text{-}\mathbf{Mod}^o$$

where $\mathscr{D}$ is the duality functor corresponding to $\omega^\bullet$.

(iii) *Let c and $\mathscr{D}$ be as in* (ii)*, $I \subset A$ a finitely generated ideal such that $V(I) = \{x\}$, and denote by $A^\wedge$ the I-adic completion of A. Then for every $i \in \mathbb{Z}$ there exists a natural isomorphism of functors :*

$$D \circ R^i\Gamma_{\{x\}} \circ \mathscr{D} \xrightarrow{\sim} A^\wedge \otimes_A R^{c-i}\Gamma \quad : \quad \mathsf{D}^b(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{coh}} \to A^\wedge\text{-}\mathbf{Mod}^o_{\mathrm{coh}}.$$

Proof. Let $C_\bullet$ be any object of $\mathsf{D}^b_{\{x\}}(A\text{-}\mathbf{Mod}_{\mathrm{coh}})$; we denote by $i : \{x\} \to X$ the natural closed immersion, and $C^\sim_\bullet$ the complex of $\mathscr{O}_X$-modules determined by $C_\bullet$. Obviously $C^\sim_\bullet = i_*i^{-1}C^\sim_\bullet$, therefore :

$$\begin{aligned}
\mathrm{Hom}^\bullet_A(C^\bullet, J(x)) &\xrightarrow{\sim} R\mathrm{Hom}^\bullet_A(C^\bullet, J(x)) && \text{by lemma 4.3.27(iii)} \\
&\xrightarrow{\sim} R\mathrm{Hom}^\bullet_{\mathscr{O}_X}(C^\sim_\bullet, R\underline{\Gamma}_{\{x\}}\omega^\bullet[c]) && \text{by theorem 5.1.27} \\
&\xrightarrow{\sim} R\mathrm{Hom}^\bullet_{\mathscr{O}_X}(i_*i^{-1}C^\sim_\bullet, \omega^\bullet[c]) && \text{by 5.4.12(i.b)} \\
&\xrightarrow{\sim} R\mathrm{Hom}^\bullet_{\mathscr{O}_X}(C^\sim_\bullet, \omega^\bullet[c])
\end{aligned}$$

which easily implies (i). Next, we compute, for any object $\mathscr{F}^\bullet$ of $\mathsf{D}^b(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{coh}}$:

$$\begin{aligned}
R^i\Gamma_{\{x\}}\mathscr{D}(\mathscr{F}^\bullet) &\xrightarrow{\sim} R^i\mathrm{Hom}^\bullet_{\mathscr{O}_X}(\mathscr{F}^\bullet, R\Gamma_{\{x\}}\omega^\bullet) && \text{by lemma 5.4.12(iii)} \\
&\xrightarrow{\sim} R^i\mathrm{Hom}^\bullet_{\mathscr{O}_X}(\mathscr{F}^\bullet, J(x)[-c]) \\
&\xrightarrow{\sim} R^i\mathrm{Hom}^\bullet_A(R\Gamma\mathscr{F}^\bullet, J(x)[-c]) && \text{by theorem 5.1.27} \\
&\xrightarrow{\sim} \mathrm{Hom}_A(R^{c-i}\Gamma\mathscr{F}^\bullet, J(x)) && \text{by lemma 4.3.27(iii)}
\end{aligned}$$

whence (ii). Finally, let $\mathscr{F}^\bullet$ be any object of $\mathsf{D}^b(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{coh}}$; we compute :

$$\begin{aligned}
A^\wedge \otimes_A R^{c-i}\Gamma\mathscr{F}^\bullet &\xrightarrow{\sim} \lim_{n\in\mathbb{N}} (A/I^n) \otimes_A R^{c-i}\Gamma\mathscr{F}^\bullet && \text{by corollary 5.7.31(ii)} \\
&\xrightarrow{\sim} \lim_{n\in\mathbb{N}} D \circ D((A/I^n) \otimes_A R^{c-i}\Gamma\mathscr{F}^\bullet) && \text{by (i)} \\
&\xrightarrow{\sim} \lim_{n\in\mathbb{N}} D(\mathrm{Hom}_A(A/I^n, R^i\Gamma_{\{x\}}\mathscr{D}(\mathscr{F}^\bullet))) && \text{by (ii)} \\
&\xrightarrow{\sim} D(\operatorname*{colim}_{n\in\mathbb{N}} \mathrm{Hom}_A(A/I^n, R^i\Gamma_{\{x\}}\mathscr{D}(\mathscr{F}^\bullet))) \\
&\xrightarrow{\sim} D(R^i\Gamma_{\{x\}}\mathscr{D}(\mathscr{F}^\bullet))
\end{aligned}$$

so (iii) holds as well. □

Corollary 5.8.17. *In the situation of* (5.8.14)*, set* $U := X\setminus\{x\}$ *and let* $\mathscr{F}$ *be a coherent* $\mathscr{O}_U$*-module, such that* $\Gamma_{\{y\}}\mathscr{F} = 0$ *for every closed point* $y \in U$*. Then* $\Gamma(U, \mathscr{F})$ *is a finitely presented* A*-module.*

Proof. By lemma 5.2.16(ii) we may find a coherent $\mathscr{O}_X$-module $\mathscr{G}$ such that $\mathscr{G}_{|U} \simeq \mathscr{F}$. Using corollary 5.7.24 we see that $\Gamma_{\{x\}}\mathscr{G}$ is a finitely generated submodule of $\mathscr{G}(X)$, hence $\overline{\mathscr{F}} := \mathscr{G}/\underline{\Gamma}_{\{x\}}\mathscr{G}$ is a coherent $\mathscr{O}_X$-module that extends $\mathscr{F}$, and clearly $\Gamma_{\{x\}}\overline{\mathscr{F}} = 0$. There follows a short exact sequence :

$$0 \to \overline{\mathscr{F}}(X) \to \Gamma(U, \mathscr{F}) \to R^1\Gamma_{\{x\}}\overline{\mathscr{F}} \to 0.$$

We are thus reduced to showing that $R^1\Gamma_{\{x\}}\overline{\mathscr{F}}$ is a finitely presented A-module. However, corollary 5.8.16(ii) yields a natural isomorphism :

$$R^1\Gamma_{\{x\}}\overline{\mathscr{F}} \xrightarrow{\sim} \mathrm{Hom}_A(\mathrm{Ext}^{c-1}_{\mathscr{O}_X}(\overline{\mathscr{F}}, \omega^\bullet), J(x))$$

where $\omega^\bullet$ is a dualizing complex for X and c and $J(x)$ are defined as in (5.8.14). In view of corollary 5.8.16(i), it then suffices to show that the finitely presented A-module $\mathrm{Ext}^{c-1}_{\mathscr{O}_X}(\overline{\mathscr{F}}, \omega^\bullet)$ is supported on $\{x\}$; equivalently, it suffices to verify that the stalks $R^{c-1}\mathscr{H}om_{\mathscr{O}_X}(\overline{\mathscr{F}}, \omega^\bullet)_y$ vanish whenever y is a closed point of U. However, by [28, Ch.0, Prop.12.3.5], the latter are naturally isomorphic to $\mathrm{Ext}^{c-1}_{\mathscr{O}_{X(y)}}(\overline{\mathscr{F}}_{|X(y)}, \omega^\bullet_{|X(y)})$, and $\omega^\bullet_{|X(y)}$ is a dualizing complex for $X(y)$. According to (5.8.14), there is a unique integer $c(y)$ such that $J(y) := R^{c(y)}\omega^\bullet_{|X(y)} \neq 0$ and in view of theorem 5.8.10 and lemma 5.8.8 we obtain :

$$c(y) = c - 1. \tag{5.8.18}$$

Finally, (5.8.18) and corollary 5.8.16(iii) yield the isomorphism :

$$A^\wedge \otimes_A \mathrm{Ext}^{c-1}_{\mathscr{O}_{X(y)}}(\overline{\mathscr{F}}_{|X(y)}, \omega^\bullet_{|X(y)}) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{O}_{X,y}}(\Gamma_{\{y\}}\overline{\mathscr{F}}, J(y)) = 0.$$

To conclude, it suffices to appeal to corollary 5.7.31(iii). □

Proposition 5.8.19. *In the situation of* (5.8.14)*, suppose additionally that the structure morphism* $f : X \to S$ *is flat, that* $f(x) = s$ *and that* $f^{-1}(s)$ *is a regular scheme. Let* $j : U := X\setminus\{x\} \to X$ *be the open immersion,* $\mathscr{F}$ *a flat quasi-coherent* $\mathscr{O}_U$*-module. Then the following two conditions are equivalent :*

(a) $\Gamma(U, \mathscr{F})$ *is a flat* A*-module.*
(b) $\delta(x, j_*\mathscr{F}) \geq \dim f^{-1}(s) + 1$.

Proof. If $d := \dim f^{-1}(s) = 0$, then A is a valuation ring (proposition 5.7.7(ii)), and U is the spectrum of the field of fractions of A, in which case both (a) and (b) hold trivially. Hence we may assume that $d \geq 1$. Suppose now that (a) holds, and set $F := \Gamma(U, \mathscr{F})$. In view of (5.4.3), we need to show that $H^i(U, \mathscr{F}) = 0$ whenever $1 \leq i \leq d - 1$. If $F^\sim$ denotes the $\mathscr{O}_X$-module determined by F, then $F^\sim_{|U} = \mathscr{F}$. Moreover, by [57, Ch.I, Th.1.2], F is the colimit of a filtered family $(L_i \mid i \in I)$ of free A-modules of finite rank, hence $H^i(U, \mathscr{F}) = \operatorname*{colim}_{i\in I} H^i(U, L_i^\sim)$ by lemma 5.1.10(ii), so we are reduced to the case where $\mathscr{F} = \mathscr{O}_U$, and therefore $j_*\mathscr{F} = \mathscr{O}_X$ (corollary 5.6.8). Since $f^{-1}(s)$ is regular, we have $\delta(x, \mathscr{O}_{f^{-1}(s)}) = d$; then, since the topological space underlying X is noetherian, lemma 5.4.19(ii) and corollary 5.4.39 imply that $\delta(x, \mathscr{O}_X) \geq d + 1$, which is (b).

Conversely, suppose that (b) holds; we shall derive (a) by induction on d; the case $d = 0$ having already been dealt with. Let $\mathfrak{m}_A$ (resp. $\overline{\mathfrak{m}}_A$) be the maximal ideal of A (resp. of $\overline{A} := A/\mathfrak{m}_K A$). Suppose then that $d \geq 1$ and that the assertion is known whenever $\dim f^{-1}(s) < d$. Pick $\bar{t} \in \overline{\mathfrak{m}}_A\setminus\overline{\mathfrak{m}}_A^2$, and let $t \in \mathfrak{m}_A$ be any lifting of $\bar{t}$. Since $f^{-1}(s)$ is regular, $\bar{t}$ is a regular element of $\overline{A}$, so that t is regular in A and the induced morphism $g : X' := \mathrm{Spec}\, A/tA \to S$ is flat ([32, Ch.IV, Th.11.3.8]). Let $j' : U' := U \cap X' \to X'$ be the restriction of j; our choice of $\bar{t}$ ensures that $g^{-1}(s)$ is a regular scheme, so the pair $(X', \mathscr{F}/t\mathscr{F}_{|U'})$ fulfills the conditions of the proposition, and $\dim g^{-1}(s) = d - 1$. Furthermore, since $\mathscr{F}$ is a flat $\mathscr{O}_U$-module, the sequence $0 \to \mathscr{F} \xrightarrow{t} \mathscr{F} \to \mathscr{F}/t\mathscr{F} \to 0$ is short exact. Assumption (b) means that

$$H^i(U, \mathscr{F}) = 0 \qquad \text{whenever } 1 \leq i \leq d-1. \tag{5.8.20}$$

Therefore, from the long exact cohomology sequence we deduce that $H^i(U, \mathscr{F}/t\mathscr{F}) = 0$ for $1 \leq i \leq d-2$, *i.e.*

$$\delta(x, j'_*\mathscr{F}/t\mathscr{F}) \geq d. \tag{5.8.21}$$

The same sequence also yields a left exact sequence :

$$0 \to H^0(U, \mathscr{F}) \otimes_A A/tA \xrightarrow{\alpha} H^0(U, \mathscr{F}/t\mathscr{F}) \to H^1(U, \mathscr{F}). \tag{5.8.22}$$

Claim 5.8.23. $H^0(U, \mathscr{F}) \otimes_A A/tA$ is a flat A/tA-module.

Proof of the claim. By (5.8.21) and our inductive assumption, $H^0(U, \mathscr{F}/t\mathscr{F})$ is a flat A/tA-module. In case $d = 1$, proposition 5.7.7(ii) shows that A/tA is a valuation ring, and then the claim follows from (5.8.22), and [14, Ch.VI, §3, n.6, Lemma 1]. If $d > 1$, (5.8.20) implies that α is an isomorphism, so the claim holds also in such case. ◊

Set $V := \mathrm{Spec}\, A[t^{-1}] = X \setminus V(t) \subset U$; since $\mathscr{F}_{|V}$ is a flat $\mathscr{O}_V$-module, the $A[t^{-1}]$-module $H^0(U, \mathscr{F}) \otimes_A A[t^{-1}] \simeq H^0(V, \mathscr{F}_{|V})$ is flat. Moreover, since t is regular on both A and $H^0(U, \mathscr{F})$, an easy calculation shows that $\mathrm{Tor}_i^A(A/tA, H^0(U, \mathscr{F})) = 0$ for $i > 0$. Then the contention follows from claim 5.8.23 and [36, Lemma 5.2.1]. □

Remark 5.8.24. Proposition 5.8.19 still holds – with analogous proof – if we replace X by the spectrum of a regular local ring A (and then $\dim f^{-1}(s) + 1$ is replaced by $\dim A$ in condition (b)). The special case where $\mathscr{F}$ is a locally free $\mathscr{O}_U$-module of finite rank has been studied in detail in [50].

We conclude this section with a result which shall be used in our discussion of almost purity.

5.8.25. For given $b \in \mathfrak{m}_K$, let $f : X \to S_{/b}$ be a morphism of finite presentation, $x \in X$ a point lying in the Cohen-Macaulay locus of f (see definition (5.6.35)(v)), $\mathscr{F}$ a finitely presented $\mathscr{O}_{X(x)}$-module. Let also $a \in \mathscr{O}_{X,x}$, and set $\mu_a := a \cdot \mathbf{1}_{\mathscr{F}} : \mathscr{F} \to \mathscr{F}$; suppose that, for every point $y \in U := X(x) \setminus \{x\}$ there is a coherent $\mathscr{O}_{X,y}$-module E, such that :

- E is $f_y^\natural$-Cohen-Macaulay and $\mathrm{Supp}\, E = X(y)$.
- $\mu_{a,y}$ factors through a $\mathscr{O}_{X,y}$-linear map $\mathscr{F}_y \to E$.

Lemma 5.8.26. *In the situation of* (5.8.25)*, the following holds :*

(i) *The $\mathscr{O}_{X,x}$-module $a \cdot R^i\Gamma_{\{x\}}\mathscr{F}$ is finitely presented for every $i < \dim \mathscr{O}_{X,x}$.*
(ii) *If $\dim \mathscr{O}_{X,x} > 1$, the $\mathscr{O}_{X,x}$-module $a \cdot \Gamma(U, \mathscr{F})$ is finitely presented.*

Proof. (i): To ease notation, set $A := \mathscr{O}_{X,x}$ and $d := \dim A$. Fix a finitely generated ideal $I \subset A$ whose radical is the maximal ideal. We may assume that X is an affine scheme and f is Cohen-Macaulay of constant fibre dimension n (lemma 5.6.36(i,ii)). In this case, $X(x)$ admits the dualizing complex

$$\omega_{X(x)} := (f^!\mathscr{O}_{S_{/b}}[-n])_{|X(x)}$$

which is a coherent $\mathscr{O}_{X(x)}$-module placed in degree zero ((5.8.14) and example 5.8.6(ii)). We may then define $c \in \mathbb{N}$ and $J(x)$ as in (5.8.14), and by inspection we find that $c = n - d(x) + 1$ if $b = 0$, and otherwise $c = n - d(x)$. To compute this quantity, pick any irreducible component

Z of the fibre $f^{-1}(fx)$ containing x, and endow it with its reduced subscheme structure; then $\dim Z = n$ (lemma 5.6.36(ii)), and since Z is a scheme of finite type over $\kappa(fx)$, we have

$$d(x) = n + 1 - \dim \mathscr{O}_{f^{-1}(fx),x} - \dim \mathscr{O}_{S,f(x)}$$

([31, Ch.IV, Prop.5.2.1]). Now, if $b = 0$, lemma 5.8.8(v) applies and we obtain $c = d$. If $b \neq 0$, then $f(x) = s$, and $\dim \mathscr{O}_{f^{-1}(s),x} = d$, hence again $c = d$. Let also D be the functor (5.8.15).

Claim 5.8.27. For every finitely presented A-module M supported at the point x, and every $t \in A$, we have a natural isomorphism : $D(tM) \simeq tD(M)$.

Proof of the claim. It is a simple application of the exactness of the functor D. Indeed, from the left exact sequence $0 \to \mathrm{Ann}_M(t) \to M \xrightarrow{t} M$ we deduce a right exact sequence :

$$D(M) \xrightarrow{t} D(M) \to D(\mathrm{Ann}_M(t)) \to 0$$

i.e. $D(\mathrm{Ann}_M(t)) \simeq D(M)/tD(M)$. Then the short exact sequence :

$$0 \to \mathrm{Ann}_M(t) \to M \xrightarrow{\tau} tM \to 0$$

yields the exact sequence $0 \to D(tM) \to D(M) \xrightarrow{D(\tau)} D(M)/tD(M) \to 0$, and by inspecting the definition one sees easily that $D(\tau)$ is the natural projection, whence the claim. $\Diamond$

We apply claim 5.8.27 and corollary 5.8.16(ii) to derive a natural isomorphism :

$$a \cdot R^i\Gamma_{\{x\}}\mathscr{F} \simeq D(a \cdot H^{d-i}(X(x), \mathscr{D}(\mathscr{F}))) \simeq D(a \cdot \mathrm{Ext}^i_{\mathscr{O}_{X(x)}}(\mathscr{F}, \omega_{X(x)})) \qquad \text{for every } i \in \mathbb{Z}.$$

However – in view of lemma 5.6.36(i) and our assumptions – for any $y \in U$ we may find an affine open neighborhood $V \subset X$ of y, a coherent $f_{|V}$-Cohen-Macaulay $\mathscr{O}_V$-module $\mathscr{E}$ with $\mathrm{Supp}\,\mathscr{E} = V$, and a factorization $\mathscr{F}_y \to \mathscr{E}_y \to \mathscr{F}_y$ of $\mu_{a,y}$; since $\omega_{X(x)}$ is a coherent $\mathscr{O}_{X(x)}$-module, there results a factorization ([28, Ch.0, Prop.12.3.3, 12.3.5])

$$\mathscr{E}xt^i_{\mathscr{O}_{X(x)}}(\mathscr{F}, \omega_{X(x)})_y \to \mathscr{E}xt^i_{\mathscr{O}_{V,y}}(\mathscr{E}_y, \omega_{X(x),y}) \to \mathscr{E}xt^i_{\mathscr{O}_{X(x)}}(\mathscr{F}, \omega_{X(x)})_y.$$

Claim 5.8.28. $\mathscr{E}xt^i_{\mathscr{O}_{V,y}}(\mathscr{E}_y, \omega_{X(x),y}) = 0$ for every $i > 0$.

Proof of the claim. Let $j : V \to X$ be the open immersion, and set $g := f \circ j : V \to S_{/b}$; by [28, Ch.0, Prop.12.3.3, 12.3.5], we are then reduced to showing that $\mathrm{Ext}^i_{\mathscr{O}_{V'}}(\mathscr{E}, g^!\mathscr{O}_{S_{/b}}[-n]) = 0$ for every $i > 0$. To this aim, pick a finitely presented S-immersion $h : V \to Y := \mathbb{A}^m_{S_{/b}}$, and let $p : Y \to S_{/b}$ be the projection; we deduce a natural isomorphism $g^!\mathscr{O}_{S_{/b}}[-n] \xrightarrow{\sim} h^!\mathscr{O}_Y[m-n]$, and by adjunction

$$\mathrm{Ext}^i_{\mathscr{O}_V}(\mathscr{E}, g^!\mathscr{O}_{S_{/b}}[-n]) = \mathrm{Ext}^i_{\mathscr{O}_Y}(h_*\mathscr{E}, \mathscr{O}_Y[m-n]). \tag{5.8.29}$$

Now, fix $v \in V$ and set $u := h(v)$; by theorem 5.6.40(ii), the complex $R\mathscr{H}om^\bullet_{\mathscr{O}_Y}(h_*\mathscr{E}, \mathscr{O}_Y)_u$ is concentrated in degree $\dim \mathscr{O}_{p^{-1}(pu),u} - \dim \mathscr{O}_{f^{-1}(fv),v}$, and since the fibres of f and p are equidimensional ([31, Ch.IV, Prop.5.2.1] and lemma 5.6.36(ii)) it is easily seen that this quantity equals $m - n$. After composing with the functor $R^i\Gamma$, we deduce that the right-hand side of (5.8.29) vanishes for $i > 0$, as required. $\Diamond$

From claim 5.8.28, it follows that $a \cdot \mathrm{Ext}^i_{\mathscr{O}_{X(x)}}(\mathscr{F}, \omega_{X(x)})$ is a coherent A-module ([28, Ch.0, Prop.12.3.3]) supported at x for $i > 0$. Assertion (i) now follows from corollary 5.8.16(i).

(ii): To start out, notice that $\underline{\Gamma}_{\{x\}}\mathscr{F}$ is an $\mathscr{O}_X$-module of finite type, by corollary 5.7.24; we may then replace $\mathscr{F}$ by $\mathscr{F}/\underline{\Gamma}_{\{x\}}\mathscr{F}$ and assume, without loss of generality, that :

$$\Gamma_{\{x\}}\mathscr{F} = 0. \tag{5.8.30}$$

Let $y_1, \dots, y_r \in U$ be the finitely many associated points of $\mathscr{F}$ that are closed in U (theorem 5.7.20(ii)), and set $Z := \{y_1, \dots, y_r\}$; notice that Z is a closed subset of U, so that the $\mathscr{O}_U$-module $\underline{\Gamma}_Z\mathscr{F}_{|U}$ is of finite type, by corollary 5.7.24. Hence $\overline{\mathscr{F}} := \mathscr{F}_{|U}/\underline{\Gamma}_Z\mathscr{F}_{|U}$ is a coherent $\mathscr{O}_U$-module, and we obtain a short exact sequence :

$$0 \to \Gamma_Z\mathscr{F}_{|U} \to \Gamma(U, \mathscr{F}) \to \Gamma(U, \overline{\mathscr{F}}) \to 0.$$

The $\mathscr{O}_U$-module $\overline{\mathscr{F}}$ fulfills the assumptions of corollary 5.8.17, so that $\Gamma(U, \overline{\mathscr{F}})$ is a finitely presented $\mathscr{O}_{X,x}$-module. On the other hand, set $F := \Gamma(X(x), \mathscr{F})$; by corollary 5.8.4 we have a natural isomorphism :

$$\operatorname*{colim}_{n\in\mathbb{N}} \mathrm{Hom}_A(I^n, F) \xrightarrow{\sim} \Gamma(U, \mathscr{F})$$

and using (5.8.30) one sees easily that the transition maps in this colimit are injective. Hence, $\Gamma(U, F)$ is an increasing union of finitely presented A-modules ([26, Ch.0, §5.3.5]). Let $M \subset \Gamma(U, \mathscr{F})$ be any finitely generated submodule that maps surjectively onto $\Gamma(U, \overline{\mathscr{F}})$; it then follows that M is finitely presented, and clearly $\Gamma(U, \mathscr{F}) = M + \Gamma_Z\mathscr{F}_{|U}$, hence $a \cdot \Gamma(U, \mathscr{F}) = aM + a \cdot \Gamma_Z\mathscr{F}_{|U}$. Finally, since aM is finitely presented, it remains only to show :

Claim 5.8.31. $a \cdot \underline{\Gamma}_Z\mathscr{F}_{|U} = 0$.

Proof of the claim. Fix any $y \in U$, and let $V \subset X$ be any open neighborhood of y with a coherent $\mathscr{O}_V$-module $\mathscr{E}$ as in the foregoing. After shrinking V, we may assume that $\mu_{a|V\cap X(x)}$ factors through $\mathscr{E}$. It follows that $a \cdot \underline{\Gamma}_{Z\cap V}\mathscr{F}_{|V}$ is a subquotient of $\underline{\Gamma}_{Z\cap V}\mathscr{E}$. We are then reduced to showing that $\underline{\Gamma}_{Z\cap V}\mathscr{E} = 0$. However, under the current assumptions $\dim U > 0$, hence

(5.8.32) $$\dim \mathscr{O}_{U,y_i} > 0 \qquad \text{for } i = 1, \dots, r$$

(lemma 5.8.8(iii)). On the other hand, since $\mathscr{E}$ is $f_{|V}$-Cohen-Macaulay with support V, its associated points are the maximal points of $V \cap f^{-1}(\eta)$. Taking into account (5.8.32), we conclude that Z does not contain any associated point of $\mathscr{E}$, as required. ☐

5.9. **Hochster's theorem and Stanley's theorem.** The two main results of this section are theorems 5.9.34 and 5.9.41, which were proved originally respectively by Hochster in [49], and by Stanley in [72]. Our proofs follow in the main the posterior methods presented by Bruns and Herzog in [22] (which in turns, partially rely on some ideas of Danilov). These results shall be used in section 6.5, in order to show that regular log schemes are Cohen-Macaulay. We begin with some preliminaries from algebraic topology, which we develop only to the extent that is required for our special purposes: a much more general theory exists, and is well known to experts (see *e.g.* [59, Ch.IX]).

For any topological space X, and any subset $T \subset X$, we denote by $\overline{T}$ the topological closure of T in X, endowed with the topology induced from X. We let $H_\bullet(X)$ (resp. $H_\bullet(X, T)$) be the singular homology groups of X (resp. of the pair (X, T)). Also, for every $n \in \mathbb{N}$ fix a Banach norm $\|\cdot\|$ on $\mathbb{R}^n$, and for every real number $\rho > 0$, let $\mathbb{B}^n(\rho) := \{v \in \mathbb{R}^n \mid \|v\| < \rho\}$, and set $\mathbb{S}^{n-1} := \overline{\mathbb{B}}^n(1) \setminus \mathbb{B}^n(1)$ (so $\mathbb{S}^{-1} = \varnothing$).

Definition 5.9.1. (i) A *finite regular cell complex* (or briefly : a *cell complex*) is the datum of a topological space X, together with a finite filtration

$$X^{-1} := \varnothing \subset X^0 \subset X^1 \subset X^2 \subset \cdots \subset X^k := X$$

consisting of closed subspaces, such that X^0 is a discrete topological space, and for every $i = 0, \dots, k$ we have a decomposition

$$X^i \setminus X^{i-1} = \bigcup_{\lambda\in\Lambda_i} e^i_\lambda$$

where :

(a) Λ_i is a finite set, and for every $\lambda \in \Lambda_i$ there exists a homeomorphism

$$f_\lambda : \overline{\mathbb{B}}^i(1) \xrightarrow{\sim} \overline{e}^i_\lambda \qquad \text{such that } f_\lambda^{-1}(e^i_\lambda) = \mathbb{B}^i(1).$$

(b) $e^i_\lambda \cap e^i_\mu = \varnothing$ for any two distinct indices $\lambda, \mu \in \Lambda_i$.

(c) $\overline{e}^i_\lambda \setminus e^i_\lambda \subset X^{i-1}$ for every $\lambda \in \Lambda_i$.

(ii) The smallest $k \in \mathbb{N}$ such that $X^k = X$ is called the *dimension* of $X^\bullet$, and is denoted $\dim X^\bullet$. For every $i = 0, \dots, k$, the subsets e^i_λ are called the *i-dimensional cells of $X^\bullet$*.

(iii) A *subcomplex* of the cell complex $X^\bullet$ is a closed subset $Y \subset X$ which is a union of cells of X. Then the filtration such that $Y^i := Y \cap X^i$ for every $i \geq -1$ defines a cell complex structure $Y^\bullet$ on Y.

Lemma 5.9.2. *With the notation of definition* 5.9.1*, we have :*

(i) *For every $i = 0, \dots, \dim X$ and $q \in \mathbb{Z}$, we have natural isomorphisms of abelian groups :*

$$H_q(X^i, X^{i-1}) \xrightarrow{\sim} \begin{cases} \mathbb{Z}^{\Lambda_i} & \textit{if } q = i \\ 0 & \textit{otherwise}. \end{cases}$$

(ii) *More precisely, let $(b_\lambda \mid \lambda \in \Lambda_i)$ be the canonical basis of the free $\mathbb{Z}$-module $\mathbb{Z}^{\Lambda_i}$; for every $\lambda \in \Lambda_i$, the isomorphism of* (i) *maps the image of the induced map*

$$H_i f_\lambda : H_i(\overline{\mathbb{B}}^i(1), \mathbb{S}^{i-1}) \to H_i(X^i, X^{i-1})$$

isomorphically onto the direct summand generated by b_λ.

Proof. (i): This is obvious for $i = 0$. For $i = 1, \dots, k := \dim X^\bullet$ and every $\lambda \in \Lambda_i$, set

$$Y^i_\lambda := f_\lambda(\overline{\mathbb{B}}^i(1/2)) \qquad Y^i := \bigcup_{\lambda \in \Lambda_i} Y^i_\lambda \qquad A^i := \bigcup_{\lambda \in \Lambda_i} \{f_\lambda(0)\}$$

and for any $q \in \mathbb{Z}$, consider the natural group homomorphisms

(5.9.3)
$$\begin{array}{ccccc} H_q(\overline{\mathbb{B}}^i(1/2), \overline{B}^i \setminus \{0\}) & \xrightarrow{\alpha'} & H_q(\overline{\mathbb{B}}^i(1), \overline{\mathbb{B}}^i(1) \setminus \{0\}) & \xleftarrow{\beta'} & H_q(\overline{\mathbb{B}}^i(1), \mathbb{S}^{i-1}) \\ {\scriptstyle\gamma}\downarrow & & \downarrow & & \downarrow \\ H_q(Y^i, Y^i \setminus A^i) & \xrightarrow{\alpha} & H_q(X^i, X^i \setminus A^i) & \xleftarrow{\beta} & H_q(X^i, X^{i-1}). \end{array}$$

The map α is an isomorphism, by excision; the same holds for β, since X^{i-1} is a deformation retract of $X^i \setminus A^i$. However, for every $\lambda \in \Lambda_i$ we have natural isomorphisms

$$H_q(Y^i_\lambda, Y^i_\lambda \setminus A) \xrightarrow{\sim} \begin{cases} \mathbb{Z} & \text{if } q = i \\ 0 & \text{otherwise}. \end{cases}$$

whence the contention.

(ii): Take $q := i$ in (5.9.3); then clearly γ is a monomorphism whose image is the direct summand $H_i(Y^i_\lambda, Y^i_\lambda \setminus A)$; on the other hand, arguing as in the foregoing, we see that both α' and β' are isomorphisms. The assertion follows easily. □

Remark 5.9.4. (i) In the situation of lemma 5.9.2, notice that there is a natural bijection from Λ_i to the set of connected components $\pi_0(X^i \setminus X^{i-1})$ of $X^i \setminus X^{i-1}$, for every $i = 0, \dots, \dim X$.

(ii) The direct sum decomposition of $H_i(X^i, X^{i-1})$ provided by lemma 5.9.2(i) depends only on the filtration $X^\bullet$ (and not on the choice of homeomorphisms f_λ). Indeed, this is clear, since the map $H_i f_\lambda$ factors as a composition :

$$H_i(\overline{\mathbb{B}}^i(1), \mathbb{S}^{i-1}) \xrightarrow{\sim} H_i(\overline{e}^i_\lambda, \overline{e}^i_\lambda \setminus e^i_\lambda) \xrightarrow{l_\lambda} H_i(X^i, X^{i-1})$$

where l_λ is the homomorphism induced by the obvious map of pairs $(\overline{e}^i_\lambda, \overline{e}^i_\lambda \setminus e^i_\lambda) \to (X^i, X^{i-1})$.

(iii) In view of (ii), the maps l_λ admit natural left inverse homomorphisms

$$p_\lambda : H_i(X^i, X^{i-1}) \to H_i(\overline{e}^i_\lambda, \overline{e}^i_\lambda \setminus e^i_\lambda)$$

such that

(5.9.5) $$p_\mu \circ l_\lambda = 0 \qquad \text{for every } \lambda, \mu \in \Lambda_i \text{ with } \lambda \neq \mu.$$

These maps can be described as follows. For any $\lambda \in \Lambda_i$, the natural map of pairs $(X^i, X^{i-1}) \to (X^i, X^i \setminus e^i_\lambda)$ induces a homomorphism

$$m_\lambda : H_i(X^i, X^{i-1}) \to H_i(X^i, X^i \setminus e^i_\lambda).$$

Arguing by excision as in the proof of lemma 5.9.2(ii), it is easily seen that $q_\lambda := m_\lambda \circ l_\lambda$ is an isomorphism (details left to the reader), and we set $p_\lambda := q_\lambda^{-1} \circ m_\lambda$. Obviously p_λ is left inverse to l_λ, and in oder to check (5.9.5) it suffices to show that $m_\mu \circ l_\lambda = 0$ for every $\mu \neq \lambda$. However, $m_\mu \circ l_\lambda$ is the homomorphism arising from the map of pairs $(\overline{e}^i_\lambda, \overline{e}^i_\lambda \setminus e^i_\lambda) \to (X^i, X^i \setminus e^i_\mu)$, so the assertion follows, after remarking that $\overline{e}^i_\lambda \subset X^i \setminus e^i_\mu$.

5.9.6. We attach to any cell complex $X^\bullet$ a complex of abelian groups $\mathscr{C}_\bullet(X^\bullet)$, as follows.

- For $i < 0$ and for $i > k := \dim X^\bullet$, we set $\mathscr{C}_i(X^\bullet) := 0$, and for $i = 1, \dots, k$ we let

$$\mathscr{C}_i(X^\bullet) := H_i(X^i, X^{i-1}).$$

Sometimes we write just $\mathscr{C}_i(X)$, unless it is useful to stress which filtration $X^\bullet$ on X we are considering. For every $i > 0$, the differential $d_i : \mathscr{C}_i(X^\bullet) \to \mathscr{C}_{i-1}(X^\bullet)$ is the composition

$$H_i(X^i, X^{i-1}) \xrightarrow{\partial_i} H_{i-1}(X^{i-1}) \xrightarrow{j_{i-1}} H_{i-1}(X^{i-1}, X^{i-2})$$

where ∂_i is the boundary operator of the long exact homology sequence associated to the pair (X^i, X^{i-1}), and j_{i-1} is the homomorphism induced by the obvious map of pairs $(X^{i-1}, \varnothing) \to (X^{i-1}, X^{i-2})$. In order to check that $d_i \circ d_{i+1} = 0$ for every $i \in \mathbb{Z}$, recall that every element of $H_{i+1}(X^{i+1}, X^i)$ is the class $\overline{c}$ of a singular $(i+1)$-chain c of X^{i+1}, whose boundary $\partial_{i+1}c$ is a singular i-chain of X^i; then $d_{i+1}(\overline{c})$ is the class of $\partial_{i+1}c$ in $H_i(X^i, X^{i-1})$, and $d_i \circ d_{i+1}(\overline{c})$ is the class of $\partial_i \circ \partial_{i+1} c$ in $H_{i-1}(X^{i-1}, X^{i-2})$, so it vanishes.

- It is also useful to consider an augmented version of the above complex; namely, let us set

$$\overline{\mathscr{C}}_{-1}(X^\bullet) := \mathbb{Z} \qquad \text{and} \qquad \overline{\mathscr{C}}_i(X^\bullet) := \mathscr{C}_i(X^\bullet) \qquad \text{for every } i \neq -1$$

with differential d_0 given by the rule : $d_0(b^0_\lambda) = 1$ for every $\lambda \in \Lambda_0$.

Proposition 5.9.7. *With the notation of* (5.9.6)*, there exist natural isomorphisms of abelian groups*

$$H_q\mathscr{C}_\bullet(X^\bullet) \xrightarrow{\sim} H_q(X) \qquad \textit{for every } q \in \mathbb{N}.$$

Proof. For every topological space T, let $C_\bullet(T)$ denote the complex of singular chains of T. The filtration $X^\bullet$ induces a finite filtration of complexes

$$C_\bullet(X^0) \subset C_\bullet(X^1) \subset C_\bullet(X^2) \subset \cdots \subset C_\bullet(X)$$

whence a convergent spectral sequence

$$E^1_{pq} := H_{p+q}(X^p, X^{p-1}) \Rightarrow H_{p+q}(X)$$

(see [75, Th.5.5.1]). By direct inspection, it is easily seen that the differential $d^1_{p,0} : E^1_{p,0} \to E^1_{p-1,0}$ agrees with the differential d_p of $\mathscr{C}_\bullet(X^\bullet)$, for every $p \in \mathbb{N}$. On the other hand, lemma 5.9.2 shows that this spectral sequence degenerates, whence the contention. $\square$

5.9.8. Keep the notation of definition 5.9.1, and set $\Lambda_0 := X^0$. For every $i = 0, \dots, \dim X$ and every $\lambda \in \Lambda_i$, the abelian group $H_{i,\lambda} := H_i(\overline{e}^i_\lambda, \overline{e}^i_\lambda \setminus e^i_\lambda)$ is free of rank one; if $i > 0$, we fix one of the two generators of this group, and we denote by b^i_λ its image in $\mathscr{C}_i(X^\bullet)$ (see remark 5.9.4(ii)). For $i = 0$, each e^0_λ is a point, hence $H_{0,\lambda}$ admits a canonical identification with $\mathbb{Z}$, and we let b^0_λ be the image of 1, under the resulting map $\mathbb{Z} \xrightarrow{\sim} H_{0,\lambda} \to \mathscr{C}_0(X^\bullet)$. The system of classes $(b^i_\lambda \mid 0 \leq i \leq \dim X, \lambda \in \Lambda_i)$ is called an *orientation* for $X^\bullet$. We may write

$$d_i(b^i_\lambda) = \sum_{\mu \in \Lambda_{i-1}} [b^i_\lambda : b^{i-1}_\mu] b^{i-1}_\mu \qquad \text{for every } i \leq \dim X^\bullet \text{ and every } \lambda \in \Lambda_i$$

for a system of uniquely determined integers $[b^i_\lambda : b^{i-1}_\mu]$, called the *incidence numbers* of the cells e^i_λ and e^{i-1}_μ (relative to the chosen orientation of $X^\bullet$). With this notation, we may state :

Lemma 5.9.9. *The incidence numbers of a cell complex $X^\bullet$ fulfill the following conditions :*

(i) $\sum_{\mu \in \Lambda_{i-1}} [b^i_\lambda : b^{i-1}_\mu] \cdot [b^{i-1}_\mu : b^{i-2}_\nu] = 0$ *for every $i \geq 2$, every $\lambda \in \Lambda_i$ and every $\nu \in \Lambda_{i-2}$.*

(ii) *Let $\lambda \in \Lambda_1$ be any index, and say that $\overline{e}^1_\lambda \setminus e^1_\lambda = e^0_\mu \cup e^0_\rho$. Then $[b^1_\lambda : b^0_\mu] + [b^1_\lambda : b^0_\rho] = 0$.*

Proof. Condition (i) translates the identity $d_{i-1} \circ d_i = 0$ for the differential $d_\bullet$ of the complex $\mathscr{C}_\bullet(X^\bullet)$. The identity of (ii) follows by a simple inspection of the definition of d_1 : details left to the reader. □

5.9.10. The generalities of the previous paragraphs shall be applied to the following situation. Let (V, σ) be a strictly convex polyhedral cone such that $\langle \sigma \rangle = V$, and set $d := \dim_{\mathbb{R}} V$. We attach to σ a cell complex $C^\bullet_\sigma$ as follows. Fix $u_0 \in \sigma^\vee$ such that $\sigma \cap \operatorname{Ker} u_0 = \{0\}$ (corollary 3.3.14), let σ° be the topological interior of σ (in V) and set

$$C_\sigma := \sigma \cap u_0^{-1}(1) \qquad C^\circ_\sigma := \sigma^\circ \cap u_0^{-1}(1).$$

Let $v_1, \dots, v_k$ be a minimal system of generators for σ; we may assume that $u_0(v_i) = 1$ for $i = 1, \dots, k$, in which case

$$C_\sigma = \left\{ \sum_{i=1}^k \lambda_i v_i \mid \lambda_1, \dots, \lambda_k \geq 0 \quad \text{and} \quad \sum_{i=1}^k \lambda_i = 1 \right\} \tag{5.9.11}$$

which shows that C_σ is a compact topological space (with the topology inherited from V).

Lemma 5.9.12. *There exists a homeomorphism $\overline{\mathbb{B}}^{d-1}(1) \xrightarrow{\sim} C_\sigma$ that maps $\mathbb{B}^{d-1}(1)$ onto C°_σ.*

Proof. Set $v_0 := k^{-1} \cdot (v_1 + \cdots + v_k)$ and $W := \operatorname{Ker} u_0$. It suffices to show that there exists a homeomorphism $\overline{\mathbb{B}}^{d-1}(1) \xrightarrow{\sim} D_\sigma := C_\sigma - v_0 \subset W$ that maps $\mathbb{B}^{d-1}(1)$ onto $D^\circ_\sigma := C^\circ_\sigma - v_0$. However, pick a minimal system $u_1, \dots, u_t$ of generators of $\sigma^\vee$ (corollary 3.3.12(i)); since σ spans V, for every $i = 1, \dots, t$ there exists $j \leq k$ such that $u_i(v_j) > 0$. Hence – after replacing the u_i by suitable scalar multiples – we may assume that $u_i(v_0) = 1$ for every $i = 1, \dots, t$, and then lemma 3.3.2 implies that

$$D_\sigma = \{v \in W \mid u_i(v) \geq -1 \quad \text{for every } i = 1, \dots, t\}.$$

Also, proposition 3.3.11(i) implies that

$$D^\circ_\sigma = \{v \in W \mid u_i(v) > -1 \quad \text{for every } i = 1, \dots, t\}.$$

For every $v \in W$, set $\mu(v) := \min(u_i(v) \mid i = 1, \dots, t)$. It is easily seen that $\mu(v) < 0$ for every $v \in W \setminus \{0\}$; indeed, if $\mu(w) \geq 0$, then the subset $\{\lambda w \mid \lambda \in \mathbb{R}_+\}$ lies in D_σ, and since the latter is compact, it follows that $w = 0$. Moreover we have

$$(\mathbb{R}_+ w) \cap D_\sigma = \{\lambda w \mid \lambda \in [0, -1/\mu(w)]\} \qquad \text{for every } w \in W \setminus \{0\}. \tag{5.9.13}$$

Fix a Banach norm $\|\cdot\|_V$ on V, and consider the mapping

$$\varphi : D_\sigma \to W \qquad \text{such that} \qquad \varphi(0) := 0 \quad \text{and} \quad \varphi(v) := -\frac{\mu(v)}{\|v\|} \cdot v \quad \text{for } v \neq 0.$$

It is easily seen that φ is injective; since μ is a continuous mapping, the same follows for φ, and since D_σ is compact, we conclude that φ induces a homeomorphism $D_\sigma \xrightarrow{\sim} \varphi(D_\sigma)$. However, from (5.9.13) it follows that

$$\varphi(D_\sigma) = \{v \in W \mid \|v\| \leq 1\} \qquad \text{and} \qquad \varphi(D_\sigma^\circ) = \{v \in W \mid \|v\| < 1\}$$

whence the claim. □

5.9.14. Keep the notation of (5.9.10); we consider the finite filtration $C_\sigma^\bullet$ of C_σ, defined as follows. For every $i = -1, \dots, d-1$, we let $C_\sigma^i \subset C_\sigma$ be the union of the subsets $\tau \cap u_0^{-1}(1)$, where τ ranges over the (finite) set of all faces of σ of dimension $i+1$. Clearly $C_\sigma^{-1} = \varnothing$, $C_\sigma^{d-1} = C_\sigma$, and C_σ^i is a closed subset of C_σ, for every $i = 0, \dots, d-1$. Moreover, it follows easily from lemma 5.9.12 and proposition 3.3.11(i) that the datum of C_σ and its filtration $C_\sigma^\bullet$ is a cell complex. With the notation of definition 5.9.1, the indexing set Λ_i can be taken to be the set of all $(i+1)$-dimensional faces of σ, for every $i = 0, \dots, d-1$: indeed, if τ is such a face, denote by τ° the relative interior of τ (see example 3.3.16(iii)); then it is clear that $e_\tau^i := \tau^\circ \cap C_\sigma \neq \varnothing$ and $\overline{e}_\tau^i = \tau \cap C_\sigma$.

Remark 5.9.15. (i) Notice that the cell complex $C_\sigma^\bullet$ is independent – up to homeomorphism – of the choice of u_0. Indeed, say that $u'_0 \in \sigma^\vee$ is any other linear form such that $\sigma \cap \mathrm{Ker}\, u'_0 = \{0\}$, and let $C'_\sigma := \sigma \cap u_0'^{-1}(1)$. We define a homeomorphism $\psi : C'_\sigma \xrightarrow{\sim} C_\sigma$, by the rule :

$$v \mapsto u_0(v)^{-1} \cdot v \qquad \text{for every } v \in C'_\sigma.$$

It is easily seen that ψ restricts to homeomorphisms $C_\sigma'^i \xrightarrow{\sim} C_\sigma^i$ for every $i = 0, \dots, d-1$.

(ii) For any integer $i = 0, \dots, d-3$, and any cells e_τ^i, e_λ^{i+2} of $C_\sigma^\bullet$ such that $e_\tau^i \subset \overline{e}_\lambda^{i+2}$, there exist exactly two $(i+1)$-dimensional cells e_μ^{i+1}, e_ρ^{i+1} such that $e_\tau^i \subset \overline{e}_\mu^{i+2} \cap \overline{e}_\rho^{i+1}$ and $e_\mu^{i+1} \cup e_\rho^{i+1} \subset \overline{e}_\lambda^{i+2}$: indeed, this assertion is a direct translation of claim 3.3.9(ii).

Proposition 5.9.16. *With the notation of* (5.9.8) *and* (5.9.14)*, the following holds for every* $i = 0, \dots, d-2$ *:*

(i) *If* $\tau \in \Lambda_{i+1}$ *and* $\mu \in \Lambda_i$ *is not a facet of* τ*, we have* $[b_\tau^{i+1} : b_\mu^i] = 0$.

(ii) *If* $\tau \in \Lambda_{i+1}$ *and* $\mu \in \Lambda_i$ *is a facet of* τ*, then* $[b_\tau^{i+1} : b_\mu^i] \in \{1, -1\}$.

(iii) *In the situation of remark* 5.9.15(ii)*, we have :*

$$[b_\lambda^{i+2} : b_\mu^{i+1}] \cdot [b_\mu^{i+1} : b_\tau^i] + [b_\lambda^{i+2} : b_\rho^{i+1}] \cdot [b_\rho^{i+1} : b_\tau^i] = 0.$$

Proof. (i): Consider the commutative diagram

$$\tag{5.9.17}\begin{array}{ccccc} H_{i+1}(\overline{e}_\tau^{i+1}, \overline{e}_\tau^{i+1} \setminus e_\tau^{i+1}) & \xrightarrow{\partial} & H_i(\overline{e}_\tau^{i+1} \setminus e_\tau^{i+1}) & \xrightarrow{j_\tau} & H_i(C_\sigma^i, C_\sigma^i \setminus e_\mu^i) \\ \downarrow{\scriptstyle l_\tau} & & \downarrow{\scriptstyle g} & & \uparrow{\scriptstyle m_\mu} \\ H_{i+1}(C_\sigma^{i+1}, C_\sigma^i) & \xrightarrow{\partial'} & H_i(C_\sigma^i) & \xrightarrow{j_i} & H_i(C_\sigma^i, C_\sigma^{i-1}) \end{array}$$

where l_τ and m_μ are defined as in remark 5.9.4(ii,iii) and g is induced by the inclusion map $\overline{e}_\tau^{i+1} \setminus e_\tau^{i+1} \to C_\sigma^i$, the maps ∂ and ∂' are the boundary operators of the long exact sequences attached to the pairs $(\overline{e}_\tau^{i+1}, \overline{e}_\tau^{i+1} \setminus e_\tau^{i+1})$ and $(C_\sigma^{i+1}, C_\sigma^i)$, and j_i (resp. j_τ) is deduced from the obvious map of pairs $(\overline{e}_\tau^{i+1} \setminus e_\tau^{i+1}, \varnothing) \to (C_\sigma^i, C_\sigma^i \setminus e_\mu^i)$ (resp. $(C_\sigma^i, \varnothing) \to (C_\sigma^i, C_\sigma^{i-1})$). In light of remark 5.9.4(iii), we need to check that $m_\mu \circ j_{i-1} \circ \partial' \circ l_\tau = 0$, and to this aim, it suffices to show that $j_\tau = 0$. However, the assumption implies that $\mu \cap \tau$ is a (proper) face of both μ and τ; this translates as the identity $(\overline{e}_\tau^{i+1} \setminus e_\tau^{i+1}) \cap e_\mu^i = \varnothing$, whence the claim.

(ii): Clearly $C^\bullet_\tau$ is a cell subcomplex of $C^\bullet_\sigma$, and e^{i+1}_τ and e^i_μ are cells of this subcomplex. Moreover, any orientation for $C^\bullet_\sigma$ restricts to an orientation for $C^\bullet_\tau$, and the resulting incidence number of e^{i+1}_τ and e^i_μ is the same for either cell complex. Thus, we may replace σ by τ, in which case the map g of (5.9.17) is the identity.

Claim 5.9.18. If $\sigma = \tau$ and $i > 0$, both ∂ and $k_\mu := m_\mu \circ j_i$ in (5.9.17) are isomorphisms.

Proof of the claim. For ∂, we use the long exact homology sequence of the pair $(\overline{e}^{i+1}_\tau, \overline{e}^{i+1}_\tau \setminus e^{i+1}_\tau)$: since $i \geq 0$, lemma 5.9.12 implies that $H_{i+1}(\overline{e}^{i+1}_\tau) = 0$, so ∂ is injective; since $i > 0$, the same argument shows that ∂ is surjective. Next, k_μ is the homomorphism deduced from the long exact homology sequence of the pair $(C^i_\tau, C^i_\tau \setminus e^i_\mu)$, and we remark that $C^i_\tau \setminus e^i_\mu$ is contractible : indeed, e^i_μ is homeomorphic to $\mathbb{B}^i(1)$ (lemma 5.9.12), hence $C^i_\tau \setminus e^i_\mu$ is a retraction of $C^i_\tau \setminus \{x\}$, for any $x \in e^i_\mu$, and the latter is homeomorphic to $\mathbb{R}^i$ (again by lemma 5.9.12). If $i > 1$, we deduce already that k_μ is an isomorphism. For $i = 1$, the same argument shows that k_μ is injective, so its cokernel is a cyclic torsion group that injects into $H_0(C^1_\tau \setminus e^1_\mu) \simeq \mathbb{Z}$, hence it must vanish as well. ◊

Since (5.9.17) commutes, claim 5.9.18 yields the assertion, in case $i > 0$. If $i = 0$, C^1_τ is isomorphic to $\overline{\mathbb{B}}^1(1)$, and e^0_μ is one of the two points of $\overline{\mathbb{B}}^1(1) \setminus \mathbb{B}^1(1)$, so the assertion can be checked easily, by inspecting the definitions.

(iii) follows from (i), lemma 5.9.9(i) and remark 5.9.15(ii). □

The following result says that the properties of proposition 5.9.16 completely characterize the incidence numbers of $C^\bullet_\sigma$.

Proposition 5.9.19. *Keep the notation of* (5.9.14)*, and consider a system of integers*

$$(\beta^i_{\lambda\mu} \mid i = 1, \dots, d-1,\ \lambda \in \Lambda_i, \mu \in \Lambda_{i-1})$$

such that :

(a) *If* $\lambda \in \Lambda_i$ *and* $\mu \in \Lambda_{i-1}$ *is not a facet of* λ*, then* $\beta^i_{\lambda\mu} = 0$.
(b) *If* $\lambda \in \Lambda_i$ *and* $\mu \in \Lambda_{i-1}$ *is a facet of* λ*, then* $\beta^i_{\lambda\mu} \in \{1, -1\}$.
(c) *If* μ *and* τ *are the two* 1*-dimensional facets of the* 2*-dimensional face* λ *of* σ*, then*

$$\beta^1_{\lambda\mu} + \beta^1_{\lambda\tau} = 0.$$

(d) *Let* $\lambda \in \Lambda_{i+1}$ *and* $\mu \in \Lambda_{i-1}$ *be two faces, such that* μ *is a face of* λ*, and denote by* τ *and* ρ *the two facets of* λ *that contain* μ*. Then*

$$\beta^{i+1}_{\lambda\tau}\beta^i_{\tau\mu} + \beta^{i+1}_{\lambda\rho}\beta^i_{\rho\mu} = 0.$$

Then there exists a unique orientation of $C^\bullet_\sigma$

$$(b^i_\lambda \mid i = 0, \dots, d-1,\ \lambda \in \Lambda_i)$$

such that for every $i = 1, \dots, d-1$ *we have :*

(5.9.20) $$[b^i_\lambda : b^{i-1}_\mu] = \beta^i_{\lambda\mu} \qquad \textit{for every } \lambda \in \Lambda_i,\ \mu \in \Lambda_{i-1}.$$

Proof. We construct the orientation classes b^i_λ fulfilling condition (5.9.20), by induction on i. For $i = 0$, the condition is empty, and the classes b^0_λ are prescribed by (5.9.8). For $i = 1$, and a given $\lambda \in \Lambda_1$, denote by τ and ρ the two faces of λ; clearly the condition $[b^1_\lambda : b^0_\tau] = \beta^1_{\lambda\tau}$ (and assumption (b)) determines b^1_λ univocally, and in view of (c) and lemma 5.9.9(ii), for this choice of orientation of e^1_λ we have as well $[b^1_\lambda : b^0_\rho] = \beta^1_{\lambda\rho}$. Moreover, if $\mu \in \Lambda_0 \setminus \{\tau, \rho\}$, then (5.9.20) is verified as well, by virtue of (a) and proposition 5.9.16(i).

Now, suppose that $i > 1$ and that we have already constructed orientation classes b^j_μ as sought, for every $j < i$ and every $\mu \in \Lambda_j$. Let $\lambda \in \Lambda_i$ be any face, and fix a facet τ of λ; again, the identity

(5.9.21) $$[b^i_\lambda : b^i_\tau] = \beta^i_{\lambda\tau}$$

determines b^i_λ, and it remains to check that – with this choice of b^i_λ – condition (5.9.20) holds for every $\mu \in \Lambda_{i-1} \setminus \{\tau\}$, and by virtue of (a) and proposition 5.9.16(i), it suffices to consider the facets μ of λ. However, notice that the system of orientation classes $(b^j_\mu \mid j = 0, \dots, i-1, \ \mu \subset \lambda)$ amounts to an orientation of C^{i-1}_λ (regarded as a cell subcomplex of $C^\bullet_\sigma$), and the incidence numbers for the complex $(C^{i-1}_\lambda)^\bullet$ relative to these classes agree with the incidence numbers of $C^\bullet_\sigma$, relative to the same classes. Especially, the sums

$$c := \sum_{\mu \in \Lambda_{i-1}} \beta^i_{\lambda\mu} b^{i-1}_\mu \qquad c' := \sum_{\mu \in \Lambda_{i-1}} [b^i_\lambda : b^{i-1}_\mu] b^{i-1}_\mu$$

are well defined elements of $\mathscr{C}_{i-1}(C^{i-1}_\lambda)$, and in fact :

$$\begin{aligned} d_{i-1}(c) &= \sum_{\mu \in \Lambda_{i-1}} \beta^i_{\lambda\mu} \cdot d_{i-1}(b^{i-1}_\mu) \\ &= \sum_{\mu \in \Lambda_{i-1}} \beta^i_{\lambda\mu} \cdot \sum_{\rho \in \Lambda_{i-2}} [b^{i-1}_\mu : b^{i-2}_\rho] b^{i-2}_\rho \\ &= \sum_{\mu \in \Lambda_{i-1}} \sum_{\rho \in \Lambda_{i-2}} \beta^i_{\lambda\mu} \beta^{i-1}_{\mu\rho} b^{i-2}_\rho && \text{(by inductive assumption)} \\ &= 0 && \text{(by (d))} \end{aligned}$$

and a similar calculation yields $d_{i-1}(c') = 0$ as well. However, C^{i-1}_λ is homeomorphic to $\mathbb{S}^{i-1}$ (lemma 5.9.12), hence

$$\mathrm{Ker}(d_{i-1} : \mathscr{C}_{i-1}(C^{i-1}_\lambda) \to \mathscr{C}_{i-2}(C^{i-1}_\lambda)) \simeq \mathbb{Z}$$

which, in view of (b), implies that $c = \pm c'$. Taking into account (5.9.21), we see that actually $c = c'$, whence the claim. The uniqueness of the orientation fulfilling condition (5.9.20) can be checked easily by induction on i : the details shall be left to the reader. $\square$

5.9.22. Let now L be a free abelian group of finite rank d, and $(L_\mathbb{R}, \sigma)$ a strictly convex L-rational polyhedral cone (see (3.3.20)), such that $\langle\sigma\rangle = L_\mathbb{R}$; set

$$P := L \cap \sigma \qquad F_\lambda := L \cap \lambda \qquad P_\lambda := F_\lambda^{-1} P \qquad \text{for every face } \lambda \text{ of } \sigma$$

so P and its localization P_λ are fine and saturated monoids (proposition 3.3.22(i)), and F_λ is a face of P, for every such λ. Let R be any ring, and if $\lambda, \mu \subset \sigma$ are any two faces with $\lambda \subset \mu$, denote by

$$j_{\lambda\mu} : R[P_\lambda] \to R[P_\mu]$$

the natural localization map; notice that, if $0 \subset \sigma$ is the unique 0-dimensional face, then $P_0 = P$, and $j_{0\mu}$ is the localization map $R[P] \to R[P_\mu]$. We attach to P the complex $\overline{\mathscr{C}}{}^\bullet_P$ of $R[P]$-modules such that :

$$\overline{\mathscr{C}}{}^0_P := R[P] \qquad \text{and} \qquad \overline{\mathscr{C}}{}^i_P := \bigoplus_{\lambda \in \Lambda_{i-1}} R[P_\lambda] \qquad \text{for every } i = 1, \dots, \dim P$$

with differentials given by the rule :

$$d^i(x_\lambda) := \sum_{\substack{\mu \in \Lambda_i \\ \lambda \subset \mu}} [b^i_\mu : b^{i-1}_\lambda] \cdot j_{\lambda\mu}(x_\lambda) \qquad \text{for every } i \geq 1,\ \lambda \in \Lambda_{i-1} \text{ and } x_\lambda \in R[P_\lambda]$$

and

$$d^0(x) := \sum_{\mu \in \Lambda_0} j_{0\mu}(x) \qquad \text{for every } x \in R[P]$$

where $(b^j_\tau \mid j = 0, \dots \dim\sigma - 1,\ \tau \in \Lambda_j)$ is a chosen orientation for $C^\bullet_\sigma$. Taking into account lemma 5.9.9(ii) and proposition 5.9.16(iii), it is easily seen that $d^{i+1} \circ d^i = 0$ for every $i \geq 0$ (essentially, the differential d^i is the transpose of the differential d_{i-1} of $\overline{\mathscr{C}}_\bullet(C^\bullet_\sigma)$).

Set as well $X := \operatorname{Spec} R[P]$, and notice that $Z := \operatorname{Spec} R\langle P/\mathfrak{m}_P\rangle \simeq \operatorname{Spec} R$ is a closed subset of X. For any $R[P]$-module M, we denote as usual by $M^\sim$ the quasi-coherent $\mathscr{O}_X$-module arising from M.

Theorem 5.9.23. *With the notation of* (5.9.22)*, for every $R[P]$-module M we have a natural isomorphism*

$$R\Gamma_Z M^\sim \xrightarrow{\sim} M \otimes_{R[P]} \overline{\mathscr{C}}{}^\bullet_P \qquad \textit{in } \mathsf{D}^+(R[P]\text{-}\mathbf{Mod}).$$

Proof. For every face λ of σ, set $U_\lambda := \operatorname{Spec} R[P_\lambda]$, and denote by $g_\lambda : U_\lambda \to X$ the open immersion. We consider the chain complex $\mathscr{R}_\bullet$ of $\mathbb{Z}_X$-modules such that :

$$\mathscr{R}_0 := \mathbb{Z}_X \qquad \mathscr{R}_i := \bigoplus_{\lambda\in\Lambda_{i-1}} g_{\lambda!}\mathbb{Z}_{U_\lambda} \qquad \text{for every } i = 1, \dots, \dim\, P.$$

The differential $d_1 : \mathscr{R}_1 \to \mathscr{R}_0$ is just the sum of the natural morphisms $g_{\lambda!}\mathbb{Z}_{U_\lambda} \to \mathbb{Z}_X$, for λ ranging over the one-dimensional faces of σ. For $i > 1$, the differential d_i is the sum of the maps

$$d_\lambda := \sum_{\substack{\mu\in\Lambda_i\\ \mu\subset\lambda}} [b^i_\lambda : b^{i-1}_\mu] \cdot d_{\lambda\mu!} : g_{\lambda!}\mathbb{Z}_{U_\lambda} \to \mathscr{R}_{n-1} \qquad \text{for every } \lambda \in \Lambda_{i-1}$$

where $d_{\lambda\mu} : g_{\lambda!}\mathbb{Z}_{U_\lambda} \to g_{\mu!}\mathbb{Z}_{U_\mu} \subset \mathscr{R}_{n-1}$ is induced by the inclusion $U_\lambda \subset U_\mu$. With this notation, a simple inspection of the definitions yields a natural identification

$$(5.9.24) \qquad M \otimes_{R[P]} \overline{\mathscr{C}}{}^\bullet_P \xrightarrow{\sim} \operatorname{Hom}^\bullet_{\mathbb{Z}}(\mathscr{R}_\bullet, M^\sim[0]) \qquad \text{in } \mathsf{C}(R[P]\text{-}\mathbf{Mod}).$$

On the other hand, let $\lambda_1, \dots, \lambda_n$ be the one-dimensional faces of σ; then λ_i is L-rational (see (3.3.20)), and F_{λ_i} is a fine, sharp and saturated monoid of dimension one (propositions 3.3.22(i) and 3.4.7(ii)), so $F_{\lambda_i} \simeq \mathbb{N}$ for $i = 1, \dots, n$ (theorem 3.4.16(ii)). For each $i = 1, \dots, n$, let y_i be the unique generator of F_{λ_i}, and denote by $I \subset P$ the ideal generated by $y_1, \dots, y_n$; we remark

Claim 5.9.25. The radical of I is $\mathfrak{m}_P$.

Proof of the claim. For any $x \in \mathfrak{m}_P$, pick a subset $S \subset \{1, \dots, n\}$ such that $x = \sum_{i\in S} a_i y_i$, with $a_i > 0$ for every $i \in S$. Let $N \in \mathbb{N}$ be large enough such that $Na_i \geq 1$ for every $i \in S$, and pick $b_i \in \mathbb{N}$ such that $Na_i \geq b_i \geq 1$ for every $i \in S$. It follows that $Nx - \sum_{i\in S} b_i y_i \in P$, and the claim follows. ◊

Now, let $i : Z \to X$ be the closed immersion; we have :

Claim 5.9.26. The natural map $\mathbb{Z}_X \to i_*\mathbb{Z}_Z$ induces an isomorphism

$$(5.9.27) \qquad \mathscr{R}_\bullet \xrightarrow{\sim} i_*\mathbb{Z}_Z[0] \qquad \text{in } \mathsf{D}^+(\mathbb{Z}_X\text{-}\mathbf{Mod}).$$

Proof of the claim. The assertion can be checked on the stalks at the points of X, hence let $x \in X$ be any such point; if $x \in Z$, claim 5.9.25 easily implies that $(\mathscr{R}_\bullet)_x$ is concentrated in degree zero, and then $(5.9.27)_x$ is clearly an isomorphism of complexes of $\mathbb{Z}$-modules. If $x \notin Z$, let $\mathfrak{p} \subset R[P]$ be the prime ideal corresponding to x, and $\lambda \subset \sigma$ the unique face such that $P \setminus F_\lambda = \mathfrak{p} \cap P$; a simple inspection of the construction shows that

$$(\mathscr{R}_\bullet)_x \xrightarrow{\sim} \overline{\mathscr{C}}_\bullet(C^\bullet_\lambda)$$

(notation of (5.9.6)). But proposition 5.9.7 and lemma 5.9.12 imply that $\overline{\mathscr{C}}_\bullet(C^\bullet_\lambda)$ is acyclic, whence the claim. ◊

In view of (5.9.24) and claim 5.9.26, we are reduced to showing that the natural map

$$\operatorname{Hom}^\bullet_{\mathbb{Z}}(\mathscr{R}_\bullet, M^\sim[0]) \to R\operatorname{Hom}^\bullet_{\mathbb{Z}}(\mathscr{R}_\bullet, M^\sim[0])$$

is an isomorphism in $\mathsf{D}^+(R[P]\text{-}\mathbf{Mod})$. This can be done by a spectral sequence argument, along the lines of remark 5.4.17 (which indeed includes a special case of the situation we are considering here, namely the case where P is a free monoid : the details shall be left to the reader). $\square$

5.9.28. Notice that $\overline{\mathscr{C}}{}^\bullet_P$ is a complex of L-graded R-modules, hence its cohomology is L-graded as well, and we wish next to compute the graded terms $\mathrm{gr}_\bullet H^\bullet(\overline{\mathscr{C}}{}^\bullet_P)$. To this aim, we make the following :

Definition 5.9.29. Let V be a finite dimensional $\mathbb{R}$-vector space, $X \subset V$ any subset, and $z \in V$ any point. Then :

(a) We say that a point $x \in X$ *is visibile from* z, if $\{tz+(1-t)x \mid 0 \leq t \leq 1\} \cap X = \{x\}$.
(b) We say that a subset $S \subset X$ *is visibile from* z, if every point of S is visible from z.

Lemma 5.9.30. *In the situation of* (5.9.14)*, let $z \in V \setminus C_\sigma$ be any point, and denote by S the set of points of C_σ that are visible from z. Then :*

(i) *S is a subcomplex of $C^\bullet_\sigma$.*
(ii) *S (with the topology induced from V) is homeomorphic to $\overline{\mathbb{B}}^e(1)$, for $e \in \{d-2, d-1\}$.*

Proof. Pick a system of generators $\rho_1, \dots, \rho_n$ for $\sigma^\vee$, and choose $\rho_0 \in \sigma^\vee$ so that $C_\sigma = \sigma \cap \rho_0^{-1}(1)$; we may assume that, for some integer $k \leq n$ we have

$$\rho_i(z) < 0 \qquad \text{if and only if} \qquad 1 \leq i \leq k.$$

Say that $y \in C_\sigma$, so that $\rho_i(y) \geq 0$ for every $i = 1, \dots, n$, and set $y_t := tz + (1-t)y$ for every $t \in [0,1]$; then clearly $\rho_i(y_t) \geq 0$ for every $i = k+1, \dots, n$. Now, if $\rho_0(z) \neq 1$, we get $\rho_0(y_t) \neq 1$ for every $t \neq 0$, therefore the whole of C_σ is visible from z, in which case (i) is trivial, and (ii) follows from 5.9.12. Hence we may assume that $\rho_0(z) = 1$, so $\rho_0(y_t) = 1$ for every $t \in [0,1]$. Suppose now that $\rho_i(y) > 0$ for some $i \leq k$; then there exists a unique $t_i \in]0,1[$ such that $\rho_i(y_{t_i}) = 0$. Hence, if $s := \min(\rho_i(y) \mid i = 1, \dots, k) > 0$, let $t := \min(t_i \mid i = 1, \dots, k)$; it follows that $\rho_i(y_t) \geq 0$ for every $i = 1, \dots, n$, so $y_t \in C_\sigma$, which says that y is not visible from z. Conversely, if $s = 0$, then it follows easily that y is visible from z. We conclude that

$$S = C_\sigma \cap \bigcup_{i=1}^k \mathrm{Ker}\,\rho_i$$

which shows (i). Next, set $W := \mathrm{Ker}\,\rho_0$, and denote by $\tau_z : \rho_0^{-1}(1) \xrightarrow{\sim} W$ the translation map given by the rule : $x \mapsto x - z$ for every $x \in \rho_0^{-1}(1)$. To conclude, it suffices to check that $S' := \tau_z(S)$ is homeomorphic to $\overline{\mathbb{B}}^{d-2}(1)$. To this aim, denote by λ the convex cone in W generated by $\tau_z(C_\sigma)$. Explicitly, if $v_1, \dots, v_k \in \sigma$ have been chosen so that (5.9.11) holds, then λ is the cone generated by $v_1 - z, \dots, v_k - z$; especially, λ is a polyhedral cone, and it is easily seen that $\langle\lambda\rangle = W$. Moreover, λ is strictly convex; indeed, otherwise there exist real numbers $a_1, \dots, a_k \geq 0$, with $a_i > 0$ for at least an index $i \leq k$, such that $\sum_{i=1}^k a_i \cdot (v_i - z) = 0$, *i.e.* $\sum_{i=1}^k a_i v_i = (\sum_{i=1}^k a_i) \cdot z$, which is absurd, since $z \notin \sigma$. Pick $u \in \lambda^\vee$ such that $\lambda \cap \mathrm{Ker}\,u = 0$, and set $C_\lambda := \lambda \cap u^{-1}(1)$; by lemma 5.9.12, the subset C_λ is homeomorphic to $\overline{\mathbb{B}}^{d-2}(1)$. Lastly, let $\pi : W \setminus \mathrm{Ker}\,u \to u^{-1}(1)$ be the radial projection (so $\pi(w)$ is the intersection point of $\mathbb{R}w$ with $u^{-1}(1)$, for every $w \in W \setminus \mathrm{Ker}\,u$). It is easily seen that π maps S' bijectively onto C_λ, so the restriction of π is a homeomorphism $S' \xrightarrow{\sim} C_\lambda$, as required. $\square$

Lemma 5.9.31. *In the situation of* (5.9.22)*, let $\lambda \subset \sigma$ be any face. For every $l \in L_\mathbb{R}$ we have:*

(i) *The set of points of σ that are visible from l is a union of faces of σ.*

(ii) *Suppose $l \in L$, and endow $R[P_\lambda]$ with its natural L-grading. Then*
$$\mathrm{gr}_l R[P_\lambda] = \begin{cases} Rl & \text{if } \lambda \subset \sigma \text{ is not visible from } l \\ 0 & \text{otherwise}. \end{cases}$$

Proof. Let $\rho_1, \dots, \rho_n$ be a system of generators of $\sigma^\vee$; we may assume that, for some integer $k \leq n$ we have
$$\rho_i(l) < 0 \qquad \text{if and only if } 1 \leq i \leq k.$$
Arguing as in the proof of lemma 5.9.30, we check easily that the set of points of σ visible from l equals $S := \sigma \cap \bigcup_{i=1}^k \mathrm{Ker}\, \rho_i$, which already shows (i).

(ii): Suppose first that $\lambda \subset \sigma$ is not visible from l; then there exists $x \in \lambda \setminus S$, and since F_λ generates λ (see (3.3.20)), we may assume that $x \in F_\lambda$ (details left to the reader). A simple inspection then shows that there exists a sufficiently large $N \in \mathbb{N}$ such that $l + Nx \in \sigma$, whence $l \in R[P_\lambda]$, and so $\mathrm{gr}_l R[P_\lambda] = Rl$. Conversely, if the latter identity holds, then there exists $x \in F_\lambda$ such that $l + x \in P$, whence $\rho_i(x) > 0$ for $i = 1, \dots, k$, so x is not visible from l. $\square$

5.9.32. In the situation of (5.9.22), let us fix a linear form $u_0 \in \sigma^\vee$ such that $\sigma \cap \mathrm{Ker}\, u_0 = 0$, and define σ° and C_σ as in (5.9.10). For any $l \in L_\mathbb{R}$, denote by S_l the set of points of σ that are visible from l, so S_l is a union of faces of σ, by lemma 5.9.31(i), and therefore $C_l := S_l \cap C_\sigma$ is a subcomplex of $C_\sigma^\bullet$.

Proposition 5.9.33. *With the notation of* (5.9.32), *suppose $l \in L$. Then the following holds :*

(i) *If $-l \in \sigma^\circ$, the complex of R-modules $\mathrm{gr}_l \overline{\mathscr{C}}{}^\bullet_P$ is isomorphic to $R[-d]$.*

(ii) *If $-l \notin \sigma^\circ$, there is a natural isomorphism of complexes of R-modules :*
$$\mathrm{gr}_l \overline{\mathscr{C}}{}^\bullet_P \xrightarrow{\sim} \mathrm{Hom}^\bullet_\mathbb{Z}(\overline{\mathscr{C}}_\bullet(C_\sigma^\bullet)/\overline{\mathscr{C}}_\bullet(C_l^\bullet), R[-1]).$$
Moreover, in this case, both $\overline{\mathscr{C}}_\bullet(C_\sigma^\bullet)$ and $\overline{\mathscr{C}}_\bullet(C_l^\bullet)$ are acyclic complexes.

Proof. (i): If $-l \in \sigma^\circ$, then $l \in P_\lambda$ if and only if $\lambda = \sigma$, so the assertion is clear.

(ii): The sought identification of complexes follows from lemma 5.9.31(ii), by a direct inspection of the constructions (and indeed, this holds even if $-l \in \sigma^\circ$: details left to the reader). Moreover, it is already known from proposition 5.9.7 and lemma 5.9.12 that $\overline{\mathscr{C}}_\bullet(C_\sigma^\bullet)$ is acyclic.

• Next, if $l \in P$, then clearly $S_l = \varnothing$, so the assertion for $\overline{\mathscr{C}}_\bullet(C_l^\bullet)$ is trivial in this case.

• Thus, suppose that $l \notin P$; if furthermore $-l \notin P$, then the convex cone τ generated by P and l is still strictly convex, so we may find $u_1 \in \tau^\vee$ such that $u_1(l) = 1$ and $\sigma \cap \mathrm{Ker}\, u_1 = 0$. Set $C'_\sigma := \sigma \cap u_1^{-1}(1)$ and $C'_l := S_l \cap C'_\sigma$. In remark 5.9.15(i) we have exhibited a homeomorphism $C'^\bullet_\sigma \xrightarrow{\sim} C^\bullet_\sigma$ that preserves the respective cell complex structures, and a simple inspection shows that this homeomorphism maps C'_l onto C_l. On the other hand, it is easily seen that C'_l is also the set of points of C'_σ that are visible from l (indeed, the segment that joins any point of C'_σ to l lies in $u_1^{-1}(1)$, so its intersection with σ equals its intersection with C'_σ : details left to the reader). By virtue of lemma 5.9.30(ii) (and proposition 5.9.7), it follows that $\overline{\mathscr{C}}_\bullet(C'^\bullet_l)$ is acyclic, so the same holds for $\overline{\mathscr{C}}_\bullet(C_l^\bullet)$.

• Lastly, suppose $-l \in P \setminus \sigma^\circ$; we let $\rho_1, \dots, \rho_n$ be a system of generators of $\sigma^\vee$, and $k \leq n$ an integer such that $\rho_i(l) < 0$ if and only if $1 \leq i \leq k$. Denote by $\tau^\vee$ (resp. $\mu^\vee$) the convex cone in $L^\vee_\mathbb{R}$ generated by $(\rho_{k+1}, \dots, \rho_n)$ (resp. by $(-\rho_1, \dots, -\rho_k)$), and let τ (resp. μ) be the dual of $\tau^\vee$ (resp. of $\mu^\vee$) in $L_\mathbb{R}$; with this notation, we have $l \in \tau \cap \mu^\circ$. Especially, $\tau \cap \mu^\circ \neq \varnothing$, and since τ° is dense in τ (by proposition 3.3.11(i)), we deduce that $\tau^\circ \cap \mu^\circ \neq \varnothing$ as well. Pick $z \in \tau^\circ \cap \mu^\circ$; by inspecting the proof of lemma 5.9.31(i), it is easily seen that $S_l = S_z$. But by construction, $-z \notin \sigma$, therefore – arguing as in the previous case – we conclude that $\overline{\mathscr{C}}_\bullet(C_z^\bullet) = \overline{\mathscr{C}}_\bullet(C_l^\bullet)$ is acyclic. $\square$

Theorem 5.9.34 (Hochster)**.** *Let R be a Cohen-Macaulay noetherian ring, P a fine, sharp and saturated monoid. Then $R[P]$ is a Cohen-Macaulay ring.*

Proof. In view of [61, p.181, Cor.] we may assume that R is a field. We argue by induction on $d := \dim P$. If $d = 0$, we have $P = 0$, and there is nothing to show. Suppose that $d > 0$ and that the assertion is already known for every Cohen-Macaulay ring R and every monoid as above, of dimension $< d$. Set $L := P^{\mathrm{gp}}$, and let $\sigma \subset L_{\mathbb{R}}$ be the unique convex polyhedral cone such that $P = L \cap \sigma$. The ideal $\mathfrak{n} := R[\mathfrak{m}_P]$ is maximal in $R[P]$, and proposition 5.9.33 and theorem 5.9.23 show that

$$\mathrm{depth}\, R[P]_{\mathfrak{n}} = \dim\, P. \tag{5.9.35}$$

On the other hand, we have the more general :

Claim 5.9.36. Let F be a field, P a fine monoid, A an integral domain which is an F-algebra of finite type, and F' the field of fractions of A. Then we have :

(i) For every maximal ideal $\mathfrak{m} \subset A$, the Krull dimension of $A_{\mathfrak{m}}$ equals the transcendence degree of F' over F.

(ii) For every maximal ideal $\mathfrak{m} \subset F[P]$, the Krull dimension of $F[P]_{\mathfrak{m}}$ equals $\mathrm{rk}_{\mathbb{Z}}\, P^{\mathrm{gp}}$.

Proof of the claim. (i): This is a straightforward consequence of [61, Th.5.6].

(ii): Choose an isomorphism : $P^{\mathrm{gp}} \xrightarrow{\sim} L \oplus T$, where T is the torsion subgroup of P^{gp}, and L is a free abelian group of finite rank; there follows an induced isomorphism (2.3.52) :

$$F[P^{\mathrm{gp}}] \xrightarrow{\sim} F[L] \otimes_F F[T].$$

Let B be the maximal reduced quotient of $F[P^{\mathrm{gp}}]$ (so the kernel of the projection $F[P^{\mathrm{gp}}] \to B$ is the nilpotent radical); we deduce that B is a direct product of the type $\prod_{i=1}^n F'_i[L]$, where each F'_i is a finite field extension of F. By (i), the Krull dimension of $F'_i[L]_{\mathfrak{m}}$ equals $r := \mathrm{rk}_{\mathbb{Z}} P^{\mathrm{gp}}$ for every maximal ideal $\mathfrak{m} \subset F'_i[L]$, hence every irreducible component of $\mathrm{Spec}\, B$ has dimension r. Let also C be the maximal reduced quotient of $F[P]$; the natural map $C \to B$ is an injective localization, obtained by inverting a finite system of generators of P, hence the induced morphism $\mathrm{Spec}\, B \to \mathrm{Spec}\, C$ is an open immersion with dense image. Let Z be any (reduced) irreducible component of $\mathrm{Spec}\, C$; again by (i) it follows that every non-empty open subset of Z has dimension equal to $\dim Z$, so necessarily the latter equals r. $\diamondsuit$

From (5.9.35), corollary 3.4.10(i) and claim 5.9.36(ii) we see already that $R[P]_{\mathfrak{n}}$ is a Cohen-Macaulay ring. Next, let $\lambda_1, \dots, \lambda_k$ be the one-dimensional faces of σ, and define P_{λ_i} as in (5.9.22), for every $i = 1, \dots, k$. In light of claim 5.9.25, we have

$$\mathrm{Spec}\, R[P] \setminus \{\mathfrak{n}\} = \bigcup_{i=1}^{k} \mathrm{Spec}\, R[P_{\lambda_i}]$$

so it remains to check that $R[P_{\lambda_i}]$ is Cohen-Macaulay for every $i = 1, \dots, k$. However, we may find a decomposition $P_{\lambda_i} \xrightarrow{\sim} Q_i \times G_i$, where G_i is a free abelian group of finite type, and Q_i is a fine, sharp and saturated monoid of dimension $d - 1$ (lemma 3.2.10). Set $S_i := R[G_i]$; then $R[P_{\lambda_i}] = S_i[Q_i]$, and S_i is a Cohen-Macaulay ring ([61, Th.17.7]), so the claim follows by inductive assumption. $\square$

5.9.37. In the situation of (5.9.22), we endow $R[P]$ with its natural L-grading, and denote by $\underline{A} := (R[P], \mathrm{gr}_\bullet R[P])$ the resulting L-graded R-algebra. Let $(M, \mathrm{gr}_\bullet M)$ be an L-graded $\underline{A}$-module (see definition 4.4.9(ii)); we set

$$M^\dagger := \bigoplus_{l \in L} \mathrm{Hom}_R(\mathrm{gr}_l M, R)$$

which is naturally an R-submodule of $M^* := \mathrm{Hom}_R(M, R)$: indeed, any R-linear map $\mathrm{gr}_l M \to R$ yields a linear form $M \to R$, after composition with the projection $M \to \mathrm{gr}_l M$. Notice that M^* is naturally an $R[P]$-module; namely for any linear form $f : M \to R$ and any

$x \in P$, one defines $x \cdot f : M \to R$ by the rule : $x \cdot f(m) := f(xm)$ for every $m \in M$. Then, it is easily seen that $M^\dagger$ is an $R[P]$-submodule of M^*; more precisely, we have

$$x \cdot \mathrm{Hom}_R(\mathrm{gr}_l M, R) \subset \mathrm{Hom}_R(\mathrm{gr}_{l-x} M, R) \qquad \text{for every } x \in P. \tag{5.9.38}$$

In light of (5.9.38), it is convenient to define an L-grading on $M^\dagger$ by the rule :

$$\mathrm{gr}_l M^\dagger := \mathrm{Hom}_R(\mathrm{gr}_{-l} M, R) \qquad \text{for every } l \in L$$

and then $(M^\dagger, \mathrm{gr}_\bullet M^\dagger)$ is naturally an L-graded $\underline{A}$-module. Clearly, the rule $M \mapsto M^\dagger$ yields a functor from the category $\underline{A}$-$\mathbf{Mod}$ of L-graded $\underline{A}$-modules to $\underline{A}$-$\mathbf{Mod}^o$.

Example 5.9.39. If $S \subset L$ is any P-submodule, notice that $L \setminus (-S)$ is also a P-submodule of L, and set

$$S^\dagger := L/(L \setminus (-S))$$

where the quotient is a pointed P-module, as in remark 2.3.17(iii). Explicitly, $S^\dagger$ is the set $(-S) \cup \{0_{S^\dagger}\}$ (where the zero element $0_{S^\dagger}$ should not be confused with the neutral element 0 of the abelian group L); the P-module structure on $S^\dagger$ is determined by the rule :

$$x \cdot s := \begin{cases} x + s & \text{if } x + s \in -S \\ 0_{S^\dagger} & \text{otherwise} \end{cases} \qquad \text{for every } s \in -S.$$

Then it is easily seen that there exists a natural isomorphism of L-graded $\underline{A}$-modules :

$$R[S]^\dagger \xrightarrow{\sim} R\langle S^\dagger \rangle$$

(notation of (3.1.31)). On the other hand, the natural map $R[S] \to (R[S]^*)^*$ induces an isomorphism $R[S] \xrightarrow{\sim} (R[S]^\dagger)^\dagger$ whence an isomorphism of L-graded $\underline{A}$-modules

$$R[S] \xrightarrow{\sim} R\langle S^\dagger \rangle^\dagger. \tag{5.9.40}$$

Theorem 5.9.41 (Stanley, Danilov)**.** *In the situation of* (5.9.22)*, take R to be any field, and set $P^\circ := L \cap \sigma^\circ$. We have :*

(i) *There exists a natural isomorphism :*

$$R\mathrm{Hom}^\bullet_{R[P]}(R, R[P^\circ]) \xrightarrow{\sim} R[-d] \qquad \textit{in } \mathsf{D}(R[P]\text{-}\mathbf{Mod})$$

(here R is regarded as an $R[P]$-module, via the augmentation map $R[P] \to R$).

(ii) *The complex of coherent $\mathscr{O}_X$-modules $R[P^\circ]^\sim[0]$ is dualizing on X.*

Proof. (i): From proposition 5.9.33 we deduce a map of complexes of L-graded $\underline{A}$-modules

$$\varphi^\bullet : \overline{\mathscr{C}}{}^\bullet_P \to R\langle (P^\circ)^\dagger \rangle[-d]$$

(notation of example 5.9.39) with $\mathrm{gr}_l \varphi^\bullet$ a quasi-isomorphism of complexes of R-modules, for every $l \in L$. Since $\mathrm{gr}_l \overline{\mathscr{C}}{}^\bullet_P$ is a finite dimensional R-vector space for every $l \in L$, there follows – in view of (5.9.40) – a quasi-isomorphism of complexes of L-graded $\underline{A}$-modules

$$R[P^\circ][0] \xrightarrow{\sim} (\overline{\mathscr{C}}{}^\bullet_P)^\dagger[-d]. \tag{5.9.42}$$

Claim 5.9.43. With the foregoing notation, we have :

(i) $\mathrm{Ext}^i_{R[P]}(R, (\overline{\mathscr{C}}{}^j_P)^\dagger) = 0$ for every $i \in \mathbb{N}$ and every $j = 1, \dots, d$.

(ii) $(\overline{\mathscr{C}}{}^0_P)^\dagger$ is the injective hull of the $R[P]$-module R.

Proof of the claim. (i): Let $\lambda \subset \sigma$ be any face with $\dim \lambda > 0$; it suffices to check that $E^i := \mathrm{Ext}^i_{R(P]}(R, R[P_\lambda]^\dagger) = 0$ for every $i \in \mathbb{N}$. To this aim, pick any $x \in F_\lambda \setminus \{0\}$; since $R[P_\lambda]^\dagger = R\langle P^\dagger_\lambda \rangle$ (example 5.9.39), we see that scalar multiplication by x is an automorphism on $R[P_\lambda]^\dagger$, hence also on E^i. On the other hand, scalar multiplication by x is the zero endomorphism on R, hence also on E^i, and the claim follows.

(ii) is a special case of example 4.3.33 : details left to the reader. ◊

From (5.9.42) we get a convergent spectral sequence

$$E_1^{pq} := \mathrm{Ext}^p_{R[P]}(R, (\overline{\mathscr{C}}^{d-q}_P)^\dagger) \Rightarrow \mathrm{Ext}^{p+q}_{R[P]}(R, R[P^\circ])$$

and claim 5.9.43 implies that $E_1^{pq} = 0$ unless $p = 0$ and $q = d$; moreover, claim 4.3.24 says that $E_1^{0,d} \simeq R$, whence the contention.

(ii): To ease notation, set $S := \mathrm{Spec}\, R$ and $\omega_P := R[P^\circ]^\sim$; in view of [46, Ch.V, Cor.2.3], it suffices to check that the complex of coherent $\mathscr{O}_{X(x)}$-modules $\omega_P(x)[0]$ is dualizing on $X(x)$, for every $x \in X$ (notation of definition 2.4.17(iii)). Hence, fix any such point x, let $\mathfrak{p} \subset R[P]$ be the prime ideal corresponding to x, and denote by $\lambda \subset \sigma$ the unique face such that $P \setminus F_\lambda = \mathfrak{p} \cap P$, so that $x \in U_\lambda := \mathrm{Spec}\, R[P_\lambda]$. We may find a decomposition $P \xrightarrow{\sim} F_\lambda^{\mathrm{gp}} \times Q$, where Q is also a fine, sharp and saturated monoid (lemma 3.2.10), whence an isomorphism of S-schemes

$$U_\lambda \xrightarrow{\sim} \mathrm{Spec}\, R[F_\lambda^{\mathrm{gp}}] \times_S Y \qquad \text{where } Y := \mathrm{Spec}\, R[Q]$$

and by construction, the induced projection $p : U_\lambda \to Y$ maps x to the maximal ideal of $R[Q]$ generated by the maximal ideal $\mathfrak{m}_Q$ of Q. Let $\tau \subset Q^{\mathrm{gp}}_{\mathbb{R}}$ be the unique polyhedral cone such that $Q = \tau \cap Q^{\mathrm{gp}}$, set $Q^\circ := Q \cap \tau^\circ$, and define the coherent $\mathscr{O}_Y$-module $\omega_Q := R[Q^\circ]^\sim$. It is easily seen that there is a natural identification

$$\omega_{P|U_\lambda} \xrightarrow{\sim} p^*\omega_Q.$$

In view of [46, Ch.V, Th.8.3], it then suffices to check that $\omega_Q(p(x))$ is dualizing on $Y(p(x))$. Thus, we may replace P by Q, and assume from start that $\mathfrak{p}$ is the augmentation ideal of $R[P]$. In this case, the assertion follows from (i) and proposition 5.6.26 (details left to the reader). $\square$

6. Logarithmic geometry

6.1. **Log topoi.** Henceforth, *all topoi under consideration will be locally ringed and with enough points, and all morphisms of topoi will be morphisms of locally ringed topoi* (see (2.4.13)). The purpose of this restriction is to insure that we obtain the right notions, when we specialize to the case of schemes.

6.1.1. Let $T := (T, \mathscr{O}_T)$ be a locally ringed topos. Recall ([52, §1.1]) that a *pre-log structure* on T is the datum of a pair $(\underline{M}, \alpha)$, where $\underline{M}$ is a T-monoid, and $\alpha : \underline{M} \to \mathscr{O}_T$ is a morphism of T-monoids, called the *structure map* of $\underline{M}$, and where the monoid structure on $\mathscr{O}_T$ is induced by the multiplication law (hence, by the multiplication in the ring $\mathscr{O}_T(U)$, for every object U of T). The generalities of [36, (6.4.1)-(6.4.8)] actually carry over *verbatim* to any locally ringed topos T, hence we shall recall briefly the main definitions and constructions that we need, and refer to *loc.cit.* for further details.

6.1.2. A morphism $(\underline{M}, \alpha) \to (\underline{N}, \beta)$ of pre-log structures on T, is a map $\gamma : \underline{M} \to \underline{N}$ of T-monoids, such that $\beta \circ \gamma = \alpha$. We denote by $\mathbf{pre\text{-}log}_T$ the category of pre-log structures on T. A morphism of locally ringed topoi $f : T \to S$ induces a pair of adjoint functors :

$$(6.1.3) \qquad f^* : \mathbf{pre\text{-}log}_S \to \mathbf{pre\text{-}log}_T \qquad f_* : \mathbf{pre\text{-}log}_T \to \mathbf{pre\text{-}log}_S.$$

A pre-log structure $(\underline{M}, \alpha)$ on T is said to be a *log structure* if α restricts to an isomorphism:

$$\alpha^{-1}\mathscr{O}_T^\times \xrightarrow{\sim} \underline{M}^\times \xrightarrow{\sim} \mathscr{O}_T^\times.$$

The datum of a locally ringed topos $(T, \mathscr{O}_T)$ and a log structure on T is also called, for short, a *log topos*. We denote by $\mathbf{log}_T$ the full subcategory of $\mathbf{pre\text{-}log}_T$ consisting of all log structures on T. When there is no danger of ambiguity, we shall often omit mentioning explicitly the map α, and therefore only write $\underline{M}$ to denote a pre-log or a log structure. The category $\mathbf{log}_T$ admits an initial object, namely the log structure $(\mathscr{O}_T^\times, j)$, where j is the natural inclusion; this is called the *trivial log structure*. $\mathbf{log}_T$ admits a final object as well : this is $(\mathscr{O}_T, \mathbf{1}_{\mathscr{O}_T})$.

Lemma 6.1.4. *Let T be a topos with enough points, $\varphi : \mathscr{M} \to \mathscr{N}$ a morphism of integral T-monoids inducing isomorphisms $\mathscr{M}^\times \xrightarrow{\sim} \mathscr{N}^\times$ and $\mathscr{M}^\sharp \xrightarrow{\sim} \mathscr{N}^\sharp$. Then φ is an isomorphism.*

Proof. This can be checked on the stalks, hence we are reduced to the corresponding assertions for a morphism $M \to N$ of monoids. Moreover, $M^\sharp$ is just the set-theoretic quotient of M by the translation action of $M^\times$ (lemma 2.3.31(iii)), so the assertion is straightforward, and shall be left as an exercise for the reader. $\square$

Definition 6.1.5. Let $\gamma : (\underline{M}, \alpha) \to (\underline{N}, \beta)$ be a morphism of pre-log structures on the locally ringed topos T, and ξ a T-point.

(i) We say that $(\underline{M}, \alpha)$ is *integral* (resp. *saturated*) if $\underline{M}$ is an integral (resp. integral and saturated) T-monoid.
(ii) We say that γ is *flat* (resp. *saturated*) *at the point* ξ, if $\gamma_\xi : \underline{M}_\xi \to \underline{N}_\xi$ is a flat morphism of monoids (resp. a saturated morphism of integral monoids) (see remark 2.3.23(vi)).
(iii) We say that γ is *flat* (resp. *saturated*), if γ is a flat morphism of T-monoids (resp. a saturated morphism of integral T-monoids) (see definition 3.2.28). In view of proposition 2.3.26 (resp. corollary 3.2.29), this is the same as saying that γ is flat (resp. saturated) at every T-point.

6.1.6. The forgetful functor :

$$\mathbf{log}_T \to \mathbf{pre\text{-}log}_T \quad : \quad \underline{M} \mapsto \underline{M}^{\text{pre-log}}$$

admits a left adjoint :

$$\mathbf{pre\text{-}log}_T \to \mathbf{log}_T \quad : \quad (\underline{M}, \alpha) \mapsto (\underline{M}, \alpha)^{\log}$$

such that the resulting diagram :

$$\begin{tikzcd} \alpha^{-1}(\mathscr{O}_T^\times) \arrow[r] \arrow[d] & \underline{M} \arrow[d] \\ \mathscr{O}_T^\times \arrow[r] & \underline{M}^{\log} \end{tikzcd} \qquad (6.1.7)$$

is cocartesian in the category of pre-log structures. One calls $\underline{M}^{\log}$ the *log structure associated* to $\underline{M}$. Composing with the adjunction (6.1.3), we deduce a pair of adjoint functors :

$$f^* : \mathbf{log}_S \to \mathbf{log}_T \qquad f_* : \mathbf{log}_T \to \mathbf{log}_S$$

for any map of locally ringed topoi $f : T \to S$. Explicitly, if $\underline{N}$ is any log structure on S, then $f^*\underline{N}$ is the push-out in the cocartesian diagram of T-monoids :

$$\begin{tikzcd} f^*\mathscr{O}_S^\times \arrow[r] \arrow[d] & f^*(\underline{N}^{\text{pre-log}}) \arrow[d] \\ \mathscr{O}_T^\times \arrow[r] & f^*\underline{N}. \end{tikzcd}$$

It follows easily that the induced map of T-monoids :

$$f^*(\underline{N}^\sharp) \to (f^*\underline{N})^\sharp \qquad (6.1.8)$$

is an isomorphism. As a consequence, for every point ξ of T, the natural map $\underline{N}_{f(\xi)} \to (f^*\underline{N})_\xi$ is a local morphism of monoids.

6.1.9. Let $(\underline{M}, \alpha)$ be a pre-log structure on T. The morphism α extends to a unique morphism of pointed T-monoids $\alpha_\circ : \underline{M}_\circ \to \mathscr{O}_T$, whence a new pre-log structure

$$(\underline{M}, \alpha)_\circ := (\underline{M}_\circ, \alpha_\circ).$$

Clearly, $(\underline{M}, \alpha)$ is a log structure if and only if the same holds for $(\underline{M}, \alpha)_\circ$. More precisely, for any pre-log structure $\underline{M}$ there is a natural isomorphism of log structures :

$$(\underline{M}_\circ)^{\log} \xrightarrow{\sim} (\underline{M}^{\log})_\circ.$$

Furthermore, for any morphism $f : T \to S$ of topoi, we have natural isomorphisms of pre-log (resp. log) structures

$$f^*(\underline{N}_\circ) \xrightarrow{\sim} (f^*\underline{N})_\circ \qquad f_*(\underline{M}_\circ) \xrightarrow{\sim} (f_*\underline{M})_\circ$$

for every pre-log (resp. log) structure $\underline{N}$ on S and $\underline{M}$ on T (details left to the reader).

Example 6.1.10. (i) Let $T \to S$ be a morphism of topoi. Since the initial object of a category is the empty coproduct, it follows formally that the inverse image $f^*(\mathscr{O}_S^\times, j)$ of the trivial log structure on S, is the trivial log structure on T.

(ii) Let T be a topos, and $j_U : T/U \to T$ an open subtopos (see example 2.2.6(ii)). Consider the subsheaf of monoids $\underline{M} \subset \mathscr{O}_T$ such that :

$$\underline{M}(V) := \{s \in \mathscr{O}_T(V) \mid s_{|U\times V} \in \mathscr{O}_T^\times(U \times V)\} \qquad \text{for every object } V \text{ of } T.$$

Then it is easily seen that the natural map $\underline{M} \to \mathscr{O}_T$ is a log structure on T. This log structure is (naturally isomorphic to) the extension $j_{U*}\mathscr{O}_U^\times$ of the trivial log structure on T/U.

(iii) Let U be any object of the topos T, and $\underline{M}$ a log structure on T. Since $\mathscr{O}_{T/U} = \mathscr{O}_{T|U}$, it is easily seen that the natural morphism of pre-log structures

$$(\underline{M}^{\text{pre-log}})_{|U} \to (\underline{M}_{|U})^{\text{pre-log}}$$

is an isomorphism.

(iv) Let $\beta : \underline{M} \to \mathscr{O}_T$ be a pre-log structure on a topos T. Then $\beta^{-1}(0) \subset \underline{M}$ is an ideal, and β factors uniquely through the natural map $\underline{M} \to \underline{M}/\beta^{-1}(0)$, and a pre-log structure

$$(\underline{M}, \beta)_{\text{red}} := (\underline{M}/\beta^{-1}(0), \overline{\beta})$$

called the *reduced pre-log structure* associated to $\underline{M}$. As usual, we shall often write just $\underline{M}_{\text{red}}$ instead of $(\underline{M}, \beta)_{\text{red}}$. We say that β is *reduced* if the induced morphism of pre-log structures $\underline{M} \to \underline{M}_{\text{red}}$ is an isomorphism.

Suppose now that $\underline{M}$ is a log structure; then it is easily seen that the same holds for $\underline{M}_{\text{red}}$. More precisely, since the tensor product is right exact (see (2.3.19)), for any pre-log structure $\underline{M}$ the natural morphism of log structures

$$(\underline{M}_{\text{red}})^{\log} \to (\underline{M}^{\log})_{\text{red}}$$

is an isomorphism.

Lemma 6.1.11. *Let $\gamma : (\underline{M}, \alpha) \to (\underline{N}, \beta)$ be a morphism of pre-log structures on T. We have :*

(i) *If $\underline{M}$ is integral (resp. saturated), then the same holds for $\underline{M}^{\log}$.*
(ii) *The unit of adjunction $\underline{M} \to \underline{M}^{\log}$ is a flat morphism.*
(iii) *If γ is flat (resp. saturated) at a T-point ξ, the same holds for the induced morphism $\gamma^{\log} : \underline{M}^{\log} \to \underline{N}^{\log}$ of log structures.*
(iv) *Especially, if γ is flat (resp. saturated), the same holds for $\gamma^{\log}$.*

Proof. In view of lemma 2.3.46(ii) and proposition 2.3.26, both (i) and (ii) can be checked on stalks. Taking into account lemma 2.3.46(i), we are reduced to showing the following. Let P be a monoid, A a ring, $\beta : P \to (A, \cdot)$ a morphism of monoids; then the natural map $P \to P' := P \otimes_{\beta^{-1}A^\times} A^\times$ is flat, and if P is integral (resp. saturated), the same holds for P'.

The first assertion follows easily from example 3.1.23(vi), and the second follows from remark 3.2.5(i) (resp. from corollary 3.2.25(ix) and proposition 3.2.26).

(iii): The map $\gamma^{\log}$ can be factored as the composition of

$$\gamma' := \gamma \otimes_{\beta^{-1}\mathscr{O}_T^\times} \mathscr{O}_T^\times : \underline{N}^{\log} \to \underline{P} := \underline{M} \otimes_{\beta^{-1}\mathscr{O}_T^\times} \mathscr{O}_T^\times$$

and the natural unit of adjunction $\gamma'' : \underline{P} \to \underline{P}^{\log} = \underline{M}^{\log}$. If γ_ξ is flat, the same clearly holds for γ'_ξ, and (ii) says that γ'' is flat, hence $\gamma_\xi^{\log}$ is flat in this case. Lastly, suppose that γ_ξ is saturated, and we wish to show that $\gamma_\xi^{\log}$ is saturated. Set $P := \alpha_\xi^{-1}\mathscr{O}_{T,\xi}^\times$ and $Q := \beta_\xi^{-1}\mathscr{O}_{T,\xi}^\times$. Then the induced map $(P^{-1}\underline{M}_\xi)^\sharp \to (Q^{-1}\underline{N}_\xi)^\sharp$ is saturated (lemma 3.2.12(ii,iii)). But the latter is the same as the morphism $(\gamma_\xi^{\log})^\sharp$, and then also γ_ξ is saturated, again by lemma 3.2.12(iii). □

Lemma 6.1.12. *Let $f : T' \to T$ be a morphism of topoi, ξ a T'-point, and $\gamma : (\underline{M}, \alpha) \to (\underline{N}, \beta)$ a morphism of integral log structures on T. The following conditions are equivalent :*

(a) *γ is flat (resp. saturated) at the T-point $f(\xi)$.*
(b) *$f^*\gamma$ is flat (resp. saturated) at the T'-point ξ.*
(c) *$\gamma_\xi^\sharp$ is a flat (resp. saturated) morphism of monoids.*

Proof. The equivalence of (a) and (c) follows from corollary 3.1.49 (resp. lemma 3.2.12(ii)). By the same token, (b) holds if and only if $(f^*\gamma)_\xi^\sharp$ is flat (resp. saturated); in light of the isomorphism (6.1.8), the latter condition is equivalent to (c). □

6.1.13. For any locally ringed topos T, let us write the objects of $\mathbf{Mnd}/\Gamma(T, \mathscr{O}_T)$ in the form (M, φ), where M is any monoid, and $\varphi : M \to \Gamma(T, \mathscr{O}_T)$ a morphism of monoids. There is an obvious global sections functor :

$$\Gamma(T, -) : \textbf{pre-log}_T \to \mathbf{Mnd}/\Gamma(T, \mathscr{O}_T) \quad : \quad (\underline{N}, \alpha) \mapsto (\Gamma(T, \underline{N}), \Gamma(T, \alpha))$$

which admits a left adjoint :

$$\mathbf{Mnd}/\Gamma(T, \mathscr{O}_T) \to \textbf{pre-log}_T \quad : \quad (M, \varphi) \mapsto (M, \varphi)_T := (M_T, \varphi_T).$$

Indeed, M_T is the constant sheaf on (T, C_T) with value M, and φ_T is the composition of the map of constant sheaves $M_T \to \Gamma(T, \mathscr{O}_T)_T$ induced by φ, with the natural map $\Gamma(T, \mathscr{O}_T)_T \to \mathscr{O}_T$. Again, we shall often just write M_T to denote this pre-log structure.

After taking associated log structures, we deduce a left adjoint :

$$\mathbf{Mnd}/\Gamma(T, \mathscr{O}_T) \to \mathbf{log}_T \quad : \quad (M, \varphi) \mapsto M_T^{\log} := (M, \varphi)_T^{\log} \tag{6.1.14}$$

to the global sections functor. $M_T^{\log}$ is called the *constant log structure* associated to (M, φ).

Definition 6.1.15. Let T be a locally ringed topos, $(\underline{M}, \alpha)$ a log structure on T.

(i) A *chart for $\underline{M}$* is an object (P, β) of $\mathbf{Mnd}/\Gamma(T, \mathscr{O}_T)$, together with a map of pre-log structures $\omega_P : (P, \beta)_T \to \underline{M}$, inducing an isomorphism on the associated log structures. (Notation of (6.1.13).)
(ii) We say that a chart (P, β) is *finite* (resp. *integral*, resp. *fine*, resp. *saturated*) if P is a finitely generated (resp. integral, resp. fine, resp. integral and saturated) monoid.
(iii) Let $\varphi : \underline{M} \to \underline{N}$ be a morphism of log structures on T. A *chart for φ* is the datum of charts :

$$\omega_P : (P, \beta)_T \to \underline{M} \quad \text{and} \quad \omega_Q : (Q, \gamma)_T \to \underline{N}$$

for $\underline{M}$, respectively $\underline{N}$, and a morphism of monoids $\vartheta : Q \to P$, fitting into a commutative diagram :

$$\begin{tikzcd} Q_T \arrow[r, "\vartheta_T^{\log}"] \arrow[d, "\omega_Q"'] & P_T^{\log} \arrow[d, "\omega_P"] \\ \underline{N} \arrow[r, "\varphi"] & \underline{M}. \end{tikzcd}$$

We say that such a chart is *finite* (resp. *integral*, resp. *fine*, resp. *saturated*) if the monoids P and Q are finitely generated (resp. integral, resp. fine, resp. integral and saturated). We say that the chart is *flat* (resp. *saturated*), if ϑ is a flat morphism of monoids (resp. a saturated morphism of integral monoids).

(iv) We say that $\underline{M}$ is *quasi-coherent* (resp. *coherent*) if there exists a covering family $(U_\lambda \to 1_T \mid \lambda \in \Lambda)$ of the final object 1_T in (T, C_T), and for every $\lambda \in \Lambda$, a chart (resp. a finite chart) $(P_\lambda, \beta_\lambda)$ for $\underline{M}_{|U_\lambda}$.

(v) We say that $(\underline{M}, \alpha)$ is *quasi-fine* (resp. *fine*) if it is integral and quasi-coherent (resp. and coherent).

(vi) Let ξ be any T-point. We say that a chart (P, β) is *local* (resp. *sharp*) *at the point* ξ, if the morphism $\beta_\xi : P \to \mathscr{O}_{T,\xi}$ is local (resp. if P is sharp and β_ξ is local).

Lemma 6.1.16. *Let $f : T \to S$ be a morphism of locally ringed topoi, $\underline{Q}$ a log structure on S, and ξ any point of S. The following holds :*

(i) *If $\underline{Q}$ is quasi-coherent (resp. coherent, resp. integral, resp. saturated, resp. quasi-fine, resp. fine), then the same holds for $f^*\underline{Q}$.*

(ii) *Suppose that $\underline{Q}$ is an integral log structure. Then $\underline{Q}^\sharp$ is an integral S-monoid, and $\underline{Q}$ is saturated if and only if $\underline{Q}^\sharp$ is a saturated S-monoid.*

(iii) *Suppose that $\underline{Q}$ is quasi-coherent. Then $\underline{Q}$ is integral (resp. integral and saturated, resp. fine, resp. fine and saturated) if and only if there exists a covering family $(U_\lambda \to 1_S \mid \lambda \in \Lambda)$ of the final object of S, and for every $\lambda \in \Lambda$, an integral (resp. integral and saturated, resp. fine, resp. fine and saturated) chart $(P_\lambda)_{U_\lambda} \to \underline{Q}_{|U_\lambda}$.*

(iv) *If $\underline{P}$ is any coherent log structure on S, and $\omega : \underline{P}_\xi \to \underline{Q}_\xi$ is a map of monoids, then :*

(a) *There exists a neighborhood U of ξ and a morphism $\vartheta : \underline{P}_{|U} \xrightarrow{\sim} \underline{Q}_{|U}$ of log structures, such that $\vartheta_\xi = \omega$.*

(b) *Moreover, for any two morphisms $\vartheta, \vartheta' : \underline{P}_{|U} \xrightarrow{\sim} \underline{Q}_{|U}$ with the property of* (a)*, we may find a smaller neighborhood $V \to U$ of ξ such that $\vartheta_{|V} = \vartheta'_{|V}$.*

(c) *Especially, if $\underline{Q}$ is also coherent, and ω is an isomorphism, we may find ϑ and a small enough U as in* (a)*, such that ϑ is an isomorphism.*

(v) *If M is a finitely generated monoid, and $\omega : M \to \mathscr{O}_{S,\xi}$ a morphism of monoids, then we may find a neighborhood U of ξ and a morphism $\vartheta : M_U \to \mathscr{O}_U$ of S/U-monoids, such that $\vartheta_\xi = \omega$.*

Proof. (i): If $\underline{Q}$ is the constant log structure associated to a map of monoids $\alpha : Q \to \Gamma(S, \mathscr{O}_S)$, then $f^*\underline{Q}$ is the constant log structure associated to $\Gamma(S, f^\natural) \circ \alpha$ (where $f^\natural : \mathscr{O}_S \to f_*\mathscr{O}_T$ is the natural map). The assertions concerning quasi-coherent or coherent log structures are a straightforward consequence. Next, suppose that $\underline{Q}$ is integral (resp. saturated); we wish to show that $f^*\underline{Q}$ is integral (resp. saturated). To this aim, let $\underline{M} := f^*(\underline{Q}^{\text{pre-log}})$; by lemma 2.3.45(i), $\underline{M}$ is an integral (resp. saturated) T-monoid; then the assertion follows from lemma 6.1.11(i).

(ii): By lemma 2.3.46(ii) the assertion can be checked on stalks. Hence, suppose that $\underline{Q}$ is integral; then $\underline{Q}_\xi$ is integral by *loc.cit.*, consequently, the same holds for $\underline{Q}_\xi/\underline{Q}_\xi^\times$ (lemma 2.3.38). The second assertion follows from lemma 3.2.9(ii).

(iii): Suppose first that $\underline{Q}$ is quasi-coherent and integral. Hence, there is a covering family $(U_\lambda \to 1_S \mid \lambda \in \Lambda)$, and for every $\lambda \in \Lambda$ a monoid M_λ, a pre-log structure $\alpha_\lambda : (M_\lambda)_{U_\lambda} \to \mathscr{O}_{U_\lambda}$, and an isomorphism $((M_\lambda)_{U_\lambda}, \alpha_\lambda)^{\log} \xrightarrow{\sim} \underline{Q}_{|U_\lambda}$; whence a cocartesian diagram of S-monoids, as in (6.1.7) :

$$
\begin{tikzcd}
\underline{N} := \alpha_\lambda^{-1}(\mathscr{O}^\times_{U_\lambda}) \arrow[r] \arrow[d] & (M_\lambda)_{U_\lambda} \arrow[d] \\
\mathscr{O}^\times_{U_\lambda} \arrow[r] & \underline{Q}_{|U_\lambda}.
\end{tikzcd}
\tag{6.1.17}
$$

The induced diagram (6.1.17)$^{\mathrm{int}}$ of integral S-monoids is still cocartesian. Moreover, since $\underline{Q}_{|U_\lambda}$ is integral, α_λ factors through a unique map $\beta_\lambda : ((M_\lambda)_{U_\lambda})^{\mathrm{int}} \to \underline{Q}_{|U_\lambda} \to \mathscr{O}_{U_\lambda}$, and the morphism in S underlying the induced morphism of S-monoids $\underline{N}^{\mathrm{int}} \to \underline{N}' := \beta_\lambda^{-1}(\mathscr{O}^\times_{U_\lambda})$ is an epimorphism (this can be checked easily on the stalks). Furthermore, $((M_\lambda)_{U_\lambda})^{\mathrm{int}} \simeq (M_\lambda^{\mathrm{int}})_{U_\lambda}$ (see (2.3.49)). It follows that the natural map

$$\underline{Q}_{|U_\lambda} \to \mathscr{O}^\times_{U_\lambda} \amalg_{\underline{N}'} (M_\lambda^{\mathrm{int}})_{U_\lambda} \simeq (M_\lambda^{\mathrm{int}})^{\log}_{U_\lambda}$$

is an isomorphism, so the claim holds with $P_\lambda := M_\lambda^{\mathrm{int}}$. If Q is fine, we can find M_λ as above which is also finitely generated, in which case the resulting P_λ shall be fine.

Suppose additionally, that $\underline{Q}$ is saturated. By the previous case, we may then find a covering family $(U_\lambda \to e_S \mid \lambda \in \Lambda)$, and for every $\lambda \in \Lambda$ an integral monoid M_λ, a pre-log structure $\alpha_\lambda : (M_\lambda)_{U_\lambda} \to \mathscr{O}_{U_\lambda}$, and an isomorphism $((M_\lambda)_{U_\lambda}, \alpha_\lambda)^{\log} \xrightarrow{\sim} \underline{Q}_{|U_\lambda}$; whence a cocartesian diagram (6.1.17) consisting of integral S-monoids. The induced diagram (6.1.17)$^{\mathrm{sat}}$ is still cocartesian; one may then argue as in the foregoing, to obtain a natural isomorphism : $\underline{Q}_{|U_\lambda} \xrightarrow{\sim} (M_\lambda^{\mathrm{sat}})^{\log}_{U_\lambda}$. Furthermore, if M_λ is finitely generated, the same holds for M_λ^{sat} (corollary 3.4.1(ii)), hence the chart thus obtained shall be fine and saturated, in this case.

Conversely, if a family $(P_\lambda \mid \lambda \in \Lambda)$ of integral (resp. saturated) monoids can be found fulfilling the condition of (iii), then $(P_\lambda)_{U_\lambda}$ is an integral (resp. saturated) pre-log structure on T/U_λ (example 2.3.47(ii)), hence the same holds for its associated log structure $\underline{Q}_{|U_\lambda}$ (lemma 6.1.11(i)), and therefore also for $\underline{Q}$ (lemmata 2.3.46(ii), 2.2.16 and example 6.1.10(iii)). Moreover, if each P_λ is fine, then $\underline{Q}$ is fine as well.

(iv.a): We may assume that $\underline{P}$ admits a finite chart $\alpha : M_S \to \underline{P}$, for some finitely generated monoid M, denote by $\beta : \underline{Q} \to \mathscr{O}_S$ the structure map of $\underline{Q}$, and set $\omega' := \omega \circ \alpha_\xi : M \to \underline{Q}_\xi$. According to lemma 3.1.7(ii), the morphism ω' factors through a map $\omega'' : M \to \Gamma(U', \underline{Q})$, for some neighborhood U' of ξ. By adjunction, ω'' determines a morphism of S/U'-monoids $\psi : M_{U'} \to \underline{Q}_{|U'}$ whence a morphism of pre-log structures

$$(M_{U'}, \beta_{|U'} \circ \psi) \to (\underline{Q}_{|U'}, \beta_{|U'}). \tag{6.1.18}$$

Let us make the following general observation :

Claim 6.1.19. Let N be a finite monoid, F any S-monoid, $f, g : N_S \to F$ two morphisms of S-monoids, such that $f_\xi = g_\xi$. Then there exists a neighborhood U of ξ in S such that $f_{|U} = g_{|U}$.

Proof of the claim. By adjunction, the morphisms f and g correspond to unique maps of monoids $\Gamma(f), \Gamma(g) : N \to \Gamma(S, F)$; since N is finite and $f_\xi = g_\xi$, we may find a neighborhood U of ξ such that the maps $N \to F(U)$ induced by $\Gamma(f)$ and $\Gamma(g)$ coincide. Again by adjunction, we deduce a unique morphism of S/U-monoids $N_U \to F_{|U}$, which by construction is just the restriction of both f and g. ◇

Let $\gamma : \underline{P} \to \mathscr{O}_S$ be the structure map of $\underline{P}$; we apply claim 6.1.19 with S replaced by S/U', to deduce that there exists a small enough neighborhood $U \to U'$ of ξ such that the restriction

$(\gamma\circ\alpha)_{|U} : M_{|U} \to \mathscr{O}_{S|U}$ agrees with $\beta_{|U}\circ\psi_{|U}$. Then it is clear that the morphism of log structure associated to (6.1.18)$_{|U}$ yields the sought extension ϑ of ω.

(iv.b): By assumption we have the identity : $\vartheta_\xi = \vartheta'_\xi$; however, any morphism of log structures $\underline{P}_{|U} \to \underline{Q}_{|U}$ is already determined by its restriction to the image of any finite local chart $M_U \to \underline{P}_{|U}$. Hence the assertion follows from claim 6.1.19.

(iv.c): We apply (iv.a) to ω^{-1} to deduce the existence of a morphism $\sigma : \underline{Q}_{|U} \to \underline{P}_{|U}$ such that $\sigma_\xi = \omega^{-1}$ on some neighborhood U of ξ. Hence, $(\vartheta\circ\sigma)_\xi = \mathbf{1}_{\underline{Q}_\xi}$ and $(\sigma\circ\vartheta)_\xi = \mathbf{1}_{\underline{P}_\xi}$. By (iv.b), these identities persist on some smaller neighborhood.

(v): The proof is similar to that of (iv.a), though simpler : we leave it as an exercise for the reader. □

Definition 6.1.20. (i) A *morphism* $(T,\underline{M}) \to (S,\underline{N})$ *of topoi with pre-log* (resp. *log*) *structures*, is a pair $f := (f, \log f)$ consisting of a morphism of locally ringed topoi $f : T \to S$, and a morphism
$$\log f : f^*\underline{N} \to \underline{M}$$
of pre-log structures (resp. log structures) on T. We say that f is *log flat* (resp. *saturated*) if $\log f$ is a flat (resp. saturated) morphism of pre-log structures.

(ii) Let $(f, \log f)$ as in (i) be a morphism of log topoi, ξ a T-point; we say that f is *strict at the point* ξ, if $\log f_\xi$ is an isomorphism. We say that f is *strict*, if it is strict at every T-point.

(iii) A *chart* for φ is the datum of charts
$$\omega_P : (P,\beta)_T \to \underline{M} \quad \text{and} \quad \omega_Q : (Q,\gamma)_S \to \underline{N}$$
for $\underline{M}$, respectively $\underline{N}$, and a morphism of monoids $\vartheta : Q \to P$, such that $(f^*\omega_Q, \omega_P, \vartheta)$ is a chart for the morphism $\log f$. We say that such a chart $(\omega_Q, \omega_P, \vartheta)$ is *finite* (resp. *fine*, resp. *flat*, resp. *saturated*) if the same holds for the corresponding chart $(f^*\omega_Q, \omega_P, \vartheta)$ of $\log f$.

Remark 6.1.21. (i) Let $f : (T,\underline{M}) \to (S,\underline{N})$ be a morphism of log topoi, $g : T' \to T$ a morphism of topoi, and $f' : (T', g^*\underline{M}) \to (S,\underline{N})$ the composition of f and the natural morphism of log topoi $(T', g^*\underline{M}) \to (T,\underline{M})$; let also ξ be a T'-point. Then f' is strict at the point ξ if and only if f is strict at the point $g(\xi)$. Indeed, f' is strict at ξ if and only if $(\log f')^\sharp_\xi$ is an isomorphism (lemma 6.1.4), if and only if $(\log f)^\sharp_{g(\xi)}$ is an isomorphism (by (6.1.8)), if and only if $\log f_{g(\xi)}$ is an isomorphism (again by lemma 6.1.4).

(ii) For any log topos $(T,\underline{M})$, let us set $(T,\underline{M})_\circ := (T,\underline{M}_\circ)$. Then $(T,\underline{M})_\circ$ is a log topos (see (6.1.9)), and clearly, every morphism $f : (T,\underline{M}) \to (S,\underline{N})$ of log topoi extends naturally to a morphism $f_\circ : (T,\underline{M})_\circ \to (S,\underline{N})_\circ$ of log topoi.

6.1.22. The category of log topoi admits arbitrary 2-limits. Indeed, if $\mathscr{T} := ((T_\lambda, \underline{M}_\lambda) \mid \lambda \in \Lambda)$ is any pseudo-functor (from a small category Λ, to the category of log topoi), the 2-limit of $\mathscr{T}$ is the pair $(T,\underline{M})$, where T is the 2-limit of the system of ringed topoi $(T_\lambda \mid \lambda\in\Lambda)$, and $\underline{M}$ is the log structure on T obtained as follows. For every $\lambda\in\Lambda$, let $p_\lambda : T \to T_\lambda$ be the natural projection; take the colimit $\underline{M}'$ of the induced system of pre-log structures $(p^*_\lambda \underline{M}_\lambda \mid \lambda\in\Lambda)$, and then let $\underline{M}$ be the log structure associated to $\underline{M}'$.

6.1.23. Consider a 2-cartesian diagram of log topoi :
$$\begin{tikzcd} (T',\underline{M}') \arrow[r, "g"] \arrow[d, "f'"'] & (T,\underline{M}) \arrow[d, "f"] \\ (S',\underline{N}') \arrow[r, "h"] & (S,\underline{N}). \end{tikzcd}$$
The following result yields a relative variant of the isomorphism (6.1.8) :

Lemma 6.1.24. *In the situation of* (6.1.23), *the morphism*

$$g^*\mathrm{Coker}(\log f) \to \mathrm{Coker}(\log f')$$

induced by $\log g$ *is an isomorphism of* T'*-monoids.*

Proof. Indeed, denote by $\beta : \underline{M} \to \mathscr{O}_T$ and $\gamma : \underline{N}' \to \mathscr{O}_{S'}$ the log structures of T and S'. Fix any T'-point ξ, let $\xi := g(\xi')$, and set

$$P := \underline{M}_\xi \otimes_{\underline{N}_{f(\xi)}} \underline{N}'_{f'(\xi')} \qquad \text{and} \qquad \rho := \beta_\xi \otimes \gamma_{f'(\xi')} : P \to \mathscr{O}_{T'}.$$

Then $\underline{M}'_{\xi'} = P \otimes_{\rho^{-1}\mathscr{O}_{T',\xi'}} \mathscr{O}^\times_{T',\xi'}$, and it is easily seen that $\rho^{-1}\mathscr{O}_{T',\xi'} = \mathscr{O}^\times_{T,\xi} \otimes_{\mathscr{O}^\times_{S,f(\xi)}} \mathscr{O}^\times_{S',f'(\xi')}$. Therefore $(\underline{M}'_{\xi'})^\sharp = P/\rho^{-1}\mathscr{O}_{T',\xi'} = \underline{M}^\sharp_\xi \otimes_{\underline{N}^\sharp_{f(\xi)}} \underline{N}'^\sharp_{f'(\xi')}$, and

$$\mathrm{Coker}(\log f'_{\xi'}) = \mathrm{Coker}(\underline{N}'^\sharp_{f'(\xi')} \to \underline{M}^\sharp_\xi \otimes_{\underline{N}^\sharp_{f(\xi)}} \underline{N}'^\sharp_{f'(\xi')}) = \mathrm{Coker}(\underline{N}^\sharp_{f(\xi)} \to \underline{M}^\sharp_\xi)$$

whence the contention. $\square$

6.1.25. We consider now a special situation, which will be encountered in proposition 6.1.28. Namely, let Q be a monoid, and $H \subset Q^\times$ a subgroup. Let also G be an abelian group, $\rho : G \to Q^{\mathrm{gp}}$ a group homomorphism, and set :

$$H_\rho := G \times_{Q^{\mathrm{gp}}} H \qquad Q_\rho := G \times_{Q^{\mathrm{gp}}} Q.$$

The natural inclusion $H \to Q$ and the projection $Q_\rho \to Q$ determine a unique morphism :

$$Q_\rho \otimes_{H_\rho} H \to Q. \tag{6.1.26}$$

Lemma 6.1.27. *In the situation of* (6.1.25), *suppose furthermore that the composition :*

$$G \xrightarrow{\rho} Q^{\mathrm{gp}} \to (Q/H)^{\mathrm{gp}}$$

is surjective. Then (6.1.26) *is an isomorphism.*

Proof. Set $G' := G \oplus H$, and let $\rho' : G' \to Q^{\mathrm{gp}}$ be the unique group homomorphism that extends ρ and the natural map $H \to Q \to Q^{\mathrm{gp}}$. Under the standing assumptions, ρ' is clearly surjective. Define $Q_{\rho'}$ and $H_{\rho'}$ as in (6.1.25); there is a natural isomorphism of monoids : $Q_{\rho'} \xrightarrow{\sim} Q_\rho \oplus H$, inducing an isomorphism $H_{\rho'} \xrightarrow{\sim} H_\rho \oplus H$, and defined as follows. To every $g \in G$, $h \in H$, $q \in Q$ such that $[(g,h),q] \in Q_{\rho'}$, we assign the element $[(g,h^{-1}q),a] \in Q_\rho \oplus H$ (details left to the reader). Under this isomorphism, the projection $H_{\rho'} \to H$ is identified with the map $H_\rho \oplus H \to H$ given by the rule : $(h_1,h_2) \mapsto \pi_H(h_1)\cdot h_2$, where $\pi_H : H_\rho \to H$ is the projection. It then follows that the natural map :

$$Q_\rho \otimes_{H_\rho} H \to Q_{\rho'} \otimes_{H_{\rho'}} H$$

is an isomorphism. Thus, we may replace G and ρ by G' and ρ', which allows to assume from start that ρ is surjective. However, we have natural isomorphisms :

$$\mathrm{Ker}(Q_\rho \xrightarrow{\pi_Q} Q) \simeq \mathrm{Ker}\,\rho \simeq \mathrm{Ker}(H_\rho \xrightarrow{\pi_H} H).$$

Moreover, the set underlying Q (resp. H) is the set-theoretic quotient of the set Q_ρ (resp. H_ρ) by the translation action of $\mathrm{Ker}\,\rho$; hence the natural maps $Q_\rho/\mathrm{Ker}\,\pi_Q \to Q$ and $H_\rho/\mathrm{Ker}\,\pi_H \to H$ are isomorphisms (lemma 2.3.31(ii)). We can then compute :

$$Q_\rho \otimes_{H_\rho} H \simeq (Q_\rho/\mathrm{Ker}\,\pi_Q) \otimes_{H_\rho/\mathrm{Ker}\,\pi_H} H \simeq Q \otimes_H H \simeq Q$$

as stated. $\square$

Proposition 6.1.28. *Let T be a locally ringed topos, ξ any T-point, and $\underline{M}$ a coherent (resp. fine) log structure on T. Suppose that G is a finitely generated abelian group with a group homomorphism $G \to \underline{M}^{\mathrm{gp}}_\xi$ such that the induced map $G \to (\underline{M}^\sharp)^{\mathrm{gp}}_\xi$ is surjective. Set*

$$P := G \times_{\underline{M}^{\mathrm{gp}}_\xi} \underline{M}_\xi.$$

Then the induced morphism $P \to \underline{M}_\xi$ extends to a finite (resp. fine) chart $P_U \to \underline{M}_{|U}$ on some neighborhood U of ξ.

Proof. We begin with the following :

Claim 6.1.29. Let Y be any locally ringed topos, ξ a Y-point, and $\alpha : Q_Y \to \mathscr{O}_Y$ the constant pre-log structure associated to a map of monoids $\vartheta : Q \to \Gamma(Y, \mathscr{O}_Y)$, where Q is finitely generated. Set $S := \alpha_\xi^{-1}\mathscr{O}^\times_{Y,\xi} \subset Q_{Y,\xi} = Q$. Then :

(i) S and $S^{-1}Q$ are finitely generated monoids.
(ii) There exists a neighborhood U of ξ such that $\alpha_{|U}$ factors as the composition of the natural map of sheaves of monoids $j_U : Q_U \to (S^{-1}Q)_U$, and a (necessarily unique) pre-log structure $\alpha_S : (S^{-1}Q)_U \to \mathscr{O}_U$.
(iii) The induced map of log structures $j_U^{\log} : Q_U^{\log} \to (S^{-1}Q)_U^{\log}$ is an isomorphism.
(iv) $\alpha_{S,\xi}^{-1}(\mathscr{O}^\times_{U,\xi}) = (S^{-1}Q)^\times$ is a finitely generated group.

Proof of the claim. (i) follows from lemma 3.1.20(iv).

(ii): Since $\mathscr{O}^\times_{Y,\xi}$ is the filtered colimit of the groups $\Gamma(U, \mathscr{O}^\times_U)$, where U ranges over the neighborhoods of ξ, lemma 3.1.7(ii) and (i) imply that the induced map $S \to \mathscr{O}^\times_{Y,\xi}$ factors through $\Gamma(U, \mathscr{O}^\times_Y)$ for some neighborhood U of ξ. Then the composition of ϑ and the natural map $\Gamma(Y, \mathscr{O}_Y) \to \Gamma(U, \mathscr{O}_U)$ extends to a unique map $S^{-1}Q \to \Gamma(U, \mathscr{O}_U)$, whence the sought pre-log structure α_S on U.

(iii): Let $\underline{N}$ be any log structure on U; it is clear that every morphism of pre-log structures $Q_U \to \underline{N}$ factors uniquely through $(S^{-1}Q)_U$, whence the contention.

(iv): Indeed, by construction we have : $\alpha^{-1}_{S,\xi}(\mathscr{O}^\times_{U,\xi}) = S^{\mathrm{gp}}$. ◊

Let Y be a neighborhood of ξ such that $\underline{M}_{|Y}$ admits a finite local chart $\alpha : Q_Y \to \mathscr{O}_Y$; we lift ξ to some Y-point, ξ_Y, and choose a neighborhood $U \in \mathrm{Ob}(T/Y)$ of ξ_Y, as provided by claim 6.1.29. We may then replace T by U, ξ by ξ_Y and α by the chart α_S of claim 6.1.29(ii), which allows to assume from start that $S := \alpha_\xi^{-1}(\mathscr{O}^\times_{T,\xi})$ is a finitely generated group. Moreover, let $H := \mathrm{Ker}(S \to \mathscr{O}^\times_{T,\xi})$; clearly $\alpha_\xi : Q \to \mathscr{O}_{T,\xi}$ factors through the quotient $Q' := Q/H$, hence we may find a neighborhood U of ξ such that $\alpha_{|U}$ factors through a (necessarily unique) map of pre-log structures $\alpha_H : Q'_U \to \mathscr{O}_U$. Furthermore, if $\underline{N}$ is any log structure on U, every map of pre-log structures $Q_U \to \underline{N}$ factors through Q'_U, so that α_H is a chart for $\underline{M}_{|U}$. We may therefore replace T by U and α by α_H, which allows to assume additionally, that α_ξ is injective on the subgroup S. Now, for any finitely generated subgroup $H \subset \mathscr{O}^\times_{T,\xi}$ with $S \subset H$, we set $\underline{M}_{\xi,H} := H \amalg_S Q$; clearly, the monoids $\underline{M}_{\xi,H}$ are finitely generated, and moreover :

$$\underline{M}_\xi = \operatorname*{colim}_{S \subset H \subset \mathscr{O}^\times_{T,\xi}} \underline{M}_{\xi,H}.$$

Furthermore, we deduce a natural sequence of maps of monoids :

$$\{1\} \to \underline{M}_{\xi,H} \xrightarrow{\varphi_H} \underline{M}_\xi \xrightarrow{\psi_H} \mathscr{O}^\times_{T,\xi}/H \to \{1\}. \tag{6.1.30}$$

Claim 6.1.31. For every subgroup H as above, the sequence (6.1.30) is exact, *i.e.* $\underline{M}_{\xi,H}$ is the kernel of ψ_H, and $\mathscr{O}^\times_{T,\xi}/H$ is the cokernel of φ_H.

Proof of the claim. By lemma 2.3.29(iii), the assertion concerning $\mathrm{Ker}\,\psi_H$ can be verified on the underlying map of sets; however, lemma 2.3.31(ii) says that the set $\underline{M}_\xi$ is the set-theoretic quotient $(Q \times \mathscr{O}_{T,\xi}^{\times})/S$, for the natural translation action of S, and a similar description holds for $\underline{M}_{\xi,H}$, therefore $\mathrm{Ker}\,\varphi_H$ is the set-theoretic quotient $(H \times Q)/S$, as required.

The assertion concerning $\mathrm{Coker}\,\varphi_H$ holds by general categorical nonsense. $\diamondsuit$

Let $\varepsilon : \underline{M}_\xi \to \underline{M}_\xi^{\mathrm{gp}}$ be the natural map; claim 6.1.31 implies that the sequence of abelian groups (6.1.30)$^{\mathrm{gp}}$ is right exact, and then a little diagram chase shows that :

(6.1.32) $$\varepsilon^{-1}(\mathrm{Im}\,\varphi_H^{\mathrm{gp}}) = \underline{M}_{\xi,H} \qquad \text{whenever } S \subset H \subset \mathscr{O}_{T,\xi}^{\times}.$$

Since G is finitely generated, we may find H as above, large enough, so that $\underline{M}_{\xi,H}^{\mathrm{gp}}$ contains the image of G. In view of (6.1.32), we deduce that the natural map

$$G \times_{\underline{M}_{\xi,H}^{\mathrm{gp}}} \underline{M}_{\xi,H} \to P$$

is an isomorphism, so P is finitely generated, by corollary 3.4.2; moreover P is integral whenever $\underline{M}_\xi$ is. Then, lemma 6.1.16(iv.a) implies that the natural map $P \to \underline{M}_\xi$ extends to a morphism of log structures $\vartheta : P_U^{\log} \to \underline{M}_{|U}$ on some neighborhood U of ξ. It remains to show that ϑ restricts to a chart for $\underline{M}_{|V}$, on some smaller neighborhood V of ξ. To this aim, it suffices to show that the map of stalks ϑ_ξ is an isomorphism (lemma 6.1.16(iv.c)). The latter assertion follows from lemma 6.1.27. $\square$

Proposition 6.1.28 is the basis of several frequently used tricks that allow to construct "good" charts for a given coherent log structure (and for a morphism of such structures), or to "improve" given charts. We conclude this section with a selection of these tricks.

Corollary 6.1.33. *Let T be a topos, ξ a T-point, $\underline{M}$ a fine log structure on T. Then there exists a neighborhood U of ξ in T, and a chart $P_U \to \underline{M}_{|U}$ such that :*

(a) *P^{gp} is a free abelian group of finite rank.*
(b) *The induced morphism of monoids $P \to \mathscr{O}_{T,\xi}$ is local.*

Proof. Choose a group homomorphism $G := \mathbb{Z}^{\oplus n} \to \underline{M}_\xi^{\mathrm{gp}}$ (for some integer $n \geq 0$), such that the induced map $G \to (\underline{M}^\sharp)_\xi^{\mathrm{gp}}$ is surjective, and set $P := G \times_{\underline{M}_\xi^{\mathrm{gp}}} \underline{M}_\xi$. By proposition 6.1.28, the induced map $P \to \underline{M}_\xi$ extends to a chart $P_U \to \underline{M}_{|V}$, for some neighborhood U of ξ. According to example 2.3.36(v), P^{gp} is a subgroup of G, whence (a). Next, claim 6.1.29 implies that, after replacing P by some localization (which does not alter P^{gp}), and U by a smaller neighborhood of ξ, we may achieve (b) as well. $\square$

Corollary 6.1.34. *Let T be a topos, ξ a T-point, $\underline{M}$ a coherent log structure on T, and suppose that $\underline{M}_\xi$ is integral and saturated. Then we have :*

(i) *There exists a neighborhood U of ξ in T, and a fine and saturated chart $P_U \to \underline{M}_{|U}$ which is sharp at the point ξ.*
(ii) *Especially, there exists a neighborhood U of ξ in T, such that $\underline{M}_{|U}$ is a fine and saturated log structure.*

Proof. (i): By lemma 3.2.10, we may find a decomposition $\underline{M}_\xi = P \times \underline{M}_\xi^{\times}$, for a sharp submonoid $P \subset \underline{M}_\xi$. Set $G := P^{\mathrm{gp}}$; we deduce an isomorphism $G \xrightarrow{\sim} \underline{M}_\xi^{\mathrm{gp}}/\underline{M}_\xi^{\times}$, and clearly $P = G \times_{\underline{M}_\xi^{\mathrm{gp}}} \underline{M}_\xi$. By proposition 6.1.28, it follows that the induced map $P \to \underline{M}_\xi$ extends to a chart $\beta : P_U \to \underline{M}_{|U}$ on a neighborhood U of ξ. By construction, β is sharp at the T-point ξ; moreover, since $\underline{M}_\xi$ is saturated, it is easily seen that the same holds for P. Finally, P is finitely generated, by corollary 3.4.2.

(ii): This follows immediately from (i) and lemma 6.1.16(iii). $\square$

Theorem 6.1.35. *Let T be a locally ringed topos, ξ a T-point, $f : \underline{M} \to \underline{N}$ a morphism of coherent (resp. fine) log structures on T. Then :*

(i) *There exists a neighborhood U of ξ, such that $f_{|U}$ admits a finite (resp. fine) chart.*

(ii) *More precisely, given a finite (resp. fine) chart $\omega_P : P_T \to \underline{M}$, we may find a neighborhood U of ξ, a finite (resp. fine) monoid Q, and a chart of $f_{|U}$ of the form*

$$(\omega_{P|U}, \omega_Q : Q_U \to \underline{N}_{|U}, \vartheta : P \to Q).$$

(iii) *Moreover, if f is a flat (resp. saturated) morphism of fine log structures and $(\omega_P, \omega_Q, \vartheta)$ is a fine chart for f, then we may find a neighborhood U of ξ, a localization map $j : Q \to Q'$, and a flat (resp. saturated) and fine chart for $f_{|U}$ of the form $(\omega_{P|U}, \omega_{Q'}, j \circ \vartheta)$, such that*

(a) $\omega_{Q|U} = \omega_{Q'} \circ j_U$.

(b) *The chart $\omega_{Q'}$ is local at the point ξ.*

Proof. Up to replacing T by T/U'_i for a covering $(U'_i \to 1_T \mid i \in I)$ of the final object, we may assume that we have finite (resp. fine) charts $\omega_P : P_T \to \underline{M}$ and $Q'_T \to \underline{N}$ (lemma 6.1.16(iii)), whence a morphism of pre-log structures :

$$\omega : P_T \xrightarrow{\omega_P} \underline{M} \xrightarrow{f} \underline{N}.$$

Let ξ be any T-point; there follow maps of monoids $\varphi : P \to \underline{M}_\xi \to \underline{N}_\xi$ and $\psi : Q' \to \underline{N}_\xi$. Set $G := (P \oplus Q')^{\mathrm{gp}}$, and apply proposition 6.1.28 to the group homomorphism $G \to \underline{N}^{\mathrm{gp}}_\xi$ induced by φ and ψ; for $Q := G \times_{\underline{N}^{\mathrm{gp}}_\xi} \underline{N}_\xi$, we obtain a finite (resp. fine) local chart $Q_U \to \underline{N}_{|U}$ on some neighborhood U of ξ. Then, φ and the natural map $P \to G$ determine a unique map $P \to Q$, whence a morphism $\omega' : P_U \to Q_U \to \underline{N}_{|U}$ of pre-log structures; by construction, $\omega'_\xi : P = P_{U,\xi} \to \underline{N}_\xi$ is none else than φ. By lemma 6.1.16(iv.b), we may then find a smaller neighborhood $V \to U$ of ξ such that $\omega'_{|V} = \omega_{|V}$. This proves (i) and (ii).

Next, we suppose that f is flat (resp. saturated) and both $\underline{M}$, $\underline{N}$ are fine, and we wish to show (iii). In view of claim 6.1.29, we may find – after replacing T by a neighborhood of ξ – a fine chart for f of the form $(\omega_{P'}, \omega_{Q'}, \vartheta')$, such that :

- P' and Q' are localizations of P and Q, and ϑ' is induced by ϑ;
- $\omega_P = \omega_{P'} \circ (j_P)_T$ and $\omega_Q = \omega_{Q'} \circ (j_Q)_T$, where $j_P : P \to P'$ and $j_Q : Q \to Q'$ are the localization maps;
- the induced maps $Q'^\sharp \to \underline{M}^\sharp_\xi$ and $P'^\sharp \to \underline{N}^\sharp_\xi$ are isomorphisms.

Now, by proposition 2.3.26 (resp. corollary 3.2.29), the map f_ξ is flat (resp. saturated), hence the same holds for the induced map $\underline{M}^\sharp_\xi \to \underline{N}^\sharp_\xi$, by corollary 3.1.49(i) (resp. by lemma 3.2.12(iii)). Then the map $P'^\sharp \to Q'^\sharp$ induced by ϑ' is flat (resp. saturated) as well, so the same holds for ϑ', by corollary 3.1.49(ii) (resp. again by lemma 3.2.12(iii)). Finally, $j_P \circ \vartheta' : P \to Q'$ is flat (resp. saturated), by example 3.1.23(iii) (resp. by lemma 3.2.12(i)), and $(\omega_P, \omega_{Q'}, j_P \circ \vartheta')$ is a chart for f with the sought properties. □

Corollary 6.1.36. *Let T be a topos, ξ a T-point, and $\varphi : \underline{M} \to \underline{N}$ a saturated morphism of fine and saturated log structures on T. We have :*

(i) *There exists a neighborhood U of ξ, and a fine and saturate chart $(\omega_P, \omega_Q, \vartheta : P \to Q)$ of $\varphi_{|U}$, such that ω_P and ω_Q are sharp at the point ξ.*

(ii) *More precisely, suppose that $(\omega_P, \omega_Q, \vartheta : P \to Q)$ is a fine and saturated chart for φ, such that $\underline{M}$ is sharp at the point ξ, and ω_Q is local at ξ. Then there exists a section $\sigma : Q^\sharp \to Q$ of the projection $Q \to Q^\sharp$, such that $(\omega_P, \omega_Q \circ \sigma_T, \vartheta^\sharp)$ is a chart for φ.*

Proof. After replacing T by some neighborhood of ξ, we may assume that $\underline{M}$ admits a chart $\omega_P : P_T \to \underline{M}$ which is fine, saturated, and sharp at the point ξ (corollary 6.1.34(i)). Then, by theorem 6.1.35(iii), we may find a neighborhood U of ξ, and a fine and saturated chart $(\omega_{P|U}, \omega_Q, \vartheta : P \to Q)$ for $\varphi_{|U}$, such that ω_Q is local at ξ, so that ϑ is also a local morphism. Hence, it suffices to show assertion (ii).

(ii): We notice the following :

Claim 6.1.37. Let $\vartheta : P \to Q$ a local and saturated morphism of fine and saturated monoids, and suppose that P is sharp. Then there exists a section $\sigma : Q^\sharp \to Q$ of the projection $Q \to Q^\sharp$, such that $\vartheta(P)$ lies in the image of σ.

Proof of the claim. Pick an isomorphism $\beta : Q \xrightarrow{\sim} Q^\sharp \times Q^\times$ as in lemma 3.2.10, and denote by $\psi : P \to Q^\times$ the composition of ϑ with the induced projection $Q \to Q^\times$. The morphism $\vartheta^\sharp$ is still local and saturated (lemma 3.2.12(iii)), hence corollary 3.2.32(ii) implies that $\vartheta^{\sharp\mathrm{gp}}$ extends to an isomorphism $P^{\mathrm{gp}} \oplus L \xrightarrow{\sim} Q^{\sharp\mathrm{gp}}$, where L is a free abelian group of finite rank. Thus, we may extend ψ^{gp} to a group homomorphism $\psi' : Q^{\sharp\mathrm{gp}} \to Q^\times$. Define an automorphism α of $Q^\sharp \times Q^\times$, by the rule : $(x, g) \mapsto (x, g \cdot \psi'(x)^{-1})$. The restriction $\sigma : Q^\sharp \to Q$ of $(\alpha \circ \beta)^{-1}$ will do. $\Diamond$

With the notation of claim 6.1.37 it is easily seen that $\omega_Q \circ \sigma_T$ is still a chart for $\underline{N}$, hence $(\omega_P, \omega_Q \circ \sigma_T, \vartheta^\sharp)$ is a chart for φ as required. $\square$

Corollary 6.1.38. *Let $f : (T, \underline{M}) \to (S, \underline{N})$ a morphism of log topoi with coherent (resp. fine) log structures, and suppose that $\underline{N}$ admits a finite (resp. fine) chart $\omega_Q : Q_S \to \underline{N}$. Let also ξ be any T-point; we have :*

(i) *There exists a neighborhood U of ξ, and a finite (resp. fine) chart $(\omega_{Q|U}, \omega_P, \vartheta : Q \to P)$ for the morphism $f_{|U}$.*
(ii) *Moreover, if $\underline{M}$ and $\underline{N}$ are fine and f is log flat (resp. saturated) then, on some neighborhood U of ξ, we may also find a chart $(\omega_{Q|U}, \omega_P, \vartheta)$ which is flat (resp. saturated) and fine.*

Proof. This is an immediate consequence of theorem 6.1.35. $\square$

6.2. **Log schemes.** We specialize now to the case of a scheme X. Whereas in [36, §6.4] we considered only pre-log structures on the Zariski site of a scheme, hereafter we shall treat uniformly the categories of log structures on the topoi $X^\sim_{\text{ét}}$ and $X^\sim_{\text{Zar}}$ (notation of (2.4.13)).

6.2.1. Henceforth, we choose $\tau \in \{\text{ét}, \text{Zar}\}$ (see (2.4.18)), and whenever we mention a topology on a scheme X, it will be implicitly meant that this is the topology X_τ (unless explicitly stated otherwise). Let X be a scheme; a *pre-log structure* (resp. *a log structure*) on X is a pre-log structure (resp. a log structure) on the topos $X^\sim_\tau$. The datum of a scheme X and a log structure on X is called briefly a *log scheme*. It is known that a morphism of schemes $X \to Y$ is the same as a morphism of locally ringed topoi $X^\sim_\tau \to Y^\sim_\tau$, hence we may define a morphism of log schemes $(X, \underline{M}) \to (Y, \underline{N})$ as a morphism of log topoi $(X^\sim_\tau, \underline{M}) \to (Y^\sim_\tau, \underline{N})$ (and likewise for morphisms of schemes with pre-log structures). We denote by $\mathbf{pre\text{-}log}_\tau$ (resp. $\mathbf{log}_\tau$) the category of schemes with pre-log structures (resp. of log schemes) on the chosen topology τ. We denote by :

$$\mathbf{int.log}_\tau \qquad \mathbf{sat.log}_\tau \qquad \mathbf{qcoh.log}_\tau \qquad \mathbf{coh.log}_\tau$$

the full subcategory of the category $\mathbf{log}_\tau$, consisting of all log schemes with integral (resp. integral and saturated, resp. quasi-coherent, resp. coherent) log structures.

A scheme with a quasi-fine (resp. fine, resp. quasi-fine and saturated, resp. fine and saturated) log structure is called, briefly, a *quasi-fine log scheme* (resp. a *fine log scheme*, resp. a *qfs log scheme*, resp. a *fs log scheme*), and we denote by

$$\mathbf{qf.log}_\tau \qquad \mathbf{f.log}_\tau \qquad \mathbf{qfs.log}_\tau \qquad \mathbf{fs.log}_\tau$$

the full subcategory of $\mathbf{log}_\tau$ consisting of all quasi-fine (resp. fine, resp. qfs, resp. fs) log schemes on the topology τ. In case it is clear (or indifferent) which topology we are dealing with, we will usually omit the subscript τ. There is an obvious (forgetful) functor :

$$F : \mathbf{log} \to \mathbf{Sch}$$

to the category of schemes, and it is easily seen that this functor is a fibration. For every scheme X, we denote by $\mathbf{log}_X$ the fibre category $F^{-1}(X)$ *i.e.* the category of all log structures on X (or $\mathbf{log}_{X_\tau}$, if we need to specify the topology τ). The same notation shall be used also for the various subcategories : so for instance we shall write $\mathbf{int}.\mathbf{log}_X$ for the full subcategory of all integral log structures on X. Moreover, we shall say that the log scheme $(X, \underline{M})$ is *locally noetherian* if the underlying scheme X is locally noetherian.

6.2.2. Most of the forthcoming assertions hold in both the étale and Zariski topoi, with the same proof. However, it may occasionally happen that the proof of some assertion concerning $X_\tau^\sim$ (for $\tau \in \{\text{ét}, \text{Zar}\}$), is easier for one choice or the other of these two topologies; in such cases, it is convenient to be able to change the underlying topology, to suit the problem at hand. This is sometimes possible, thanks to the following general considerations.

The morphism of locally ringed topoi $\widetilde{u}$ of (2.4.15) induces a pair of adjoint functors :

$$\widetilde{u}^* : \mathbf{log}_{\mathrm{Zar}} \to \mathbf{log}_{\text{ét}} \qquad \widetilde{u}_* : \mathbf{log}_{\text{ét}} \to \mathbf{log}_{\mathrm{Zar}}$$

as well as analogous adjoint pairs for the corresponding categories of sheaves of monoids (resp. of pre-log structures) on the two sites. It follows formally that $\widetilde{u}^*$ sends constant log structures to constant log structures, *i.e.* for every scheme X, and every object $M := (M, \varphi)$ of $\mathbf{Mnd}/\Gamma(X, \mathscr{O}_X)$ we have a natural isomorphism :

$$\widetilde{u}^*(X_{\mathrm{Zar}}, M^{\log}_{X_{\mathrm{Zar}}}) \simeq (X_{\text{ét}}, M^{\log}_{X_{\text{ét}}}).$$

More generally, lemma 6.1.16(i) shows that $\widetilde{u}^*$ preserves the subcategories of quasi-coherent (resp. coherent, resp. integral, resp. fine, resp. fine and saturated) log structures.

Proposition 6.2.3. (i) *The functor $\widetilde{u}^*$ on log structures is faithful and conservative.*

(ii) *The functor $\widetilde{u}^*$ restricts to a fully faithful functor :*

$$\widetilde{u}^* : \mathbf{int}.\mathbf{log}_{\mathrm{Zar}} \to \mathbf{int}.\mathbf{log}_{\text{ét}}.$$

(iii) *Let $(X_{\text{ét}}, \underline{M})$ be any log scheme. Then the counit of adjunction $\widetilde{u}^*\widetilde{u}_*(X_{\text{ét}}, \underline{M}) \to (X_{\text{ét}}, \underline{M})$ is an isomorphism if and only if the same holds for the counit of adjunction $\widetilde{u}^*\widetilde{u}_*(\underline{M}^\sharp) \to \underline{M}^\sharp$.*

Proof. (i): Let $(X_{\mathrm{Zar}}, \underline{M})$ be any log structure; set $(X_{\text{ét}}, \underline{M}_{\text{ét}}) := \widetilde{u}^*(X, \underline{M})$, and denote by $\widetilde{u}^\natural_X : \widetilde{u}^*\mathscr{O}_{X_{\mathrm{Zar}}} \to \mathscr{O}_{X_{\text{ét}}}$ the natural map of structure rings. Since $\widetilde{u}$ is a morphism of locally ringed topoi, we have :

$$(\widetilde{u}^\natural_X)^{-1}\mathscr{O}^\times_{X_{\text{ét}}} = \widetilde{u}^*(\mathscr{O}^\times_{X_{\mathrm{Zar}}}).$$

It follows easily that $\underline{M}_{\text{ét}}$ is the push-out in the cocartesian diagram :

$$\begin{tikzcd} \widetilde{u}^*\underline{M}^\times \arrow[r, "\alpha"] \arrow[d] & \mathscr{O}^\times_{X_{\text{ét}}} \arrow[d] \\ \widetilde{u}^*\underline{M} \arrow[r, "\beta"] & \underline{M}_{\text{ét}}. \end{tikzcd}$$

However, for every geometric point ξ of X, the natural map $\mathscr{O}_{X,|\xi|} \to \mathscr{O}_{X,\xi}$ is faithfully flat, hence α_ξ is injective, and therefore also β_ξ, in light of lemma 2.3.31(ii). The faithfulness of $\widetilde{u}^*$ is an easy consequence. Next, let $f : \underline{M} \to \underline{N}$ be a morphism of log structures on X_{Zar}, set $(X_{\text{ét}}, \underline{N}_{\text{ét}}) := \widetilde{u}^*(X, \underline{N})$, and suppose that $\widetilde{u}^*f : \underline{M}_{\text{ét}} \to \underline{N}_{\text{ét}}$ is an isomorphism; we wish to show that f is an isomorphism. However, β induces an isomorphism of monoids :

$$\widetilde{u}^*(\underline{M}^\sharp) \xrightarrow{\sim} \underline{M}^\sharp_{\text{ét}} \tag{6.2.4}$$

and likewise for $\underline{N}$; it follows already that f induces an isomorphism $\underline{M}^\sharp \xrightarrow{\sim} \underline{N}^\sharp$. To conclude, it suffices to invoke lemma 6.1.4.

(ii): Let us suppose that $\underline{M}$ is integral. According to [10, Prop.3.4.1], it suffices to show that the unit of adjunction $(X, \underline{M}) \to \widetilde{u}_*(X_{\text{ét}}, \underline{M}_{\text{ét}})$ is an isomorphism. Now, from the isomorphism (6.2.4) we deduce a commutative diagram :

$$\begin{tikzcd} \underline{M}^\sharp \arrow[r] \arrow[d] & (\widetilde{u}_*\underline{M}_{\text{ét}})^\sharp \arrow[d] \\ \widetilde{u}_*\widetilde{u}^*(\underline{M}^\sharp) \arrow[r] & \widetilde{u}_*(\underline{M}^\sharp_{\text{ét}}) \end{tikzcd}$$

whose bottom arrow is an isomorphism, and whose left vertical arrow is an isomorphism as well, by lemma 2.4.26(iii) (and again [10, Prop.3.4.1]). We claim that also the right vertical arrow is an isomorphism. Indeed, since $\widetilde{u}_*$ is left exact, the latter arrow is a monomorphism, hence it suffices to show it is an epimorphism; however, since $\underline{M}_{\text{ét}}$ is an integral log structure (lemma 6.1.16(i)), it is easily seen that the projection $\underline{M}_{\text{ét}} \to \underline{M}^\sharp_{\text{ét}}$ is a $\mathscr{O}^\times_{X_{\text{ét}}}$-torsor (in the topos $X^\sim_{\text{ét}}/\underline{M}^\sharp_{\text{ét}}$). Then the contention follows from the exact sequence of pointed sheaves (2.4.12), and the vanishing result of lemma 2.4.26(iv).

Summing up, we conclude that the top horizontal arrow in the above diagram is an isomorphism, so the assertion follows from lemma 6.1.4.

(iii): Set $(X_{\text{ét}}, (\widetilde{u}_*\underline{M})_{\text{ét}}) := \widetilde{u}^*\widetilde{u}_*(X_{\text{ét}}, \underline{M})$. To begin with, lemma 2.4.26(iv) easily implies that the natural morphism $(\widetilde{u}_*\underline{M})^\sharp \to \widetilde{u}_*(\underline{M}^\sharp)$ is an isomorphism; together with the general isomorphism (6.1.8), this yields a short exact sequence of $X_{\text{ét}}$-monoids :

$$0 \to \mathscr{O}^\times_{X_{\text{ét}}} \to (\widetilde{u}_*\underline{M})_{\text{ét}} \to \widetilde{u}^*\widetilde{u}_*(\underline{M}^\sharp) \to 0$$

which easily implies the assertion : the details shall be left to the reader. □

We shall prove later on some more results in the same vein (see corollary 6.2.22).

6.2.5. Arguing as in (6.1.22), we see easily that all finite limits are representable in the category of log schemes. The rule $X \mapsto (X, \mathscr{O}^\times_X)$ defines a fully faithful inclusion of the category of schemes into the category of log schemes. Hence, we shall regard a scheme as a log scheme with trivial log structure. Especially, if $(X, \underline{M})$ is any log scheme, and $Y \to X$ is any morphism of schemes, we shall often use the notation :

(6.2.6) $$Y \times_X (X, \underline{M}) := (Y, \mathscr{O}^\times_Y) \times_{(X, \mathscr{O}^\times_X)} (X, \underline{M}).$$

Especially, if ξ is any τ-point of X, the *localization* of $(X, \underline{M})$ at ξ is the log scheme

$$(X(\xi), \underline{M}(\xi)) := X(\xi) \times_X (X, \underline{M})$$

(see definition 2.4.17(ii,iii)). If $\tau = \text{ét}$, this operation is also called the *strict henselization* of X at ξ.

Definition 6.2.7. (i) For every integer $n \in \mathbb{N}$, we have the subset :

$$(X, \underline{M})_n := \{x \in X \mid \dim \underline{M}_\xi \leq n \text{ for every } \tau\text{-point } \xi \to X \text{ localized at } x\}.$$

Especially $(X, \underline{M})_0$ consists of all points $x \in X$ such that $\underline{M}_\xi = \mathscr{O}^\times_{X,\xi}$ for every τ-point ξ of X localized at x; this subset is called the *trivial locus* of $(X, \underline{M})$, and is also denoted $(X, \underline{M})_{\text{tr}}$.

(ii) If $f : (X, \underline{M}) \to (Y, \underline{N})$ is a morphism of log schemes, we denote by $\text{Str}(f) \subset X$ the *strict locus* of f, which is the subset consisting of all points $x \in X$ such that f is strict at every τ-point localized at x (see definition 6.1.20(ii)).

Remark 6.2.8. Let $f : (X, \underline{M}) \to (Y, \underline{N})$ be any morphism of log schemes.

(i) Since $\log f$ induces local morphisms on stalks, it is easily seen that f restricts to a map

$$f_{\text{tr}} : (X, \underline{M})_{\text{tr}} \to (Y, \underline{N})_{\text{tr}}.$$

(ii) Especially, we have $(X, \underline{M})_{\text{tr}} \subset \text{Str}(f)$.

6.2.9. Let $\underline{X} := (X_i \mid i \in I)$ be a cofiltered family of quasi-separated schemes, with affine transition morphisms $f_\varphi : X_j \to X_i$, for every morphism $\varphi : j \to i$ in I. Denote by X the limit of $\underline{X}$, and for each $i \in I$ let $\pi_i : X \to X_i$ be the natural projection.

Lemma 6.2.10. *In the situation of* (6.2.9)*, let* $\mathscr{X} := ((X_i, \underline{M}_i) \mid i \in I)$ *be a cofiltered system of log schemes, with transition morphisms* $(f_\varphi, \log f_\varphi) : (X_i, \underline{M}_i) \to (X_j, \underline{M}_j)$ *for every morphism* $\varphi : i \to j$ *in* I*. We have :*

(i) *The limit of the system* $\mathscr{X}$ *exists in the category* **log**.

(ii) *Let* $(X, \underline{M})$ *denote the limit of the system* $\mathscr{X}$*. If* X_i *is quasi-compact for every* $i \in I$*, then the natural map*

$$\operatorname*{colim}_{i\in I} \Gamma(X_i, \underline{N}_i) \to \Gamma(X, \underline{N})$$

is an isomorphism.

Proof. (i): Let X be the limit of the system of schemes $\underline{X}$, and endow X with the sheaf of monoids $\underline{M} := \operatorname*{colim}_{i\in I} \pi_i^* \underline{M}_i$, where π^* is the pull-back functor for sheaves of monoids (see (2.3.43)), and the transition maps $\pi_j^* \underline{M}_j \to \pi_i^* \underline{M}_i$ are induced by the morphisms $\log f_\varphi : f_\varphi^* \underline{M}_j \to \underline{M}_i$, for every $\varphi : i \to j$ in I. Then the structure maps of the log structures $\underline{M}_i$ induce a well defined morphism of X-monoids $\underline{M} \to \mathscr{O}_X$, and we claim that the resulting scheme with pre-log structure $(X, \underline{M})$ is actually a log scheme. Indeed, the assertion can be checked on the stalks, and notice that, for every τ-point ξ of X we have a natural identification

$$\mathscr{O}_{X,\xi} \xrightarrow{\sim} \operatorname*{colim}_{i\in I} \mathscr{O}_{X_i,\pi_i(\xi_i)}.$$

(This is clear for $\tau = \mathrm{Zar}$, and for $\tau = \text{ét}$ one uses [33, Ch.IV, Prop.18.8.18(ii)]); it then suffices to invoke lemma 2.3.46(i). Lastly, it is easily seen that $(X, \underline{M})$ is a limit of the system $\mathscr{X}$: the details shall be left to the reader.

(ii): In view of the explicit construction in (i), the assertion follows immediately from proposition 5.1.15. □

Example 6.2.11. Let X be a scheme. For the following example we choose to work with the étale topology $X_{\text{ét}}$ on X. A *divisor* on X is a closed subscheme $D \subset X$ which is regularly embedded in X and of codimension 1 ([33, Ch.IV, Déf.19.1.3, §21.2.12]). Suppose moreover that X is noetherian, let $D \subset X$ be a divisor, and denote by $(D_i \mid i \in I)$ the irreducible reduced components of D. We say that D is a *strict normal crossing* divisor, if :

- $\mathscr{O}_{X,x}$ is a regular ring, for every $x \in D$.
- D is a reduced subscheme.
- For every subset $J \subset I$, the (scheme theoretic) intersection $\bigcap_{j\in J} D_j$ is regular of pure codimension $\sharp J$ in X.

A closed subscheme D of a noetherian scheme X is a *normal crossing divisor* if, for every $x \in X$ there exists an étale neighborhood $f : U \to X$ of x such that $f^{-1}D$ is a strict normal crossing divisor in U.

Suppose that D is a normal crossing divisor of a noetherian scheme X, and let $j : U := X \setminus D \to X$ be the natural open immersion. We claim that the log structure $j_* \mathscr{O}_U^\times$ is fine (this is the direct image of the trivial log structure on $U_{\text{ét}}$: see example 6.1.10(ii)). To see this, let ξ be any geometric point of X localized at a point of D; up to replacing X by an étale neighborhood of ξ, we may assume that D is a strict normal crossing divisor; we can assume as well that X is affine and small enough, so that the irreducible components $(D_\lambda \mid \lambda \in \Lambda)$ are of the form $V(I_\lambda)$, where $I_\lambda \subset A := \Gamma(X, \mathscr{O}_X)$ is a principal divisor, say generated by an element $x_\lambda \in A$, for every $\lambda \in \Lambda$. We claim that $j_* \mathscr{O}_U^\times$ is the constant log structure associated to the pre-log structure :

$$\alpha : \mathbb{N}_X^{(\Lambda)} \to \mathscr{O}_X \quad : \quad e_\lambda \mapsto x_\lambda$$

where $(e_\lambda \mid \lambda \in \Lambda)$ is the standard basis of $\mathbb{N}^{(\Lambda)}$. Indeed, let $S \subset \Lambda$ be the largest subset such that the image of ξ lies in $D_S := \bigcap_{\lambda\in S} D_\lambda$, we have $x_\lambda \in \mathscr{O}^\times_{X,x}$ for all $\lambda \notin S$, so that the push-out of the induced diagram of stalks $\mathscr{O}^\times_{X,\xi} \leftarrow \alpha^{-1}\mathscr{O}^\times_{X,\xi} \to \mathbb{N}^{(\Lambda)}$ is the same as the push-out P_S of the analogous diagram $\mathscr{O}^\times_{X,\xi} \leftarrow \alpha_S^{-1}\mathscr{O}^\times_{X,\xi} \to \mathbb{N}^{(S)}$, where $\alpha_S : \mathbb{N}^{(S)}_X \to \mathscr{O}_X$ is the restriction of α. Suppose that $a \in \mathscr{O}_{X,\xi}$ and a is invertible on $X(\xi)\setminus D_S$; the minimal associated primes of $A/(a)$ are all of height one, and they must therefore be found among the prime ideals Ax_λ, with $\lambda \in S$. It follows easily that a is of the form $u \cdot \prod_{\lambda\in S} x_\lambda^{k_\lambda}$ for certain $k_\lambda \in \mathbb{N}$ and $u \in \mathscr{O}^\times_{X,\xi}$. Therefore, the natural map $\beta_\xi : P_S \to (j_*\mathscr{O}^\times_U)_\xi$ is surjective. Moreover, the family $(x_\lambda \mid \lambda \in S)$ is a regular system of parameters of $\mathscr{O}_{X,\xi}$ ([30, Ch.0, Prop.17.1.7]), therefore the natural map $\mathrm{Sym}^n_{\kappa(\xi)}(\mathfrak{m}_\xi/\mathfrak{m}_\xi^2) \to \mathfrak{m}_\xi^n/\mathfrak{m}_\xi^{n+1}$ is an isomorphism for every $n \in \mathbb{N}$ (here $\mathfrak{m}_\xi \subset \mathscr{O}_{X,\xi}$ is the maximal ideal); it follows easily that β_ξ is also injective.

Example 6.2.12. Suppose that X is a regular noetherian scheme, and $D \subset X$ a divisor on X; let $U := X\setminus D$. If D is not a normal crossing divisor, the log structure $\underline{M} := j_*\mathscr{O}^\times_U$ on $X_{\text{ét}}$ is not necessarily fine. For a counterexample, let K be an algebraically closed field, $C \subset \mathbb{A}^2_K$ a nodal cubic; take $D \subset X := \mathbb{A}^3_K$ to be the (reduced, affine) cone over the cubic C, with vertex $x_0 \in \mathbb{A}^3$, and pick a geometric point ξ localized at x_0. It is easily seen that, away from the vertex, D is a normal crossing divisor, hence $\underline{M}_{|X\setminus\{x_0\}}$ is a fine log structure on $X\setminus\{x_0\}$, by example 6.2.11. More precisely, let $y_0 \in C$ be the unique singular point, $L \subset D$ the line spanned by x_0 and y_0, and η a geometric point localized at the generic point of L. By inspecting the argument in example 6.2.11, we find that :

$$\underline{M}_\eta \simeq \mathbb{N}^{\oplus 2} \oplus \mathscr{O}^\times_{X,\eta}$$

(indeed, an isomorphism is obtained by choosing $a, b \in \mathscr{O}_{C,y_0}$ such that $V(a)$ and $V(b)$ are the two branches of the cubic C in an étale neighborhood of y_0). On the other hand, let $\mathfrak{p} \subset A := K[T_1, T_2, T_3]$ be the prime ideal corresponding to x_0, and $I \subset A_\mathfrak{p}$ the ideal defining the closed subscheme $X(x_0)\cap D$ in $X(x_0)$; we claim that $I\cdot\mathscr{O}_{X,\xi}$ is still a prime ideal, necessarily of height one. Indeed, let $B := A_\mathfrak{p}/I$, and denote by $A^\wedge$ (resp. $B^\wedge$) the $\mathfrak{p}$-adic completion of $A_\mathfrak{p}$ (resp. of B); then $B^\wedge$ is also the completion of the reduced ring $\mathscr{O}_{X,\xi}/I$, hence it suffices to show that $\mathrm{Spec}\, B^\wedge$ is irreducible. However, we may assume that $C \subset \mathrm{Spec}\, K[T_1, T_2]$ is the affine cubic defined by the ideal $J \subset K[T_1, T_2]$ generated by $T_1^3 - T_2^2 + T_1T_2$. Then I is generated by the element $f := T_1^3 - T_2^2T_3 + T_1T_2T_3$; also, $\mathfrak{p} = (T_1, T_2, T_3)$, so that $A^\wedge \simeq K[[T_1, T_2, T_3]]$ and $B^\wedge \simeq A^\wedge/(f)$. Suppose $\mathrm{Spec}\, B^\wedge$ is not irreducible. This means that $V(f) \subset \mathrm{Spec}\, A^\wedge$ is a union $V(f) = Z_1 \cup \cdots \cup Z_n$ of $n \geq 2$ irreducible components Z_i. Since $A^\wedge$ is a local regular ring, each such irreducible component Z_i is a divisor, defined by some principal prime ideal $\mathfrak{q}_i$ in $A^\wedge$. Let a_i be a generator for $\mathfrak{q}_i$; then f admits factorizations of the form $f = a_ib_i$ for some non-invertible $b_i \in A^\wedge$. Fix some i, and set $a := a_i$, $b := b_i$; since f is homogeneous of degree 3, we must have $a \in \mathfrak{p}^k\setminus\mathfrak{p}^{k+1}$ for either $k = 1$ or $k = 2$ ($k \neq 0$ since a is not a unit, and $k \neq 3$, since b is not a unit); then $b \in \mathfrak{p}^{3-k}\setminus\mathfrak{p}^{4-k}$. Write $a = a' + a''$ and $b = b' + b''$, where $a'' \in \mathfrak{p}^{k+1}$, $b'' \in \mathfrak{p}^{4-k}$ and a' (resp. b') is homogeneous of degree k (resp. $3-k$). Then $f = ab = a'b' + c$, where $c \in \mathfrak{p}^4$ and $a'b'$ is homogeneous of degree 3. This means that $f = a'b'$ is a factorization of f in A. However, f is irreducible in A, a contradiction. (Instead of this elementary argument, one can appeal to [63, Th.43.20], which runs as follows. If R is a local domain, then there is a natural bijection between the set of minimal prime ideals of the henselization R^{h} of R and the set of maximal ideals of the normalization of R in its ring of fractions. In our case, the normalization D^ν of D is the cone over the normalization of C, hence the only point of D^ν lying over x_0 is the vertex of D^ν.)

It follows that any choice of a generator of I yields an isomorphism :

$$\underline{M}_\xi \simeq \mathbb{N} \oplus \mathscr{O}^\times_{X,\xi}.$$

Suppose now that – in an étale neighborhood V of ξ – the log structure $\underline{M}$ is associated to a pre-log structure $\alpha : P_V \to \mathscr{O}_V$, for some monoid P; hence $\underline{M}_{|V}$ is the push-out of the diagram $\mathscr{O}_V^\times \leftarrow \alpha^{-1}(\mathscr{O}_V^\times) \to P_V$, whence isomorphisms :

$$P/\alpha_\xi^{-1}(\mathscr{O}_{X,\xi}^\times) \simeq \underline{M}_\xi/\mathscr{O}_{X,\xi}^\times \simeq \mathbb{N} \qquad P/\alpha_\eta^{-1}(\mathscr{O}_{X,\eta}^\times) \simeq \underline{M}_\eta/\mathscr{O}_{X,\eta}^\times \simeq \mathbb{N}^{\oplus 2}.$$

But clearly $\alpha_\xi^{-1}(\mathscr{O}_{X,\xi}^\times) \subset \alpha_\eta^{-1}(\mathscr{O}_{X,\eta}^\times)$, so we would have a surjection of monoids $\mathbb{N} \to \mathbb{N}^{\oplus 2}$, which is absurd.

On the other hand, we remark that the log structure $j_*\mathscr{O}_U^\times$ *on the Zariski site* X_{Zar} is fine : indeed, one has a global chart $\mathbb{N}_{X_{\mathrm{Zar}}} \to j_*\mathscr{O}_U^\times$, provided by the equation defining the divisor D.

6.2.13. Let R be a ring, M a monoid, and set $S := \operatorname{Spec} R[M]$. The unit of adjunction $\varepsilon_M : M \to R[M]$ can be regarded as an object (M, ε_M) of $\mathbf{Mnd}/\Gamma(S, \mathscr{O}_S)$, whence a constant log structure $M_S^{\log}$ on S (see (6.1.13)). The rule

$$M \mapsto \operatorname{Spec}(R, M) := (S, M_S^{\log})$$

is clearly functorial in M. Namely, to any morphism $\lambda : M \to N$ of monoids, we attach the morphism of log schemes

$$\operatorname{Spec}(R, \lambda) := (\operatorname{Spec} R[\lambda], \lambda^{\log}_{\operatorname{Spec} R[N]}) : \operatorname{Spec}(R, N) \to \operatorname{Spec}(R, M).$$

Likewise, if P is a pointed monoid, $\operatorname{Spec} R\langle P\rangle$ is a closed subscheme of $\operatorname{Spec} R[P]$, and we may define

$$\operatorname{Spec}\langle R, P\rangle := \operatorname{Spec}(R, P) \times_{\operatorname{Spec} R[P]} \operatorname{Spec} R\langle P\rangle.$$

Lastly, if M is a non-pointed monoid, notice the natural isomorphism of log schemes

$$\operatorname{Spec}\langle R, M_\circ\rangle \xrightarrow{\sim} \operatorname{Spec}(R, M)_\circ.$$

(Notation of remark 6.1.21(ii) : the details shall be left to the reader.)

Lemma 6.2.14. *With the notation of* (6.2.13)*, let $a \in M$ be any element, and set $M_a := S_a^{-1}M$, where $S_a := \{a^n \mid n \in \mathbb{N}\}$. Then $U_a := \operatorname{Spec} R[M_a]$ is an open subscheme of S, and the induced morphism of log schemes :*

$$\operatorname{Spec}(R, M_a) \to \operatorname{Spec}(R, M) \times_S U_a$$

is an isomorphism.

Proof. Let $\beta_S : M_S \to \mathscr{O}_S$ and $\beta_{U_a} : (M_a)_{U_a} \to \mathscr{O}_{U_a}$ be the natural charts, and denote by $\varphi : M \to M_a$ the localization map. For every τ-point ξ of U_a, we have the identity :

$$\beta_{S,\xi} = \beta_{U_a,\xi} \circ \varphi : M \to \mathscr{O}_{S,\xi}.$$

Let $Q := \beta_{U_a,\xi}^{-1}\mathscr{O}_{U_a,\xi}^\times$. The assertion is a straightforward consequence of the following :

Claim 6.2.15. The induced commutative diagram of monoids :

$$\begin{tikzcd} \varphi^{-1}Q \arrow[r] \arrow[d] & M \arrow[d] \\ Q \arrow[r] & M_a \end{tikzcd}$$

is cocartesian.

Proof of the claim. Let $b \in M$, and suppose that $\beta_{U_a,\xi}(a^{-1}b) \in \mathscr{O}_{U_a,\xi}^\times$. Since $\beta_{U_a,\xi}(a) \in \mathscr{O}_{U_a,\xi}^\times$, we deduce that the same holds for $\beta_{U_a,\xi}(b)$, *i.e.* $b \in \varphi^{-1}Q$. Let $Q' := \varphi(\varphi^{-1}Q)$; we conclude that $Q = S_a^{-1}Q'$, the submonoid of M_a generated by Q' and a^{-1}. The claim follows easily. $\square$

6.2.16. In the same vein, let X be a R-scheme, and (M,φ) any object of $\mathbf{Mnd}/\Gamma(X,\mathscr{O}_X)$. The map φ induces, via the adjunction of (2.3.50), a homomorphism of R-algebras $R[M]\to\Gamma(X,\mathscr{O}_X)$, whence a map of schemes $f:X\to S:=\operatorname{Spec}R[M]$, inducing a morphism

$$X\times_S \operatorname{Spec}(R,M)\to(X,(M,\varphi)^{\log}_X) \tag{6.2.17}$$

of log schemes.

Lemma 6.2.18. *In the situation of* (6.2.16)*, we have :*

(i) *The map* (6.2.17) *is an isomorphism.*
(ii) *The log scheme* $\operatorname{Spec}(R,M)$ *represents the functor*

$$\mathbf{log}\to\mathbf{Set}\quad:\quad(Y,\underline{N})\mapsto\operatorname{Hom}_{\mathbb{Z}\text{-}\mathbf{Alg}}(R,\Gamma(Y,N))\times\operatorname{Hom}_{\mathbf{Mnd}}(P,\Gamma(Y,\underline{N})).$$

Proof. (i): The log structure $f^*(M^{\log}_S)$ of $\operatorname{Spec}(R,M)\times_S X$ represents the functor :

$$F:\mathbf{log}_X\to\mathbf{Set}\quad:\quad\underline{N}\mapsto\operatorname{Hom}_{\mathbf{Mnd}/\Gamma(S,\mathscr{O}_S)}((M,\varepsilon_M),\Gamma(S,f_*\underline{N})).$$

However, if $\underline{N}$ is any log structure on X, the pre-log structure $(f_*\underline{N})^{\text{pre-log}}$ is the same as $f_*(\underline{N}^{\text{pre-log}})$ (see [36, (6.4.8)]). From the explicit construction of direct images for pre-log structures, and since the global sections functor is left exact (because it is a right adjoint), we deduce a cartesian diagram of monoids :

$$\begin{tikzcd}
\Gamma(S,f_*\underline{N}) \arrow[r] \arrow[d] & \Gamma(S,\mathscr{O}_S) \arrow[d] \\
\Gamma(X,\underline{N}) \arrow[r] & \Gamma(X,\mathscr{O}_X).
\end{tikzcd}$$

It follows easily that F is naturally isomorphic to the functor given by the rule :

$$\underline{N}\mapsto\operatorname{Hom}_{\mathbf{Mnd}/\Gamma(X,\mathscr{O}_X)}((M,\varphi),\Gamma(X,\underline{N})).$$

The latter is of course the functor represented by $(M,\varphi)^{\log}_X$.

(ii) can now be deduced formally from (i), or proved directly by inspecting the definitions. □

6.2.19. From (6.2.13) it is also clear that the rule $(R,M)\mapsto\operatorname{Spec}(R,M)$ defines a functor

$$\mathbb{Z}\text{-}\mathbf{Alg}^o\times\text{-}\mathbf{Mon}^o\to\mathbf{log}$$

which commutes with fibre products; namely, say that

$$(R',M')\leftarrow(R,M)\to(R'',M'')$$

are two morphisms of $\mathbb{Z}\text{-}\mathbf{Alg}\times\text{-}\mathbf{Mon}$; then there is a natural isomorphism of log schemes :

$$\operatorname{Spec}(R'\otimes_R R'',M'\otimes_M M'')\xrightarrow{\sim}\operatorname{Spec}(R',M')\times_{\operatorname{Spec}(R,M)}\operatorname{Spec}(R'',M'').$$

For the proof, one compares the universal properties characterizing these log schemes, using lemma 6.2.18(ii) : details left to the reader.

6.2.20. Let X be a scheme, $\underline{M}$ a sheaf of monoids on X_τ. We say that $\underline{M}$ is *locally constant* if there exists a covering family $(U_\lambda\to X\mid\lambda\in\Lambda)$ and for every $\lambda\in\Lambda$, a monoid P_λ and an isomorphism of sheaves of monoids $\underline{M}_{|U_\lambda}\simeq(P_\lambda)_{U_\lambda}$. We say that $\underline{M}$ is *constructible* if, for every affine open subset $U\subset X$ we can find finitely many constructible subsets $Z_1,\dots,Z_n\subset U$ such that :

- $U=Z_1\cup\cdots\cup Z_n$
- $\underline{M}_{|Z_i}$ is a locally constant sheaf of monoids, for every $i=1,\dots,n$.

If moreover, $\underline{M}_\xi$ is a finitely generated monoid for every τ-point ξ of X, we say that $\underline{M}$ is *of finite type*. If $\varphi : \underline{M} \to \underline{N}$ is a morphism of constructible sheaves of monoids on X_τ, then it is easily seen that $\mathrm{Ker}\,\varphi$, $\mathrm{Coker}\,\varphi$ and $\mathrm{Im}\,\varphi$ are also constructible. Moreover, if $\underline{M}$ and $\underline{N}$ are of finite type, then the same holds for $\mathrm{Coker}\,\varphi$ and $\mathrm{Im}\,\varphi$.

Suppose that $\underline{M}$ is a sheaf of monoids on X_τ, and for every $x \in X$ choose a τ-point $\overline{x}$ localized at x; the *rank* of $\underline{M}$ is the function

$$\mathrm{rk}_{\underline{M}} : X \to \mathbb{N} \cup \{\infty\} \qquad x \mapsto \dim_{\mathbb{Q}} \underline{M}^{\mathrm{gp}}_{\overline{x}} \otimes_{\mathbb{Z}} \mathbb{Q}.$$

It is clear from the definitions that the rank function of a constructible sheaf of monoid of finite type is constructible on X.

Lemma 6.2.21. *Let X be a scheme, $\varphi : \underline{Q} \to \underline{Q}'$ a morphism of coherent log structures on X_τ. Then :*

(i) *The sheaves $\underline{Q}^\sharp$ and* $\mathrm{Coker}\,\varphi$ *are constructible of finite type.*
(ii) *$(X, \underline{Q})_n$ is an open subset of X, for every integer $n \geq 0$. (See definition 6.2.7(i).)*
(iii) *The rank functions of $\underline{Q}^\sharp$ and* $\mathrm{Coker}\,\varphi$ *are (constructible and) upper semicontinuous.*

Proof. Suppose that $\underline{Q}$ admits a finite chart $\alpha : M_X \to \underline{Q}$. Pick a finite system of generators $\Sigma \subset M$, and for every $S \subset \Sigma$, set :

$$Z_S := \bigcap_{s \in S} D(s) \cap \bigcap_{t \in \Sigma \setminus S} V(t)$$

where, as usual $D(s)$ (resp. $V(s)$) is the open (resp. closed) subset of the points $x \in X$ such that the image $s(x) \in \kappa(x)$ of s is invertible (resp. vanishes). Clearly each Z_S is a constructible subset of X, and their union equals X. Moreover, for every $S \subset \Sigma$, and every τ-point ξ supported on Z_S, the submonoid $N_\xi := \alpha_\xi^{-1}(\mathscr{O}^\times_{X,\xi}) \subset M$ is a face of M (lemma 3.1.20(i)), hence it is the submonoid $\langle S \rangle$ generated by $\Sigma \cap N_\xi = S$ (lemma 3.1.20(ii)). It follows easily that $\underline{Q}^\sharp_{|Z_S} \simeq (M/\langle S \rangle)_{|Z_S}$.

More generally, suppose that $\underline{Q}$ is coherent. We may assume that X is quasi-compact. Then, by the foregoing, we may find a finite set Λ and a covering family $(f_\lambda : U_\lambda \to X \mid \lambda \in \Lambda)$ of X, such that $\underline{Q}^\sharp_{|U_\lambda}$ is a constructible sheaf of monoids on U_λ. Since f_λ is finitely presented, it maps constructible subsets to constructible subsets ([33, Ch.IV, Th.1.8.4]); it follows easily that the restriction of $\underline{Q}^\sharp$ to $f_\lambda(U_\lambda)$ is constructible, therefore $\underline{Q}^\sharp$ is constructible. Next, notice that $\mathrm{Coker}\,\varphi = \mathrm{Coker}\,\varphi^\sharp$; by the foregoing, $\underline{Q}'^\sharp$ is constructible as well, so the same holds for $\mathrm{Coker}\,\varphi$.

Next, Let $x \in X$ be any point, and ξ a τ-point of X localized at x; by theorem 6.1.35(i) we may find a neighborhood $f : U \to X$ of ξ in X_τ and a finite chart $(\omega_P, \omega_{P'}, \vartheta)$ for $\varphi_{|U}$. By claim 6.1.29, we may assume, after replacing U by a smaller neighborhood of ξ, that ω_P and $\omega_{P'}$ are local at the point ξ, in which case $\underline{Q}^\sharp_\xi = P^\sharp$ and $\mathrm{Coker}\,\varphi_\xi = \mathrm{Coker}\,\vartheta$. Let $r : X \to \mathbb{N}$ (resp. $r' : U \to \mathbb{N}$) denote the rank function of $\underline{Q}^\sharp$ (resp. of $\underline{Q}^\sharp_{|U}$); then it is clear that $r' = r \circ f$. On the other hand, for every $y \in U$ and every τ-point η of U localized at y, the stalk $\underline{Q}^\sharp_\eta$ is a quotient of $P^\sharp$, hence $r'(y) \leq r(x)$ and $\dim \underline{Q}_\eta \leq \dim \underline{Q}_\xi$. Since f is an open mapping, this shows (ii), and also that the rank function of $\underline{Q}^\sharp$ is upper semicontinuous. Likewise, let s (resp. s') denote the rank function of $\mathrm{Coker}\,\varphi$ (resp. $\mathrm{Coker}\,\varphi_{|U}$); then $s' = s \circ f$, and $\mathrm{Coker}\,\varphi_\eta$ is a quotient of $\mathrm{Coker}\,\vartheta$ for every τ-point η of U; the latter implies that $s'(y) \leq s(x)$ for every $y \in U$, which shows that s is upper semicontinuous. $\square$

Corollary 6.2.22. *Let X be a scheme, $\underline{M}$ a log structure on X_{Zar}, and set*

$$(X_{\mathrm{\acute{e}t}}, \underline{M}_{\mathrm{\acute{e}t}}) := \widetilde{u}^*(X_{\mathrm{Zar}}, \underline{M}).$$

Then $\underline{M}$ is integral (resp. coherent, resp. fine, resp. fine and saturated) if and only if the same holds for $\underline{M}_{\text{ét}}$.

Proof. It has already been remarked that $(X_{\text{ét}}, \underline{M}_{\text{ét}})$ is coherent (resp. fine, resp. fine and saturated) whenever the same holds for $(X_{\text{Zar}}, \underline{M})$; furthermore, the proof of proposition 6.2.3(i) shows that the natural map on stalks $\underline{M}_{|\xi|} \to \underline{M}_{\text{ét},\xi}$ is injective for every geometric point ξ of X, therefore $\underline{M}$ is integral whenever the same holds for $\underline{M}_{\text{ét}}$.

Next, we suppose that $\underline{M}_{\text{ét}}$ is coherent, and we wish to show that $\underline{M}$ is coherent.

Let $x \in X$ be any point, ξ a geometric point localized at x. By assumption there exists a finitely generated monoid P', with a morphism $\alpha : P' \to \underline{M}_{\text{ét},\xi}$ inducing an isomorphism :

$$P' \otimes_{\beta^{-1}\underline{M}^{\times}_{\text{ét},\xi}} \underline{M}^{\times}_{\text{ét},\xi} \to \underline{M}_{\text{ét},\xi} \simeq \underline{M}_x \otimes_{\underline{M}^{\times}_x} \mathscr{O}^{\times}_{X,\xi}.$$

It follows easily that we may find a finitely generated submonoid $Q \subset \underline{M}_x$, such that the image of α lies in $(Q \cdot \underline{M}^{\times}_x) \otimes_{\underline{M}^{\times}_x} \mathscr{O}^{\times}_{X,\xi}$, and therefore the natural map :

$$(Q \cdot \underline{M}^{\times}_x) \otimes_{\underline{M}^{\times}_x} \mathscr{O}^{\times}_{X,\xi} \to \underline{M}_{\text{ét},\xi}$$

is surjective. Then lemma 2.3.31(ii) implies that $Q \cdot \underline{M}^{\times}_x = \underline{M}_x$, in other words, the induced map $Q \to \underline{M}^{\sharp}_x = \underline{M}^{\sharp}_{\text{ét},\xi}$ is surjective. Set

$$P := Q^{\text{gp}} \times_{\underline{M}^{\text{gp}}_{\text{ét},\xi}} \underline{M}_{\text{ét},\xi}.$$

In this situation, proposition 6.1.28 tells us that the induced map $\beta_\xi : P \to \underline{M}_{\text{ét},\xi}$ extends to an isomorphism of log structures $\beta^{\log} : P^{\log}_{U_{\text{ét}}} \to \underline{M}_{\text{ét}|U_{\text{ét}}}$ on some étale neighborhood $U \to X$ of ξ. On the other hand, it is easily seen that the diagram of monoids :

$$\begin{tikzcd} \underline{M}_x \arrow[r] \arrow[d] & \underline{M}^{\text{gp}}_x \arrow[d] \\ \underline{M}_{\text{ét},\xi} \arrow[r] & \underline{M}^{\text{gp}}_{\text{ét},\xi} \end{tikzcd}$$

is cartesian, therefore β_ξ factors uniquely through a morphism $\beta'_x : P \to \underline{M}_x$. The latter extends to a morphism of log structures $\beta'^{\log} : P^{\log}_{U'_{\text{Zar}}} \to \underline{M}_{|U'_{\text{Zar}}}$ on some (Zariski) open neighborhood U' of x in X (lemma 6.1.16(iv.a),(v)). By inspecting the construction, we find a commutative diagram of monoids :

$$\begin{tikzcd} (\widetilde{u}^* P^{\log}_{U'_{\text{Zar}}})_\xi \arrow[r] \arrow[d, "(\widetilde{u}^*\beta'^{\log})_\xi"'] & P^{\log}_{U_{\text{ét}},\xi} \arrow[d, "\beta^{\log}_\xi"] \\ (\widetilde{u}^*\underline{M})_\xi \arrow[r, equal] & \underline{M}_{\text{ét},\xi} \end{tikzcd}$$

(where $\widetilde{u}^*$ is the functor on log structures of (6.2.2)) whose horizontal arrows and right vertical arrow are isomorphisms; it follows that $(\widetilde{u}^*\beta'^{\log})_\xi$ is an isomorphism as well, therefore $\widetilde{u}^*\beta'^{\log}$ restricts to an isomorphism on some smaller étale neighborhood $f : U'' \to U'$ of ξ (lemma 6.1.16(iv.c)). Since f is an open morphism, we deduce that the restriction of $(\widetilde{u}^*\beta'^{\log})_\xi$ is already an isomorphism on $f(U'')_{\text{ét}}$, and $f(U'')$ is a (Zariski) open neighborhood of x in X. Finally, in light of proposition 6.2.3(i), we conclude that the restriction of $\beta'^{\log}$ is an isomorphism on $f(U'')_{\text{Zar}}$, so $\underline{M}$ is coherent, as stated.

Lastly, we suppose that $\underline{M}_{\text{ét}}$ is fine and saturated, and we wish to show that $\underline{M}$ is saturated. However, the assertion can be checked on the stalks, hence let ξ be any geometric point of X; the proof of proposition 6.2.3 shows that the natural map $\underline{M}^{\sharp}_{|\xi|} \to \underline{M}^{\sharp}_{\text{ét},\xi}$ is bijective, so the assertion follows from lemma 3.2.9(ii). $\square$

Corollary 6.2.23. *The category* $\mathbf{coh.log}$ *admits all finite limits. More precisely, the limit (in the category of log structures) of a finite system of coherent log structures, is coherent.*

Proof. Let Λ be a finite category, $\mathscr{X} := ((X_\lambda, \underline{M}_\lambda) \mid \lambda \in \Lambda)$ an inverse system of schemes with coherent log structures indexed by Λ. Denote by Y the limit of the system $(X_\lambda \mid \lambda \in \Lambda)$ of underlying schemes, and by $\pi_\lambda : Y \to X_\lambda$ the natural morphism, for every $\lambda \in \Lambda$. It is easily seen that the limit of $\mathscr{X}$ is naturally isomorphic to the limit of the induced system $\mathscr{Y} := ((Y, \pi_\lambda^* \underline{M}_\lambda) \mid \lambda \in \Lambda)$, and in view of lemma 6.1.16(i), we may therefore replace $\mathscr{X}$ by $\mathscr{Y}$, and assume that $\mathscr{X}$ is a finite inverse system in the category $\mathbf{log}_X$, for some scheme X (especially, the underlying maps of schemes are $\mathbf{1}_X$, for every morphism $\lambda \to \mu$ in Λ).

It suffices then to show that the push-out of two morphisms $g : \underline{N} \to \underline{M}$, $h : \underline{N} \to \underline{M}'$ of coherent log structures on X, is coherent. However, notice that the assertion is local on the site X_τ, hence we may assume that $\underline{N}$ admits a finite chart $Q_X \to \underline{N}$. Then, thanks to theorem 6.1.35(ii), we may further assume that both f and g admit finite charts of the form $Q_X \to P_X$ and respectively $Q_X \to P'_X$, for some finitely generated monoids P and P'. We deduce a natural map $(P \amalg_Q P')_X \xrightarrow{\sim} P_X \amalg_{Q_X} P'_X \to \underline{M} \amalg_{\underline{N}} \underline{M}'$, which is the sought finite chart. $\square$

Lemma 6.2.24. *In the situation of* (6.2.9)*, suppose that X_i is quasi-compact for every $i \in I$, and that there exists $i \in I$, and a coherent log structure $\underline{N}_i$ on X_i, such that the log structure $\underline{N} := \pi_i^* \underline{N}_i$ on X_τ admits a finite chart $\beta : Q_X \to \underline{N}$. Then there exists a morphism $\varphi : j \to i$ in I and a chart $\beta_j : Q_{X_j} \to \underline{N}_j := f_\varphi^* \underline{N}_i$ for $\underline{N}_j$, such that $\pi_j^* \beta_j = \beta$.*

Proof. After replacing I by I/i, we may assume that $\underline{N}_j$ is well defined for every $j \in I$, and i is the final object of I. In this case, notice that $(X, \underline{N})$ is the limit of the cofiltered system of log schemes $((X_j, \underline{N}_j) \mid j \in I)$. We begin with the following :

Claim 6.2.25. In order to prove the lemma, it suffices to show that, for every τ-point ξ of X there exists $j(\xi) \in I$, a neighborhood $U_\xi \to X_{j(\xi)}$ of $\pi_{j(\xi)}(\xi)$ in $X_{j(\xi),\tau}$, and a chart $\beta_{j(\xi)} : Q_{U_\xi} \to \underline{N}_{j(\xi)|U_\xi}$ such that $(\mathbf{1}_{U_\xi} \times_{X_{j(\xi)}} \pi_{j(\xi)})^* \beta_{j(\xi)} = \beta$.

Proof of the claim. Clearly we may assume that U_ξ is an affine scheme, for every τ-point ξ of X. Under the stated assumptions, X is quasi-compact, hence we may find a finite set $\{\xi_1, \dots, \xi_n\}$ of τ-points of X, such that $(U_{\xi_k} \times_{X_{j(\xi_k)}} X \to X \mid k = 1, \dots, n)$ is covering in X_τ. Since I is cofiltered, we may then find $j \in I$ and morphisms $\varphi_k : j \to j(\xi_k)$ for every $k = 1, \dots, n$. After replacing each $\beta_{j(\xi_k)}$ by $f_{\varphi_k}^* \beta_{j(\xi_k)}$, we may assume that $j(\xi_k) = j$ for every $k \leq n$.

In this case, set $\beta_k := \beta_{j(\xi_k)}$, and $U_k := U_{\xi_k}$ for every $k \leq n$; let also $U_{kl} := U_k \times_{X_j} U_l$, $U_{kl}^\sim := U_{kl} \times_{X_j} X$ for every $k, l \leq n$, and denote by $\pi_{kl} : U_{kl}^\sim \to U_{kl}$ the natural projection. Hence, for every $k, l \leq n$ we have a morphism of U_{kl}-monoids :

$$\beta_{kl} := \beta_{k|U_{kl}} : Q_{U_{kl}} \to \underline{N}_{j|U_{kl}}$$

and by construction we have $\pi_{kl}^* \beta_{kl} = \pi_{lk}^* \beta_{lk}$ under the natural identification : $U_{kl}^\sim \xrightarrow{\sim} U_{lk}^\sim$. Notice now that U_{kl} is quasi-compact and quasi-separated for every $k, l \leq n$, hence the natural map

$$\operatorname*{colim}_{i \in I} \Gamma(U_{kl} \times_{X_j} X_i, \underline{N}_i) \to \Gamma(U_{kl}^\sim, \underline{N})$$

is an isomorphism (lemma 6.2.10(ii)). On the other hand, β_{kl} is given by a morphism of monoids $b_{kl} : Q \to \Gamma(U_{kl}, \underline{N}_j)$, and likewise $\pi_{kl}^* \beta_{kl}$ is given by the morphism $Q \to \Gamma(U_{kl}^\sim, \underline{N})$ obtained by composiiton of b_{kl} and the natural map $\Gamma(U_{kl}, \underline{N}_j) \to \Gamma(U_{kl}^\sim, \underline{N})$. By lemma 3.1.7(ii), we may then find a morphism $j' \to j$ in I such that the following holds. Set

$$V_k := U_k \times_{X_j} X_{j'} \qquad V_{kl} := V_k \times_{X_{j'}} V_l \qquad \text{for every } k, l \leq n$$

and let $p_k : V_k \to U_k$ be the projection for every $k \leq n$; let also $\beta'_k := p_k^*\beta_k$, which is a chart $Q_{V_k} \to \underline{N}_{j'|V_k}$ for the restriction of $\underline{N}_{j'}$. Then

$$\beta'_{k|V_{kl}} = \beta'_{l|V_{kl}} \qquad \text{for every } k, l \leq n \tag{6.2.26}$$

under the natural identification $V_{kl} \xrightarrow{\sim} V_{lk}$. By construction, the system of morphisms $(V_k \times_{X_{j'}} X \to X \mid k = 1, \dots, n)$ is covering in X_τ; after replacing j' by a larger index, we may then assume that the system of morphisms $(V_k \to X_{j'} \mid k = 1, \dots, n)$ is covering in $X_{j',\tau}$ ([32, Ch.IV, Th.8.10.5]). In this case, (6.2.26) implies that the local charts β'_k glue to a well defined chart $\beta_{j'} : Q_{X_{j'}} \to \underline{N}_{j'}$, and a direct inspection shows that we have indeed $\pi^*_{j'}\beta_{j'} = \beta$. ◊

Now, denote by $\alpha : Q_X \to \mathscr{O}_X$ the composition of β and the structure map of $\underline{N}$, and let ξ be a τ-point of X; we have a natural isomorphism

$$\underline{N}_\xi = \underline{N}_{i,\pi_i(\xi)} \otimes_{\underline{N}^\times_{i,\pi_i(\xi)}} \mathscr{O}^\times_{X,\xi} \xrightarrow{\sim} \operatorname*{colim}_{j\in I} \underline{N}_{i,\pi_i(\xi)} \otimes_{\underline{N}^\times_{i,\pi_i(\xi)}} \mathscr{O}^\times_{X_j,\pi_j(\xi_j)} \xrightarrow{\sim} \operatorname*{colim}_{inI} \underline{N}_{j,\pi_j(\xi)}$$

([33, Ch.IV, Prop.18.8.18(ii)]). It follows that β_ξ and α_ξ factor through morphisms of monoids $\beta_{\xi,j} : Q \to \underline{N}_{j,\pi_j(\xi)}$ and $\alpha_{\xi,j} : Q \to \mathscr{O}_{X_j,\pi_j(\xi)}$ for some $j \in I$ (lemma 3.1.7(ii)), and again we may replace I by I/j, after which we may assume that $\beta_{\xi,j}$ and $\alpha_{\xi,j}$ are defined for every $j \in I$. Then $\alpha_{\xi,j}$ extends to a pre-log structure $\alpha_j : Q_{U_j} \to \mathscr{O}_{U_j}$ on some neighborhood $U_j \to X_j$ of $\pi_j(\xi)$ in $X_{j,\tau}$ (lemma 6.1.16(v)), and we may also assume that $\beta_{\xi,j}$ extends to a morphism of pre-log structures $\beta_j : (Q_{U_j}, \alpha_j) \to \underline{N}_{j|U_j}$ (lemma 6.1.16(iv.a)). Notice as well that

$$\underline{N}^\sharp_{j,\xi} \simeq \underline{N}^\sharp_\xi \quad \text{and} \quad \beta^{-1}_{\xi,j}\underline{N}^\times_{j,\pi_j(\xi)} = \beta^{-1}_\xi \underline{N}^\times_\xi \qquad \text{for every } j \in I$$

and since β_ξ induces an isomorphism $Q/\beta^{-1}_\xi \underline{N}^\times_\xi \xrightarrow{\sim} \underline{N}^\sharp_\xi$, we deduce that $\beta_{j,\xi}$ (which is the same as $\beta_{\xi,j}$) induces an isomorphism $Q/\alpha^{-1}_{j,\xi}\mathscr{O}^\times_{X_j,\pi_j(\xi)} \xrightarrow{\sim} \underline{N}^\sharp_{j,\xi}$ for every $j \in I$. In turn, it then follows from lemma 6.1.4 that $\beta_{j,\xi}$ induces an isomorphism $(Q_{U_j}, \alpha_j)^{\log}_{\pi_j(\xi)} \xrightarrow{\sim} \underline{N}_{j,\pi_j(\xi)}$.

Next, by lemma 6.1.16(iv.b,c) we may find, for every $j \in I$, a neighborhood $U'_j \to U_j$ of ξ in X_τ, such that the restriction $(Q_{U'_j}, \alpha_{j|U'_j}) \to \underline{N}_{j|U'_j}$ of β_j is a chart for $\underline{N}_{j|U'_j}$.

By construction, the morphism $((\mathbf{1}_{U_j} \times_{X_j} \pi_j)^*\beta_j)_\xi : Q \to \underline{N}_\xi$ is the same as β_ξ, so by lemma 6.1.16(iv.b) we may find a neighborhood $V_j \to U'_j \times_{X_j} X$ of ξ in X_τ, such that

$$((\mathbf{1}_{U_j} \times_{X_j} \pi_j)^*\beta_j)_{|V_j} = \beta_{|V_j} \qquad \text{and} \qquad ((\mathbf{1}_{U_j} \times_{X_j} \pi_j)^*\alpha_j)_{|V_j} = \alpha_{|V_j}.$$

Claim 6.2.27. In the situation of (6.2.9), let $Y \to X$ be an object of the site X_τ, with Y quasi-compact and quasi-separated. We have :

(i) There exists $i \in I$, an object $Y_i \to X_i$ of $X_{i,\tau}$, and an isomorphism of X-schemes $Y \xrightarrow{\sim} Y_i \times_{X_i} X$.
(ii) Moreover, if $Y \to X$ is covering in X_τ, then we may find $i \in I$ and $Y_i \to X_i$ as in (i), which is covering in $X_{i,\tau}$.

Proof of the claim. (i) is obtained by combining [32, Ch.IV, Th.8.8.2(ii)] and [33, Ch.IV, Prop.17.7.8(ii)] (and [32, Ch.IV, Cor.8.6.4] if $\tau = \mathrm{Zar}$). Assertion (ii) follows from (i) and [32, Ch.IV, Th.8.10.5]. ◊

By claim 6.2.27(i), we may assume that $V_j = U''_j \times_{X_j} X$ for some neighborhood $U''_j \to U'_j$ of $\pi_j(\xi)$ in $X_{j,\tau}$. To conclude, it suffices to invoke claim 6.2.25. □

Proposition 6.2.28. *In the situation of* (6.2.9)*, suppose that X_i is quasi-compact for every $i \in I$. Then the natural functor :*

$$2\text{-}\operatorname*{colim}_{I} \mathbf{coh.log}_{X_i} \to \mathbf{coh.log}_X \tag{6.2.29}$$

is an equivalence.

Proof. To begin with, we show :

Claim 6.2.30. The functor (6.2.29) is faithful. Namely, for a given $i \in I$, let $\underline{M}_i$ and $\underline{N}_i$ be two coherent log structures on X_i, and $f_i, g_i : \underline{M}_i \to \underline{N}_i$ two morphisms, such that $\pi_i^* f_i$ agrees with $\pi_i^* g_i$. Then there exists a morphism $\psi : j \to i$ in I such that $f_\psi^* f_i = f_\psi^* g_i$.

Proof of the claim. Set $\underline{M} := \pi_i^* \underline{M}_i$, and define likewise the coherent log structure $\underline{N}$ on X. For any τ-point ξ of X, pick a neighborhood $U_\xi \to X_i$ of $\pi_i(\xi)$ in X_i, and a finite chart $\beta : P_{U_\xi} \to \underline{M}_{i|U_\xi}$ for the restriction of $\underline{M}_i$. The morphisms $f_{i|U_\xi}$ and $g_{i|U_\xi}$ are determined by the induced maps $\varphi := \Gamma(U_\xi, f_i) \circ \beta$ and $\gamma := \Gamma(U_\xi, g_i) \circ \beta$, and the assumption means that the composition of φ and the natural map $\Gamma(U_\xi, \underline{N}_i) \to \Gamma(U_\xi \times_{X_i} X, \underline{N})$ equals the composition of γ with the same map.

It then follows from lemmata 6.2.10(ii) and 3.1.7(ii) that there exists a morphism $\psi : i(\xi) \to i$ in I such that the composition of φ with the natural map $\Gamma(U_\xi, \underline{N}_i) \to \Gamma(U_\xi \times_{X_i} X_{i(\xi)}, f_\psi^* \underline{N})$ equals the composition of γ with the same map; in other words, if we set $U'_\xi := U_\xi \times_{X_i} X_{i(\xi)}$, we have $f_\psi^* f_{i|U'_\xi} = f_\psi^* g_{i|U'_\xi}$. Next, since X is quasi-compact, we may find finitely many τ-points $\xi_1, \dots, \xi_n$ such that the family $(U'_{\xi_k} \times_{X_{i(\xi_k)}} X \to X)$ is covering in X_τ. Then, by [32, Ch.IV, Th.8.10.5] we may find $j \in I$ and morphisms $\psi_k : j \to i(\xi_k)$ in I, for $k = 1, \dots, n$, such that the induced family $(U''_k := U'_{\xi_k} \times_{X_{i(\xi_k)}} X_j \to X_j)$ is covering in $X_{j,\tau}$. By construction we have

$$f^*_{\psi_k \circ \psi} f_{i|U''_k} = f^*_{\psi_k \circ \psi} g_{i|U''_k} \qquad \text{for } k = 1, \dots, n$$

therefore $f^*_{\psi_k \circ \psi} f_i = f^*_{\psi_k \circ \psi} g_i$, as required. $\diamondsuit$

Claim 6.2.31. The functor (6.2.29) is full. Namely, let $i \in I$ and $\underline{M}_i$, $\underline{N}_i$ as in claim 6.2.30, and suppose that $f : \pi_i^* \underline{M}_i \to \pi_i^* \underline{N}_i$ is a given morphism of log structures; then there exists a morphism $\psi : j \to i$ in I, and a morphism of log structures $f_j : f_\psi^* \underline{M}_i \to f_\psi^* \underline{N}_i$ such that $\pi_j^* f_j = f$.

Proof of the claim. Indeed, by theorem 6.1.35(i) we may find a covering family $(U_\lambda \to X \mid \lambda \in \Lambda)$ in X_τ, and for each $\lambda \in \Lambda$ a finite chart

$$\beta_\lambda : P_{\lambda, U_\lambda} \to \underline{M}_{|U_\lambda} \quad \gamma_\lambda : Q_{\lambda, U_\lambda} \to \underline{N}_{|U_\lambda} \quad \vartheta_\lambda : P_\lambda \to Q_\lambda$$

for the restriction $f_{|U_\lambda}$. Clearly we may assume that each U_λ is affine, and since X is quasi-compact, we may assume as well that Λ is a finite set; in this case, claim 6.2.27 implies that there exists a morphism $j \to i$ in I, a covering family $(U_{j,\lambda} \to X_j \mid \lambda \in \Lambda)$, and isomorphism of X-schemes $U_\lambda \xrightarrow{\sim} U_{j,\lambda} \times_{X_j} X$ for every $\lambda \in \Lambda$. Next, lemma 6.2.24 says that, after replacing j by some larger index, we may assume that for every $\lambda \in \Lambda$ there exist charts

$$\beta_{j,\lambda} : P_{\lambda, U_{j,\lambda}} \to \underline{M}_j := f_\psi^* \underline{M}_i \qquad \text{and} \qquad \gamma_{j,\lambda} : Q_{\lambda, U_{j,\lambda}} \to \underline{N}_j := f_\psi^* \underline{N}_i$$

with $\pi_j^* \beta_{j,\lambda} = \beta_\lambda$ and $\pi_j^* \gamma_{j,\lambda} = \gamma_\lambda$. Set $f_{j,\lambda} := \vartheta^{\log}_{\lambda, U_{j,\lambda}} : \underline{M}_j \to \underline{N}_j$; by construction we have :

$$(\mathbf{1}_{U_{j,\lambda}} \times_{X_j} \pi_j)^* f_{j,\lambda} = f_{|U_\lambda} \qquad \text{for every } \lambda \in \Lambda.$$

Next, for every $\lambda, \mu \in \Lambda$, let $U_{j,\lambda\mu} := U_{j,\lambda} \times_{X_j} U_{j,\mu}$; we deduce, for every $\lambda, \mu \in \Lambda$, two morphisms of log structures $f_{j,\lambda|U_{j,\lambda\mu}}, f_{j,\mu|U_{\lambda\mu}} : \underline{M}_{j|U_{\lambda\mu}} \to \underline{N}_{j|U_{\lambda\mu}}$, which agree after pull back to $U_{j,\lambda\mu} \times_{X_j} X$. By applying claim 6.2.30 to the cofiltered system of schemes $(U_{j,\lambda\mu} \times_{X_j} X_{j'} \mid j' \in I/j)$, we may then achieve – after replacing j by some larger index – that $f_{j,\lambda|U_{j,\lambda\mu}} = f_{j,\mu|U_{\lambda\mu}}$, in which case the system $(f_{j,\lambda} \mid \lambda \in \Lambda)$ glues to a well defined morphism f_j as sought. $\diamondsuit$

Finally, let us show that (6.2.29) is essentially surjective. Indeed, let $\underline{M}$ be a coherent log structure on X; let us pick a covering family $\mathscr{U} := (U_\lambda \to X \mid \lambda \in \Lambda)$ and finite charts $\beta_\lambda : P_{\lambda, U_\lambda} \to \underline{M}_{|U_\lambda}$ for every $\lambda \in \Lambda$. As in the foregoing, we may assume that each U_λ is affine, and Λ is a finite set, in which case, according to claim 6.2.27(ii) we may find $i \in I$ and a covering family $(U_{i,\lambda} \to X_i \mid \lambda \in \Lambda)$ with isomorphisms of X-schemes $U_{i,\lambda} \times_{X_i} X \xrightarrow{\sim} U_\lambda$ for every $\lambda \in \Lambda$; for every $j \in I/i$ and every $\lambda \in \Lambda$, let us set $U_{j,\lambda} := U_{i,\lambda} \times_{X_i} X_j$. The composition $\alpha_\lambda : P_{\lambda,U_\lambda} \to \mathscr{O}_{U_\lambda}$ of β_λ and the structure map of $\underline{M}_{|U_\lambda}$ is determined by a morphism of monoids

$$P_\lambda \to \Gamma(U_\lambda, \mathscr{O}_{U_\lambda}) = \operatorname*{colim}_{i \in I/i} \Gamma(U_{j,\lambda}, \mathscr{O}_{U_{j,\lambda}}).$$

Then, as usual, lemma 3.1.7(ii) implies that there exists $j \in I/i$ such that α_λ descends to a pre-log structure $P_{\lambda,U_{j,\lambda}} \to \mathscr{O}_{U_{j,\lambda}}$ on $U_{j,\lambda}$, whose associated log structure we denote by $\underline{M}_{j,\lambda}$. For every $\lambda, \mu \in \Lambda$, let $U_{j,\lambda\mu} := U_{j,\lambda} \times_{X_j} U_{j,\mu}$; by construction we have isomorphisms

$$(6.2.32) \qquad (\mathbf{1}_{U_{j,\lambda\mu}} \times_{X_j} \pi_j)^* \underline{M}_{j,\lambda|U_{j,\lambda\mu}} \xrightarrow{\sim} (\mathbf{1}_{U_{j,\mu\lambda}} \times_{X_j} \pi_j)^* \underline{M}_{j,\mu|U_{j,\mu\lambda}} \qquad \text{for every } \lambda, \mu \in \Lambda.$$

By applying claim 6.2.31 to the cofiltered system $(U_{j,\lambda\mu} \times_{X_j} X_{j'} \mid j' \in I/j)$, we can then obtain – after replacing j by a larger index – isomorphisms $\omega_{\lambda\mu} : \underline{M}_{j,\lambda|U_{j,\lambda\mu}} \xrightarrow{\sim} M_{j,\mu|U_{j,\mu\lambda}}$ for every $\lambda, \mu \in \Lambda$, whose pull-back to $U_{j,\lambda\mu} \times_{X_j} X$ are the isomorphisms (6.2.32). Lastly, in view of claim 6.2.30, we may achieve – after further replacement of j by a larger index – that the system $\mathscr{D} := (\underline{M}_{j,\lambda}, \omega_{\lambda\mu} \mid \lambda, \mu \in \Lambda)$ is a descent datum for the fibration F of (6.2.1), whose pull-back to X is isomorphic to the natural descent datum for $\underline{M}$, associated to the covering family $\mathscr{U}$. Then $\mathscr{D}$ glues to a log structure $\underline{M}_j$, such that $\pi_j^* \underline{M}_j \simeq \underline{M}$. ☐

Corollary 6.2.33. *In the situation of* (5.1.14)*, suppose that X_0 is quasi-compact, and let*

$$(g_\infty, \log g_\infty) : (Y_\infty, \underline{N}_\infty) \to (X_\infty, \underline{M}_\infty)$$

be a morphism of log schemes with coherent log structures. Then there exists $\lambda \in \Lambda$, and a morphism

$$(g_\lambda, \log g_\lambda) : (Y_\lambda, \underline{N}_\lambda) \to (X_\lambda, \underline{M}_\lambda)$$

of log schemes with coherent log structures, such that $\log g_\infty = \psi_\lambda^* \log g_\lambda$.

Proof. If X_0 is quasi-compact, the same holds for all X_λ and Y_λ, so the assertion is an immediate consequence of proposition 6.2.28. ☐

Corollary 6.2.34. *In the situation of* (5.1.14)*, let $\underline{N}_0$ and $\underline{M}_0$ be coherent log structures on Y_0, and respectively X_0, and $(g_0, \log g_0) : (Y_0, \underline{N}_0) \to (X_0, \underline{M}_0)$ a morphism of log schemes. Suppose also that X_0 is quasi-compact (as well as quasi-separated); then we have :*

(i) *If the morphism of log schemes*

$$(g_\infty, \psi_0^* \log g_0) : Y_\infty \times_{Y_0} (Y_0, \underline{N}_0) \to X_\infty \times_{X_0} (X_0, \underline{M}_0)$$

admits a chart $(\omega_\infty, \omega'_\infty, \vartheta : P \to Q)$, there exists a morphism $u : \lambda \to 0$ in Λ such that the morphism of log schemes

$$(g_\lambda, \psi_u^* \log g_0) : Y_\lambda \times_{Y_0} (Y_0, \underline{N}_0) \to X_\lambda \times_{X_0} (X_0, \underline{M}_0)$$

admits a chart $(\omega_\lambda, \omega'_\lambda, \vartheta)$.

(ii) *If $(g_\infty, \psi_0^* \log g_0)$ is a log flat (resp. saturated) morphism of fine log schemes, there exists a morphism $u : \lambda \to 0$ in Λ such that $(g_\lambda, \psi_u^* \log g_0)$ is a log flat (resp. saturated) morphism of fine log schemes.*

Proof. (i): Since X_0 is quasi-compact, the same holds for every X_λ and Y_λ, as well as for X_∞ and Y_∞. In view of lemma 6.2.24, we may then find a morphism $v : \mu \to 0$, such that ω_∞ and ω'_∞ descend to charts $\omega_v : P_{X_\mu} \to \varphi_v^* \underline{M}_0$ and $\omega'_v : Q_{Y_\mu} \to \psi_v^* \underline{N}_0$. We deduce two morphisms of pre-log structures $P_{Y_\mu} \to \psi_v^* \underline{N}_0$, namely

$$\beta_1 := \psi_v^* (\log g_0) \circ g_\mu^* \omega_v \qquad \text{and} \qquad \beta_2 := \omega'_v \circ \vartheta_{Y_\mu}$$

and by construction we have $\psi_\mu^* \beta_1^{\log} = \psi_\mu^* \beta_2^{\log}$. By proposition 6.2.28 it follows that there exists $w : \lambda \to \mu$ such that $\psi_w^* \beta_1^{\log} = \psi_w^* \beta_2^{\log}$. The latter means that $(\varphi_w^* \omega_v, \psi_w^* \omega'_v, \vartheta)$ is a chart for the morphism of log schemes $(g_\lambda, \psi^*_{vow} \log g_0) : (Y_\lambda, \psi^*_{vow}\underline{N}_0) \to (X_\lambda, \varphi^*_{vow}\underline{M}_0)$, *i.e.* the claim holds with $u := v \circ w$.

(ii): Suppose therefore that $(g_\infty, \psi^*_0 \log g_0)$ is a log flat (resp.saturated) morphism, and let $\mathscr{U} := (U_i \to X_\infty \mid i \in I)$ be a covering family for $X_{\infty,\tau}$, such that $U_i \times (X_0, \underline{M}_0)$ admits a finite (resp. fine) chart, and U_i is affine for every $i \in I$. Since X_∞ is quasi-compact, we may assume that I is a finite set, and then there exists $\lambda \in \Lambda$ such that $\mathscr{U}$ descends to a covering family $\mathscr{U}_\lambda := (U_{\lambda,i} \to X_\lambda \mid i \in I)$ for $X_{\lambda,\tau}$ (claim 6.2.27(ii)). After replacing Λ by Λ/λ, we may assume that $\lambda = 0$, in which case we set $U_{\lambda,i} := U_{0,i} \times_{X_0} X_\lambda$ for every object λ of Λ, and every $i \in I$. Clearly, it suffices to show that there exists $u : \lambda \to 0$ such that $U_{\lambda,i} \times_{X_\lambda} (g_\lambda, \psi^*_u \log g_0)$ is flat (resp. saturated) for every $i \in I$. Set $Y'_{\lambda,i} := Y_\lambda \times_{X_\lambda} U_{\lambda,i}$ for every $\lambda \in \Lambda$; we may then replace the cofiltered system $\underline{X}$ and $\underline{Y}$, by respectively $(U_{\lambda,i} \mid \lambda \in \Lambda)$ and $(Y'_{\lambda,i} \mid \lambda \in \Lambda)$, which allows to assume from start, that $X_\infty \times_{X_0} (X_0, \underline{M}_0)$ admits a finite (resp. fine) chart. In this case, lemma 6.2.24 allows to further reduce to the case where $\underline{M}_0$ admits a finite (resp. fine) chart.

In this case, by theorem 6.1.35(iii), we may find a covering family $\mathscr{V} := (V_j \to Y_\infty \mid j \in J)$ for $Y_{\infty,\tau}$, consisting of finitely many affine schemes V_j, and for every $j \in J$, a flat (resp. saturated) and fine chart for the induced morphism $V_j \times_{Y_0} (Y_0, \underline{N}_0) \to X_\infty \times_{X_0} (X_0, \underline{M}_0)$. As in the foregoing, after replacing Λ by some category Λ/λ, we may assume that $\mathscr{V}$ descends to a covering family $\mathscr{V}_0 := (V_{0,j} \to Y_0 \mid j \in J)$ for $Y_{0,\tau}$, in which case we set $V_{\lambda,j} := V_{0,j} \times_{Y_0} Y_\lambda$ for every $\lambda \in \Lambda$. Clearly, it suffices to show that there exists $\lambda \in \Lambda$ such that the induced morphism $V_{\lambda,j} \times_{Y_0} (Y_0, \underline{N}_0) \to X_\lambda \times_{X_0} (X_0, \underline{M}_0)$ is log flat (resp. saturated). Thus, we may replace $\underline{Y}$ by the cofiltered system $(V_{\lambda,j} \mid \lambda \in \Lambda)$, which allows to assume that $(g_\infty, \psi^*_0 \log g_0)$ admits a flat (resp. saturated) and fine chart. In this case, the assertion follows from (i). $\square$

Proposition 6.2.35. *The inclusion functors :*

$$\mathbf{qf}.\mathbf{log} \to \mathbf{qcoh}.\mathbf{log} \qquad \mathbf{qfs}.\mathbf{log} \to \mathbf{qf}.\mathbf{log}$$

admit right adjoints :

$$\mathbf{qcoh}.\mathbf{log} \to \mathbf{qf}.\mathbf{log} \;:\; (X, \underline{M}) \mapsto (X, \underline{M})^{\mathrm{qf}} \qquad \mathbf{qf}.\mathbf{log} \to \mathbf{qfs}.\mathbf{log} \;:\; (X, \underline{M}) \mapsto (X, \underline{M})^{\mathrm{qfs}}.$$

Proof. Let $(X, \underline{M})$ be a scheme with quasi-coherent (resp. quasi-fine) log structure. We need to construct a morphism of log schemes

$$\varphi : (X, \underline{M})^{\mathrm{qf}} \to (X, \underline{M}) \qquad (\text{resp. } \varphi : (X, \underline{M})^{\mathrm{qfs}} \to (X, \underline{M}))$$

such that $(X, \underline{M})^{\mathrm{qf}}$ (resp. $(X, \underline{M})^{\mathrm{qfs}}$) is a quasi-fine (resp. qfs) log scheme, and the following holds. Every morphism of log schemes $\psi : (Y, \underline{N}) \to (X, \underline{M})$ with $(Y, \underline{N})$ quasi-fine (resp. qfs), factors uniquely through φ. To this aim, suppose first that $\underline{M}$ admits a chart (resp. a quasi-fine chart) $\alpha : P_X \to \underline{M}$. By lemma 6.2.18(i), α determines an isomorphism

$$(X, \underline{M}) \xrightarrow{\sim} \mathrm{Spec}(\mathbb{Z}, P) \times_{\mathrm{Spec}\,\mathbb{Z}[P]} X.$$

Since $\underline{N}$ is integral (resp. and saturated), the morphism $(Y, \underline{N}) \to \mathrm{Spec}(\mathbb{Z}, P)$ induced by ψ factors uniquely through $\mathrm{Spec}(\mathbb{Z}, P^{\mathrm{int}})$ (resp. $\mathrm{Spec}(\mathbb{Z}, P^{\mathrm{sat}})$) (lemma 6.2.18(ii)). Taking into account lemma 6.1.16(iii), it follows easily that we may take

$$(X, \underline{M})^{\mathrm{qf}} := \mathrm{Spec}(\mathbb{Z}, P^{\mathrm{int}}) \times_{\mathbb{Z}[P]} X \qquad (X, \underline{M})^{\mathrm{qfs}} := \mathrm{Spec}(\mathbb{Z}, P^{\mathrm{sat}}) \times_{\mathbb{Z}[P]} X$$

Next, notice that the universal property of $(X', \underline{M}') := (X, \underline{M})^{\mathrm{qf}}$ (resp. of $(X', \underline{M}') := (X, \underline{M})^{\mathrm{qfs}}$) is local on X_τ : namely, suppose that $(X', \underline{M}')$ has already been found, and let $U \to X$ be an object of X_τ, with a morphism $(Y, \underline{N}) \to (U, \underline{M}_{|U})$ from a quasi-fine (resp. qfs) log scheme; there follows a unique morphism

$$(Y, \underline{N}) \to (U, \underline{M}_{|U}) \times_{(X, \underline{M})} (X', \underline{M}') \xrightarrow{\sim} U \times_X (X', \underline{M}')$$

(notation of (6.2.6)). Thus $(U, \underline{M}_{|U})^{\mathrm{qf}} \simeq U \times_X (X, \underline{M})^{\mathrm{qf}}$ (resp. $(U, \underline{M}_{|U})^{\mathrm{qfs}} \simeq U \times_X (X, \underline{M})^{\mathrm{qfs}}$,) since the latter is a quasi-fine (resp. qfs) log scheme. Therefore, for a general quasi-coherent (resp. quasi-fine) log structure $\underline{M}$, choose a covering family $(U_\lambda \to X \mid \lambda \in \Lambda)$ such that $(U_\lambda, \underline{M}_{|U_\lambda})$ admits a chart for every $\lambda \in \Lambda$; it follows that the family $\mathscr{U} := ((U_\lambda, \underline{M}_{|U_\lambda})^{\mathrm{qf}} \mid \lambda \in \Lambda)$, together with the natural isomorphisms :

$$U_\mu \times_X (U_\lambda, \underline{M}_{|U_\lambda})^{\mathrm{qf}} \simeq U_\lambda \times_X (U_\mu, \underline{M}_{|U_\mu})^{\mathrm{qf}} \qquad \text{for every } \lambda, \mu \in \Lambda$$

is a descent datum for the fibred category over X_τ, whose fibre over any object $U \to X$ is the category of affine U-schemes endowed with a log structure (resp. likewise for the family $\mathscr{U} := ((U_\lambda, \underline{M}_{|U_\lambda})^{\mathrm{qfs}} \mid \lambda \in \Lambda)$). Using faithfully flat descent ([42, Exp.VIII, Th.2.1]), one sees that $\mathscr{U}$ comes from a quasi-fine (resp. qfs) log scheme which enjoys the sought universal property. $\square$

Remark 6.2.36. (i) By inspecting the proof of proposition 6.2.35, we see that the quasi-fine log scheme associated to a coherent log scheme, is actually fine, and the qfs log scheme associated to a fine log scheme, is a fs log scheme (one applies corollary 3.4.1(i)). Hence we obtain functors

$$\mathbf{coh}.\mathbf{log} \to \mathbf{f}.\mathbf{log} \;:\; (X, \underline{M}) \mapsto (X, \underline{M})^{\mathrm{f}} \qquad \mathbf{f}.\mathbf{log} \to \mathbf{fs}.\mathbf{log} \;:\; (X, \underline{M}) \mapsto (X, \underline{M})^{\mathrm{fs}}$$

which are right adjoint to the inclusion functors $\mathbf{f}.\mathbf{log} \to \mathbf{coh}.\mathbf{log}$ and $\mathbf{fs}.\mathbf{log} \to \mathbf{f}.\mathbf{log}$.

(ii) Notice as well that, for every log scheme $(X, \underline{M})$ with quasi-coherent (resp. fine) log structure, the morphism of schemes underlying the counit of adjunction $(X, \underline{M})^{\mathrm{qf}} \to (X, \underline{M})$ (resp. $(X, \underline{M})^{\mathrm{fs}} \to (X, \underline{M})$) is a closed immersion (resp. is finite).

(iii) Furthermore, let $f : Y \to X$ be any morphism of schemes; the proof of proposition 6.2.35 also yields a natural isomorphism of $(Y, f^*\underline{M})$-schemes :

$$(Y, f^*\underline{M})^{\mathrm{qf}} \xrightarrow{\sim} Y \times_X (X, \underline{M})^{\mathrm{qf}} \qquad (\text{resp. } (Y, f^*\underline{M})^{\mathrm{qfs}} \xrightarrow{\sim} Y \times_X (X, \underline{M})^{\mathrm{qfs}}).$$

(iv) Let $(X, \underline{M})$ be a quasi-fine log scheme, and suppose that X is a normal, irreducible scheme, and $(X, \underline{M})_{\mathrm{tr}}$ is a dense subset of X. Denote by X^{qfs} the scheme underlying $(X, \underline{M})^{\mathrm{qfs}}$; then we claim that the projection $X^{\mathrm{qfs}} \to X$ (underlying the counit of adjunction) admits a natural section :

$$\sigma_X : X \to X^{\mathrm{qfs}}.$$

The naturality means that, if $f : (X, \underline{M}) \to (Y, \underline{N})$ is any morphism of quasi-fine log schemes, where Y is also normal and irreducible, and $(Y, \underline{N})_{\mathrm{tr}}$ is dense in Y, then the induced diagram of schemes :

(6.2.37)
$$\begin{tikzcd} X \arrow[r, "\sigma_X"] \arrow[d, "f"'] & X^{\mathrm{qfs}} \arrow[d, "f^{\mathrm{qfs}}"] \\ Y \arrow[r, "\sigma_Y"] & Y^{\mathrm{qfs}} \end{tikzcd}$$

commutes, and therefore it is cartesian, by virtue of (iii). Indeed, suppose first that X is affine, say $X = \operatorname{Spec} A$ for some normal domain A, and $\underline{M}$ admits an integral chart, given by a morphism $\beta : P \to A$, for some integral monoid P; we have to exhibit a ring homomorphism $P^{\mathrm{sat}} \otimes_P A \to A$, whose composition with the natural map $A \to P^{\mathrm{sat}} \otimes_P A$ is the identity of A. The latter is the same as the datum of a morphism of monoids $P^{\mathrm{sat}} \to A$ whose restriction to P agrees with β. However, since the trivial locus of $\underline{M}$ is dense in X, the image of P in A does not contain 0, hence β extends to a group homomorphism $\beta^{\mathrm{gp}} : P^{\mathrm{gp}} \to \operatorname{Frac}(A)^\times$; since A is integrally closed in $\operatorname{Frac}(A)$, we have $\beta^{\mathrm{gp}}(P^{\mathrm{sat}}) \subset A$, as required. Next, suppose that $U \to X$ is an object of X_τ, with U also affine and irreducible; then U is normal and the trivial locus of

$(U, \underline{M}_{|U})$ is dense in U. Thus, the foregoing applies to $(U, \underline{M}_{|U})$ as well, and by inspecting the constructions we deduce a natural identification :

$$\sigma_U = \mathbf{1}_U \times_X \sigma_X.$$

Lastly, for a general $(X, \underline{M})$, we can find a covering family $(U_\lambda \to X \mid \lambda \to \Lambda)$ in X_τ, such that U_λ is affine, and $(U_\lambda, \underline{M}_{|U_\lambda})$ admits an integral chart for every $\lambda \in \Lambda$; proceeding as above, we obtain a system of morphisms $(\sigma_\lambda : U_\lambda \to U^{\mathrm{qfs}} \mid \lambda \in \Lambda)$, as well as natural identifications :

$$\mathbf{1}_{U_\mu} \times_X \sigma_\lambda = \mathbf{1}_{U_\lambda} \times_X \sigma_\mu \qquad \text{for every } \lambda, \mu \in \Lambda.$$

In other words, we have defined a descent datum for the category fibred over X_τ, whose fibre over any object $U \to X$ is the category of all morphisms of schemes $U \to U^{\mathrm{qfs}}$. By invoking faithfully flat descent ([42, Exp.VIII, Th.2.1]), we see that this descent datum yields a morphism $\sigma_X : X \to X^{\mathrm{qfs}}$ such that $\mathbf{1}_{U_\lambda} \times_X \sigma_X = \sigma_\lambda$ for every $\lambda \in \Lambda$. The verification that σ_X is a section of the projection $X^{\mathrm{qfs}} \to X$, and that (6.2.37) commutes, can be carried out locally on X_τ, in which case we can assume that $\underline{M}$ admits a chart as above, and one can check explicitly these assertions, by inspecting the constructions.

6.3. **Logarithmic differentials and smooth morphisms.** In this section we introduce the logarithmic version of the usual sheaves of relative differentials, and we study some special classes of morphisms of log schemes.

Definition 6.3.1. Let $(X, \underline{M} \xrightarrow{\alpha} \mathscr{O}_X)$ and $(Y, \underline{N} \xrightarrow{\beta} \mathscr{O}_Y)$ be two schemes with pre-log structures, and $f : (X, \underline{M}) \to (Y, \underline{N})$ a morphism of schemes with pre-log structures. Let also $\mathscr{F}$ be an $\mathscr{O}_X$-module. An *f-linear derivation of $\underline{M}$ with values in $\mathscr{F}$* is a pair $(\partial, \log \partial)$ consisting of maps of sheaves :

$$\partial : \mathscr{O}_X \to \mathscr{F} \qquad \log \partial : \log \underline{M} \to \mathscr{F}$$

such that :

- ∂ is a derivation (in the usual sense).
- $\log \partial$ is a morphism of sheaves of (additive) monoids on X_τ.
- $\partial \circ f^\natural = 0$ and $\log \partial \circ \log f = 0$.
- $\partial \circ \alpha(m) = \alpha(m) \cdot \log \partial(m)$ for every object U of X_τ, and every $m \in \underline{M}(U)$.

The set of all f-linear derivations with values in f shall be denoted by :

$$\mathrm{Der}_{(Y, \underline{N})}((X, \underline{M}), \mathscr{F}).$$

The f-linear derivations shall also be called $(Y, \underline{N})$*-linear derivations*, when there is no danger of ambiguity.

6.3.2. The set $\mathrm{Der}_{(Y, \underline{N})}((X, \underline{M}), \mathscr{F})$ is clearly functorial in $\mathscr{F}$, and moreover, for any object U of X_τ, any $s \in \mathscr{O}_X(U)$, and any f-linear derivation $(\partial, \log \partial)$, the restriction $(s \cdot \partial_{|U}, s \cdot \log \partial_{|U})$ is an element of $\mathrm{Der}_{(Y, \underline{N})}((U, \underline{M}_{|U}), \mathscr{F})$, hence the rule $U \mapsto \mathrm{Der}_{(Y, \underline{N})}((U, \underline{M}_{|U}), \mathscr{F})$ defines an $\mathscr{O}_X$-module :

$$\mathscr{D}er_{(Y, \underline{N})}((X, \underline{M}), \mathscr{F}).$$

In case $\underline{M} = \mathscr{O}_X^\times$ and $\underline{N} = \mathscr{O}_Y^\times$ are the trivial log structures, the f-linear derivations of $\underline{M}$ are the same as the usual f-linear derivations, *i.e.* the natural map

$$\mathscr{D}er_{(Y, \mathscr{O}_Y^\times)}((X, \mathscr{O}_X^\times), \mathscr{F}) \to \mathscr{D}er_Y(X, \mathscr{F}) \tag{6.3.3}$$

is an isomorphism. In the category of usual schemes, the functor of derivations is represented by the module of relative differentials. This construction extends to the category of schemes with pre-log structures. Namely, let us make the following :

Definition 6.3.4. Let $(X, \underline{M} \xrightarrow{\alpha} \mathscr{O}_X)$ and $(Y, \underline{N} \xrightarrow{\beta} \mathscr{O}_Y)$ be two schemes with pre-log structures, and $f : (X, \underline{M}) \to (Y, \underline{N})$ a morphism of schemes with pre-log structures. The *sheaf of logarithmic differentials* of f is the $\mathscr{O}_X$-module :

$$\Omega^1_{X/Y}(\log \underline{M}/\underline{N}) := (\Omega^1_{X/Y} \oplus (\mathscr{O}_X \otimes_{\mathbb{Z}} \log \underline{M}^{\mathrm{gp}}))/R$$

where R is the $\mathscr{O}_X$-submodule generated locally on X_τ by local sections of the form :

- $(d\alpha(a), -\alpha(a) \otimes \log a)$ with $a \in \underline{M}(U)$
- $(0, 1 \otimes \log a)$ with $a \in \mathrm{Im}((f^{-1}\underline{N})(U) \to \underline{M}(U))$

where U ranges over all the objects of X_τ (here we use the notation of (3.1)). For arbitrary $a \in \underline{M}(U)$, the class of $(0, 1 \otimes \log a)$ in $\Omega^1_{X/Y}(\log \underline{M}/\underline{N})$ shall be denoted by $d \log a$.

6.3.5. It is easy to verify that $\Omega^1_{X/Y}(\log \underline{M}/\underline{N})$ represents the functor

$$\mathscr{F} \mapsto \mathrm{Der}_{(Y,\underline{N})}((X, \underline{M}), \mathscr{F})$$

on $\mathscr{O}_X$-modules. Consequently, (6.3.3) translates as a natural isomorphism of $\mathscr{O}_X$-modules :

$$\Omega^1_{X/Y} \xrightarrow{\sim} \Omega^1_{X/Y}(\log \mathscr{O}_X^\times / \mathscr{O}_Y^\times). \tag{6.3.6}$$

Furthermore, let us fix a scheme with pre-log structure $(S, \underline{N})$, and define the category :

$$\mathbf{pre\text{-}log}/(S, \underline{N})$$

as in (1.1.2). Also, let $\mathbf{Mod}.\mathbf{pre\text{-}log}/(S, \underline{N})$ be the category whose objects are all the pairs $((X, \underline{M}), \mathscr{F})$, where $(X, \underline{M})$ is a $(S, \underline{N})$-scheme, and $\mathscr{F}$ is an $\mathscr{O}_X$-module. The morphisms

$$((X, \underline{M}), \mathscr{F}) \to ((Y, \underline{N}), \mathscr{G})$$

in $\mathbf{Mod}.\mathbf{pre\text{-}log}/(S, \underline{N})$ are the pairs (f, φ) consisting of a morphism $f : (X, \underline{M}) \to (Y, \underline{N})$ of $(S, \underline{N})$-schemes, and a morphism $\varphi : f^*\mathscr{G} \to \mathscr{F}$ of $\mathscr{O}_X$-modules. With this notation, we claim that the rule :

$$(X, \underline{M}) \mapsto ((X, \underline{M}), \Omega^1_{X/S}(\log \underline{M}/\underline{N})) \tag{6.3.7}$$

defines a functor $\mathbf{pre\text{-}log}/(S, \underline{N}) \to \mathbf{Mod}.\mathbf{pre\text{-}log}/(S, \underline{N})$. Indeed, slightly more generally, consider a commutative diagram of schemes with pre-log structures :

$$\begin{array}{ccc} (X, \underline{M}) & \xrightarrow{g} & (X', \underline{M}') \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ (S, \underline{N}) & \xrightarrow{h} & (S', \underline{N}'). \end{array} \tag{6.3.8}$$

An $\mathscr{O}_X$-linear map :

$$g^*\Omega^1_{X'/S'}(\log \underline{M}'/\underline{N}') \xrightarrow{dg} \Omega_{X/S}(\log \underline{M}/\underline{N}) \tag{6.3.9}$$

is the same as a natural transformation of functors :

$$\mathscr{D}er_{(S,\underline{N})}((X, \underline{M}), \mathscr{F}) \to \mathscr{D}er_{(S',\underline{N}')}((X', \underline{M}'), g_*\mathscr{F}) \tag{6.3.10}$$

on all $\mathscr{O}_X$-modules $\mathscr{F}$. The latter can be defined as follows. Let $(\partial, \log \partial)$ be an $(S, \underline{N})$-linear derivation of $\underline{M}$ with values in $\mathscr{F}$; then we deduce morphisms :

$$\partial' : \mathscr{O}_{X'} \xrightarrow{g^\natural} g_*\mathscr{O}_X \xrightarrow{g_*\partial} g_*\mathscr{F} \qquad \log \partial' : \underline{M}' \xrightarrow{\log g} g_*\underline{M} \xrightarrow{g_* \log \partial} g_*\mathscr{F}$$

and it is easily seen that $(\partial', \log \partial')$ is a $(S', \underline{N}')$-linear derivation of $\underline{M}'$ with values in $g_*\mathscr{F}$.

6.3.11. Consider two morphisms $(X,\underline{M}) \xrightarrow{f} (Y,\underline{N}) \xrightarrow{g} (Z,\underline{P})$ of schemes with pre-log structures. A direct inspection of the definitions shows that :

$$\mathscr{D}er_{(Y,\underline{N})}((X,\underline{M}),\mathscr{F}) = \mathrm{Ker}(\mathscr{D}er_{(Z,\underline{P})}((X,\underline{M}),\mathscr{F}) \to \mathscr{D}er_{(Z,\underline{P})}((Y,\underline{N}),f_*\mathscr{F}))$$

for every $\mathscr{O}_X$-module $\mathscr{F}$, whence a right exact sequence of $\mathscr{O}_X$-modules :

$$f^*\Omega^1_{Y/Z}(\log \underline{N}/\underline{P}) \xrightarrow{df} \Omega^1_{X/Z}(\log \underline{M}/\underline{P}) \to \Omega^1_{X/Y}(\log \underline{M}/\underline{N}) \to 0 \tag{6.3.12}$$

extending the standard right exact sequence for the usual sheaves of relative differentials.

Proposition 6.3.13. *Suppose that the diagram* (6.3.8) *is cartesian. Then the map* (6.3.9) *is an isomorphism.*

Proof. If (6.3.8) is cartesian, X is the scheme $X' \times_{S'} S$, and $\underline{M}$ is the push-out of the diagram :

$$f^{-1}\underline{N} \leftarrow (f' \circ g)^{-1}\underline{N}' \xrightarrow{\varphi} g^*\underline{M}'.$$

Suppose now that $\log\partial : \underline{M}' \to g_*\mathscr{F}$ and $\partial : \mathscr{O}_{X'} \to g_*\underline{\mathscr{F}}$ define a f'-linear derivation of $\underline{M}'$. By adjunction, we deduce morphisms $\alpha : g^{-1}\mathscr{O}_{X'} \to \mathscr{F}$ and $\beta : g^{-1}\underline{M}' \to \mathscr{F}$. By construction, we have $\beta \circ \varphi = 0$, hence β extends uniquely to a morphism $\log\partial' : \underline{M} \to \mathscr{F}$ such that $\log\partial' \circ \log f = 0$. Likewise, α extends by linearity to a unique f-linear derivation $\partial : \mathscr{O}_X \to \mathscr{F}$. One checks easily that $(\partial', \log\partial')$ is a f-linear derivation of $\underline{M}$, and that every f-linear derivation of $\underline{M}$ with values in $\mathscr{F}$ is obtained in this fashion. □

6.3.14. The functor (6.3.7) admits a left adjoint. Indeed, let $((X, \underline{M} \xrightarrow{\alpha} \mathscr{O}_X), \mathscr{F})$ be any object of $\mathbf{Mod.pre\text{-}log}/(S,\underline{N})$; we define an $(S,\underline{N})$-scheme $(X \oplus \mathscr{F}, \underline{M} \oplus \mathscr{F})$ as follows. $X \oplus \mathscr{F}$ is the spectrum of the $\mathscr{O}_X$-algebra $\mathscr{O}_X \oplus \mathscr{F}$, whose multiplication law is given by the rule :

$$(s,t)\cdot(s',t') := (ss', st' + s't) \qquad \text{for every local section } s \text{ of } \mathscr{O}_X \text{ and } t \text{ of } \mathscr{F}.$$

Likewise, we define a composition law on the sheaf $\underline{M} \oplus \mathscr{F}$, by the rule :

$$(m,t)\cdot(m',t') := (mm', \alpha(m)\cdot t' + \alpha(m')\cdot t) \qquad \text{for every local section } m \text{ of } \underline{M} \text{ and } t \text{ of } \mathscr{F}.$$

Then $\underline{M} \oplus \mathscr{F}$ is a sheaf of monoids, and α extends to a pre-log structure $\alpha \oplus \mathbf{1}_{\mathscr{F}} : \underline{M} \oplus \mathscr{F} \to \mathscr{O}_X \oplus \mathscr{F}$. The natural map $\mathscr{O}_X \to \mathscr{O}_X \oplus \mathscr{F}$ is a morphism of algebras, whence a natural map of schemes $\pi : X \oplus \mathscr{F} \to X$, which extends to a morphism of schemes with pre-log structures $(X \oplus \mathscr{F}, \underline{M} \oplus \mathscr{F}) \to (X,\underline{M})$, by letting $\log\pi : \pi^*\underline{M} \to \underline{M} \oplus \mathscr{F}$ be the map induced by the natural monomorphism (notice that π^* induces an equivalence of sites $X_\tau \xrightarrow{\sim} (X \oplus \mathscr{F})_\tau$).

Now, let $(Y,\underline{P})$ be any $(S,\underline{N})$-scheme, and :

$$\varphi : \mathscr{F} := ((X,\underline{M}),\mathscr{F}) \to \Omega := ((Y,\underline{P}), \Omega^1_{Y/S}(\log \underline{P}/\underline{N}))$$

a morphism in $\mathbf{Mod.pre\text{-}log}/(S,\underline{N})$. By definition, φ consists of a morphism $f : (X,\underline{M}) \to (Y,\underline{P})$ and an $\mathscr{O}_X$-linear map $f^*\Omega^1_{Y/S}(\log \underline{P}/\underline{N}) \to \mathscr{F}$, which is the same as a $(S,\underline{N})$-linear derivation :

$$\partial : \mathscr{O}_Y \to f_*\mathscr{F} \qquad \log\partial : \underline{P} \to f_*\mathscr{F}.$$

In turns, the latter yields a morphism of $(S,\underline{N})$-schemes :

$$(Y \oplus f_*\mathscr{F}, \underline{P} \oplus f_*\mathscr{F}) \to (Y,\underline{P}) \tag{6.3.15}$$

determined by the map of algebras :

$$\mathscr{O}_Y \to \mathscr{O}_Y \oplus f_*\mathscr{F} \qquad s \mapsto (s, \partial s) \quad \text{for every local section } s \text{ of } \mathscr{O}_Y$$

and the map of monoids :

$$\underline{P} \mapsto \underline{P} \oplus f_*\mathscr{F} \qquad p \mapsto (p, \log\partial p) \quad \text{for every local section } p \text{ of } \underline{P}.$$

Finally, we compose (6.3.15) with the natural morphism

$$(X \oplus \mathscr{F}, \underline{M} \oplus \mathscr{F}) \to (Y \oplus f_*\mathscr{F}, \underline{P} \oplus f_*\mathscr{F})$$

that extends f, to obtain a morphism $g_\varphi : (X \oplus \mathscr{F}, \underline{M} \oplus \mathscr{F}) \to (Y, \underline{P})$. We leave to the reader the verification that the rule $\varphi \mapsto g_\varphi$ establishes a natural bijection :

$$\mathrm{Hom}_{\mathbf{Mod.pre\text{-}log}/(S,\underline{N})}(\mathscr{F}, \Omega) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{pre\text{-}log}/(S,\underline{N})}((X \oplus \mathscr{F}, \underline{M} \oplus \mathscr{F}), (Y, \underline{P})).$$

6.3.16. In the situation of definition 6.3.4, let $(\partial, \log \partial)$ be an f-linear derivation of $\underline{M}$ with values in an $\mathscr{O}_X$-module $\mathscr{F}$. Consider the map :

$$\partial' : \mathscr{O}_X^\times \to \mathscr{F} \quad : \quad u \mapsto u^{-1} \cdot \partial u \quad \text{for all local sections } u \text{ of } \mathscr{O}_X^\times.$$

By definition, $\partial' \circ \alpha : \alpha^{-1}\underline{M} \to \mathscr{F}$ agrees with the restriction of $\log \partial$; in view of the cocartesian diagram (6.1.7), we deduce that $\log \partial$ extends uniquely to an f-linear derivation $\log \partial^{\log}$ of $\underline{M}^{\log}$. Let $f^{\log} : (X, \underline{M}^{\log}) \to (Y, \underline{N}^{\log})$ be the map deduced from f; a similar direct verification shows that $\log \partial^{\log}$ is a $f^{\log}$-linear derivation. There follow natural identifications :

$$\Omega^1_{X/Y}(\log(\underline{M}/\underline{N})) = \Omega^1_{X/Y}(\log(\underline{M}^{\log}/\underline{N})) = \Omega^1_{X/Y}(\log(\underline{M}^{\log}/\underline{N}^{\log})). \tag{6.3.17}$$

Moreover, if $\underline{M}$ and $\underline{N}$ are log structures, the natural map :

$$\mathscr{O}_X \otimes_{\mathbb{Z}} \log \underline{M}^{\mathrm{gp}} \to \Omega^1_{X/Y}(\log(\underline{M}/\underline{N})) \qquad a \otimes b \mapsto a \cdot d\log(b) \tag{6.3.18}$$

is an epimorphism. Indeed, we have $da = d(a+1)$ for every local section $a \in \mathscr{O}_X(U)$ (for any étale X-scheme U), and locally on X_τ, either a or $1 + a$ is invertible in $\mathscr{O}_X$ (this holds certainly on the stalks, hence on appropriate small neighborhoods $U' \to U$); hence da lies in the image of (6.3.18).

Example 6.3.19. Let R be a ring, $\varphi : N \to M$ be any map of monoids, and set :

$$S := \mathrm{Spec}\, R \qquad S[M] := \mathrm{Spec}\, R[M] \qquad S[N] := \mathrm{Spec}\, R[N].$$

Also, let $f : \mathrm{Spec}(R, M) \to \mathrm{Spec}(R, N)$ be the morphism of log schemes induced by φ (see (6.2.13)). With this notation, we claim that (6.3.18) induces an isomorphism :

$$\mathscr{O}_{S[M]} \otimes_{\mathbb{Z}} \mathrm{Coker}\, \varphi^{\mathrm{gp}} \xrightarrow{\sim} \Omega^1_{S[M]/S[N]}(\log M^{\log}_{S[M]}/N^{\log}_{S[N]})$$

To see this, we may use (6.3.12) to reduce to the case where $N = \{1\}$. Next, notice that the functor

$$\mathbf{Mnd}^o \to \mathbf{pre\text{-}log}/(S, \mathscr{O}_S^\times) \quad : \quad M \mapsto \mathrm{Spec}(R, M)$$

commutes with limits, and the same holds for the functor (6.3.7), since the latter is a right adjoint. Hence, we may assume that M is finitely generated; then lemma 3.1.7(i) further reduces to the case where $M = \mathbb{N}^{\oplus n}$ for some integer $n \geq 0$, and even to the case where $n = 1$. Set $X := S[\mathbb{N}]$; it is easy to see that a S-linear derivation of $\mathbb{N}_X^{\log}$ with values in an $\mathscr{O}_X$-module $\mathscr{F}$, is completely determined by a map of additive monoids $\mathbb{N} \to \Gamma(X, \mathscr{F})$, and the latter is the same as an $\mathscr{O}_X$-linear map $\mathscr{O}_X \to \mathscr{F}$, whence the contention.

Lemma 6.3.20. *Let $f : (X, \underline{M}) \to (Y, \underline{N})$ be a morphism of schemes with quasi-coherent log structures. Then :*

(i) *The $\mathscr{O}_X$-module $\Omega^1_{X/Y}(\log \underline{M}/\underline{N})$ is quasi-coherent.*
(ii) *If $\underline{M}$ is coherent, X is noetherian, and $f : X \to Y$ is locally of finite type, then $\Omega^1_{X/Y}(\log \underline{M}/\underline{N})$ is a coherent $\mathscr{O}_X$-module.*
(iii) *If both $\underline{M}$ and $\underline{N}$ are coherent, and $f : X \to Y$ is locally of finite presentation, then $\Omega^1_{X/Y}(\log \underline{M}/\underline{N})$ is a quasi-coherent $\mathscr{O}_X$-module of finite presentation.*

Proof. Applying the right exact sequence (6.3.12) to the sequence $(X, \underline{M}) \xrightarrow{f} (Y, \underline{N}) \to (Y, \mathscr{O}_Y^\times)$, we may easily reduce to the case where $\underline{N} = \mathscr{O}_Y^\times$ is the trivial log structure on Y. In this case, $\underline{N}$ admits the chart given by the unique map of monoids : $\{1\} \to \Gamma(Y, \mathscr{O}_Y)$, and f admits the chart $\{1\} \to P$, whenever $\underline{M}$ admits a chart $P \to \Gamma(X, \mathscr{O}_X)$.

Hence, everything follows from the following assertion, whose proof shall be left to the reader. Suppose that $\varphi : A \to B$ is a ring homomorphism, $\underline{M}$ (resp. $\underline{N}$) is the constant log structure on $X := \operatorname{Spec} B$ (resp. on $Y := \operatorname{Spec} A$) associated to a map of monoids $\alpha : P \to B$ (resp. $\beta : Q \to A$), and $f : (X, \underline{M}) \to (Y, \underline{N})$ is defined by φ admits a chart $\vartheta : Q \to P$. Then $\Omega^1_{X/Y}(\log \underline{M}/\underline{N})$ is the quasi-coherent $\mathscr{O}_X$-module $L^\sim$, associated to the B-module $L := (\Omega^1_{B/A} \oplus (B \otimes_{\mathbb{Z}} P^{\mathrm{gp}}))/R$, where R is the submodule generated by the elements of the form $(0, 1 \otimes \log \vartheta(q))$ for all $q \in Q$, and those of the form $(d\alpha(m), -\alpha(m) \otimes \log m)$, for all $m \in M$. $\square$

6.3.21. Let us fix a log scheme $(Y, \underline{N})$. To any pair of $(Y, \underline{N})$-schemes $X := (X, \underline{M})$, $X' := (X', \underline{M}')$, we attach a contravariant functor

$$\mathscr{H}_Y(X', X) : (X'_\tau)^o \to \mathbf{Set}$$

by assigning, to every object U of X'_τ, the set of all morphisms $(U, \underline{M}'_{|U}) \to (X, \underline{M})$ of $(Y, \underline{N})$-schemes. It is easily seen that $\mathscr{H}_Y(X', X)$ is a sheaf on X'_τ. Any morphism $\varphi : (X'', \underline{M}'') \to (X', \underline{M}')$ (resp. $\psi : (X, \underline{M}) \to (X'', \underline{M})$) of $(Y, \underline{N})$-schemes induces a map of sheaves :

$$\varphi^* : \varphi^* \mathscr{H}_Y(X', X) \to \mathscr{H}_Y(X'', X) \qquad \text{(resp. } \psi_* : \mathscr{H}_Y(X', X) \to \mathscr{H}_Y(X', X''))$$

in the obvious way.

Definition 6.3.22. With the notation of (6.3.21) :

(i) We say that a morphism $i : (T', \underline{L}') \to (T, \underline{L})$ of log schemes is a *closed immersion* (resp. an *exact closed immersion*) if the underlying morphism of schemes $T' \to T$ is a closed immersion, and $\log i : i^* \underline{L} \to \underline{L}'$ is an epimorphism (resp. an isomorphism) of T'_τ-monoids.

(ii) We say that a morphism $i : (T', \underline{L}') \to (T, \underline{L})$ of log schemes is an *exact nilpotent immersion* if i is an exact closed immersion, and the ideal $\mathscr{I} := \operatorname{Ker}(\mathscr{O}_T \to i_* \mathscr{O}_{T'})$ is nilpotent.

(iii) We say that a morphism $f : (X, \underline{M}) \to (Y, \underline{N})$ of log schemes is *formally smooth* (resp. *formally unramified*, resp. *formally étale*) if, for every exact nilpotent immersion $i : T' \to T$ of fine $(Y, \underline{N})$-schemes, the induced map of sheaves $i^* : i^* \mathscr{H}_Y(T, X) \to \mathscr{H}_Y(T', X)$ is an epimorphism (resp. a monomorphism, resp. an isomorphism).

(iv) We say that a morphism $f : (X, \underline{M}) \to (Y, \underline{N})$ of log schemes is *smooth* (resp. *unramified*, resp. *étale*) if the underlying morphism $X \to Y$ is locally of finite presentation, and f is formally smooth (resp. formally unramified, resp. formally étale).

Example 6.3.23. Let $(S, \underline{P})$ be a log scheme, $(f, \log f) : (X, \underline{M}) \to (Y, \underline{N})$ be a morphism of S-schemes, such that Y is a separated S-scheme. The pair $(\mathbf{1}_{(X,\underline{M})}, (f, \log f))$ induces a morphism

$$\Gamma_f : (X, \underline{M}) \to (X', \underline{M}') := (X, \underline{M}) \times_S (Y, \underline{N})$$

the *graph* of f. Then it is easily seen that Γ_f is a closed immersion of log schemes. Indeed, the morphism of schemes underlying Γ_f is a closed immersion ([26, Ch.I, 5.4.3]) and it is easily seen that the morphism $\log \Gamma_f : \Gamma_f^* \underline{M}' \to \underline{M}$ is an epimorphism on the underlying sheaves of sets, so it is *a fortiori* an epimorphism of X_τ-monoids.

Proposition 6.3.24. *Let $f : (X, \underline{M}) \to (Y, \underline{N})$, $g : (Y, \underline{N}) \to (Z, \underline{P})$, and $h : (Y', \underline{N}') \to (Y, \underline{N})$ be morphisms of log schemes. Denote by* $\mathbf{P}$ *either one of the properties : "formally* *smooth", "formally unramified", "formally étale", "smooth", "unramified", "étale". The following holds :*

(i) *If f and g enjoy the property $\mathbf{P}$, then the same holds for $g \circ f$.*
(ii) *If $(f, \log f)$ enjoys the property $\mathbf{P}$, then the same holds for*

$$(f, \log f) \times_{(Y,\underline{N})} (Y', \underline{N}') : (X, \underline{M}) \times_{(Y,\underline{N})} (Y', \underline{N}') \to (Y', \underline{N}').$$

(iii) *Let $(j_\lambda : U_\lambda \to X \mid \lambda \in \Lambda)$ be a covering family in X_τ; endow U_λ with the log structure $(\underline{M})_{|U_\lambda}$ and suppose that $f_\lambda := (f \circ j_\lambda, (\log f)_{|U_\lambda}) : (U_\lambda, (\underline{M})_{|U_\lambda}) \to (Y, \underline{N})$ enjoys the property $\mathbf{P}$, for every $\lambda \in \Lambda$. Then f enjoys the property $\mathbf{P}$ as well.*
(iv) *An open immersion of log schemes is étale.*
(v) *A closed immersion of log schemes is formally unramified.*

Proof. (i), (ii) and (iv) are left to the reader.

(iii): To begin with, if each f_λ is locally of finite presentation, the same holds for f, by [33, Ch.IV, lemme 17.7.5], hence we may assume that $\mathbf{P}$ is either "formally smooth" or "formally unramified". (Clearly, the case where $\mathbf{P}$ is "formally étale" will follow.)

Now, let $i : T' \to T$ be an exact nilpotent immersion of $(Y, \underline{N})$-schemes, and ξ a τ-point of T'. By inspecting the definitions, it is easily seen that the stalk $\mathscr{H}_Y(T, X)_\xi$ is the union of the images of the stalks $\mathscr{H}_Y(T, U_\lambda)_\xi$, for every $\lambda \in \Lambda$, and likewise for $\mathscr{H}_Y(T', X)_\xi$. It readily follows that f is formally smooth whenever all the f_λ are formally smooth.

Lastly, suppose that all the f_λ are formally unramified, and let $s_\xi, t_\xi \in \mathscr{H}_Y(T, X)_{i(\xi)}$ be two sections whose images in $\mathscr{H}_Y(T', X)_\xi$ agree; after replacing T by a neighborhood of $i(\xi)$ in T_τ, we may assume that s_ξ, t_ξ are represented by two $(Y, \underline{N})$-morphisms $s, t : T \to (X, \underline{M})$ such that $s \circ i = t \circ i$. Choose $\lambda \in \Lambda$ such that U_λ is a neighborhood of $s \circ i(\xi) = t \circ i(\xi)$; this means that there exists a neighborhood $p' : U' \to T'$ of ξ, and a morphism $s_{U'} : U' \to U_\lambda$ lifiting $s \circ i$ (and thus, also $t \circ i$). Then we may find a neighborhood $p : U \to T$ of $i(\xi)$ which identifies p' with $p \times_T \mathbf{1}_{T'}$ ([33, Ch.IV, Th.18.1.2]), and since j_λ is étale, we may furthermore find morphisms $s_U, t_U : U \to U_\lambda$ such that $j_\lambda \circ s_U = s \circ i = t \circ i = j_\lambda \circ t_U$. Set $i_U := i \times_T \mathbf{1}_U : U' \to U$; by construction, s_U and t_U yield two sections of $i_U^* \mathscr{H}_Y(U, U_\lambda)_\xi$, whose images in $\mathscr{H}_Y(U', U_\lambda)_\xi$ coincide. Since f_λ is formally unramified, it follows that – up to replacing U by a neighborhood of $i(\xi)$ in U_τ – we must have $s_U = t_U$, so $s_\xi = t_\xi$, and we conclude that f is formally unramified.

(v): Consider a commutative diagram of log schemes :

$$\begin{tikzcd} (T', \underline{L}') \arrow[r, "h'"] \arrow[d, "i"'] & (X, \underline{M}) \arrow[d, "f"] \\ (T, \underline{L}) \arrow[r, "h"] & (Y, \underline{N}) \end{tikzcd}$$

where f is a closed immersion, and i is an exact closed immersion. We are easily reduced to showing that there exists at most a morphism $(g, \log) : (T, \underline{L}) \to (X, \underline{M})$ such that $f \circ g = h$ and $h' = g \circ i$. Since $f : X \to Y$ is a closed immersion of schemes, there exists at most one morphism of schemes $g : T \to X$ lifting h and extending h' ([33, Ch.IV, Prop.17.1.3(i)]). Hence we may assume that such a g is given, and we need to check that there exists at most one morphism $\log g : g^*\underline{M} \to \underline{L}$ whose composition with $g^*(\log f) : h^*\underline{N} \to g^*\underline{M}$ equals $\log h$. However, by assumption $\log f$ is an epimorphism, hence the same holds for $g^*(\log f)$ (see (1.1.31)), whence the contention. $\square$

Corollary 6.3.25. *Let f and g be as proposition* 6.3.24. *We have :*

(i) *If $g \circ f$ is formally unramified, the same holds for f.*
(ii) *If $g \circ f$ is formally smooth (resp. formally étale) and g is formally unramified, then f is formally smooth (resp. formally étale).*

(iii) *Suppose that g is formally étale. Then f is formally smooth (resp. formally unramified, resp. formally étale) if and only if the same holds for $g \circ f$.*

Proof. (i): Let $Y = \bigcup_{\lambda \in \Lambda}$ be an affine open covering of Y, and for each $\lambda \in \Lambda$, let $f_\lambda : U_\lambda \times_Y (X, \underline{M}) \to (U_\lambda, \underline{N}_{|U_\lambda})$ be the restriction of f; in light of proposition 6.3.24(iii) it suffices to show that each f_λ is formally unramified, and on the other hand, the restriction $g \circ f_\lambda : U_\lambda \times_Y (X, \underline{M}) \to (Z, \underline{P})$ of $g \circ f$ is formally unramified, by proposition 6.3.24(i),(iv). It follows that we may replace f and g respectively by f_λ and g_λ, which allows to assume that Y is affine, especially separated, so that g is a separated morphism of schemes. In such situation, one may – in view of example 6.3.23 – argue as in the proof of [33, Ch.IV, Prop.17.1.3] : the details shall be left to the reader.

(ii) is a formal consequence of the definitions (cp. the proof of [33, Ch.IV, Prop.17.1.4]), and (iii) follows from (ii) and proposition 6.3.24(i). $\square$

Proposition 6.3.26. *Let $f : (X, \underline{M}) \to (Y, \underline{N})$ be a morphism of log schemes, and i an exact closed immersion of $(Y, \underline{N})$-schemes, defined by an ideal $\mathscr{I} := \mathrm{Ker}(\mathscr{O}_T \to i_*\mathscr{O}_{T'})$ with $\mathscr{I}^2 = 0$. For any global section $s : T' \to \mathscr{H}_Y(T', X)$, denote by $\mathscr{T}_s$ the morphism :*

$$i^*\mathscr{H}_Y(T, X) \times_{\mathscr{H}_Y(T', X)} T' \to T'$$

deduced from $i^* : i^*\mathscr{H}_Y(T, X) \to \mathscr{H}_Y(T', X)$. Let also $U \subset T'$ be the image *of $\mathscr{T}_s$* (i.e. *the subset of all $t' \in T'$ such that $\mathscr{T}_{s,\xi} \neq \varnothing$ for every τ-point ξ localized at t'), and suppose that $U \neq \varnothing$. We have :*

(i) *U is an open subset of T', and $\mathscr{T}_{s|U}$ is a torsor for the abelian sheaf*

$$\mathscr{G} := \mathscr{H}om_{\mathscr{O}_{T'}}(s^*\Omega^1_{X/Y}(\log(\underline{M}/\underline{N})), \mathscr{I})_{|U}.$$

(ii) *If f is a smooth morphism of log schemes with coherent log structures, we have :*
 (a) *The $\mathscr{O}_X$-module $\Omega^1_{X/Y}(\log(\underline{M}/\underline{N}))$ is locally free of finite type.*
 (b) *If T' is affine, $\mathscr{T}_s$ is a trivial $\mathscr{G}$-torsor.*

Proof. (i): To any τ-point ξ of U, and any two given local sections h and g of $\mathscr{T}_{s,\xi}$, we assign the f-linear derivation of $\underline{M}_{s(\xi)}$ with values in $s_*\mathscr{I}_\xi$ given by the rule :

$$\begin{aligned} a &\mapsto h^*(a) - g^*(a) && \text{for every } a \in \mathscr{O}_{X,s(\xi)} \\ d\log m &\mapsto \log u(m) := u(m) - 1 && \text{for every } m \in \underline{M}_{s(\xi)} \end{aligned}$$

where $u(m)$ is the unique local section of $\mathrm{Ker}(\mathscr{O}_{T,\xi} \to i_*\mathscr{O}_{T',\xi})$ such that

$$\log(g(m) \cdot u(m)) = \log h(m).$$

We leave to the reader the laborious, but straightforward verification that the above map is well-defined, and yields the sought bijection between $\mathscr{T}_{s,\xi}$ and $\mathscr{G}_\xi$.

(ii.b) is standard : first, since f is smooth, we have $U = T'$; by lemma 6.3.20(iii), the $\mathscr{O}_X$-module $\Omega^1_{X/Y}(\log(\underline{M}/\underline{N}))$ is quasi-coherent and finitely presented, hence $\mathscr{G}$ is quasi-coherent; however, the obstruction to gluing local sections of a $\mathscr{G}$-torsor lies in $H^1(T_\tau, \mathscr{G})$ (see (2.4.11)); the latter vanishes whenever T is affine.

(ii.a) We may assume that X is affine, say $X = \mathrm{Spec}\, A$; then $\Omega^1_{X/Y}(\log(\underline{M}/\underline{N}))$ is the quasi-coherent $\mathscr{O}_X$-module arising from a finitely presented A-module Ω (lemma 6.3.20(iii)). Let I be any A-module, and set :

$$T_I := X \oplus I^\sim \qquad \underline{L}_I := \underline{M} \oplus I^\sim$$

(notation of (6.3.14)). Let $i : X \to T_I$ (resp. $\pi : T_I \to X$) be the natural closed immersion (resp. the natural projection); then $(T_I, \underline{L}_I)$ is a fine log scheme, and π (resp. i) extends to a morphism of log schemes $(\pi, \log \pi) : (T_I, \underline{L}_I) \to (X, \underline{M})$ (resp. $(i, \log i) : (X, \underline{M}) \to (T_I, \underline{L}_I)$) with $\log \pi : \pi^*\underline{M} \to \underline{L}_I$ (resp. $\log i : i^*\underline{L}_I \to \underline{M}$) induced by the obvious inclusion (resp.

projection) map. Notice that $(i, \log i)$ is an exact immersion and $(I^\sim)^2 = 0$, hence (i) says that the set $\mathscr{T}(T_I)$ of all morphisms $g : (T_I, \underline{L}_I) \to (X, \underline{M})$ such that

$$f \circ g = f \circ \pi \qquad g \circ i = \mathbf{1}_{(X,\underline{M})}$$

is in bijection with $\mathrm{Hom}_A(\Omega, I)$. Moreover, any map $\varphi : I \to J$ of A-modules induces a morphism $i_\varphi : (T_J, \underline{L}_J) \to (T_I, \underline{L}_I)$ of log schemes, and if φ is surjective, i_φ is an exact nilpotent immersion. Furthermore, we have a commutative diagram of sets :

$$\begin{tikzcd} \mathscr{T}(T_I) \arrow[r, "i_\varphi^*"] \arrow[d] & \mathscr{T}(T_J) \arrow[d] \\ \mathrm{Hom}_A(\Omega, I) \arrow[r, "\varphi_*"] & \mathrm{Hom}_A(\Omega, J) \end{tikzcd}$$

whose vertical arrows are bijections, and where i_φ^* (resp. φ_*) is given by the rule $g \mapsto g \circ i_\varphi$, (resp. $\psi \mapsto \varphi \circ \psi$). Assertion (ii.b) implies that ι_φ^* is surjective when f is smooth and φ is surjective, hence the same holds for φ_*, *i.e.* Ω is a projective A-module, as stated. □

Corollary 6.3.27. *Let $(f, \log f) : (X, \underline{M}) \to (Y, \underline{N})$ be a morphism of log schemes. We have :*

(i) *If $\underline{M}$ and $\underline{N}$ are fine log structures, and f is strict (see definition* 6.1.20(ii)*), then $(f, \log f)$ is smooth (resp. étale) if and only if the underlying morphism of schemes $f : X \to Y$ is smooth (resp. étale).*

(ii) *Suppose that $(f, \log f)$ is a smooth (resp. étale) morphism of log schemes on the Zariski sites of X and Y, and either :*
   (a) *$\underline{M}$ and $\underline{N}$ are both integral log structures,*
   (b) *or else $\underline{N}$ is coherent (on Y_{Zar}) and $\underline{M}$ is fine (on X_{Zar}).*
   Then the induced morphism of log schemes on étale sites :
$$\widetilde{u}^*(f, \log f) := (f, \widetilde{u}_X^* \log f) : \widetilde{u}_X^*(X, \underline{M}) \to \widetilde{u}_Y^*(Y, \underline{N})$$
   is smooth (resp. étale). (Notation of (6.2.2)*.)*

(iii) *Suppose that $(f, \log f)$ is a morphism of log schemes on the Zariski sites of X and Y, and $\underline{M}$ is an integral log structure on X_{Zar}. Suppose also that $\widetilde{u}^*(f, \log f)$ is smooth (resp. étale). Then the same holds for $(f, \log f)$.*

Proof. (i): Suppose first that $(f, \log f)$ is smooth (resp. étale). Let $i : T' \to T$ be a nilpotent immersion of affine schemes, defined by an ideal $\mathscr{I} \subset \mathscr{O}_T$ such that $\mathscr{I}^2 = 0$; let $s : T' \to X$ and $t : T \to Y$ be morphisms of schemes such that $f \circ s = t \circ i$. By lemma 6.1.16(i), the log structures $\underline{L} := t^*\underline{N}$ and $\underline{L}' := s^*\underline{M}$ are fine, and by choosing the obvious maps $\log s$ and $\log t$, we deduce a commutative diagram :

$$\begin{tikzcd} (T', \underline{L}') \arrow[r, "s"] \arrow[d, "i"'] & (X, \underline{M}) \arrow[d, "f"] \\ (T, \underline{L}) \arrow[r, "t"] & (Y, \underline{N}). \end{tikzcd} \tag{6.3.28}$$

Then proposition 6.3.26(ii.b) says that there exists a morphism (resp. a unique morphism) of schemes $g : T \to X$ such that $g \circ i = s$ and $f \circ g = t$, *i.e.* f is smooth (resp. étale).

The converse is easy, and shall be left as an exercise for the reader.

(ii): Suppose first that $(f, \log f)$ is smooth, $\underline{N}$ is coherent and $\underline{M}$ is fine. By proposition 6.3.24(iii), the assertion to prove is local on $X_{\mathrm{ét}}$, hence we may assume that f admits a chart, given by a morphism of finite monoids $P \to P'$ and commutative diagrams :

$$P'_{X_{\mathrm{Zar}}} \to \underline{M} \qquad \omega : P_{Y_{\mathrm{Zar}}} \to \underline{N}$$

(theorem 6.1.35(i)). Now, consider a commutative diagram of log schemes on étale sites :

$$\begin{tikzcd}(T',\underline{L}') \arrow[r, "s"] \arrow[d, "i"'] & \widetilde{u}^*_X(X,\underline{M}) \arrow[d, "\widetilde{u}^*f"] \\ (T,\underline{L}) \arrow[r, "t"] & \widetilde{u}^*_Y(Y,\underline{N})\end{tikzcd} \tag{6.3.29}$$

where i is an exact nilpotent immersion of fine log schemes, defined by an ideal $\mathscr{I} \subset \mathscr{O}_T$ such that $\mathscr{I}^2 = 0$. Since the assertion to prove is local on $T_{\text{ét}}$, we may assume that T is affine and – in view of theorem 6.1.35(ii) – that t admits a chart, given by a morphism of finite monoids $\varphi : P \to Q$, and commutative diagrams :

$$\begin{tikzcd}P_{T_{\text{ét}}} = t^*P_{Y_{\text{ét}}} \arrow[r, "t^*u^*_Y\omega"] \arrow[d, "\varphi_T"'] & t^*\widetilde{u}^*_Y\underline{N} \arrow[d, "\log t"] \\ Q_{T_{\text{ét}}} \arrow[r, "\beta"] & \underline{L}\end{tikzcd} \qquad \begin{tikzcd}P \arrow[r] \arrow[d, "\varphi"'] & \Gamma(Y,\mathscr{O}_Y) \arrow[d, "t^\natural"] \\ Q \arrow[r, "\psi"] & \Gamma(T,\mathscr{O}_T)\end{tikzcd} \tag{6.3.30}$$

Since i is an exact closed immersion, it follows that the morphism :

$$Q_{T'_{\text{ét}}} \xrightarrow{i^*\beta} i^*\underline{L} \xrightarrow{\log i} \underline{L}'$$

is a chart for $\underline{L}'$. Especially, $(T',\underline{L}')$ is isomorphic to $(T'_{\text{ét}}, Q^{\log}_{T'})$, the constant log structure deduced from the morphism $i^\natural \circ \psi : Q \to \Gamma(T',\mathscr{O}_{T'})$. The latter is also the log scheme $\widetilde{u}^*_{T'}(T_{\text{Zar}}, Q^{\log}_{T'})$. From proposition 6.2.3(ii) we deduce that there exists a unique morphism $s_{\text{Zar}} : (T_{\text{Zar}}, Q^{\log}_{T'}) \to (X,\underline{M})$, such that $\widetilde{u}^*s_{\text{Zar}} = s$. On the other hand, by inspecting (6.3.30) we find that there exists a unique morphism $t_{\text{Zar}} : (T_{\text{Zar}}, Q^{\log}_T) \to (Y,\underline{N})$ such that $\widetilde{u}^*t_{\text{Zar}} = t$. These morphisms can be assembled into a diagram :

$$\begin{tikzcd}(T'_{\text{Zar}}, Q^{\log}_{T'}) \arrow[r, "s_{\text{Zar}}"] \arrow[d, "i_{\text{Zar}}"'] & (X,\underline{M}) \arrow[d, "f"] \\ (T_{\text{Zar}}, Q^{\log}_T) \arrow[r, "t_{\text{Zar}}"] & (Y,\underline{N})\end{tikzcd} \tag{6.3.31}$$

where i_{Zar} is an exact closed immersion. By construction, the diagram $\widetilde{u}^*$(6.3.31) is naturally isomorphic to (6.3.29); especially, (6.3.31) commutes (proposition 6.2.3(i)). Now, since f is smooth, proposition 6.3.26(ii.b) implies that there exists a morphism $v : (T_{\text{Zar}}, Q^{\log}_T) \to (X,\underline{M})$ such that $f \circ v = t_{\text{Zar}}$ and $v \circ i_{\text{Zar}} = s_{\text{Zar}}$; then $\widetilde{u}^*v$ provides an appropriate lifting of t, which allows to conclude that $\widetilde{u}^*(f, \log f)$ is smooth.

Next, suppose that $(f, \log f)$ is étale (and we are still in case (b) of the corollary); then there exists a unique morphism v with the properties stated above. However, from proposition 6.2.3(i),(ii) we deduce easily that the natural map :

$$\mathscr{H}_Y((T_{\text{Zar}}, Q^{\log}_T), (X,\underline{M}))(T) \to \mathscr{H}_{\widetilde{u}^*(Y,\underline{N})}((T,\underline{L}), \widetilde{u}^*(X,\underline{M}))(T)$$

is a bijection, and the same holds for the analogous map for T'. In view of the foregoing, this shows that the map $\mathscr{H}_{\widetilde{u}^*(Y,\underline{N})}((T,\underline{L}), \widetilde{u}^*(X,\underline{M}))(T) \to \mathscr{H}_{\widetilde{u}^*(Y,\underline{N})}((T',\underline{L}'), \widetilde{u}^*(X,\underline{M}))(T')$ is bijective, whenever t admits a chart and T is affine. We easily conclude that $\widetilde{u}^*(f, \log f)$ is étale.

The case where both $\underline{N}$ and $\underline{M}$ are both integral, is similar, though easier : we may assume that $\underline{L}$ admits a chart, in which case $(T,\underline{L})$ is of the form $\widetilde{u}^*(T_{\text{Zar}}, Q^{\log}_T)$ for some finite integral monoid Q; then $(T',\underline{L}')$ admits a similar description, and again, by appealing to proposition 6.2.3(ii) we deduce that (6.3.29) is of the form $\widetilde{u}^*$(6.3.31), in which case we conclude as in the foregoing.

Conversely, suppose that $\widetilde{u}^*(f, \log f)$ is smooth, and consider a commutative diagram (6.3.28) of log schemes on Zariski sites, with i an exact closed immersion, defined by a sheaf of ideals $\mathscr{I} \subset \mathscr{O}_T$ such that $\mathscr{I}^2 = 0$. By applying everywhere the pull-back functors from Zariski to étale sites, we deduce a commutative diagram $\widetilde{u}^*$(6.3.28) of log schemes on étale sites, and it is easy to see that $\widetilde{u}^*(i, \log i)$ is again an exact nilpotent immersion. According to proposition 6.3.26(ii.b), after replacing T by any affine open subset, we may find a morphism of log schemes $g : \widetilde{u}^*_T(T, \underline{L}) \to \widetilde{u}^*_X(X, \underline{M})$, such that $g \circ \widetilde{u}^* i = \widetilde{u}^* s$ and $\widetilde{u}^* f \circ g = t$. By proposition 6.2.3(i),(ii) there exists a unique morphism $g' : (T, \underline{L}) \to (X, \underline{M})$ such that $\widetilde{u}^* g' = g$, and necessarily $g' \circ i = s$ and $f \circ g' = t$. We conclude that $(f, \log f)$ is smooth.

Finally, if $\widetilde{u}^*(f, \log f)$ is étale, the morphism g exhibited above is unique, and therefore the same holds for g', so $(f, \log f)$ is étale as well. $\square$

Proposition 6.3.32. *In the situation of* (6.3.11)*, suppose that the log structures $\underline{M}, \underline{N}, \underline{P}$ are coherent, and consider the following conditions :*

(a) *f is smooth (resp. étale).*

(b) *df is a locally split monomorphism (resp. an isomorphism).*

Then (a)⇒(b)*, and if $g \circ f$ is smooth, then* (b)⇒(a).

Proof. We may assume that the schemes under consideration are affine, say $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$, $Z = \operatorname{Spec} C$; then (6.3.12) amounts to an exact sequence of A-modules :

$$A \otimes_B \Omega(g) \xrightarrow{df} \Omega(g \circ f) \xrightarrow{\omega} \Omega(f) \to 0.$$

On the other hand, let $i : (T'\underline{L}') \to (T, \underline{L})$ be an exact closed immersion of $(Y, \underline{N})$-schemes, defined by an ideal $\mathscr{I} \subset \mathscr{O}_T$ with $\mathscr{I}^2 = 0$. Suppose that $s : T' \to \mathscr{H}_Y(T', X)$ is a global section, *i.e.* a given morphism of $(Y, \underline{N})$-schemes $(T'\underline{L}') \to (X, \underline{M})$. In this situation, we have a natural sequence of morphisms :

$$\mathscr{H}_1 := \mathscr{H}_Y(T, X) \xrightarrow{\alpha} \mathscr{H}_2 := \mathscr{H}_Z(T, X) \xrightarrow{f_*} \mathscr{H}_3 := \mathscr{H}_Z(T, Y)$$

where α is the obvious monomorphism. Let us consider the pull-back of these sheaves along the global section s :

$$\mathscr{T} := T' \times_{\mathscr{H}_Y(T',X)} i^*\mathscr{H}_1 \qquad \mathscr{T}' := T' \times_{\mathscr{H}_Z(T',X)} i^*\mathscr{H}_2 \qquad \mathscr{T}'' := T' \times_{\mathscr{H}_Z(T',Y)} i^*\mathscr{H}_3.$$

By proposition 6.3.26(i), any choice of a global section of $\mathscr{T}(T)$ determines a commutative diagram of sets :

$$\begin{tikzcd}
\operatorname{Hom}_A(\Omega(f), \mathscr{I}(T)) \arrow[r, "\omega^*"] \arrow[d] & \operatorname{Hom}_A(\Omega(g \circ f), \mathscr{I}(T)) \arrow[r, "df^*"] \arrow[d] & \operatorname{Hom}_B(\Omega(g), \mathscr{I}(T)) \arrow[d] \\
\mathscr{T}(T) \arrow[r] & \mathscr{T}'(T) \arrow[r, "f_*"] & \mathscr{T}''(T)
\end{tikzcd} \tag{6.3.33}$$

whose vertical arrows are bijections.

In order to show that df is a locally split monomorphism (resp. an isomorphism), it suffices to show that the map $df^* := \operatorname{Hom}_A(df, I)$ is surjective for every A-module I. To this aim, we take $(T, L) := (T_I, \underline{L}_I)$, $(T', \underline{L}') := (X, \underline{M})$, and let i be the nilpotent immersion defined by the ideal $I \subset A \oplus I$, as in the proof of proposition 6.3.26(ii.a). If $\mathscr{I} \subset \mathscr{O}_T$ is the corresponding sheaf of ideals, then $\mathscr{I}(T) = I$; moreover, the inclusions $A \to A \oplus I$ and $\underline{M} \subset \underline{M} \oplus \mathscr{I}$ determine a morphism $h : (T_I, \underline{L}_I) \to (X, \underline{M})$, which is a global section of $\mathscr{T}(T)$. Having made these choices, consider the resulting diagram (6.3.33) : if f is étale, f_* is an isomorphism, and when f is smooth, f_* is surjective (proposition 6.3.26(ii.b)), whence the contention.

For the converse, we may suppose that df is a split monomorphism, hence df^* in (6.3.33) is a split surjection. We have to show that $\mathscr{T}(T)$ is not empty, and by assumption (and proposition

6.3.26(ii.b)) we know that $\mathscr{T}'(T) \neq \varnothing$. Choose any $\widetilde{h} \in \mathscr{T}'(T)$; then t and $f_*\widetilde{h}$ are two elements of $\mathscr{T}''(T)$, so we may find $\varphi \in \mathrm{Hom}_B(\Omega(g), \mathscr{I}(T))$ such that $\varphi + f_*\widetilde{h} = t$ (where the sum denotes the action of $\mathrm{Hom}_B(\Omega(g), \mathscr{I}(T))$ on its torsor $\mathscr{T}''$). Then we may write $\varphi = \psi \circ df$ for some $\psi : \Omega(g \circ f) \to \mathscr{I}(Y)$, and it follows easily that $\psi + \widetilde{h}$ lies in $\mathscr{T}(T)$. Finally, if df is an isomorphism, we have $\Omega(f) = 0$, hence $\mathscr{T}(T)$ contains exactly one element. $\square$

Proposition 6.3.34. *Let R be a ring, $\varphi : P \to Q$ a morphism of finitely generated monoids, such that $\mathrm{Ker}\,\varphi^{\mathrm{gp}}$ and the torsion subgroup of $\mathrm{Coker}\,\varphi^{\mathrm{gp}}$ (resp. $\mathrm{Ker}\,\varphi^{\mathrm{gp}}$ and $\mathrm{Coker}\,\varphi^{\mathrm{gp}}$) are finite groups whose orders are invertible in R. Then, the induced morphism*

$$\mathrm{Spec}(R, \varphi) : \mathrm{Spec}(R, Q) \to \mathrm{Spec}(R, P)$$

is smooth (resp. étale).

Proof. To ease notation, set

$$f := \mathrm{Spec}(R, \varphi) \qquad (X, \underline{M}) := \mathrm{Spec}(R, Q) \qquad (Y, \underline{N}) := \mathrm{Spec}(R, P).$$

Clearly f is finitely presented. We have to show that, for every commutative diagram like (6.3.28) with i an exact nilpotent immersion of fine log schemes, there is, locally on T_τ at least one morphism (resp. a unique morphism) $h : (T, \underline{L}) \to (X, \underline{M})$ such that $f \circ h = h \circ i$.

Let $\mathscr{I} := \mathrm{Ker}(\mathscr{O}_T \to i_*\mathscr{O}_{T'})$; by considering the $\mathscr{I}$-adic filtration on $\mathscr{O}_T$, we reduce easily to the case where $\mathscr{I}^2 = 0$, and then we may embed $\mathscr{I}$ in $\underline{L}$ via the morphism :

$$\mathscr{I} \to \mathscr{O}_T^\times \subset \underline{L} \qquad x \mapsto 1 + x.$$

(Here $\mathscr{I}$ is regarded as a sheaf of abelian groups, via its addition law.) Since i is exact, the natural morphism $\underline{L}/\mathscr{I} \to i_*\underline{L}'$ is an isomorphism, whence a commutative diagram :

$$\begin{tikzcd} \underline{L} \arrow[r, "\log i"] \arrow[d] & i_*\underline{L}' \arrow[d] \\ \underline{L}^{\mathrm{gp}} \arrow[r, "(\log i)^{\mathrm{gp}}"] & i_*(\underline{L}')^{\mathrm{gp}} \simeq \underline{L}^{\mathrm{gp}}/\mathscr{I} \end{tikzcd} \tag{6.3.35}$$

and since $\underline{L}$ is integral, one sees easily that (6.3.35) is cartesian (it suffices to consider the stalks over the τ-points).

• On the other hand, suppose first that both $\mathrm{Ker}\,\varphi^{\mathrm{gp}}$ and $\mathrm{Coker}\,\varphi^{\mathrm{gp}}$ are finite groups whose order is invertible in R, hence in $\mathscr{I}$; then a standard diagram chase shows that we may find a unique map $g : P_T^{\mathrm{gp}} \to L^{\mathrm{gp}}$ of abelian sheaves that fits into a commutative diagram :

$$\begin{tikzcd} Q_T^{\mathrm{gp}} \arrow[r] \arrow[d, "\varphi_T^{\mathrm{gp}}"'] & t^*\underline{N}^{\mathrm{gp}} \arrow[r, "(\log t)^{\mathrm{gp}}"] & \underline{L}^{\mathrm{gp}} \arrow[d, "(\log i)^{\mathrm{gp}}"] \\ P_T^{\mathrm{gp}} \arrow[r] \arrow[urr, "g"] & i_*s^*\underline{M}^{\mathrm{gp}} \arrow[r, "i_*(\log s)^{\mathrm{gp}}"] & i_*(\underline{L}')^{\mathrm{gp}}. \end{tikzcd} \tag{6.3.36}$$

• More generally, we may write : $\mathrm{Coker}\,\varphi^{\mathrm{gp}} \simeq G \oplus H$, where H is a finite group with order invertible in R, and G is a free abelian group of finite rank. The direct summand G lifts to a direct summand $G' \subset P^{\mathrm{gp}}$. Extend φ^{gp} to a map $\psi : Q^{\mathrm{gp}} \oplus G' \to P^{\mathrm{gp}}$, by the rule : $(x, g) \mapsto \varphi^{\mathrm{gp}}(x) \cdot g$. Given any τ-point ξ of T, we may extend the morphism $Q^{\mathrm{gp}} = Q_{T,\xi}^{\mathrm{gp}} \to \underline{L}_\xi^{\mathrm{gp}}$ in (6.3.36)$_\xi$, to a map $\omega : Q^{\mathrm{gp}} \oplus G' \to \underline{L}_\xi^{\mathrm{gp}}$ whose composition with $(\log i)_\xi^{\mathrm{gp}}$ agrees with the composition of ψ and the bottom map $P^{\mathrm{gp}} = P_{T,\xi}^{\mathrm{gp}} \to i_*(\underline{L}')_\xi^{\mathrm{gp}}$ of (6.3.36)$_\xi$. By the usual arguments, ω extends to a map of abelian sheaves $\vartheta : (Q^{\mathrm{gp}} \oplus G')_U \to (\underline{L}^{\mathrm{gp}})_{|U}$ on some neighborhood $U \to T$ of ξ, and if U is small enough, the composition $(\log i)_{|U}^{\mathrm{gp}} \circ \vartheta$ agrees with the composition of ψ_U and the bottom map $\beta : P_U^{\mathrm{gp}} \to i_*(\underline{L}'^{\mathrm{gp}})_{|U}$ of (6.3.36)$_{|U}$. We may then replace T by U, and since $\mathrm{Ker}\,\psi = \mathrm{Ker}\,\varphi^{\mathrm{gp}}$, and $\mathrm{Coker}\,\psi = H$, the same diagram chase as in the foregoing shows that we may again find a morphism $g : P_T^{\mathrm{gp}} \to \underline{L}^{\mathrm{gp}}$ fitting into a commutative diagram :

$$\begin{tikzcd}
(Q^{\mathrm{gp}} \oplus G')_T \arrow[r, "\vartheta"] \arrow[d, "\psi_T"'] & \underline{L}^{\mathrm{gp}} \arrow[d, "(\log i)^{\mathrm{gp}}"] \\
P_T^{\mathrm{gp}} \arrow[r, "\beta"] \arrow[ur, "g"] & i_*\underline{L}'^{\mathrm{gp}}
\end{tikzcd}$$

In either case, in view of (6.3.35), the morphisms

$$P_T \to P_T^{\mathrm{gp}} \xrightarrow{g} \underline{L}^{\mathrm{gp}} \qquad \text{and} \qquad P_T \to i_*s^*\underline{M} \xrightarrow{i_*\log s} i_*\underline{L}'$$

determine a unique morphism $P_T \to \underline{L}$, which induces a morphism of log schemes $(T, \underline{L}) \to (X, \underline{M})$ with the sought property. $\square$

Theorem 6.3.37. *Let $f : (X, \underline{M}) \to (Y, \underline{N})$ be a morphism of fine log schemes. Assume we are given a fine chart $\beta : Q_Y \to \underline{N}$ of $\underline{N}$. Then the following conditions are equivalent :*

(a) *f is smooth (resp. étale).*

(b) *There exists a covering family $(g_\lambda : U_\lambda \to X \mid \lambda \in \Lambda)$ in $X_{\mathrm{\acute{e}t}}$, and for every $\lambda \in \Lambda$, a fine chart $(\beta, (P_\lambda)_{U_\lambda} \to g_\lambda^*\underline{M}, \varphi_\lambda : Q \to P_\lambda)$ of the induced morphism of log schemes*

$$f_{|U_\lambda} := (f \circ g_\lambda, g_\lambda^* \log f) : (U_\lambda, g_\lambda^*\underline{M}) \to (Y, \underline{N})$$

such that :

(i) $\mathrm{Ker}\,\varphi_\lambda^{\mathrm{gp}}$ *and the torsion subgroup of* $\mathrm{Coker}\,\varphi_\lambda^{\mathrm{gp}}$ *(resp.* $\mathrm{Ker}\,\varphi_\lambda^{\mathrm{gp}}$ *and* $\mathrm{Coker}\,\varphi_\lambda^{\mathrm{gp}}$*) are finite groups of orders invertible in $\mathscr{O}_{U_\lambda}$.*

(ii) *The natural morphism of Y-schemes $p_\lambda : U_\lambda \to Y \times_{\mathrm{Spec}\,\mathbb{Z}[Q]} \mathrm{Spec}\,\mathbb{Z}[P_\lambda]$ is étale.*

Proof. Suppose first that $\tau = \mathrm{\acute{e}t}$, so $\underline{M}$ and $\underline{N}$ are log structures on étale sites. Then the log structure $g_\lambda^*\underline{M}$ on $(U_\lambda)_{\mathrm{\acute{e}t}}$ is just the restriction of $\underline{M}$, and $g_\lambda^* \log f$ is the restriction of $\log f$ to $(U_\lambda)_{\mathrm{\acute{e}t}}$. In case $\tau = \mathrm{Zar}$, the log structure $g_\lambda^*\underline{M}$ can be described as follows. Form the log scheme $(X, \underline{M}') := \widetilde{u}_X^*(X, \underline{M})$ (notation of (6.2.2)), take the restriction $\underline{M}'_{|U_\lambda}$ of $\underline{M}'$ to $(U_\lambda)_{\mathrm{\acute{e}t}}$, and push forward to the Zariski site to obtain $\widetilde{u}_{X*}(\underline{M}'_{|U_\lambda}) = g_\lambda^*\underline{M}$ (by proposition 6.2.3(ii)). Since f is smooth (resp. étale) if and only if $\widetilde{u}^*f$ is (corollary 6.3.27(ii),(iii)), we conclude that the assertion concerning f holds if and only if the corresponding assertion for $\widetilde{u}^*f$ does. Hence, it suffices to consider the case of log structures on étale sites. Set $S := \mathrm{Spec}\,\mathbb{Z}[Q]$.

(b)⇒(a): Taking into account lemma 6.2.18(i), we deduce a commutative diagram of log schemes :

$$\begin{tikzcd}
(U_\lambda, \underline{M}_{|U_\lambda}) \arrow[r, "\sim"] \arrow[d, "g_\lambda"'] & \mathrm{Spec}(\mathbb{Z}, P_\lambda) \times_{\mathrm{Spec}\,\mathbb{Z}[P_\lambda]} U_\lambda \arrow[r, "p_\lambda"] & \mathrm{Spec}(\mathbb{Z}, P_\lambda) \times_{\mathrm{Spec}\,\mathbb{Z}[Q]} Y \arrow[d, "\pi_\lambda"] \\
(X, \underline{M}) \arrow[r, "f"] & (Y, \underline{N}) \arrow[r, "\sim"] & \mathrm{Spec}(\mathbb{Z}, Q) \times_{\mathrm{Spec}\,\mathbb{Z}[Q]} Y.
\end{tikzcd}$$

It follows by corollary 6.3.27(i) (resp. by propositions 6.3.24(ii) and 6.3.34) that p_λ (resp. π_λ) is smooth. Hence $f \circ g_\lambda$ is smooth, by proposition 6.3.24(i). Finally, f is smooth, by proposition 6.3.24(iii).

(a)⇒(b): Suppose that f is smooth, and fix a geometric point ξ of X. Since $(6.3.18)_\xi$ is a surjection, we may find elements $t_1, \dots, t_r \in \underline{M}_\xi$ such that $(d \log t_i \mid i = 1, \dots, r)$ is a basis of the free $\mathscr{O}_{X,\xi}$-module $\Omega^1_{X/Y}(\log \underline{M}/\underline{N})_\xi$ (proposition 6.3.26(ii.a)). Moreover, the kernel of $(6.3.18)_\xi$ is generated by sections of the form :

- $1 \otimes \log a$ where $a \in N' := \mathrm{Im}(\log f_\xi : \underline{N}_{f(\xi)} \to \underline{M}_\xi)$
- $\sum_{j=1}^s \alpha(m_i) \otimes \log m_i$ where $m_1, \dots, m_s \in \underline{M}_\xi$ and $\sum_{j=1}^s d\alpha(m_j) = 0$ in $\Omega^1_{X/Y}$

whence a well-defined $\mathscr{O}_{X,\xi}$-linear map :

$$\Omega^1_{X/Y}(\log \underline{M}/\underline{N})_\xi \to \kappa(\xi) \otimes_{\mathbb{Z}} (\underline{M}^{\mathrm{gp}}_\xi/(\underline{M}^\times_\xi \cdot N')) \quad : \quad d\log a \mapsto 1 \otimes a. \tag{6.3.38}$$

Consider the map of monoids :

$$\varphi : P_1 := \mathbb{N}^{\oplus r} \oplus Q \to \underline{M}_\xi$$

which is given by the rule : $e_i \mapsto t_i$ on the canonical basis $e_1, \dots, e_r$ of $\mathbb{N}^{\oplus r}$, and on the summand Q it is given by the map $Q \xrightarrow{\beta_\xi} \underline{N}_{f(\xi)} \xrightarrow{\log f_\xi} \underline{M}_\xi$. Since (6.3.38) is a surjection, we see that the same holds for the induced map

$$\kappa(\xi) \otimes_{\mathbb{Z}} P_1^{\mathrm{gp}} \xrightarrow{\mathbf{1}_{\kappa(\xi)} \otimes_{\mathbb{Z}} \varphi^{\mathrm{gp}}} \kappa(\xi) \otimes_{\mathbb{Z}} \underline{M}^{\mathrm{gp}}_\xi \to \kappa(\xi) \otimes_{\mathbb{Z}} (\underline{M}^{\mathrm{gp}}_\xi/\underline{M}^\times_\xi).$$

It follows that the cokernel of the map $\overline{\varphi} : P_1^{\mathrm{gp}} \to \underline{M}^{\mathrm{gp}}_\xi/\underline{M}^\times_\xi$ induced by φ^{gp}, is a finite group (lemma 6.2.21(i)) annihilated by an integer n which is invertible in $\mathscr{O}_{X,\xi}$. Let $m_1, \dots, m_s \in \underline{M}^{\mathrm{gp}}_\xi$ be finitely many elements, whose images in $\underline{M}^{\mathrm{gp}}_\xi/\underline{M}^\times_\xi$ generate $\mathrm{Coker}\,\overline{\varphi}$; therefore we may find $u_1, \dots, u_s \in \underline{M}^\times_\xi$, and $x_1, \dots, x_s \in P_1^{\mathrm{gp}}$, such that $m_i^n \cdot u_i = \overline{\varphi}(x_i)$ for every $i \leq s$. However, since $\mathscr{O}_{X,\xi}$ is strictly henselian, $\underline{M}^\times_\xi \simeq \mathscr{O}^\times_{X,\xi}$ is n-divisible, hence we may find $v_1, \dots, v_s$ in $\underline{M}^\times_\xi$ such that $u_i = v_i^n$ for $i = 1, \dots, n$. Define group homomorphisms :

$$\mathbb{Z}^{\oplus s} \xrightarrow{\gamma} \mathbb{Z}^{\oplus s} \oplus P_1^{\mathrm{gp}} \xrightarrow{\delta} \underline{M}^{\mathrm{gp}}_\xi$$

by the rules : $\gamma(e_i) = (-ne_i, x_i)$ and $\delta(e_i, y) = m_i v_i \cdot \varphi(y)$ for every $i = 1, \dots, s$ and every $y \in P_1^{\mathrm{gp}}$. It is easily seen that δ factors through a group homomorphism $h : G := \mathrm{Coker}\,\gamma \to \underline{M}_\xi$; moreover the natural map $P_1^{\mathrm{gp}} \to G$ is injective, and its cokernel is annihilated by n; furthermore, the induced map $G \to \underline{M}^\sharp_\xi$ is surjective. Let $P := h^{-1}\underline{M}_\xi$. Then, the natural map $Q^{\mathrm{gp}} \to P^{\mathrm{gp}}$ is injective, and the torsion subgroup of $P^{\mathrm{gp}}/Q^{\mathrm{gp}}$ is annihilated by n. We deduce that the rule : $x \mapsto d\log h(x)$ for every $x \in P^{\mathrm{gp}}$, induces an isomorphism :

$$\mathscr{O}_{X,\xi} \otimes_{\mathbb{Z}} (P^{\mathrm{gp}}/Q^{\mathrm{gp}}) \xrightarrow{\sim} \Omega^1_{X/Y}(\log \underline{M}/\underline{N})_\xi. \tag{6.3.39}$$

It follows that we may find an étale neighborhood $U \to X$ of ξ, such that (6.3.39) extends to an isomorphism of $\mathscr{O}_U$-modules :

$$\mathscr{O}_U \otimes_{\mathbb{Z}} (P^{\mathrm{gp}}/Q^{\mathrm{gp}}) \xrightarrow{\sim} \Omega^1_{U/Y}(\log \underline{M}_{|U}/\underline{N}). \tag{6.3.40}$$

Next, proposition 6.1.28 says that, after replacing U by a smaller étale neighborhood of ξ, the restriction $h_{|P} : P \to \underline{M}_\xi$ extends to a chart $P_U \to \underline{M}_{|U}$, whence a strict morphism

$$p : (U, \underline{M}_{|U}) \to (Y', P^{\log}_{Y'}) := (Y, \underline{N}) \times_{\mathrm{Spec}(\mathbb{Z},Q)} \mathrm{Spec}(\mathbb{Z}, P).$$

as sought. Taking into account corollary 6.3.27(i), it remains only to show :

Claim 6.3.41. p is étale.

Proof of the claim. By proposition 6.3.13 and example 6.3.19, we have natural isomorphisms

$$\Omega^1_{Y'/Y}(\log P^{\log}_{Y'}/\underline{N}) \xrightarrow{\sim} \mathscr{O}_{Y'} \otimes_{\mathbb{Z}} (P^{\mathrm{gp}}/Q^{\mathrm{gp}}).$$

In view of (6.3.40), it follows that the map

$$dp : p^*\Omega^1_{Y'/Y}(\log P^{\log}_{Y'}/\underline{N}) \to \Omega^1_{U/Y}(\log \underline{M}_{|U}/\underline{N})$$

is an isomorphism, and then the claim follows from proposition 6.3.32. ☐

Corollary 6.3.42. *Keep the notation of theorem* 6.3.37. *Suppose that f is a smooth morphism of fs log schemes, and that Q is fine, sharp and saturated. Then there exists a covering family $(g_\lambda : U_\lambda \to X)$ in $X_{\text{ét}}$, and fine and saturated charts $(\beta, (P_\lambda)_{U_\lambda} \to g^*_\lambda \underline{M}, \varphi_\lambda : Q \to P_\lambda)$ of $f_{|U_\lambda}$ fulfilling conditions* (b.i) *and* (b.ii) *of the theorem, and such that moreover φ_λ is injective, and $P^\times_\lambda$ is a torsion-free abelian group, for every $\lambda \in \Lambda$.*

Proof. Notice that, under the stated assumptions, Q^{gp} is a torsion-free abelian group; hence theorem 6.3.37 already implies the existence of a covering family $(g_\lambda : U_\lambda \to X)$ in $X_{\mathrm{\acute{e}t}}$, and of fine charts $(\beta, \omega'_\lambda : (P'_\lambda)_{U_\lambda} \to g^*_\lambda \underline{M}, \varphi'_\lambda : Q \to P'_\lambda)$ such that φ'_λ is injective and $\mathrm{Coker}(\varphi'_\lambda)^{\mathrm{gp}}$ is a finite group of order invertible in $\mathscr{O}_{U_\lambda}$, for every $\lambda \in \Lambda$. Since $g^*_\lambda \underline{M}$ is saturated (lemma 6.1.16(i)), it is clear that ω'_λ factors through a morphism $(P'_\lambda)^{\mathrm{sat}}_{U_\lambda} \to g^*_\lambda \underline{M}$ of pre-log structures, which is again a chart, so we may assume that P'_λ is fine and saturated, for every $\lambda \in \Lambda$ (corollary 3.4.1(ii)), in which case we may find an isomorphism of monoids $P'_\lambda = P_\lambda \times G$, where $P^\times_\lambda$ is torsion-free, and G is a finite group (lemma 3.2.10). Let d be the order of $\mathrm{Coker}(\varphi'_\lambda)^{\mathrm{gp}}$, and denote by φ_λ the composition of φ'_λ and the projection $P'_\lambda \to P_\lambda$; since Q^{gp} is a torsion-free abelian group, we deduce a short exact sequence :

$$0 \to G \to \mathrm{Coker}(\varphi'_\lambda)^{\mathrm{gp}} \to \mathrm{Coker}\, \varphi^{\mathrm{gp}}_\lambda \to 0 \qquad (6.3.43)$$

and notice that φ_λ is also injective. We may assume that U_λ and Y are affine, say $U_\lambda = \mathrm{Spec}\, B_\lambda$ and $Y = \mathrm{Spec}\, A$, and since d is invertible in $\mathscr{O}_{U_\lambda}$, we reduce easily – via base change by a finite morphism $Y' \to Y$ – to the case where A contains the subgroup $\boldsymbol{\mu}_d \subset \overline{\mathbb{Q}}^\times$ of d-th power roots of 1. The chart ω'_λ determines a morphism of monoids $P'_\lambda \to B_\lambda$, and the map $f^\sharp : A \to B_\lambda$ factors through the natural ring homomorphism

$$A \to A \otimes_{R[Q]} R[P'_\lambda] \xrightarrow{\sim} A \otimes_{R[Q]} (R[P_\lambda] \otimes_R R[G]) \qquad \text{where} \qquad R := \mathbb{Z}[d^{-1}, \boldsymbol{\mu}_d].$$

On the other hand, let $\Gamma := \mathrm{Hom}_{\mathbb{Z}}(G, \boldsymbol{\mu}_d)$; then we have a natural decomposition

$$R[G] \simeq \prod_{\chi \in \Gamma} e_\chi R[G]$$

where e_χ is the idempotent of $R[G]$ defined as in (8.6.11) (cp. the proof of theorem 8.6.23(i)); each factor is a ring isomorphic to R, whence a corresponding decomposition of U_λ as a disjoint union of Y-schemes $U_\lambda = \coprod_{\chi \in \Gamma} U_{\lambda,\chi}$. We are then further reduced to the case where $U_\lambda = U_{\lambda,\chi}$ for some character χ of G. In view of (6.3.43), it is easily seen that the composition

$$Q^{\mathrm{gp}} \xrightarrow{(\varphi'_\lambda)^{\mathrm{gp}}} (P'_\lambda)^{\mathrm{gp}} \to G \xrightarrow{\chi} \boldsymbol{\mu}_d$$

extends to a well defined group homomorphism $\overline{\chi} : P^{\mathrm{gp}}_\lambda \to \boldsymbol{\mu}_d$, whence a map $\overline{\chi}_{U_\lambda} : P_{\lambda,U_\lambda} \to \mu_{d,U_\lambda} \subset g^*_\lambda \underline{M}$ of sheaves on $U_{\lambda,\tau}$. Define $\omega_\lambda : P_{\lambda,U_\lambda} \to g^*_\lambda \underline{M}$ by the rule : $s \mapsto \omega'_\lambda(s) \cdot \overline{\chi}_{U_\lambda}(s)$ for every local section s of P_{λ,U_λ}. It is easily seen that ω_λ is again a chart for $g^*_\lambda \underline{M}$ (*e.g.* one may apply lemma 6.1.4). Lastly, a direct inspection shows that $(\beta, \omega_\lambda, \varphi_\lambda)$ is a chart of $f_{|U_\lambda}$ with the sought properties. □

6.3.44. Let $(Y_i \mid i \in I)$ be a cofiltered system of quasi-compact and quasi-separated schemes, with affine transition morphisms, and suppose that 0 is an initial object of the indexing category I. Suppose also that $g_0 : X_0 \to Y_0$ is a finitely presented morphism of schemes. Let $X_i := X_0 \times_{Y_0} Y_i$ for every $i \in I$, and denote $g_i : X_i \to Y_i$ the induced morphism. Let also $g : X \to Y$ be the limit of the family of morphisms $(g_i \mid i \in I)$.

Corollary 6.3.45. *In the situation of* (6.3.44)*, suppose that* $(g, \log g) : (X, \underline{M}) \to (Y, \underline{N})$ *is a smooth morphism of fine log schemes. Then there exists* $i \in I$*, and a smooth morphism* $(g_i, \log g_i) : (X_i, \underline{M}_i) \to (Y_i, \underline{N}_i)$ *of fine log schemes, such that* $\log g = \pi^*_i \log g_i$*.*

Proof. First, using corollary 6.2.33, we may find $i \in I$ such that $(g, \log g)$ descends to a morphism $(g_i, \log g_i)$ of log schemes with coherent log structures. After replacing I by I/i, we may then suppose that $i = 0$, in which case we set $(X_i, \underline{M}_i) := X_i \times (X_0, \underline{M}_0)$, and define likewise $(Y_i, \underline{N}_i)$ for every $i \in I$.

Next, arguing as in the proof of corollary 6.2.34(ii), we may assume that $\underline{N}_0$ admits a fine chart. In this case, also $\underline{N}$ admits a fine chart, and then we my find a covering family $\mathscr{U} := (U_\lambda \to X \mid \lambda \in \Lambda)$ for X_τ, such that the induced morphism $(U_\lambda, \underline{M}_{|U_\lambda}) \to (Y, \underline{N})$ admits a chart fulfilling conditions (i) and (ii) of theorem 6.3.37. Moreover, under the current assumptions, X is quasi-compact, hence we may assume that Λ is a finite set, in which case there exists $i \in I$ such that $\mathscr{U}$ descends to a covering family $(U_{i,\lambda} \to X_i \mid \lambda \in \Lambda)$ for $X_{i,\tau}$ (claim 6.2.27(ii)), and as usual, we may then reduce to the case where $i = 0$. It then suffices to show that there exists $i \in I$ such that the induced morphism $U_{0,\lambda} \times_{X_0} (X_i, \underline{M}_i) \to (Y_i, \underline{N}_i)$ is smooth (proposition 6.3.24(iii)). Thus, we may replace the system $(X_i \mid i \in I)$ by the system of schemes $(U_{0,\lambda} \times_{X_0} X_i \mid i \in I)$, and assume from start that $(g, \log g)$ admits a fine chart fulfilling conditions (i) and (ii) of theorem 6.3.37. In this case, corollary 6.2.34(i) and [32, Ch.IV, Prop.17.7.8(ii)] imply more precisely that there exists $i \in I$ such that the morphism $(X_i, \underline{M}_i) \to (Y_i, \underline{N}_i)$ fulfills conditions (i) and (ii) of theorem 6.3.37, whence the contention. □

6.4. **Logarithmic blow up of a coherent ideal.** This section introduces the logarithmic version of the scheme-theoretic blow up of a coherent ideal. We begin by recalling a few generalities on graded algebras and their homogeneous prime spectra; next we introduce the logarithmic variants of these constructions. Finally, the logarithmic blow up shall be exhibited as the logarithmic homogeneous spectrum of a certain graded algebra, naturally attached to any sheaf of ideals in a log structure.

6.4.1. Let $A := \oplus_{n\in\mathbb{N}} A_n$ be a $\mathbb{N}$-graded ring, and set $A_+ := \oplus_{n>0} A_n$, which is an ideal of A. Following [27, Ch.II, (2.3.1)], one denotes by $\mathrm{Proj}\, A$ the set consisting of all *graded prime ideals* of A that do not contain A_+, and one endows $\mathrm{Proj}\, A$ with the topology induced from the Zariski topology of $\mathrm{Spec}\, A$. For every homogeneous element $f \in A_+$, set :

$$D_+(f) := D(f) \cap \mathrm{Proj}\, A$$

where as usual, $D(f) := \mathrm{Spec}\, A_f \subset \mathrm{Spec}\, A$. The system of open subsets $D_+(f)$, for f ranging over the homogeneous elements of A_+, is a basis of the topology of $\mathrm{Proj}\, A$ ([27, Ch.II, Prop.2.3.4]). Clearly :

$$D_+(fg) = D_+(f) \cap D_+(g) \qquad \text{for every homogeneous } f, g \in A_+. \tag{6.4.2}$$

For any homogeneous element $f \in A_+$, let also $A_{(f)} \subset A_f$ be the subring consisting of all elements of degree zero (for the natural $\mathbb{Z}$-grading of A_f); in other words :

$$A_{(f)} := \sum_{k\in\mathbb{N}} A_k \cdot f^{-k} \subset A_f.$$

The topological space $\mathrm{Proj}\, A$ carries a sheaf of rings $\mathscr{O}$, with isomorphisms of ringed spaces :

$$\omega_f : (D_+(f), \mathscr{O}_{|D_+(f)}) \xrightarrow{\sim} \mathrm{Spec}\, A_{(f)} \qquad \text{for every homogeneous } f \in A_+.$$

and the system of isomorphisms ω_f is compatible, in an obvious way, with the inclusions :

$$j_{f,g} : D_+(fg) \subset D_+(f)$$

as in (6.4.2), and with the natural homomorphisms $A_{(f)} \to A_{(fg)}$. Especially, the locally ringed space $(\mathrm{Proj}\, A, \mathscr{O})$ is a separated scheme.

6.4.3. Let $A' := \oplus_{n\in\mathbb{N}} A'_n$ be another $\mathbb{N}$-graded ring, and $\varphi : A \to A'$ a homomorphism of graded rings (*i.e.* $\varphi(A_n) \subset A'_n$ for every $n \in \mathbb{N}$). Following [27, Ch.II, (2.8.1)], we let :

$$G(\varphi) := \mathrm{Proj}\, A' \setminus V(\varphi(A_+)).$$

This open subset of $\mathrm{Proj}\, A'$ is also the same as the union of all the open subsets of the form $D_+(\varphi(f))$, where f ranges over the homogeneous elements of A_+. The restriction to $G(\varphi)$ of $\mathrm{Spec}\, \varphi : \mathrm{Spec}\, A' \to \mathrm{Spec}\, A$, is a continuous map ${}^a\varphi : G(\varphi) \to \mathrm{Proj}\, A$. Moreover, we have the identity :

$${}^a\varphi^{-1}(D_+(f)) = D_+(\varphi(f)) \qquad \text{for every homogeneous } f \in A_+.$$

Furthermore, the homomorphism φ induces a homomorphism $\varphi_{(f)} : A_{(f)} \to A'_{(\varphi(f))}$, whence a morphism of schemes :

$$\Phi_f : D_+(\varphi(f)) \to D_+(f).$$

Let $g \in A_+$ be another homogeneous element; it is easily seen that :

$$j_{f,g} \circ \Phi_{fg} = (\Phi_f)_{|D_+(\varphi(fg))}.$$

It follows that the locally defined morphisms Φ_f glue to a unique morphism of schemes :

$$\mathrm{Proj}\,\varphi : G(\varphi) \to \mathrm{Proj}\,A$$

such that the diagram of schemes :

(6.4.4)
$$\begin{tikzcd}
\mathrm{Spec}\,A'_{(\varphi(f))} \arrow[r, "\mathrm{Spec}\,\varphi_{(f)}"] \arrow[d, "\omega_{\varphi(f)}"'] & \mathrm{Spec}\,A_{(f)} \arrow[d, "\omega_f"] \\
D_+(\varphi(f)) \arrow[r, "(\mathrm{Proj}\,\varphi)_{|D_+(\varphi(f))}"] & D_+(f)
\end{tikzcd}$$

commutes for every homogeneous $f \in A_+$ ([27, Ch.II, Prop.2.8.2]). Notice that $G(\varphi) = \mathrm{Proj}\,A'$, whenever $\varphi(A_+)$ generates the ideal A'_+.

6.4.5. To ease notation, set $Y := \mathrm{Proj}\,A$. Let $M := \oplus_{n\in\mathbb{Z}} M_n$ be a $\mathbb{Z}$-graded A-module (*i.e.* $A_k \cdot M_n \subset M_{k+n}$ for every $k \in \mathbb{N}$ and $n \in \mathbb{Z}$); for every homogeneous $f \in A_+$, denote by $M_{(f)} \subset M_f$ the submodule consisting of all elements of degree zero (for the natural $\mathbb{Z}$-grading of M_f). Clearly $M_{(f)}$ is a $A_{(f)}$-module in a natural way, whence a quasi-coherent $\mathscr{O}_{D_+(f)}$-module $M^\sim_{(f)}$; these modules glue to a unique quasi-coherent $\mathscr{O}_Y$-module $M^\sim$ ([27, Ch.II, Prop.2.5.2]). Especially, for every $n \in \mathbb{Z}$, let $A(n)$ be the $\mathbb{Z}$-graded A-module such that $A(n)_k := A_{n+k}$ for every $k \in \mathbb{Z}$ (with the convention that $A_k = 0$ if $k < 0$). We set :

$$\mathscr{O}_Y(n) := A(n)^\sim.$$

Any element $f \in A_n$ induces a natural isomorphism of $D_+(f)$-modules :

$$\mathscr{O}_Y(n)_{|D_+(f)} \xrightarrow{\sim} \mathscr{O}_{D_+(f)}$$

([27, Ch.II, Prop.2.5.7]). Hence, on the open subset

$$U_n(A) := \bigcup_{f\in A_n} D_+(f)$$

the sheaf $\mathscr{O}_Y(n)$ restricts to an invertible $\mathscr{O}_{U_n(A)}$-module. Especially, if A_1 generates the ideal A_+, the $\mathscr{O}_Y$-modules $\mathscr{O}_Y(n)$ are invertible, for every $n \in \mathbb{Z}$.

6.4.6. In the situation of (6.4.3), let $M := \oplus_{n\in\mathbb{Z}} M_n$ be a $\mathbb{Z}$-graded A-module. Then $M' := M \otimes_A A'$ is a $\mathbb{Z}$-graded A'-module, with the grading defined by the rule :

$$M'_n := \sum_{j+k=n} \mathrm{Im}(M_j \otimes_{\mathbb{Z}} A'_k \to M') \qquad \text{for every } n \in \mathbb{Z}$$

([27, Ch.II, (2.1.2)]). Then [27, Ch.II, Prop.2.8.8] yields a natural morphism of $\mathscr{O}_{G(\varphi)}$-modules:

$$\nu_M : (\mathrm{Proj}\,\varphi)^* M^\sim \to (M')^\sim_{|G(\varphi)}.$$

Moreover, set :

$$G_1(\varphi) := \bigcup_{f\in A_1} D_+(\varphi(f))$$

and notice that $G_1(\varphi) \subset U_1(A') \cap G(\varphi)$; by inspecting the proof of *loc.cit.* we see that the restriction $\nu_{M|G_1(\varphi)}$ is an isomorphism. Especially, ν_M is an isomorphism whenever A_1 generates the ideal A_+. It is also easily seen that $G_1(\varphi) = U_1(A')$ if $\varphi(A_+)$ generates A'_+.

For any $f \in A_1$, the restriction $(\nu_M)_{|D_+(\varphi(f))}$ can be described explicitly : namely, we have natural identifications

$$\omega_f^*(M^\sim_{|D_+(f)}) \xrightarrow{\sim} M^\sim_{(f)} \qquad \omega^*_{\varphi(f)}(M')^\sim_{|D_+(\varphi(f))} \xrightarrow{\sim} (M')^\sim_{(\varphi(f))}$$

and in view of (6.4.4), the morphism $(\nu_M)_{|D_+(\varphi(f))}$ is induced by the $A'_{(\varphi(f))}$-linear map :

$$M_{(f)} \otimes_{A_{(f)}} A'_{(\varphi(f))} \to M'_{(\varphi(f))}$$

given by the rule :

$$(m_k \cdot f^{-k}) \otimes (a'_j \cdot \varphi(f)^{-j}) \mapsto (m_k \otimes a'_j) \cdot \varphi(f)^{-j-k} \qquad \text{for all } k, j \in \mathbb{Z}, m_k \in M_k, a'_j \in A'_j.$$

6.4.7. The foregoing results can be improved somewhat, in the following special situation. Let $R \to R'$ be a ring homomorphism, A a $\mathbb{N}$-graded R-algebra (hence the structure morphism $R \to A$ is a ring homomorphism $R \to A_0$); the ring $A' := R' \otimes_R A$ is naturally a $\mathbb{N}$-graded R'-algebra, and the induced map $\varphi : A \to A'$ is a homomorphism of graded rings. In this case, obviously $\varphi(A_+)$ generates the ideal A'_+, hence $G(\varphi) = \mathrm{Proj}\, A'$, and indeed, $\mathrm{Proj}\,\varphi$ induces an isomorphism of $\mathrm{Spec}\, R'$-schemes :

$$Y' \xrightarrow{\sim} \mathrm{Spec}\, R' \times_{\mathrm{Spec}\, R} Y$$

where again $Y := \mathrm{Proj}\, A$ and $Y' := \mathrm{Proj}\, A'$. Moreover, for every $\mathbb{Z}$-graded A-module M, the corresponding morphism ν_M is an isomorphism, regardless of whether or not A_1 generates A_+ ([27, Ch.II, Prop.2.8.10]). Especially, $\nu_{A(n)}$ is a natural identification ([27, Ch.II, Cor.2.8.11]) :

$$(\mathrm{Proj}\,\varphi)^*\mathscr{O}_Y(n) \xrightarrow{\sim} \mathscr{O}_{Y'}(n) \qquad \text{for every } n \in \mathbb{Z}.$$

6.4.8. Let X be a scheme, $\mathscr{A} := \oplus_{n\in\mathbb{N}}\mathscr{A}_n$ a $\mathbb{N}$-graded quasi-coherent $\mathscr{O}_X$-algebra on the Zariski site of X; we let $\mathscr{A}_+ := \oplus_{n>0}\mathscr{A}_n$. According to [27, Ch.II, Prop.3.1.2], there exists an X-scheme $\pi : \mathrm{Proj}\,\mathscr{A} \to X$, with natural isomorphisms of U-schemes :

$$\psi_U : U \times_X \mathrm{Proj}\,\mathscr{A} \xrightarrow{\sim} \mathrm{Proj}\,\mathscr{A}(U)$$

for every affine open subset $U \subset X$, and the system of isomorphisms ψ_U is compatible, in an obvious way, with inclusions $U' \subset U$ of affine open subsets. For any integer $d > 0$, every $f \in \Gamma(X, \mathscr{A}_d)$ defines an open subset $D_+(f) \subset \mathrm{Proj}\,\mathscr{A}$, such that :

$$D_+(f) \cap \pi^{-1}U = D_+(f_{|U}) \subset \mathrm{Proj}\,\mathscr{A}(U) \qquad \text{for every affine open subset } U \subset X.$$

6.4.9. To ease notation, set $Y := \mathrm{Proj}\,\mathscr{A}$, and let again $\pi : Y \to X$ be the natural morphism. Let $\mathscr{M} := \oplus_{n\in\mathbb{Z}}\mathscr{M}_n$ be a $\mathbb{Z}$-graded $\mathscr{A}$-module, quasi-coherent as a $\mathscr{O}_X$-module; for every affine open subset $U \subset X$, the graded $\mathscr{A}(U)$-module $\mathscr{M}(U)$ yields a quasi-coherent $\mathscr{O}_{\pi^{-1}U}$-module $\mathscr{M}^\sim_U$, and every inclusion of affine open subsets $U' \subset U$ induces a natural isomorphism of $\mathscr{O}_{\pi^{-1}U'}$-modules : $\mathscr{M}^\sim_{U|U'} \xrightarrow{\sim} \mathscr{M}^\sim_{U'}$. Therefore the locally defined modules $\mathscr{M}^\sim_U$ glue to a well defined quasi-coherent $\mathscr{O}_Y$-module $\mathscr{M}^\sim$.

Especially, for every $n \in \mathbb{Z}$, denote by $\mathscr{A}(n)$ the $\mathbb{Z}$-graded $\mathscr{A}$-module such that $\mathscr{A}(n)_k := \mathscr{A}_{n+k}$ for every $k \in \mathbb{Z}$, with the convention that $\mathscr{A}_n = 0$ whenever $n < 0$. We set:

$$\mathscr{O}_Y(n) := \mathscr{A}(n)^\sim \qquad \text{and} \qquad \mathscr{M}^\sim(n) := \mathscr{M}^\sim \otimes_{\mathscr{O}_Y} \mathscr{O}_Y(n).$$

Denote by $U_n(\mathscr{A}) \subset Y$ the union of the open subsets $U_n(\mathscr{A}(U))$, for U ranging over the affine open subsets of Y; from the discussion in (6.4.5), it clear that the restriction $\mathscr{O}_Y(n)_{|U_n(\mathscr{A})}$ is an invertible $\mathscr{O}_{U_n(\mathscr{A})}$-module. This open subset can be described as follows. For every $x \in X$, let :

$$\mathscr{A}_n(x) := \mathscr{A}_{n,x} \otimes_{\mathscr{O}_{X,x}} \kappa(x) \qquad \text{and set} \qquad \mathscr{A}(x) := \oplus_{n\in\mathbb{N}}\mathscr{A}_n(x) \tag{6.4.10}$$

which is a $\mathbb{N}$-graded $\kappa(x)$-algebra; then :

$$U_n(\mathscr{A}) = \{y \in Y \mid \mathscr{A}_n(\pi(y)) \not\subset \mathfrak{p}(y)\} \tag{6.4.11}$$

where $\mathfrak{p}(y) \subset \mathscr{A}(\pi(y))$ denotes the prime ideal corresponding to the point y.

6.4.12. Moreover, for every $\mathbb{Z}$-graded $\mathscr{A}$-module $\mathscr{M}$, and every $n \in \mathbb{Z}$, there exists a natural morphism of $\mathscr{O}_Y$-modules :

$$\mathscr{M}^\sim(n) \to \mathscr{M}(n)^\sim \tag{6.4.13}$$

where $\mathscr{M}(n)$ is the $\mathbb{Z}$-graded $\mathscr{A}$-module given by the rule : $\mathscr{M}(n)_k := \mathscr{M}_{n+k}$ for every $k \in \mathbb{N}$ ([27, Prop.3.2.16]). The restriction of (6.4.13) to the open subset $U_1(\mathscr{A})$ is an isomorphism ([27, Ch.II, Cor.3.2.8]). Especially, we have natural morphisms of $\mathscr{O}_Y$-modules :

$$\mathscr{O}_Y(n) \otimes_{\mathscr{O}_Y} \mathscr{O}_Y(m) \to \mathscr{O}_Y(n+m) \qquad \text{for every } n, m \in \mathbb{Z} \tag{6.4.14}$$

whose restrictions to $U_1(\mathscr{A})$ are isomorphisms. Furthermore, we have a natural morphism $\mathscr{M}_0 \to \pi_*\mathscr{M}^\sim$ of $\mathscr{O}_X$-modules ([27, Ch.II, (3.3.2.1)]), whence, by adjunction, a morphism of $\mathscr{O}_Y$-modules :

$$\pi^*\mathscr{M}_0 \to \mathscr{M}^\sim. \tag{6.4.15}$$

Applying (6.4.15) to the modules $\mathscr{M}_n = \mathscr{M}(n)_0$, and taking into account the isomorphism (6.4.13), we deduce a natural morphism of $\mathscr{O}_{U_1(\mathscr{A})}$-modules :

$$(\pi^*\mathscr{M}_n)_{|U_1(\mathscr{A})} \to \mathscr{M}^\sim(n)_{|U_1(\mathscr{A})} \tag{6.4.16}$$

which can be described as follows. Let $U \subset X$ be any affine open subset; for every $f \in \mathscr{A}_1(U)$, the restriction of (6.4.16) to $D_+(f) \subset \pi^{-1}U$ is given by the morphisms

$$\mathscr{M}_n(U) \otimes_{\mathscr{O}_X(U)} \mathscr{A}(U)_{(f)} \to \mathscr{M}(n)(U)_{(f)} := \sum_{k\in\mathbb{Z}} \mathscr{M}_{k+n}(U) \cdot f^{-k} \subset \mathscr{M}(U)_f.$$

induced by the scalar multiplication $\mathscr{M}_n \otimes_{\mathscr{O}_X} \mathscr{A}_k \to \mathscr{M}_{n+k}$. Especially, we have natural morphisms of $\mathscr{O}_Y$-modules :

$$\pi^*\mathscr{A}_n \to \mathscr{O}_Y(n) \qquad \text{for every } n \in \mathbb{N} \tag{6.4.17}$$

whose restrictions to $U_1(\mathscr{A})$ are epimorphisms. An inspection of the definition, also shows that the diagram of $\mathscr{O}_Y$-modules :

$$\begin{tikzcd}
\pi^*\mathscr{A}_n \otimes_{\mathscr{O}_Y} \pi^*\mathscr{A}_m \arrow[r] \arrow[d] & \pi^*\mathscr{A}_{n+m} \arrow[d] \\
\mathscr{O}_Y(n) \otimes_{\mathscr{O}_Y} \mathscr{O}_Y(m) \arrow[r] & \mathscr{O}_Y(n+m)
\end{tikzcd} \tag{6.4.18}$$

commutes for every $n, m \in \mathbb{N}$, where the top horizontal arrow is induced by the graded multiplication $\mathscr{A}_n \otimes_{\mathscr{O}_X} \mathscr{A}_m \to \mathscr{A}_{n+m}$, the vertical arrows are the maps (6.4.17), and the bottom horizontal arrow is the map (6.4.14).

6.4.19. Next, let $\mathscr{A}' := \oplus_{n\in\mathbb{N}}\mathscr{A}'_n$ be another $\mathbb{N}$-graded quasi-coherent $\mathscr{O}_X$-algebra on the Zariski site of X, and $\varphi : \mathscr{A} \to \mathscr{A}'$ a morphism of graded $\mathscr{O}_X$-algebras; for every affine open subset $U \subset X$, we deduce a morphism $\varphi_U : \mathscr{A}(U) \to \mathscr{A}'(U)$ of graded $\mathscr{O}_X(U)$-algebras, whence an open subset $G(\varphi_U) \subset \mathrm{Proj}\,\mathscr{A}'(U)$ as in (6.4.3). If $V \subset U$ is a smaller affine open subset, the natural isomorphism

$$V \times_U \mathrm{Proj}\,\mathscr{A}'(U) \xrightarrow{\sim} \mathrm{Proj}\,\mathscr{A}'(V)$$

induces an identification $V \times_U G(\varphi_U) \xrightarrow{\sim} G(\varphi_V)$, hence there exists a well defined open subset $G(\varphi) \subset \mathrm{Proj}\,\mathscr{A}'$ such that the morphisms $\mathrm{Proj}\,\varphi_U$ glue to a unique morphism of X-schemes :

$$\mathrm{Proj}\,\varphi : G(\varphi) \to \mathrm{Proj}\,\mathscr{A}.$$

If $\mathscr{A}'_+$ is generated – locally on X – by $\varphi(\mathscr{A}_+)$, we have $G(\varphi) = \mathrm{Proj}\,\mathscr{A}'$.

Moreover, if $\mathscr{M}$ is a $\mathbb{Z}$-graded quasi-coherent $\mathscr{A}$-module, the morphisms $\nu_{\mathscr{M}(U)}$ assemble into a well defined morphism of $\mathscr{O}_{G(\varphi)}$-modules :

$$\nu_{\mathscr{M}} : (\mathrm{Proj}\,\varphi)^*\mathscr{M}^\sim \to (\mathscr{M}')^\sim_{|G(\varphi)}$$

where the grading of $\mathscr{M}' := \mathscr{M} \otimes_{\mathscr{A}} \mathscr{A}'$ is defined as in (6.4.6). Likewise, the union of the subsets $G_1(\varphi_U)$, for U ranging over the affine open subsets of X, is an open subset :

$$G_1(\varphi) \subset U_1(\mathscr{A}') \cap G(\varphi) \tag{6.4.20}$$

such that the restriction $\nu_{\mathscr{M}|G_1(\varphi)}$ is an isomorphism. Especially, set $Y' := \mathrm{Proj}\,\mathscr{A}'$; we have a natural morphism :

$$\nu_{\mathscr{A}(n)} : (\mathrm{Proj}\,\varphi)^*\mathscr{O}_Y(n) \to \mathscr{O}_{Y'}(n)_{|G(\varphi)} \tag{6.4.21}$$

which is an isomorphism, if $\mathscr{A}_1$ generates $\mathscr{A}_+$ locally on X. Again, we have $G_1(\varphi) = U_1(\mathscr{A}')$ whenever $\varphi(\mathscr{A}_+)$ generates $\mathscr{A}'_+$, locally on X.

6.4.22. The discussion in (6.4.7) implies that any morphism of schemes $f : X' \to X$ induces a natural isomorphism of X'-schemes ([27, Ch.II, Prop.3.5.3]) :

$$\mathrm{Proj}\,f^*\mathscr{A} \xrightarrow{\sim} X' \times_X \mathrm{Proj}\,\mathscr{A} \tag{6.4.23}$$

and the description (6.4.11) implies that (6.4.23) restricts to an isomorphism :

$$U_n(f^*\mathscr{A}) \xrightarrow{\sim} X' \times_X U_n(\mathscr{A}) \qquad \text{for every } n \in \mathbb{N}.$$

Furthermore, set $Y' := \mathrm{Proj}\,f^*\mathscr{A}$, and let $\pi_Y : Y' \to Y$ be the morphism deduced from (6.4.23); the discussion in (6.4.7) implies as well that, for any $\mathbb{Z}$-graded quasi-coherent $\mathscr{A}$-module $\mathscr{M}$, there is a natural isomorphism :

$$(f^*\mathscr{M})^\sim \xrightarrow{\sim} \pi_Y^*\mathscr{M}^\sim$$

([27, Ch.II, Prop.3.5.3]). Especially, f induces a natural identification ([27, Ch.II, Cor.3.5.4]) :

$$\mathscr{O}_{Y'}(n) \xrightarrow{\sim} \pi_Y^*\mathscr{O}_Y(n) \qquad \text{for every } n \in \mathbb{Z}. \tag{6.4.24}$$

6.4.25. Keep the notation of (6.4.9), and let $\mathscr{C}_X$ be the category whose objects are all the pairs $(\psi : Z \to X, \mathscr{L})$, where ψ is a morphism of schemes and $\mathscr{L}$ is an invertible $\mathscr{O}_Z$-module on the Zariski site of Z; the morphisms $(\psi : Z \to X, \mathscr{L}) \to (\psi' : Z' \to X, \mathscr{L}')$ are the pairs (β, h), where $\beta : Z \to Z'$ is a morphism of X-schemes, and $h : \beta^*\mathscr{L}' \xrightarrow{\sim} \mathscr{L}$ is an isomorphism of $\mathscr{O}_Z$-modules (with composition of morphisms defined in the obvious way). Consider the functor:

$$F_{\mathscr{A}} : \mathscr{C}_X^o \to \mathbf{Set}$$

which assigns to any object $(\psi, \mathscr{L})$ of $\mathscr{C}_X$, the set consisting of all homomorphisms of graded $\mathscr{O}_Z$-algebras :

$$g : \psi^*\mathscr{A} \to \mathrm{Sym}^\bullet_{\mathscr{O}_Z}\mathscr{L}$$

which are epimorphisms on the underlying $\mathscr{O}_Z$-modules (here $\mathrm{Sym}^\bullet_{\mathscr{O}_Z}\mathscr{L}$ denotes the symmetric $\mathscr{O}_Z$-algebra on the $\mathscr{O}_Z$-module $\mathscr{L}$); on a morphism (β, h) as in the foregoing, and an element $g' \in F_{\mathscr{A}}(\psi', \mathscr{L}')$, the functor acts by the rule :

$$F_{\mathscr{A}}(\beta, h)(g') := (\mathrm{Sym}^\bullet_{\mathscr{O}_Z} h) \circ \beta^* g'.$$

Lemma 6.4.26. *The object* $(\pi : U_1(\mathscr{A}) \to X, \mathscr{O}_Y(1)_{|U_1(\mathscr{A})})$ *of* $\mathscr{C}_X$ *represents the functor* $F_{\mathscr{A}}$.

Proof. Given an object $(\psi : Z \to X, \mathscr{L})$ of $\mathscr{C}_X$, and $g \in F_{\mathscr{A}}(\psi, \mathscr{L})$, set :

$$\mathbb{P}(\mathscr{L}) := \mathrm{Proj}\,\mathrm{Sym}^\bullet_{\mathscr{O}_Z}\mathscr{L}.$$

According to [27, Ch.II, Cor.3.1.7, Prop.3.1.8(iii)], the natural morphism $\pi_Z : \mathbb{P}(\mathscr{L}) \to Z$ is an isomorphism. On the other hand, since g is an epimorphism, we have $G(g) = \mathbb{P}(\mathscr{L})$; taking (6.4.23) into account, we deduce a morphism of Z-schemes :

$$\mathrm{Proj}\, g : \mathbb{P}(\mathscr{L}) \to Y' := Z \times_X \mathrm{Proj}\,\mathscr{A}$$

which is the same as a morphism of X-schemes :

$$\mathbb{P}(g) : Z \to \mathrm{Proj}\,\mathscr{A}.$$

We need to show that the image of $\mathbb{P}(g)$ lies in the open subset $U_1(\mathscr{A})$; to this aim, we may assume that both X and Z are affine, say $X = \mathrm{Spec}\,R$, $Z = \mathrm{Spec}\,S$, in which case $\mathscr{A}$ is the quasi-coherent algebra associated to a $\mathbb{N}$-graded R-algebra A, $\mathscr{L}$ is the invertible module associated to a projective rank one S-module L, and $g : S \otimes_R A \to \mathrm{Sym}^\bullet_S L$ is a surjective homomorphism of R-algebras. Then locally on Z, $\mathscr{L}$ is generated by elements of the form $g(1 \otimes t)$, for some local sections t of $\mathscr{A}_1$, and up to replacing Z by an affine open subset, we may assume that $t \in A_1$ is an element such that $t' := g(1 \otimes t)$ generates the free S-module L. In this situation, we have $\mathbb{P}(\mathscr{L}) = D_+(t')$, and the isomorphism $\pi_Z : \mathbb{P}(\mathscr{L}) \xrightarrow{\sim} Z$ is induced by the natural identification:

$$S = S[t']_{(t')}. \tag{6.4.27}$$

Likewise, $\mathscr{O}_{\mathbb{P}(\mathscr{L})}(n)$ is the $\mathscr{O}_{\mathbb{P}(\mathscr{L})}$-module associated to the $S[t']$-module $L^{\otimes n} \otimes_S S[t'] \xrightarrow{\sim} S[t'](n)$, hence (6.4.27) induces a natural identification :

$$\pi_Z^* \mathscr{L}^{\otimes n} \xrightarrow{\sim} \mathscr{O}_{\mathbb{P}(\mathscr{L})}(n) \qquad \text{for every } n \in \mathbb{N}. \tag{6.4.28}$$

Moreover, $\mathbb{P}(g)$ is the same as the morphism $\Phi_t : D_+(t') \to D_+(t)$ (notation of (6.4.3)); especially the image of $\mathbb{P}(g)$ lies in $U_1(\mathscr{A})$, as required. From this description, we also can extract an explicit expression for Φ_t; namely, it is induced by the map of R-algebras :

$$A_{(t)} \to S \qquad \text{such that} \qquad a_k \cdot t^{-k} \mapsto g(1 \otimes a_k) \cdot t'^{-k}$$

for every $k \in \mathbb{N}$, and every $a_k \in A_k$. Next, letting $n := 1$ in (6.4.21) and (6.4.24), we obtain a natural isomorphism of $\mathscr{O}_{\mathbb{P}(\mathscr{L})}$-modules :

$$\mathscr{O}_{\mathbb{P}(\mathscr{L})}(1) \xrightarrow{\sim} (\mathrm{Proj}\, g)^* \mathscr{O}_{Y'}(1) \xrightarrow{\sim} (\mathrm{Proj}\, g)^* \circ \pi_Y^* \mathscr{O}_Y(1) \xrightarrow{\sim} \pi_Z^* \circ \mathbb{P}(g)^* \mathscr{O}_Y(1)$$

(notice that, since by assumption g is an epimorphism, we have $G_1(g) = U_1(\mathrm{Sym}^\bullet_{\mathscr{O}_Z}\mathscr{L}) = \mathbb{P}(\mathscr{L})$, hence $\nu_{\varphi^*\mathscr{A}(1)}$ is an isomorphism). Composition with (6.4.28) yields an isomorphism :

$$h(g) : \mathbb{P}(g)^* \mathscr{O}_Y(1) \xrightarrow{\sim} \mathscr{L}$$

of $\mathscr{O}_Z$-modules, whence a morphism in $\mathscr{C}_X$

$$(\mathbb{P}(g), h(g)) : (\psi, \mathscr{L}) \to (\pi_{|U_1(\mathscr{A})}, \mathscr{O}_Y(1)_{|U_1(\mathscr{A})}).$$

In case X and Z are affine, and $\mathscr{L}$ is associated to a free module L, generated by an element of the form $t' := g(1 \otimes t)$ as in the foregoing, we can describe explicitly $h(g)$; namely, a direct inspection of the construction shows that in this case $h(g)$ is induced by the map of S-modules

$$S \otimes_{A_{(t)}} A(1)_{(t)} \to L \quad : \quad s \otimes a_k \cdot t^{1-k} \mapsto s \cdot g(1 \otimes a_k) \cdot (t')^{1-k} \qquad \text{for every } s \in S,\ a_k \in A_k.$$

Conversely, let $\beta : Z \to U_1(\mathscr{A})$ be a morphism of X-schemes, and $h : \beta^* \mathscr{O}_Y(1)_{|U_1(\mathscr{A})} \xrightarrow{\sim} \mathscr{L}$ an isomorphism of $\mathscr{O}_Z$-modules. In view of the natural isomorphisms (6.4.14), we deduce, for every $n \in \mathbb{N}$, an isomorphism :

$$h^{\otimes n} : \beta^* \mathscr{O}_Y(n)_{|U_1(\mathscr{A})} \xrightarrow{\sim} \mathscr{L}^{\otimes n}.$$

Combining with the epimorphisms (6.4.17) :

$$\omega_n : (\pi^*\mathscr{A}_n)_{|U_1(\mathscr{A})} \to \mathscr{O}_Y(n)_{|U_1(\mathscr{A})}$$

we may define the epimorphism of $\mathscr{O}_Z$-modules :

$$g(\beta,h) := \bigoplus_{n\in\mathbb{N}} h^{\otimes n} \circ \beta^*(\omega_n) : \psi^*\mathscr{A} \to \mathrm{Sym}^\bullet_{\mathscr{O}_Z}\mathscr{L} \tag{6.4.29}$$

which, in view of (6.4.18), is a homomorphism of graded $\mathscr{O}_Z$-algebras, *i.e.* $g(\beta,h) \in F(\psi,\mathscr{L})$. This homomorphism can be described explicitly, locally on Z : namely, say again that $X = \mathrm{Spec}\, R$, $Z = \mathrm{Spec}\, S$, $\mathscr{L} = L^\sim$ for a free S-module of rank one, and $\mathscr{A} = A^\sim$ for some $\mathbb{N}$-graded R-algebra A; suppose moreover that the image of β lies in an open subset $D_+(t) \subset U_1(\mathscr{A})$, for some $t \in A_1$. Then β comes from a ring homomorphism $\beta^\natural : A_{(t)} \to S$, h is an S-linear isomorphism $S \otimes_{A_{(t)}} A(1)_{(t)} \xrightarrow{\sim} L$, and $t' := h(1 \otimes t)$ is a generator of L; moreover, ω_n is the epimorphism deduced from the map :

$$A_n \otimes_R A_{(t)} \to A(n)_{(t)} \quad : \quad a_n \otimes b_k \cdot t^{-k} \mapsto a_n b_k \cdot t^{-k} \qquad \text{for every } a_n \in A_n,\ b_k \in A_k$$

By inspecting the construction, we see therefore that g is the direct sum of the morphisms :

$$g_n : S \otimes_R A_n \to L^{\otimes n} \quad : \quad s \otimes a_n \mapsto s \cdot \beta^\natural(a_n \cdot t^{-n}) \cdot t'^{\otimes n} \qquad \text{for every } s \in S,\ a_n \in A_n.$$

Finally, it is easily seen that the natural transformations :

$$g \mapsto (\mathbb{P}(g), h(g)) \qquad \text{and} \qquad (\beta,h) \mapsto g(\beta,h) \tag{6.4.30}$$

are inverse to each other : indeed, the verification can be made locally on Z, hence we may assume that X and Z are affine, and $\mathscr{L}$ is free, in which case one may use the explicit formulae provided above. $\square$

6.4.31. We wish now to consider the logarithmic counterparts of the notions introduced in the foregoing. To begin with, let X be any scheme, N a monoid, and $\underline{P}$ a (commutative) N-graded monoid of the topos X_τ (see definition 2.3.8). Notice that $\underline{P}^\times = \coprod_{n\in N}(\underline{P}^\times \cap \underline{P}_n)$, hence the sheaf of invertible sections of a N-graded monoid on X is a N-graded abelian sheaf.

Definition 6.4.32. Let X be a scheme, and $\beta : \underline{M} \to \mathscr{O}_X$ a log structure on X_τ.

(i) A $\mathbb{N}$-*graded* $\mathscr{O}_{(X,\underline{M})}$-*algebra* is a datum $(\mathscr{A}, \underline{P}, \alpha, \beta_{\mathscr{A}})$ consisting of a $\mathbb{N}$-graded $\mathscr{O}_X$-algebra $\mathscr{A} := \oplus_{n\in\mathbb{N}}\mathscr{A}_n$ (on the site X_τ), a graded monoid $\underline{P}$ on X_τ, and a commutative diagram :

$$\begin{tikzcd} \underline{M} \arrow[r, "\alpha"] \arrow[d, "\beta"'] & \underline{P} \arrow[d, "\beta_{\mathscr{A}}"] \\ \mathscr{O}_X \arrow[r] & \mathscr{A} \end{tikzcd}$$

where $\beta_{\mathscr{A}}$ restricts to a morphism of graded monoids $\underline{P} \to \coprod_{n\in\mathbb{N}} \mathscr{A}_n$ (and the composition law on the target is induced by the multiplication law of $\mathscr{A}$), α is a morphism of monoids $\underline{M} \to (\underline{P})_0$, and the bottom map is the natural morphism $\mathscr{O}_X \to \mathscr{A}_0$. We say that the $\mathbb{N}$-graded $\mathscr{O}_{(X,\underline{M})}$-algebra $(\mathscr{A}, \underline{P}, \alpha, \beta_{\mathscr{A}})$ is *quasi-coherent*, if $\mathscr{A}$ is a quasi-coherent $\mathscr{O}_X$-algebra.

(ii) A *morphism* $(g, \log g) : (\mathscr{A}, \underline{P}, \alpha, \beta_{\mathscr{A}}) \to (\mathscr{A}', \underline{P}', \alpha', \beta'_{\mathscr{A}})$ *of* $\mathbb{N}$-*graded* $\mathscr{O}_{(X,\underline{M})}$-*algebras* is a commutative diagram :

$$\begin{tikzcd} \underline{P} \arrow[r, "\log g"] \arrow[d, "\beta_{\mathscr{A}}"'] & \underline{P}' \arrow[d, "\beta'_{\mathscr{A}}"] \\ \mathscr{A} \arrow[r, "g"] & \mathscr{A}' \end{tikzcd}$$

where $\log g$ is a morphism of $\mathbb{N}$-graded monoid such that $\log g \circ \alpha = \alpha'$, and g is a morphism of $\mathscr{O}_Z$-algebras.

6.4.33. Let M be a monoid, S an M-module; we say that an element $s \in S$ is *invertible*, if the translation map $M \to S : m \mapsto m \cdot s$ (for all $m \in M$) is an isomorphism. It is easily seen that the subset $S^\times$ consisting of all invertible elements of S, is naturally an $M^\times$-module.

Let $\underline{M}$ be a sheaf of monoids on X_τ, and $\mathscr{N}$ an $\underline{M}$-module; by restriction of scalars, $\mathscr{N}$ is naturally an $\underline{M}^\times$-module. We define the $\underline{M}^\times$-submodule $\mathscr{N}^\times \subset \mathscr{N}$, by the rule :

$$\mathscr{N}^\times(U) := \mathscr{N}(U)^\times \qquad \text{for every object } U \text{ of } X_\tau.$$

Conversely, for any $\underline{M}^\times$-module $\mathscr{P}$, the extension of scalars $\mathscr{N} \otimes_{\underline{M}^\times} \underline{M}$ defines a functor $\underline{M}^\times\text{-}\mathbf{Mod} \to \underline{M}\text{-}\mathbf{Mod}$. It is easily seen that the latter restricts to an equivalence from the full subcategory $\underline{M}^\times\text{-}\mathbf{Inv}$ of $\underline{M}^\times$-torsors to the subcategory $(\underline{M}\text{-}\mathbf{Inv})^\times$ of $\underline{M}\text{-}\mathbf{Mod}$ whose objects are all invertible $\underline{M}$-modules, and whose morphisms are the isomorphisms of $\underline{M}$-modules (see definition 2.3.6(iv)); the functor $\mathscr{N} \mapsto \mathscr{N}^\times$ provides a quasi-inverse $(\underline{M}\text{-}\mathbf{Inv})^\times \to \underline{M}^\times\text{-}\mathbf{Inv}$.

Especially, if $(X, \underline{M})$ is a log scheme, we see that the category of invertible $\underline{M}$-modules is equivalent to that of invertible $\mathscr{O}_X^\times$-modules, hence also to that of invertible $\mathscr{O}_X$-modules.

If $\varphi : (Z, \underline{N}) \to (X, \underline{M})$ is a morphism of log schemes, and $\mathscr{N}$ a $\underline{M}$-module, we let :

$$\varphi^*\mathscr{N} := \varphi^{-1}\mathscr{N} \otimes_{\varphi^{-1}\underline{M}} \underline{N}. \tag{6.4.34}$$

Clearly, $\varphi^*\mathscr{N}$ is an invertible $\underline{N}$-module, whenever $\mathscr{N}$ is an invertible $\underline{M}$-module.

6.4.35. Keep the notation of definition 6.4.32(i); by faithfully flat descent, the restriction of $\mathscr{A}$ to the Zariski site of X is again a quasi-coherent $\mathscr{O}_X$-algebra, which we denote again by $\mathscr{A}$. We may then set $Y := \operatorname{Proj} \mathscr{A}$, and let $\pi : Y \to X$ be the natural morphism. The composition of $\pi^{-1}\beta_{\mathscr{A}}$ and the morphism (6.4.17), yields a map on the site Y_τ :

$$\pi^{-1}\underline{P}_n \to \mathscr{O}_Y(n) \qquad \text{for every } n \in \mathbb{N}. \tag{6.4.36}$$

Set $\mathscr{O}_Y(\bullet) := \coprod_{n\in\mathbb{Z}} \mathscr{O}_Y(n)$; it is easily seen that the morphisms (6.4.14) induce a natural $\mathbb{Z}$-graded monoid structure on $\mathscr{O}_Y(\bullet)$, and the coproduct of the maps (6.4.36) amounts to a morphism of graded monoids :

$$\omega_\bullet : \pi^{-1}\underline{P} \to \mathscr{O}_Y(\bullet).$$

We let $\underline{Q}$ be the push-out in the cocartesian diagram :

$$\begin{array}{ccc} \omega_\bullet^{-1}(\mathscr{O}_Y(\bullet)^\times) & \longrightarrow & \pi^{-1}\underline{P} \\ \downarrow & & \downarrow \\ \mathscr{O}_Y(\bullet)^\times & \longrightarrow & \underline{Q}. \end{array} \tag{6.4.37}$$

Clearly $\underline{Q}$ is naturally a $\mathbb{Z}$-graded monoid, in such a way that all the arrows in (6.4.37) are morphisms of $\mathbb{Z}$-graded monoids. For every $n \in \mathbb{Z}$, let $\underline{Q}_n$ be the degree n subsheaf of $\underline{Q}$; the map $\omega_\bullet$ and the natural inclusion $\mathscr{O}_Y(\bullet)^\times \to \mathscr{O}_Y(\bullet)$ determine a unique morphism $\underline{Q} \to \mathscr{O}_Y(\bullet)$, whose restriction in degree zero is a pre-log structure :

$$\beta_{\mathscr{A}}^\sim : \underline{Q}_0 \to \mathscr{O}_Y.$$

Clearly α induces a unique morphism $\alpha^\sim : \pi^{-1}\underline{M} \to \underline{Q}_0$, such that the diagram of monoids :

$$\begin{array}{ccc} \pi^{-1}\underline{M} & \xrightarrow{\alpha^\sim} & \underline{Q}_0 \\ {\scriptstyle \pi^{-1}\beta}\downarrow & & \downarrow{\scriptstyle \beta_{\mathscr{A}}^\sim} \\ \pi^{-1}\underline{\mathscr{O}}_X & \xrightarrow{\pi^\natural} & \mathscr{O}_Y \end{array}$$

commutes. Denote by $\underline{P}^{\sim}$ the log structure associated to $\beta_{\mathscr{A}}^{\sim}$; the *homogeneous spectrum* of the quasi-coherent $\mathbb{N}$-graded algebra $(\mathscr{A}, \underline{P}, \alpha, \beta_{\mathscr{A}})$ is defined as the $(X, \underline{M})$-scheme :

$$\mathrm{Proj}(\mathscr{A}, \underline{P}) := (Y, \underline{P}^{\sim}).$$

We also let :

$$U_1(\mathscr{A}, \underline{P}) := U_1(\mathscr{A}) \times_Y \mathrm{Proj}(\mathscr{A}, \underline{P}).$$

Furthermore, for every $n \in \mathbb{Z}$ we have a natural morphism of $\underline{Q}_0$-monoids :

$$\underline{Q}_0 \otimes_{\mathscr{O}_Y^{\times}} \mathscr{O}_Y(n)^{\times} \to \underline{Q}_n \tag{6.4.38}$$

and it is easily seen that $(6.4.38)_{|U_1(\mathscr{A})}$ is an isomorphism. We set :

$$\underline{P}^{\sim}(n) := (\underline{P}^{\sim} \otimes_{\underline{Q}_0} \underline{Q}_n)_{|U_1(\mathscr{A})} \qquad \text{for every } n \in \mathbb{Z}.$$

Hence (6.4.38) induces a natural isomorphism :

$$(\underline{P}^{\sim} \otimes_{\mathscr{O}_Y^{\times}} \mathscr{O}_Y(n)^{\times})_{|U_1(\mathscr{A})} \xrightarrow{\sim} \underline{P}^{\sim}(n). \tag{6.4.39}$$

Especially, $\underline{P}^{\sim}(n)$ is an invertible $\underline{P}^{\sim}_{|U_1(\mathscr{A})}$-module, for every $n \in \mathbb{Z}$. From (6.4.39), we also deduce natural isomorphisms of $\underline{P}^{\sim}_{|U_1(\mathscr{A})}$-modules :

$$\underline{P}^{\sim}(n) \otimes_{\underline{P}^{\sim}} \underline{P}^{\sim}(m) \xrightarrow{\sim} \underline{P}^{\sim}(n+m) \qquad \text{for every } n, m \in \mathbb{Z} \tag{6.4.40}$$

and of $\mathscr{O}_{U_1(\mathscr{A})}$-modules :

$$\underline{P}^{\sim}(n) \otimes_{\underline{P}^{\sim}} \mathscr{O}_{U_1(\mathscr{A})} \xrightarrow{\sim} \mathscr{O}_Y(n)_{|U_1(\mathscr{A})} \qquad \text{for every } n \in \mathbb{Z}. \tag{6.4.41}$$

Additionally, the morphism $\pi^{-1}\underline{P}_n \to \underline{Q}_n$ deduced from (6.4.37), yields a natural map of $\underline{P}^{\sim}_{|U_1(\mathscr{A})}$-modules :

$$\lambda_n : (\pi^* \underline{P}_n)_{|U_1(\mathscr{A})} \to \underline{P}^{\sim}(n) \qquad \text{for every } n \in \mathbb{N}. \tag{6.4.42}$$

Example 6.4.43. Let $(Z, \gamma : \underline{N} \to \mathscr{O}_Z)$ be a log scheme, $\mathscr{L}$ an invertible $\underline{N}$-module, and set :

$$\beta_{\mathscr{A}(\mathscr{L})} := \mathrm{Sym}^{\bullet}_{\underline{N}} \mathscr{L} \otimes_{\underline{N}} \gamma : \mathrm{Sym}^{\bullet}_{\underline{N}} \mathscr{L} \to \mathscr{A}(\mathscr{L}) := \mathrm{Sym}^{\bullet}_{\mathscr{O}_Z}(\mathscr{L} \otimes_{\underline{N}} \mathscr{O}_Z)$$

which is a morphisms of $\mathbb{N}$-graded monoids (notation of example 2.3.10). Clearly $\mathscr{A}(\mathscr{L})$ is also a $\mathbb{N}$-graded quasi-coherent $\mathscr{O}_Z$-algebra. Denote also :

$$\alpha_{\mathscr{L}} : \underline{N} \to \mathrm{Sym}^{\bullet}_{\underline{N}} \mathscr{L}$$

the natural morphism that identifies $\underline{N}$ to $\mathrm{Sym}^{0}_{\underline{N}} \mathscr{L}$; then the datum

$$(\mathscr{A}(\mathscr{L}), \mathrm{Sym}^{\bullet}_{\underline{N}} \mathscr{L}, \alpha_{\mathscr{L}}, \beta_{\mathscr{A}(\mathscr{L})})$$

is a quasi-coherent $\mathscr{O}_{(Z,\underline{N})}$-algebra, and a direct inspection of the definitions shows that the induced morphism of log schemes :

$$\pi_{(Z,\underline{N})} : \mathbb{P}(\mathscr{L}) := \mathrm{Proj}(\mathscr{A}(\mathscr{L}), \mathrm{Sym}^{\bullet}_{\underline{N}} \mathscr{L}) \to (Z, \underline{N}) \tag{6.4.44}$$

is an isomorphism. Furthermore, we have natural isomorphisms as in (6.4.28) :

$$\pi^*_{(Z,\underline{N})}(\mathscr{L}^{\otimes n} \otimes_{\underline{N}} \mathscr{O}_Z) \xrightarrow{\sim} \mathscr{O}_{\mathbb{P}(\mathscr{L})}(n) \qquad \text{for every } n \in \mathbb{Z}.$$

Let $\underline{P}^{\sim}_{\mathscr{L}} \to \mathscr{O}_{\mathbb{P}(\mathscr{L})}$ be the log structure of $\mathbb{P}(\mathscr{L})$; there follows a natural identification :

$$\pi^*_{(Z,\underline{N})} \mathscr{L}^{\otimes n} \xrightarrow{\sim} \underline{P}^{\sim}_{\mathscr{L}} \otimes_{\mathscr{O}^{\times}_{\mathbb{P}(\mathscr{L})}} \mathscr{O}_{\mathbb{P}(\mathscr{L})}(n)^{\times} \xrightarrow{\sim} \underline{P}^{\sim}_{\mathscr{L}}(n) \qquad \text{for every } n \in \mathbb{Z} \tag{6.4.45}$$

where the last isomorphism is (6.4.39), in view of the fact that $U_1(\mathscr{A}(\mathscr{L})) = \mathrm{Proj}\,\mathscr{A}(\mathscr{L})$.

Example 6.4.46. (i) Let $(Z, \underline{N})$ be a log scheme, and $n \in \mathbb{N}$ any integer. We define an $\mathbb{N}$-grading on $\mathbb{N}^{\oplus n}$, by setting

$$\mathrm{gr}^k \mathbb{N}^{\oplus n} := \{a_\bullet := (a_1, \dots, a_n) \mid a_1 + \cdots + a_n = k\} \qquad \text{for every } k \in \mathbb{N}.$$

We then define the $\mathbb{N}$-graded monoid

$$\mathrm{Sym}^\bullet_{\underline{N}} \underline{N}^{\oplus n} := \mathbb{N}_Z^{\oplus n} \times \underline{N}$$

whose $\mathbb{N}$-grading is deduced in the obvious way from the foregoing grading of $\mathbb{N}^{\oplus n}$. The log structure $\gamma : \underline{N} \to \mathscr{O}_Z$ extends naturally to a map of $\mathbb{N}$-graded Z-monoids :

$$\mathrm{Sym}^\bullet_{\underline{N}} \gamma^{\oplus n} : \mathrm{Sym}^\bullet_{\underline{N}} \underline{N}^{\oplus n} \to \mathrm{Sym}^\bullet_{\mathscr{O}_Z} \mathscr{O}_Z^{\oplus n}.$$

Namely, if $e_1, \dots, e_n$ is the canonical basis of the free $\mathscr{O}_Z$-module $\mathscr{O}_Z^{\oplus n}$, then $\mathrm{Sym}^k_{\mathscr{O}_Z} \mathscr{O}_Z^{\oplus n}$ is a free $\mathscr{O}_Z$-module with basis

$$\{e_{a_\bullet} := e_1^{a_1} \cdots e_n^{a_n} \mid a_\bullet \in \mathrm{gr}^k \mathbb{N}^{\oplus n}\}$$

and $\mathrm{Sym}^\bullet_{\underline{N}} \gamma^{\oplus n}$ is given by the rule : $(a_\bullet, x) \mapsto \gamma(x) \cdot e_{a_\bullet}$ for every $a_\bullet \in \mathbb{N}^{\oplus n}$, every τ-open subset of Z, and every section $x \in \underline{N}(U)$. Clearly $\mathrm{Sym}^\bullet_{\underline{N}} \gamma^{\oplus n}$ defines an $\mathbb{N}$-graded $\mathscr{O}_{(Z,\underline{N})}$-algebra

$$\mathrm{Sym}^\bullet_{(\mathscr{O}_Z, \underline{N})} (\mathscr{O}_Z, \underline{N})^{\oplus n} := (\mathrm{Sym}^\bullet_{\mathscr{O}_Z} \mathscr{O}_Z^{\oplus n}, \mathrm{Sym}^\bullet_{\underline{N}} \underline{N}^{\oplus n}) \qquad \text{for every } n \in \mathbb{N}.$$

We set :

$$\mathbb{P}^n_{(Z,\underline{N})} := \mathrm{Proj}\, \mathrm{Sym}^\bullet_{(\mathscr{O}_Z, \underline{N})} (\mathscr{O}_Z, \underline{N})^{\oplus n+1}$$

and we call it the *projective n-dimensional space* over $(Z, \underline{N})$.

(ii) Denote by $\underline{N}^\sim$ the log structure of $\mathbb{P}^n_{(Z,\underline{N})}$, and by $\underline{N}^\sim(k)$ the $\underline{N}^\sim$-modules defined as in (6.4.39), for every $k \in \mathbb{Z}$. By simple inspection we get a commutative diagram of monoids :

$$\begin{array}{ccc}
\Gamma(Z, \mathrm{Sym}^k_{\underline{N}} \underline{N}^{\oplus n+1}) & \xrightarrow{\Gamma(Z, \mathrm{Sym}^k_{\underline{N}} \gamma^{\oplus n+1})} & \Gamma(Z, \mathrm{Sym}^k_{\mathscr{O}_Z} \mathscr{O}_Z^{\oplus n+1}) \\
\downarrow & & \downarrow \\
\Gamma(\mathbb{P}^n_{(Z,\underline{N})}, \underline{N}^\sim(k)) & \longrightarrow & \Gamma(\mathbb{P}^n_{(Z,\underline{N})}, \mathscr{O}_{\mathbb{P}^n_{(Z,\underline{N})}}(k))
\end{array} \qquad \text{for every } k \in \mathbb{N}$$

whose right vertical arrow is an isomorphism. Especially, the natural basis of the free $\underline{N}(Z)$-module $\Gamma(Z, \mathrm{Sym}^1_{\underline{N}} \underline{N}^{\oplus n+1}) = \underline{N}(Z)^{\oplus n+1}$ yields a distinguished system of $n+1$ elements

$$\varepsilon_0, \dots, \varepsilon_n \in \Gamma(\mathbb{P}^n_{(Z,\underline{N})}, \underline{N}^\sim(1)).$$

On the other hand, we have as well the distinguished system of global sections

$$T_0, \dots, T_n \in \Gamma(\mathbb{P}^n_{(Z,\underline{N})}, \mathscr{O}_{\mathbb{P}^n_{(Z,\underline{N})}}(1))$$

corresponding to the natural basis of $\Gamma(Z, \mathrm{Sym}^1_{\mathscr{O}_Z} \mathscr{O}_Z^{\oplus n+1}) = \mathscr{O}_Z(Z)^{\oplus n+1}$. For each $i = 0, \dots, n$, the largest open subset $U_i \subset \mathbb{P}^n_{(Z,\underline{N})}$ such that $T_i \in \Gamma(U_i, \mathscr{O}_{\mathbb{P}^n_{(Z,\underline{N})}}(1)^\times)$ is the complement of the hyperplane where T_i vanishes. Moreover, notice that

$$T_i^{-1} T_j \in \underline{N}^\sim(U_i) \qquad \text{for every } i, j = 0, \dots, n.$$

With this notation, the isomorphism (6.4.39) yields the identification :

$$\varepsilon_j = T_i^{-1} T_j \otimes T_i \qquad \text{on } U_i \qquad \text{for every } i, j = 0, \dots, n$$

from which we also see that, for every $i = 0, \dots, n$, the open subset U_i is the largest such that $\varepsilon_i \in \Gamma(U_i, \underline{N}^\sim(1)^\times)$. By the same token, we obtain :

$$(6.4.47) \qquad (\mathbb{P}^n_{(Z,\underline{N})}, \underline{N}^\sim)_{\mathrm{tr}} = U_0 \cap \cdots \cap U_n \simeq \mathbb{G}^n_{m,Z}.$$

6.4.48. Let $(Z,\underline{N})$ be a log scheme, $(\varphi,\log\varphi):(\mathscr{A},\underline{P})\to(\mathscr{A}',\underline{P}')$ a morphism of quasi-coherent $\mathbb{N}$-graded $\mathscr{O}_{(Z,\underline{N})}$-algebras. We let (notation of (6.4.19)) :

$$G(\varphi,\log\varphi):=G(\varphi)\times_{\mathrm{Proj}\,\mathscr{A}'}\mathrm{Proj}(\mathscr{A}',\underline{P}').$$

Denote also by $\pi:Y:=\mathrm{Proj}\,\mathscr{A}\to Z$ and $\pi':Y':=\mathrm{Proj}\,\mathscr{A}'\to Z'$ the natural projections; there follows, on the one hand, a morphism of $\mathbb{N}$-graded monoids :

$$\pi'^{-1}(\log\varphi):(\mathrm{Proj}\,\varphi)^{-1}(\pi^{-1}\underline{P})\to(\pi'^{-1}\underline{P}')_{|G(\varphi)} \tag{6.4.49}$$

and on the other hand, a morphism of $\mathbb{Z}$-graded monoids :

$$(\mathrm{Proj}\,\varphi)^{-1}\mathscr{O}_Y(\bullet)^\times\to\mathscr{O}_{Y'}(\bullet)^\times_{|G(\varphi)} \tag{6.4.50}$$

deduced from (6.4.21). Define the $\mathbb{Z}$-graded monoid $\underline{Q}$ on Y_τ as in (6.4.37), and the analogous $\mathbb{Z}$-graded monoid $\underline{Q}'$ on Y'_τ. Then (6.4.49) and (6.4.50) determine a unique morphism of $\mathbb{Z}$-graded monoids :

$$\vartheta:(\mathrm{Proj}\,\varphi)^{-1}\underline{Q}\to\underline{Q}'_{|G(\varphi)}$$

and by construction, the restriction of ϑ in degree zero is a morphism of pre-log structures :

$$\vartheta_0:(\mathrm{Proj}\,\varphi)^*((\underline{Q})_0,\beta^\sim_{\mathscr{A}})\to((\underline{Q}')_0,\beta^\sim_{\mathscr{A}'})$$

whence a morphism of $(Z,\underline{N})$-schemes :

$$\mathrm{Proj}(\varphi,\log\varphi):G(\varphi,\log\varphi)\to\mathrm{Proj}(\mathscr{A},\underline{P}).$$

Moreover, on the one hand, (6.4.21) induces an isomorphism of $\mathscr{O}^\times_{Y'}$-modules :

$$\nu^\times_{\mathscr{A}(n)}:(\mathscr{O}^\times_{Y'}\otimes_{(\mathrm{Proj}\,\varphi)^{-1}\mathscr{O}^\times_Y}(\mathrm{Proj}\,\varphi)^{-1}\mathscr{O}_Y(n)^\times)_{|G_1(\varphi)}\xrightarrow{\sim}\mathscr{O}_{Y'}(n)^\times_{|G_1(\varphi)}\qquad\text{for every } n\in\mathbb{Z}$$

(notation of (6.4.20)). On the other hand, for every $n\in\mathbb{Z}$, the morphism of $(\mathrm{Proj}\,\varphi)^{-1}(\underline{Q})_0$-modules ϑ_n determines a morphism of $\underline{P}'^\sim$-modules :

$$\vartheta^\sim_n:\mathrm{Proj}(\varphi,\log\,\varphi)^*\underline{P}^\sim(n)_{|G_1(\varphi)}\to\underline{P}'^\sim(n)_{|G_1(\varphi)} \tag{6.4.51}$$

and by inspecting the construction, it is easily seen that the isomorphism (6.4.39) (and the corresponding one for $\underline{P}'^\sim(n)$) identifies $\vartheta^\sim_n$ with $\nu^\times_{\mathscr{A}(n)}\otimes_{\mathscr{O}^\times_{Y'}}\underline{P}'^\sim$; especially, $\vartheta^\sim_n$ is an isomorphism.

6.4.52. Let $\psi:(Z',\underline{N}')\to(Z,\underline{N})$ be a morphism of log schemes, and $(\mathscr{A},\underline{P},\alpha,\beta_{\mathscr{A}})$ a $\mathbb{N}$-graded quasi-coherent $\mathscr{O}_{(Z,\underline{N})}$-algebra. We may view $\underline{P}$ as a $\underline{N}$-module, via the morphism α, hence we may form the $\underline{N}'$-module $\psi^*\underline{P}$, as in (6.4.34). Moreover, by remark (3.1.25)(i), $\psi^*\underline{P}$ is a $\mathbb{N}$-graded sheaf of monoids on Z'_τ, such that

$$\psi^*(\mathscr{A},\underline{P}):=(\psi^*\mathscr{A},\psi^*\underline{P},\psi^*\alpha,\psi^*\beta_{\mathscr{A}}) \tag{6.4.53}$$

is a $\mathbb{N}$-graded quasi-coherent $\mathscr{O}_{(Z',\underline{N}')}$-algebra, and in view of the isomorphism (3.1.26), we obtain a natural isomorphism of $(Z',\underline{N}')$-schemes:

$$\mathrm{Proj}\,\psi^*(\mathscr{A},\underline{P})\xrightarrow{\sim}(Z',\underline{N}')\times_{(Z,\underline{N})}\mathrm{Proj}(\mathscr{A},\underline{P}). \tag{6.4.54}$$

Furthermore, denote by $\pi_{(\mathscr{A},\underline{P})}:U_1(\psi^*(\mathscr{A},\underline{P}))\to U_1(\mathscr{A},\underline{P})$ the morphism deduced from (6.4.54), and by $\pi_Y:Y':=\mathrm{Proj}\,\psi^*\mathscr{A}\to Y:=\mathrm{Proj}\,\mathscr{A}$ the underlying morphism of schemes. From (6.4.24) we obtain natural isomorphisms :

$$(\mathscr{O}^\times_{Y'}\otimes_{\pi_Y^{-1}\mathscr{O}^\times_Y}\pi_Y^{-1}\mathscr{O}_Y(n)^\times)_{|U_1(\psi^*\mathscr{A})}\xrightarrow{\sim}\mathscr{O}_{Y'}(n)^\times_{|U_1(\psi^*\mathscr{A})}\qquad\text{for every } n\in\mathbb{Z}$$

and the latter induce natural identifications :

$$\pi^*_{(\mathscr{A},\underline{P})}\underline{P}^\sim(n)\xrightarrow{\sim}(\psi^*\underline{P})^\sim(n)\qquad\text{for every } n\in\mathbb{Z}. \tag{6.4.55}$$

6.4.56. Keep the notation of (6.4.35), and let $\log\mathscr{C}_{(X,\underline{M})}$ be the category whose objects are the pairs $((Z,\underline{N}),\mathscr{L})$, where $(Z,\underline{N})$ is a $(X,\underline{M})$-scheme, and $\mathscr{L}$ is an invertible $\underline{N}$-module. The morphisms $((Z,\underline{N}),\mathscr{L})\to((Z',\underline{N}'),\mathscr{L}')$ are the pairs (φ,h), where $\varphi:(Z,\underline{N})\to(Z',\underline{N}')$ is a morphism of $(X,\underline{M})$-schemes, and $h:\varphi^*\mathscr{L}'\xrightarrow{\sim}\mathscr{L}$ is an isomorphism of $\underline{N}$-modules (with composition of morphisms defined in the obvious way). There is an obvious forgetful functor :

$$\mathsf{p}:\log\mathscr{C}_{(X,\underline{M})}\to\mathscr{C}_X \quad : \quad ((Z,\underline{N}),\mathscr{L})\mapsto(Z,\mathscr{L}\otimes_{\underline{N}}\mathscr{O}_Z)$$

and the functor $F_{\mathscr{A}}$ can be lifted to a functor :

$$F_{(\mathscr{A},\underline{P})}:\log\mathscr{C}_{(X,\underline{M})}\to\mathbf{Set}$$

which assigns to any object $((Z,\gamma:\underline{N}\to\mathscr{O}_Z),\mathscr{L})$ the set consisting of all morphisms of $\mathbb{N}$-graded quasi-coherent $\mathscr{O}_{(Z,\underline{N})}$-algebras :

$$\begin{tikzcd}
\psi^*\underline{P} \arrow[r, "\log g"] \arrow[d, "\psi^*\beta_{\mathscr{A}}"'] & \mathrm{Sym}^\bullet_{\underline{N}}\mathscr{L} \arrow[d, "\beta_{\mathscr{A}(\mathscr{L})}"] \\
\psi^*\mathscr{A} \arrow[r, "g"] & \mathscr{A}(\mathscr{L})
\end{tikzcd}$$

where $\psi:(Z,\underline{N})\to(X,\underline{M})$ is the structural morphism, and g is an epimorphism on the underlying $\mathscr{O}_Z$-modules.

Proposition 6.4.57. *The object* $(U_1(\mathscr{A},\underline{P}),\underline{P}^\sim(1))$ *of* $\log\mathscr{C}_{(X,\underline{M})}$ *represents the functor* $F_{(\mathscr{A},\underline{P})}$.

Proof. Given an object $(\psi:(Z,\underline{N})\to(X,\underline{M}),\mathscr{L})$ of $\log\mathscr{C}_{(X,\underline{M})}$, and any element $(g,\log g)\in F_{(\mathscr{A},\underline{P})}((Z,\underline{N}),\mathscr{L})$, define $\mathbb{P}(\mathscr{L})$ as in (6.4.44); there follows a morphism of $(Z,\underline{N})$-schemes :

$$\mathrm{Proj}(g,\log g):\mathbb{P}(\mathscr{L})\to\mathrm{Proj}\,\psi^*(\mathscr{A},\underline{P}).$$

In view of (6.4.44) and (6.4.54), this is the same as a morphism of $(X,\underline{M})$-schemes :

$$\mathbb{P}(g,\log g):(Z,\underline{N})\to\mathrm{Proj}(\mathscr{A},\underline{P})$$

and arguing as in the proof of lemma 6.4.26, we see that the image of $\mathbb{P}(g,\log g)$ lands in $U_1(\mathscr{A},\underline{P})$. Next, combining (6.4.45), (6.4.51) and (6.4.55), we deduce a natural isomorphism :

$$\begin{aligned}
\pi^*_{(Z,\underline{N})}\circ\mathbb{P}(g,\log g)^*\underline{P}^\sim(1) &\xrightarrow{\sim}\mathrm{Proj}(g,\log g)^*\circ\pi^*_{(\mathscr{A},\underline{P})}\underline{P}^\sim(1)\\
&\xrightarrow{\sim}\mathrm{Proj}(g,\log g)^*(\psi^*\underline{P})^\sim(1)\\
&\xrightarrow{\sim}\underline{P}^\sim_{\mathscr{L}}(1)\\
&\xrightarrow{\sim}\pi^*_{(Z,\underline{N})}\mathscr{L}
\end{aligned}$$

whence an isomorphism $h(g,\log g):\mathbb{P}(g,\log g)^*\underline{P}^\sim(1)\xrightarrow{\sim}\mathscr{L}$, and the datum :

$$(\mathbb{P}(g,\log g),h(g,\log g)):((Z,\underline{N}),\mathscr{L})\to(U_1(\mathscr{A},\underline{P}),\underline{P}^\sim(1))$$

is a well defined morphism of $\log\mathscr{C}_{(X,\underline{M})}$.

Conversely, let $\varphi:=(\beta,\log\beta):(Z,\underline{N})\to U_1(\mathscr{A},\underline{P})$ be a morphism of $(X,\underline{M})$-schemes, and $h:\varphi^*\underline{P}^\sim(1)\xrightarrow{\sim}\mathscr{L}$ an isomorphism of $\underline{N}$-modules. In view of (6.4.40), we deduce an isomorphism :

$$h^{\otimes n}:\varphi^*\underline{P}^\sim(n)\xrightarrow{\sim}\mathscr{L}^{\otimes n}\qquad\text{for every } n\in\mathbb{Z}.$$

Combining with (6.4.42), we may define the map of $\underline{N}$-modules :

$$\log\widehat{g}(\varphi,h):=\bigoplus_{n\in\mathbb{N}}h^{\otimes n}\circ\beta^*(\lambda_n):\psi^*\underline{P}\to\mathrm{Sym}^\bullet_{\underline{N}}\mathscr{L}.$$

On the other hand, in view of (6.4.41), we have an isomorphism of $\mathscr{O}_Z$-modules :

$$h\otimes_{\underline{N}}\mathscr{O}_Z:\beta^*\mathscr{O}_Y(1)_{|U_1(\mathscr{A})}\xrightarrow{\sim}\varphi^*\underline{P}^\sim(1)\otimes_{\underline{N}}\mathscr{O}_Z\xrightarrow{\sim}\mathscr{L}\otimes_{\underline{N}}\mathscr{O}_Z$$

(where, as usual, $Y := \mathrm{Proj}\,\mathscr{A}$). We let (notation of (6.4.29)) :

$$\widehat{g}(\varphi, h) := g(\beta, h \otimes_{\underline{N}} \mathscr{O}_Z)$$

and notice that the pair $(\widehat{g}(\varphi, h), \log \widehat{g}(\varphi, h))$ is an element of $F_{(\mathscr{A},\underline{P})}((Z, \underline{N}), \mathscr{L})$. Summing up, we have exhibited two natural transformations :

$$\vartheta : F_{(\mathscr{A},\underline{P})} \Rightarrow \mathrm{Hom}_{\log \mathscr{C}(X,\underline{M})}(-, (U_1(\mathscr{A}, \underline{P}), \underline{P}^{\sim}(1))) \quad (g, \log g) \mapsto \mathbb{P}(g, \log g)$$
$$\sigma : \mathrm{Hom}_{\log \mathscr{C}(X,\underline{M})}(-, (U_1(\mathscr{A}, \underline{P}), \underline{P}^{\sim}(1))) \Rightarrow F_{(\mathscr{A},\underline{P})} \qquad (\varphi, h) \mapsto (\widehat{g}(\varphi, h), \log \widehat{g}(\varphi, h))$$

and it remains to show that these transformations are isomorphisms of functors. However, the latter fit into an essentially commutative diagram of natural transformations :

$$\begin{array}{ccccc} F_{(\mathscr{A},\underline{P})} & \xRightarrow{\vartheta} & \mathrm{Hom}_{\log \mathscr{C}(X,\underline{M})}(-, (U_1(\mathscr{A}, \underline{P}), \underline{P}^{\sim}(1))) & \xRightarrow{\sigma} & F_{(\mathscr{A},\underline{P})} \\ \Big\Downarrow & & \Big\Downarrow & & \Big\Downarrow \\ F_{\mathscr{A}} \circ \mathsf{p} & \Longrightarrow & \mathrm{Hom}_{\mathscr{C}_X}(\mathsf{p}(-), (U_1(\mathscr{A}), \mathscr{O}_Y(1)_{|U_1(\mathscr{A})})) & \Longrightarrow & F_{\mathscr{A}} \circ \mathsf{p} \end{array}$$

whose bottom line is given by the natural transformations (6.4.30). Moreover, given an isomorphism h of invertible $\underline{N}$-modules, the discussion in (6.4.33) leads to the identity :

$$h = (h \otimes_{\underline{N}} \mathscr{O}_Z)^{\times} \otimes_{\mathscr{O}_Z^{\times}} \underline{N}. \tag{6.4.58}$$

Likewise, we have natural identifications :

$$\mathrm{Sym}^n_{\underline{N}} \mathscr{L} = (\mathrm{Sym}^n_{\mathscr{O}_Z} \mathscr{L} \otimes_{\underline{N}} \mathscr{O}_Z)^{\times} \otimes_{\mathscr{O}_Z^{\times}} \underline{N} \qquad \text{for every } n \in \mathbb{Z}$$

which show that $\beta_{\mathscr{A}}$ and g determine uniquely $\log g$. This – and an inspection of the proof of lemma 6.4.26 – already implies that $\sigma \circ \vartheta$ is the identity automorphism of the functor $F_{(\mathscr{A},\underline{P})}$. Finally, let $\varphi = (\beta, \log \beta)$ and h as in the foregoing, so that (φ, h) is a morphism in $\log \mathscr{C}_{(X,\underline{M})}$; to conclude we have to show that $(\varphi', h') := \vartheta \circ \sigma(\varphi, h)$ equals (φ, h). Say that $\varphi' := (\beta', \log \beta')$; by the above (and by the proof of lemma 6.4.26) we already know that $\beta = \beta'$, and (6.4.58) implies that $h = h'$. Hence, it remains only to show that $\log \beta = \log \beta'$, which can be checked directly on the stalks over the τ-points of Z : we leave the details to the reader. ☐

Example 6.4.59. (i) Let $\psi : (Z', \underline{N}') \to (Z, \underline{N})$ be a morphism of log schemes, $n \in \mathbb{N}$ any integer, and $\mathbb{P}^n_{(Z,\underline{N})}$ the n-dimensional projective space over $(Z, \underline{N})$, defined as in example 6.4.46. A simple inspection of the definitions yields a natural isomorphism of $\mathscr{O}_{(Z',\underline{N}')}$-algebras

$$\mathrm{Sym}^{\bullet}_{(\mathscr{O}_{Z'},\underline{N}')}(\mathscr{O}_{Z'}, \underline{N}')^{\oplus n} \xrightarrow{\sim} \psi^* \mathrm{Sym}^{\bullet}_{(\mathscr{O}_Z,\underline{N})}(\mathscr{O}_Z, \underline{N})^{\oplus n} \qquad \text{for every } n \in \mathbb{N}$$

whence a natural isomorphism of $(Z', \underline{N}')$-schemes :

$$\mathbb{P}^n_{(Z',\underline{N}')} \xrightarrow{\sim} (Z', \underline{N}') \times_{(Z,\underline{N})} \mathbb{P}^n_{(Z,\underline{N})} \qquad \text{for every } n \in \mathbb{N}.$$

(ii) Let $\mathscr{L}$ be any invertible $\underline{N}$-module; notice that a morphism of $\mathbb{N}$-graded monoids

$$\mathrm{Sym}^{\bullet}_{\underline{N}} \underline{N}^{\oplus n} \to \mathrm{Sym}^{\bullet}_{\underline{N}} \mathscr{L}$$

which is the identity map in degree zero, is the same as the datum of a sequence

$$(\beta_i : \underline{N} \to \mathscr{L} \mid i = 1, , \dots, n)$$

of morphisms of $\underline{N}$-modules, and the latter is the same as a sequence $(b_1, \dots, b_n)$ of global sections of $\mathscr{L}$. Since $\mathrm{Sym}^1_{\mathscr{O}_Z} \mathscr{O}_Z^{\oplus n+1}$ generates $\mathrm{Sym}^{\bullet}_{\mathscr{O}_Z} \mathscr{O}_Z^{n+1}$, proposition 6.4.57 and (i) imply that $\mathbb{P}^n_{(Z,\underline{N})}$ represents the functor $\log \mathscr{C}_{(Z,\underline{N})} \to \mathbf{Set}$ that assigns to any pair $((X, \underline{M}), \mathscr{L})$ the set of all sequences $(b_0, \dots, b_n)$ of global sections of $\mathscr{L}$. The bijection

$$\mathrm{Hom}_{\log \mathscr{C}_{(Z,\underline{N})}}(((X, \underline{M}), \mathscr{L}), (\mathbb{P}^n_{(Z,\underline{N})}, \underline{N}^{\sim}(1))) \xrightarrow{\sim} \Gamma(X, \mathscr{L})^{n+1} \tag{6.4.60}$$

can be explicited as follows. Let $(\varphi, h) : ((X, \underline{M}), \mathscr{L}) \to (\mathbb{P}^n_{(Z,\underline{N})}, \underline{N}^\sim(1))$ be a given morphism in $\log \mathscr{C}_{(Z,\underline{N})}$; then $h : \varphi^*\underline{N}^\sim(1) \to \mathscr{L}$ is an isomorphism of $\underline{M}$-modules, which induces a map on global sections

$$\Gamma(h) : \Gamma(\mathbb{P}^n_{(Z,\underline{N})}, \underline{N}^\sim(1))) \to \Gamma(X, \mathscr{L}).$$

However, $\underline{N}^\sim(1)$ admits a distiguished system of global sections $\varepsilon_0, \dots, \varepsilon_n$ (example 6.4.46(ii)), and the bijection (6.4.60) assigns to (φ, h) the sequence $(\Gamma(h)(\varepsilon_0), \dots, \Gamma(h)(\varepsilon_n))$.

(iii) Given a sequence $b_\bullet := (b_0, \dots, b_n)$ as in (ii), denote by

$$f_{b_\bullet} : (X, \underline{M}) \to \mathbb{P}^n_{(Z,\underline{N})}$$

the corresponding morphism. A direct inspection of the definitions shows that the formation of $f_{b_\bullet}$ is compatible with arbitrary base changes $h : (Z', \underline{N}') \to (Z, \underline{N})$. Namely, set $(X', \underline{M}') := (Z', \underline{N}') \times_{(Z,\underline{N})} (X, \underline{M})$, let $g : (X', \underline{M}') \to (X, \underline{M})$ be the induced morphism, $\mathscr{L}' := g^*\mathscr{L}$, and suppose that $\mathscr{L}'$ is also invertible (which always holds, if $\underline{M}'$ is an integral log structure); the sequence $b_\bullet$ pulls back to a corresponding sequence $b'_\bullet := (b'_0, \dots, b_n)$ of global sections of $\mathscr{L}'$, and there follows a cartesian diagram of log schemes :

$$\begin{tikzcd}
(X', \underline{M}') \arrow[r, "f_{b'_\bullet}"] \arrow[d, "g"'] & \mathbb{P}^n_{(Z',\underline{N}')} \arrow[d, "\mathbb{P}^n_h"] \\
(X, \underline{M}) \arrow[r, "f_{b_\bullet}"] & \mathbb{P}^n_{(Z,\underline{N})}.
\end{tikzcd}$$

The details shall be left to the reader.

Definition 6.4.61. Let $(X, \underline{M})$ be a log scheme, and $\mathscr{I} \subset \underline{M}$ an ideal of $\underline{M}$ (see (1.2.22)).

(i) Let $f : (Y, \underline{N}) \to (X, \underline{M})$ be a morphism of log schemes; then $f^{-1}\mathscr{I}$ is an ideal in the sheaf of monoids $f^{-1}\underline{M}$; we let :

$$\mathscr{I}\underline{N} := \log f(f^{-1}\mathscr{I}) \cdot \underline{N}$$

which is the smallest ideal of $\underline{N}$ containing the image of $f^{-1}\mathscr{I}$.

(ii) A *logarithmic blow up* of the ideal $\mathscr{I}$ is a morphism of log schemes

$$\varphi : (X', \underline{M}') \to (X, \underline{M})$$

which enjoys the following universal property. The ideal $\mathscr{I}\underline{M}'$ is invertible, and every morphism of log schemes $(Y, \underline{N}) \to (X, \underline{M})$ such that $\mathscr{I}\underline{N} \subset \underline{N}$ is invertible, factors uniquely through φ.

Remark 6.4.62. (i) Keep the notation of definition 6.4.61. By the usual general nonsense, it is clear that the blow up $(X', \underline{M}')$ is determined up to unique isomorphism of $(X, \underline{M})$-schemes.

(ii) Moreover, let $f : Y \to X$ be a morphism of schemes. Then we claim that the natural projection :

$$(Y', \underline{M}'_Y) := Y \times_X (X', \underline{M}') \to (Y, \underline{M}_Y) := Y \times_X (X, \underline{M})$$

is a logarithmic blow up of $\mathscr{I}\underline{M}_Y$, provided $\mathscr{I}\underline{M}'_Y$ is an invertible ideal; especially, this holds whenever $\underline{M}'$ is an integral log structure (lemma 6.1.16(i)). The proof is left as an exercise for the reader.

6.4.63. Let $(X, \underline{M})$ be a log scheme, $\mathscr{I} \subset \underline{M}$ an ideal; we shall show the existence of the logarithmic blow up of $\mathscr{I}$, under fairly general conditions. To this aim, we introduce the graded *blow up $\mathscr{O}_X$-algebra* :

$$\mathsf{B}(X, \underline{M}, \mathscr{J}) := \bigoplus_{n \in \mathbb{N}} \mathscr{I}^n \otimes_{\underline{M}} \mathscr{O}_X$$

where $\mathscr{I}^n \otimes_{\underline{M}} \mathscr{O}_X$ is the sheaf associated to the presheaf $U \mapsto \mathscr{I}^n(U) \otimes_{\underline{M}(U)} \mathscr{O}_X(U)$ on X_τ. Here $\mathscr{I}^0 := \underline{M}$, and for every $n > 0$ we let $\mathscr{I}^n$ be the sheaf associated to the presheaf $U \mapsto \mathscr{I}(U)^n$ on X_τ. The graded multiplication law of the blow up $\mathscr{O}_X$-algebra is induced by the multiplication $\mathscr{I}^n \times \mathscr{I}^m \to \mathscr{I}^{n+m}$, for every $n, m \in \mathbb{N}$.

The natural map $\mathscr{I}^n \to \mathscr{I}^n \otimes_{\underline{M}} \mathscr{O}_X$ induces a morphism of sheaves of monoids :

$$\coprod_{n\in\mathbb{N}} \mathscr{I}^n \to \mathsf{B}(X, \underline{P}, \mathscr{J}).$$

The latter defines a $\mathbb{N}$-graded $\mathscr{O}_{(X,\underline{M})}$-algebra, which we denote $\mathscr{B}(X, \underline{M}, \mathscr{I})$.

6.4.64. Suppose first that $\mathscr{I}$ is invertible; then it is easily seen that, locally on X_τ, $\mathscr{I}$ is generated by a regular local section (see example 2.3.36(i)), hence the same holds for the power $\mathscr{I}^n$, for every $n \in \mathbb{N}$. Therefore $\mathscr{I}^n$ is a locally free $\underline{M}$-module of rank one, and we have a natural isomorphism :

$$\mathscr{B}(X, \underline{M}, \mathscr{I}) \xrightarrow{\sim} (\mathscr{A}(\mathscr{I}), \mathrm{Sym}^\bullet_{\underline{M}} \mathscr{I})$$

(notation of example (6.4.43)). It follows that in this case, the natural projection :

$$\pi_{(X,\underline{M},\mathscr{I})} : \mathrm{Proj}\, \mathscr{B}(X, \underline{M}, \mathscr{I}) \to (X, \underline{M})$$

is an isomorphism of log schemes.

6.4.65. The formation of $\mathscr{B}(X, \underline{M}, \mathscr{I})$ is obviously functorial with respect to morphisms of log schemes; more precisely, such a morphism $f : (Y, \underline{N}) \to (X, \underline{M})$ induces a morphism of $\mathbb{N}$-graded $\mathscr{O}_{(Y,\underline{N})}$-algebras:

$$\mathscr{B}(f, \mathscr{I}) : f^*\mathscr{B}(X, \underline{M}, \mathscr{I}) \to \mathscr{B}(Y, \underline{N}, \mathscr{I}\underline{N})$$

(notation of (6.4.53)) which is an epimorphism on the underlying $\mathscr{O}_Y$-modules. Moreover, if $g : (Z, \underline{Q}) \to (Y, \underline{N})$ is another morphism, we have the identity :

$$\mathscr{B}(f \circ g, \mathscr{I}) = \mathscr{B}(g, \mathscr{I}\underline{N}) \circ g^*\mathscr{B}(f, \mathscr{I}). \quad (6.4.66)$$

Furthermore, the construction of the blow up algebra is local for the topology of X_τ : if $U \to X$ is any object of X_τ, we have a natural identification

$$\mathscr{B}(U, \underline{M}_{|U}, \mathscr{I}_{|U}) \xrightarrow{\sim} \mathscr{B}(X, \underline{M}, \mathscr{I})_{|U}.$$

In the presence of charts for the log structure $\underline{M}$, we can give a handier description for the blow up algebra; namely, we have the following :

Lemma 6.4.67. *Let X be a scheme, $\alpha : \underline{P} \to \mathscr{O}_X$ a pre-log structure, $\beta : \underline{P} \to \underline{P}^{\log}$ the natural morphism of pre-log structures. Let also $\mathscr{I} \subset \underline{P}$ be an ideal, and set $\mathscr{I}\underline{P}^{\log} := \beta(\mathscr{I}) \cdot \underline{P}^{\log}$ (which is the ideal of $\underline{P}^{\log}$ generated by the image of $\mathscr{I}$). Then :*

(i) *There is a natural isomorphism of graded $\mathscr{O}_X$-algebras :*

$$\bigoplus_{n\in\mathbb{N}} \mathscr{I}^n \otimes_{\underline{P}} \mathscr{O}_X \xrightarrow{\sim} \mathsf{B}(X, \underline{P}^{\log}, \mathscr{I}\underline{P}^{\log}).$$

(ii) *Especially, suppose $(X, \underline{M})$ is a log scheme that admits a chart $\beta : P_X \to \underline{M}$, let $I \subset P$ be an ideal, and set $I\underline{M} := \beta(I_X) \cdot \underline{M}$. Then $\mathscr{B}(X, \underline{M}, I\underline{M})$ is a $\mathbb{N}$-graded quasi-coherent $\mathscr{O}_{(X,\underline{M})}$-algebra.*

(iii) *In the situation of* (ii)*, set $(S, P_S^{\log}) := \mathrm{Spec}(\mathbb{Z}, P)$ (see* (6.2.13)*), and denote by $f : (X, \underline{M}) \to (S, P_S^{\log})$ the natural morphism. Then the map :*

$$\mathscr{B}(f, IP_S^{\log}) : f^*\mathscr{B}(S, P_S^{\log}, IP_S^{\log}) \to \mathscr{B}(X, \underline{M}, I\underline{M})$$

is an isomorphism of $\mathbb{N}$-graded $\mathscr{O}_{(X,\underline{M})}$-algebras.

Proof. (i): Since the functor (2.3.51) commutes with all colimits, we have a natural isomorphism of sheaves of rings on X_τ :

$$\mathbb{Z}[\underline{P}^{\log}] \xrightarrow{\sim} \mathbb{Z}[\underline{P}] \otimes_{\mathbb{Z}[\alpha^{-1}\mathscr{O}^\times_X]} \mathbb{Z}[\mathscr{O}^\times_X].$$

We are therefore reduced to showing that the natural morphism

$$\mathbb{Z}[\mathscr{I}^n] \otimes_{\mathbb{Z}[\alpha^{-1}\mathscr{O}^\times_X]} \mathbb{Z}[\mathscr{O}^\times_X] \to \mathscr{I}^n\mathbb{Z}[\underline{P}^{\log}]$$

is an isomorphism for every $n \in \mathbb{N}$. The latter assertion can be checked on the stalks over the τ-points of X; to this aim, we invoke the more general :

Claim 6.4.68. Let G be an abelian group, $\varphi : H \to P'$ and $\psi : H \to G$ two morphisms of monoids, $I \subset P'$ an ideal. Then the natural map

$$\mathbb{Z}[I] \otimes_{\mathbb{Z}[H]} \mathbb{Z}[G] \to I\mathbb{Z}[P' \otimes_H G] \tag{6.4.69}$$

is an isomorphism.

Proof of the claim. Recall that $\mathbb{Z}[I] = I\mathbb{Z}[P']$, and the map (6.4.69) is induced by the natural identification: $\mathbb{Z}[P'] \otimes_{\mathbb{Z}[H]} \mathbb{Z}[G] \xrightarrow{\sim} \mathbb{Z}[P' \otimes_H G]$ (see (2.3.52)). Especially, (6.4.69) is clearly surjective, and it remains to show that it is also injective. To this aim, notice first that ψ factors through the unit of adjunction $\eta : H \to H^{\mathrm{gp}}$; the morphism $\mathbb{Z}[\eta] : \mathbb{Z}[H] \to \mathbb{Z}[H^{\mathrm{gp}}] = H^{-1}\mathbb{Z}[H]$ is a localization map (see (2.3.53)), especially it is flat, hence (6.4.69) is injective when $G = H^{\mathrm{gp}}$ and $\psi = \eta$. It follows easily that we may replace H by H^{gp}, P' by $P' \otimes_H H^{\mathrm{gp}}$, I by $I \cdot (P' \otimes_H H^{\mathrm{gp}})$, and therefore assume that H is a group. Let $L := \psi(H)$; arguing as in the foregoing, we may consider separately the case where ψ is the surjection $H \to L$ and the case where ψ is the injection $i : L \to G$. However, one sees easily that $\mathbb{Z}[i] : \mathbb{Z}[L] \to \mathbb{Z}[G]$ is a flat morphism, hence it suffices to consider the case where ψ is a surjective group homomorphism. Set $K := \mathrm{Ker}\,\psi$; we have a natural identification : $P' \otimes_H G \simeq P' \otimes_K \{1\}$, hence we may further reduce to the case where $G = \{1\}$. Then the contention is that the augmentation map $\mathbb{Z}[K] \to \mathbb{Z}$ induces an isomorphism $\omega : \mathbb{Z}[I] \otimes_{\mathbb{Z}[K]} \mathbb{Z} \xrightarrow{\sim} I\mathbb{Z}[P'/K]$. From lemma 2.3.31(ii), we derive easily that $I\mathbb{Z}[P'/K] = \mathbb{Z}[I/K]$, where I/K is the set-theoretic quotient of I for the K-action deduced from φ. However, any set-theoretic section $I/K \to I$ of the natural projection $I \to I/K$ yields a well-defined surjection $\mathbb{Z}[I/K] \to \mathbb{Z}[I] \otimes_{\mathbb{Z}[K]} \mathbb{Z}$ whose composition with ω is the identity map. The claim follows. ◊

(ii): Let U be an affine object of X_τ, say $U = \mathrm{Spec}\, A$; from (i), we see that $\mathscr{B}(X, \underline{M}, I\underline{M})_{|U}$ is the quasi-coherent $\mathscr{O}_U$-algebra associated to the A-algebra $\oplus_{n\in\mathbb{N}}\mathbb{Z}[I^n] \otimes_{\mathbb{Z}[P]} A$.

(iii): In view of (i) we know already that $\mathscr{B}(f, IP^{\log}_S)$ induces an isomorphism on the underlying $\mathscr{O}_X$-algebras; hence, by lemma 6.2.18(i), it remains to show that $\mathscr{B}(f, IP^{\log}_S)$ induces an isomorphism :

$$f^*(I^n P^{\log}_S) \xrightarrow{\sim} I^n f^*(P^{\log}_S) \qquad \text{for every } n \in \mathbb{N}.$$

Let $\gamma : f^{-1}P^{\log}_S \to f^{-1}\mathscr{O}_S \to \mathscr{O}_X$ be the natural map; after replacing I by I^n, we come down to showing that the natural map :

$$f^{-1}(IP^{\log}_S) \otimes_{\gamma^{-1}\mathscr{O}^\times_X} \mathscr{O}^\times_X \to I \cdot (f^{-1}P^{\log}_S \otimes_{\gamma^{-1}\mathscr{O}^\times_X} \mathscr{O}^\times_X)$$

is an isomorphism. This assertion can be checked on the stalks over the τ-points of X; if x is such a point, let $G := \mathscr{O}^\times_{X,x}$, $H := \gamma_x^{-1}G$ and $P' := P^{\log}_{S,f(x)}$. The map under consideration is the natural morphism of P'-modules :

$$\omega : (IP') \otimes_H G \to I(P' \otimes_H G)$$

and it suffices to show that $\mathbb{Z}[\omega]$ is an isomorphism; however, the latter is none else than (6.4.69), so we may appeal to claim 6.4.68 to conclude. □

6.4.70. We wish to generalize lemma 6.4.67(ii) to log structures that do not necessarily admit global charts. Namely, suppose now that $\underline{M}$ is a quasi-coherent log structure on X, and $\mathscr{I} \subset \underline{M}$ is a coherent ideal (see definition 2.3.6(v)). For every τ-point ξ of X, we may then find a neighborhood U of ξ in X_τ, a chart $\beta : P_U \to \underline{M}_{|U}$, and local sections $s_1, \dots, s_n \in \mathscr{I}(U)$ which form a system of generators for $\mathscr{I}_{|U}$. We may then write $s_{i,\xi} = u_i \cdot \beta(x_i)$ for certain $x_1, \dots, x_n \in P$ and $u_1, \dots, u_n \in \mathscr{O}^\times_{X,\xi}$. Up to shrinking U, we may assume that $u_1, \dots, u_n \in \mathscr{O}^\times_X(U)$, and it follows that $\mathscr{I}_{|U} = I \cdot \underline{M}_{|U}$, where $I \subset P$ is the ideal generated by $x_1, \dots, x_n$. In other words, locally on X_τ, the datum $(X, \underline{M}, \mathscr{I})$ is of the type considered in lemma 6.4.67(ii); therefore the blow up $\mathscr{O}_X$-algebra $\mathscr{B}(X, \underline{M}, \mathscr{I})$ is quasi-coherent. We may then consider the natural projection :

$$\pi_{(X,\underline{M},\mathscr{I})} : \mathrm{Bl}_{\mathscr{I}}(X, \underline{M}) := \mathrm{Proj}\, \mathscr{B}(X, \underline{M}, \mathscr{I}) \to (X, \underline{M}). \tag{6.4.71}$$

Next, let $f : (Y, \underline{N}) \to (X, \underline{M})$ be a morphism of log schemes; it is easily seen that $\mathscr{I}\underline{N}$ is a coherent ideal of $\underline{N}$, hence $\mathscr{B}(Y, \underline{N}, \mathscr{I}\underline{N})$ is quasi-coherent as well, and since the map $\mathscr{B}(f, \mathscr{I})$ of (6.4.65) is an epimorphism on the underlying $\mathscr{O}_Y$-modules, we have :

$$G(\mathscr{B}(f, \mathscr{I})) = \mathrm{Bl}_{\mathscr{I}\underline{N}}(Y, \underline{N})$$

(notation of (6.4.48)) whence a closed immersion of $(Y, \underline{N})$-schemes :

$$\mathrm{Proj}\, \mathscr{B}(f, \mathscr{I}) : \mathrm{Bl}_{\mathscr{I}\underline{N}}(Y, \underline{N}) \to (Y, \underline{N}) \times_{(X,\underline{M})} \mathrm{Bl}_{\mathscr{I}}(X, \underline{M})$$

([27, Ch.II, Prop.3.6.2(i)] and (6.4.54)), which is the same as a morphism of $(X, \underline{M})$-schemes :

$$\varphi(f, \mathscr{I}) : \mathrm{Bl}_{\mathscr{I}\underline{N}}(Y, \underline{N}) \to \mathrm{Bl}_{\mathscr{I}}(X, \underline{M}).$$

Moreover, if $g : (Z, \underline{Q}) \to (Y, \underline{N})$ is another morphism, (6.4.66) induces the identity :

$$\varphi(f, \mathscr{I}) \circ \varphi(g, \mathscr{I}\underline{N}) = \varphi(f \circ g, \mathscr{I}). \tag{6.4.72}$$

Example 6.4.73. In the situation of lemma 6.4.67(iii), notice that $\mathscr{O}_S$ is a flat $\mathbb{Z}[P_S]$-algebra, and therefore :

$$\mathsf{B}(S, P^{\log}_S, IP^{\log}_S) = \bigoplus_{n \in \mathbb{N}} I^n \mathscr{O}_S.$$

Moreover, (6.4.54) specializes to a natural isomorphism of $(X, \underline{M})$-schemes :

$$\mathrm{Bl}_{I\underline{M}}(X, \underline{M}) \xrightarrow{\sim} X \times_S \mathrm{Bl}_{IP^{\log}_S}(S, P^{\log}_S). \tag{6.4.74}$$

We wish to give a more explicit description of the log structure of $\mathrm{Bl}_{IP^{\log}_S}(S, P^{\log}_S)$. To begin with, recall that $S' := \mathrm{Proj}\, \mathsf{B}(S, P^{\log}_S, IP^{\log}_S)$ admits a distinguished covering by (Zariski) affine open subsets : namely, for every $a \in I$, consider the localization

$$P_a := T_a^{-1} P \qquad \text{where} \qquad T_a := \{a^n \mid n \in \mathbb{N}\}$$

and let $Q_a \subset P_a$ be the submonoid generated by the image of P and the subset $\{a^{-1}b \mid b \in I\}$; then

$$S' = \bigcup_{a \in I} \mathrm{Spec}\, \mathbb{Z}[Q_a].$$

Hence, let us set

$$U_a := \mathrm{Spec}(\mathbb{Z}, Q_a) \qquad \text{for every } a \in I.$$

We claim that these locally defined log structures glue to a well defined log structure $\underline{Q}$ on the whole of S'_τ. Indeed, let $a, b \in I$; we have

$$U_{a,b} := \mathrm{Spec}\, \mathbb{Z}[Q_a] \cap \mathrm{Spec}\, \mathbb{Z}[Q_b] = \mathrm{Spec}\, \mathbb{Z}[Q_a \otimes_P Q_b]$$

and it is easily seen that $Q_a \otimes_P Q_b = Q_a[b^{-1}a]$, *i.e.* the localization of Q_a obtained by inverting its element $a^{-1}b$, and this is of course the same as $Q_b[a^{-1}b]$. Then lemma 6.2.14 implies that the log structures of U_a and U_b agree on $U_{a,b}$, whence the contention. It is easy to check that the resulting log scheme is precisely $\mathrm{Bl}_{IP_S^{\log}}(S, P_S^{\log})$: the details shall be left to the reader.

Proposition 6.4.75. *Let* $(X, \underline{M})$ *be a log scheme with quasi-coherent log structure, and* $\mathscr{I} \subset \underline{M}$ *a coherent ideal. Then, the morphism* (6.4.71) *is a logarithmic blow up of* $\mathscr{I}$*.*

Proof. Let $f : (Y, \underline{N}) \to (X, \underline{M})$ be a morphism of log schemes, and suppose that $\mathscr{I}\underline{N}$ is an invertible ideal of $\underline{N}$; in this case, we have already remarked (see (6.4.64)) that the projection $\pi_{(Y,\underline{N},\mathscr{I}\underline{N})} : \mathrm{Bl}_{\mathscr{I}\underline{N}}(Y, \underline{N}) \to (Y, \underline{N})$ is an isomorphism. We deduce a morphism :

$$\varphi(f, \mathscr{I}) \circ \pi^{-1}_{(Y,\underline{N},\mathscr{I}\underline{N})} : (Y, \underline{N}) \to \mathrm{Bl}_{\mathscr{I}}(X, \underline{M}). \tag{6.4.76}$$

To conclude, it remains to show that (6.4.76) is the only morphism of log schemes whose composition with $\pi_{(X,\underline{M},\mathscr{I})}$ equals f. The latter assertion can be checked locally on X_τ, hence we may assume that $\underline{M}$ admits a chart $P_X \to \underline{M}$, such that $\mathscr{I} = I\underline{M}$ for some finitely generated ideal $I \subset P$. In this case, in view of (6.4.74), the set of morphisms of $(X, \underline{M})$-schemes $(Y, \underline{N}) \to \mathrm{Bl}_{\mathscr{I}}(X, \underline{M})$ is in natural bijection with the set of $(S, P_S^{\log})$-morphisms $(Y, \underline{N}) \to B := \mathrm{Bl}_{IP_S^{\log}}(S, P_S^{\log})$ (notation of example 6.4.73). In other words, we may assume that $(X, \underline{M}) = (S, P_S^{\log})$, and $\mathscr{I} = IP_S^{\log}$. Then f is determined by $\log f$, *i.e.* by a map $\beta : P \to \underline{N}(Y)$. Let $a_1, \dots, a_k$ be a system of generators for I; for each $i = 1, \dots, k$, we let $U_i \subset Y$ be the subset of all $y \in Y$ for which there exists a τ-point ξ of Y with $|\xi| = y$ and

$$a_i\underline{N}_\xi = I\underline{N}_\xi. \tag{6.4.77}$$

Notice that, if $y \in U_i$, then (6.4.77) holds for every τ-point ξ of Y localized at y (details left to the reader).

Claim 6.4.78. The subset U_i is open in Y_i for every $i = 1, \dots, k$, and $Y = \bigcup_{i=1}^k U_i$.

Proof of the claim. Say that $y \in U_i$, and let ξ be a τ-point of Y localized at y, such that (6.4.77) holds. This means that, for every $j = 1, \dots, k$, there exists $u_j \in \underline{N}_\xi$ such that $a_j = u_j a_i$. Then, we may find a τ-neighborhood $h : U' \to$ of ξ such that this identity persists in $\underline{N}(U')$; thus, $h(U') \subset U_i$. Since h is an open map, this shows that U_i is an open subset.

Next, let ξ be any τ-point of Y; by assumption, we have $I\underline{N}_\xi = b\underline{N}_\xi$ for some $b \in \underline{N}_\xi$; this means that for every $i = 1, \dots, k$ there exists $u_i \in \underline{N}_\xi$ such that $a_i = u_i b$. Since $a_1, \dots, a_k$ generate I, we must have $u_i \in \underline{N}_\xi^\times$ for at least an index $i \le k$, in which case $|\xi| \in U_i$, and this shows that the U_i cover the whole of Y, as claimed. ◊

It is easily seen that, for every $i = 1, \dots, k$, any morphism $(U_i, \underline{N}_{|U_i}) \to B$ of $(S, P_S^{\log})$-schemes factors through the open immersion $\mathrm{Spec}\,(Z, Q_{a_i}) \to B$ (where Q_a, for an element $a \in P$, is defined as in example 6.4.73). Conversely, by construction β extends to a unique morphism of monoids $Q_{a_i} \to \underline{N}(U_i)$. Summing up, there exists at most one morphism of $(S, P_S^{\log})$-schemes $(U_i, \underline{N}_{|U_i}) \to B$. In light of claim 6.4.78, the proposition follows. □

6.4.79. Keep the notation of proposition 6.4.75; by inspecting the construction, it is easily seen that the log structure $\underline{M}'$ of $\mathrm{Bl}_{\mathscr{I}}(X, \underline{M})$ is quasi-coherent, and if $\underline{M}$ is coherent (resp. quasi-fine, resp. fine), then the same holds for $\underline{M}'$. However, simple examples show that $\underline{M}'$ may fail to be saturated, even in cases where $(X, \underline{M})$ is a fs log scheme. Due to the prominent role played by fs log schemes, it is convenient to introduce the special notation :

$$\mathrm{sat.Bl}_{\mathscr{I}}(X, \underline{M}) := (\mathrm{Bl}_{\mathscr{I}}(X, \underline{M}))^{\mathrm{qfs}}$$

for the *saturated logarithmic blow up* of a coherent ideal $\mathscr{I}$ in a quasi-fine log structure $\underline{M}$ (notation of proposition 6.2.35). Clearly the projection $\mathrm{sat.Bl}_{\mathscr{I}}(X, \underline{M}) \to (X, \underline{M})$ is a final object of the category of saturated $(X, \underline{M})$-schemes in which the preimage of $\mathscr{I}$ is invertible. If $(X, \underline{M})$ is a fine log scheme, $\mathrm{sat.Bl}_{\mathscr{I}}(X, \underline{M})$ is a fs log scheme. Moreover, for any morphism of schemes $f : Y \to X$, let $\underline{M}_Y := f^*\underline{M}$; from remarks (6.4.62)(ii) and 6.2.36(iii), we deduce a natural isomorphism of $(Y, \underline{M}_Y)$-schemes :

$$\text{(6.4.80)} \qquad \mathrm{sat.Bl}_{\mathscr{I}\underline{M}_Y}(Y, \underline{M}_Y) \xrightarrow{\sim} Y \times_X \mathrm{sat.Bl}_{\mathscr{I}}(X, \underline{M}).$$

Theorem 6.4.81. *Let $(X, \underline{M})$ be a quasi-fine log scheme with saturated log structure, $\mathscr{I} \subset \underline{M}$ an ideal, ξ a τ-point of X. Suppose that, in a neighborhood of ξ, the ideal $\mathscr{I}$ is generated by at most two sections, and denote :*

$$\varphi : \mathrm{Bl}_{\mathscr{I}}(X, \underline{M}) \to (X, \underline{M}) \qquad \textit{(resp. } \varphi_{\mathrm{sat}} : \mathrm{sat.Bl}_{\mathscr{I}}(X, \underline{M}) \to (X, \underline{M})\textit{)}$$

the logarithmic (resp. saturated logarithmic) blow up of $\mathscr{I}$. Then :

(i) *If $\mathscr{I}_\xi$ is an invertible ideal of $\underline{M}_\xi$, the natural morphisms*

$$\varphi^{-1}(\xi) \to \mathrm{Spec}\, \kappa(\xi) \qquad \varphi^{-1}_{\mathrm{sat}}(\xi) \to \mathrm{Spec}\, \kappa(\xi)$$

are isomorphisms.

(ii) *Otherwise, $\varphi^{-1}(\xi)$ is a $\kappa(\xi)$-scheme isomorphic to $\mathbb{P}^1_{\kappa(\xi)}$; furthermore, the same holds for the reduced fibre $\varphi^{-1}_{\mathrm{sat}}(\xi)_{\mathrm{red}}$, provided $\underline{M}$ is fine.*

Proof. After replacing X by a τ-neighborhood of ξ, we reduce to the case where $\underline{M}$ admits an integral and saturated chart $\alpha : P_X \to \underline{M}$ (lemma 6.1.16(iii)), and if $\underline{M}$ is a fs log structure, we may also assume that the chart α is fine and sharp at ξ (corollary 6.1.34(i)). Furthermore, we may assume that $\mathscr{I}$ is generated by at most two elements of $\underline{M}(X)$, and if $\mathscr{I}_\xi$ is principal, we may assume that the same holds for $\mathscr{I}$. In the latter case, since $\underline{M}$ is integral, $\mathscr{I}$ is invertible, hence φ and φ_{sat} are isomorphisms, so (i) follows already.

Now, suppose that $\mathscr{I}_\xi$ is not invertible, and let $a', b' \in \underline{M}(X)$ be a system of generators for $\mathscr{I}$; we can write $a' = \alpha(a) \cdot u$, $b' = \alpha(b) \cdot v$ for some $a, b \in P$ and $u, v \in \kappa^\times$. Set $t := a^{-1}b$, and let $J \subset P$ be the ideal generated by a and b; clearly $J\underline{M} = \mathscr{I}$, and $Pa, Pb \neq J$.

Consider first the case where $X = \mathrm{Spec}\, \kappa$, where κ is a field (resp. a separably closed field, in case $\tau = \text{ét}$). In this situation, a pre-log structure on X_τ is the same as a morphism of monoids $\beta : P \to \kappa$, the associated log structure is the induced map of monoids

$$\beta^{\log} : P \otimes_{P_0} \kappa^\times \to \kappa \qquad \text{where} \qquad P_0 := \beta^{-1}\kappa^\times$$

and α is the natural map $P \to P \otimes_{P_0} \kappa^\times$. After replacing P by its localization $P_0^{-1}P$, we may also assume that $P_0 = P^\times$. Let

$$(S, P_S^{\log}) := \mathrm{Spec}(\kappa, P) \qquad J^\sim := JP_S^{\log} \qquad (Y, \underline{N}) := \mathrm{Bl}_{J^\sim}(S, P_S^{\log}).$$

Denote by $\varepsilon : \kappa[P] \to \kappa$ the homomorphism of κ-algebras induced by β via the adjunction (2.3.50), and set $I := \mathrm{Ker}\, \varepsilon$. In view of lemma 6.4.67(i) and (6.4.80), we have natural cartesian diagrams of κ-schemes :

$$\text{(6.4.82)} \qquad \begin{tikzcd} \varphi^{-1}(\xi) \arrow[r] \arrow[d] & (Y, \underline{N}) \arrow[d] \\ {|\xi|} \arrow[r, "\mathrm{Spec}\,\varepsilon"] & S \end{tikzcd} \qquad \begin{tikzcd} \varphi^{-1}_{\mathrm{sat}}(\xi) \arrow[r] \arrow[d] & (Y, \underline{N})^{\mathrm{sat}} \arrow[d] \\ {|\xi|} \arrow[r, "\mathrm{Spec}\,\varepsilon"] & S. \end{tikzcd}$$

On the other hand, (a, b) can be regarded as a pair of global sections of $J^\sim\underline{N}$, so the universal property of example 6.4.59(ii) yields a morphism of $(S, P_S^{\log})$-schemes :

$$f_{(a,b)} : (Y, \underline{N}) \to \mathbb{P}^1_{(S, P_S^{\log})}.$$

In light of example 6.4.59(i), the assertion concerning $\varphi^{-1}(\xi)$ will then follow from the :

Claim 6.4.83. The morphism $|\xi| \times_S f_{(a,b)}$ is an isomorphism of κ-schemes.

Proof of the claim. Let $Q_a \subset P^{\mathrm{gp}}$ (resp. $Q_b \subset P^{\mathrm{gp}}$) be the submonoid generated by P and t (resp. by P and t^{-1}); by inspecting example 6.4.73, we see that Y is covered by two affine open subsets :

$$U_a := \mathrm{Spec}\,\kappa[Q_a] \qquad U_b := \mathrm{Spec}\,\kappa[Q_b]$$

and $U_a \cap U_b = \mathrm{Spec}\,\kappa[Q_a \otimes_P Q_b]$. On the other hand, $\mathbb{P}^1_{(S,P_S^{\log})}$ is covered as well by two affine open subsets U'_0 and U'_∞, both isomorphic to $\mathrm{Spec}\,\kappa[P \oplus \mathbb{N}]$, and such that $U'_0 \cap U'_\infty = \mathrm{Spec}\,\kappa[P \oplus \mathbb{Z}]$, as usual. Moreover, a direct inspection shows that $f_{(a,b)}$ restricts to morphisms

$$U_a \to U'_0 \qquad U_b \to U'_\infty$$

induced respectively by the maps of κ-algebras

$$\omega_0 : \kappa[P \oplus \mathbb{N}] \to \kappa[Q_a] \qquad \omega_\infty : \kappa[P \oplus \mathbb{N}] \to \kappa[Q_b]$$

such that $\omega_0(x,n) = x \cdot t^n$ and $\omega_\infty(x,n) = x \cdot t^{-n}$ for every $(x,n) \in P \oplus \mathbb{N}$. We show that $\overline{\omega}_0 := \omega_0 \otimes_{\kappa[P]} \kappa[P]/I$ is an isomorphism; the same argument will apply also to ω_∞, so the claim shall follow.

Indeed, clearly the $\kappa[P]$-algebra $\kappa[Q_a]$ is generated by t, hence ω_0 is surjective, and then the same holds for $\overline{\omega}_0$. Next, set $H_a := I \cdot \kappa[Q_a]$; it is easily seen that H_a consists of all sums of the form $\sum_{j=0}^n c_j t^j$, for arbitrary $n \in \mathbb{N}$, with $c_j \in I$ for every $j = 0, \dots, n$. Clearly, an element $p(T) \in \kappa[\mathbb{N}] = \kappa[T]$ lies in $\mathrm{Ker}\,\overline{\omega}_0$ if and only if $p(t) \in H_a$, so we come down to the following assertion. Let $c_0, \dots, c_n \in \kappa[P]$ such that

$$\sum_{j=0}^{n} c_j t^j = 0 \tag{6.4.84}$$

in $\kappa[Q_a]$; then c_j lies in the ideal $\kappa[\beta^{-1}(0)]$ of $\kappa[P]$, for every $j = 0, \dots, n$. Since P is integral, (6.4.84) is equivalent to the identity : $\sum_{j=0}^n c_j a^{n-j} b^j = 0$ in $\kappa[P]$. For every $x \in P$, denote by $\pi_x : \kappa[P] \to \kappa$ the κ-linear projection such that $\pi_x(x) = 1$, and $\pi_x(y) = 0$ for every $y \in P \setminus \{x\}$.

Suppose, by way of contradiction, that $c_i \notin \kappa[\beta^{-1}(0)]$ for some $i \leq n$, hence $\pi_x(c_i) \neq 0$ for some $x \in P_0$; since $P_0 = P^\times$, we may replace c_j by $x^{-1}c_j$, for every $j \leq n$, and assume that $\pi_1(c_i) \neq 0$, hence $\pi_{a^{n-i}b^i}(c_i a^{n-i} b^i) \neq 0$ (again, using the assumption that P is integral). Thus, there exists $j \neq i$ with $j \leq n$, such that $\pi_{a^{n-i}b^i}(c_j a^{n-j} b^j) \neq 0$, and we may then find an element $c \in P$ such that $\pi_{a^{n-i}b^i}(c a^{n-j} b^j) = 1$, *i.e.* $c a^{n-j} b^j = a^{n-i} b^i$; up to swapping the roles of a and b, we may assume that $i > j$, in which case we may write $t^{i-j} = c$; since P is saturated, it follows that $t \in P$, hence J is generated by a, which is excluded. $\Diamond$

Next, we assume that $\underline{M}$ is a fs log structure (and X is still $\mathrm{Spec}\,\kappa$), and we consider the morphism φ_{sat}. As already remarked, we may assume that α is sharp at ξ, and P fine and saturated; the sharpness condition amounts to saying that $\beta(x) = 0$ for every $x \in P \setminus \{1\}$, therefore I is the augmentation ideal of the graded κ-algebra $\kappa[P]$. By inspecting the proof of proposition 6.2.35, we see that $(Y, \underline{N})^{\mathrm{sat}}$ is covered by two affine open subsets

$$U_a^{\mathrm{fs}} := \mathrm{Spec}\,\kappa[Q_a^{\mathrm{sat}}] \qquad U_b^{\mathrm{fs}} := \mathrm{Spec}\,\kappa[Q_b^{\mathrm{sat}}]$$

and $U_a^{\mathrm{fs}} \cap U_b^{\mathrm{fs}} = \mathrm{Spec}\,\kappa[Q_a^{\mathrm{sat}} \otimes_P Q_b^{\mathrm{sat}}]$. Since J is not principal, we have $t \notin P$, and since P is saturated, we deduce that t is not a torsion element of P^{gp}; as the latter is a finitely generated abelian group, it follows that we may find a *unimodular* element $u \in P^{\mathrm{gp}}$ such that t lies in the submonoid $\mathbb{N}u \subset P^{\mathrm{gp}}$ generated by u; this condition means that $t = u^k$ for some $k \in \mathbb{N}$, and $\mathbb{N}u$ is not properly contained in another rank one free submonoid of P^{gp}. Write $u = a'^{-1}b'$ for some $a', b' \in P$, let $J' \subset P$ be the ideal generated by a' and b', and $R_{a'}$ (resp. $R_{b'}$) the submonoid of P^{gp} generated by P and u (resp. by P and u^{-1}); clearly $R_{a'}^{\mathrm{sat}} = Q_a^{\mathrm{sat}}$, and $R_{b'}^{\mathrm{sat}} = Q_b^{\mathrm{sat}}$. Denote by $\underline{N}'$ the log structure of $(Y, \underline{N})^{\mathrm{sat}}$, and $\mathscr{J}' := J'\underline{N}'$; it is easily seen that

$$\mathscr{J}'_{|U_a^{\mathrm{fs}}} = a'\underline{N}'_{|U_a^{\mathrm{fs}}} \qquad \mathscr{J}'_{|U_b^{\mathrm{fs}}} = b'\underline{N}'_{|U_b^{\mathrm{fs}}}$$

hence $\mathscr{J}'$ is invertible, and example 6.4.59(ii) yields a morphism of $(S, P_S^{\log})$-schemes

$$f_{(a',b')} : (Y, \underline{N})^{\mathrm{sat}} \to \mathbb{P}^1_{(S,P_S^{\log})}.$$

In light of example 6.4.59(i), the assertion concerning $\varphi_{\mathrm{sat}}^{-1}(\xi)$ will then follow from the :

Claim 6.4.85. $(|\xi| \times_S f_{(a',b')})_{\mathrm{red}} : \varphi_{\mathrm{sat}}^{-1}(\xi)_{\mathrm{red}} \to \mathbb{P}^1_\kappa$ is an isomorphism of κ-schemes.

Proof of the claim. As in the proof of claim 6.4.83, the morphism $f_{(a',b')}$ restricts to morphisms $U_a^{\mathrm{fs}} \to U'_0$ and $U_b^{\mathrm{fs}} \to U'_\infty$ induced by maps of κ-algebras :

$$\omega_0 : \kappa[P \oplus \mathbb{N}] \to \kappa[Q_a^{\mathrm{sat}}] \qquad \omega_\infty : \kappa[P \oplus \mathbb{N}] \to \kappa[Q_b^{\mathrm{sat}}]$$

such that $\omega_0(x,n) = x \cdot u^n$ and $\omega_\infty(x,n) = x \cdot u^{-n}$ for every $(x,n) \in P \oplus \mathbb{N}$, and again, it suffices to show that

$$(\omega_0 \otimes_{\kappa[P]} \kappa[P]/I)_{\mathrm{red}} : \kappa[T] \to (\kappa[R_a^{\mathrm{sat}}]/I\kappa[R_a^{\mathrm{sat}}])_{\mathrm{red}}$$

is an isomorphism (where for any ring A, we denote A_{red} be the maximal reduced quotient of A, *i.e.* $A_{\mathrm{red}} := A/\mathrm{nil}(A)$, where $\mathrm{nil}(A)$ is the nilpotent radical of A). We break the latter verification in two steps : first, let us check that the map

$$\overline{\omega}'_0 : \kappa[T] \to \kappa[R_a]/I\kappa[R_a] \qquad p(T) \mapsto p(u) \pmod{I\kappa[R_a]}$$

is an isomorphism. Indeed, $\overline{\omega}'_0$ is induced by the map of monoids $\varphi : \mathbb{N} \to R_a$ such that $n \mapsto u^n$ for every $n \in \mathbb{N}$; if $t = u^k$, the map φ fits into the cocartesian diagram :

$$\begin{tikzcd} \mathbb{N} \arrow[r, "\psi"] \arrow[d, "\boldsymbol{k}_{\mathbb{N}}"'] & Q_a \arrow[d, "j"] \\ \mathbb{N} \arrow[r, "\varphi"] & R_a \end{tikzcd}$$

where $\boldsymbol{k}_{\mathbb{N}}$ is the k-Frobenius map, ψ is given by the rule : $n \mapsto t^n$ for every $n \in \mathbb{N}$, and j is the natural inclusion. Hence :

$$\overline{\omega}'_0 = (\kappa[\psi] \otimes_{\kappa[P]} \kappa[P]/I) \otimes_{\kappa[T^k]} \kappa[T].$$

However, the proof of claim 6.4.83 shows that $\kappa[\psi] \otimes_{\kappa[P]} \kappa[P]/I$ is an isomorphism, whence the contention. Lastly, let show that the natural map

$$\overline{\omega}''_0 : \kappa[R_a]/I\kappa[R_a] \to (\kappa[R_a^{\mathrm{sat}}]/I\kappa[R_a^{\mathrm{sat}}])_{\mathrm{red}}$$

is an isomorphism. Indeed, it is clear that the natural map $\omega''_0 : \kappa[R_a] \to \kappa[R_a^{\mathrm{sat}}]$ is integral and injective, hence $\mathrm{Spec}\,\omega''_0$ is surjective; therefore $\mathrm{Spec}\,\overline{\omega}''_0$ is still surjective and integral. However, the foregoing shows that $\kappa[R_a]/I\kappa[R_a]$ is reduced, so we deduce that $\overline{\omega}''_0$ is injective. To show that $\overline{\omega}''_0$ is surjective, it suffices to show that the classes of the generating system $R_a^{\mathrm{sat}} \subset \kappa[R_a^{\mathrm{sat}}]$ lie in the image of $\overline{\omega}''_0$. Hence, let $x \in R_a^{\mathrm{gp}}$, with $x^m \in R_a$ for some $m > 0$, so that $x^m = y \cdot u^n$ for some $n \in \mathbb{N}$ and $y \in P$. If $y \neq 1$, we have $y \in I$, hence the image of x^m vanishes in $\kappa[R_a^{\mathrm{sat}}]/I\kappa[R_a^{\mathrm{sat}}]$, and the image of x vanishes in the reduced quotient; finally, if $y = 1$, the identity $x^m = u^n$ implies that m divides n, since u is unimodular; hence $x = u^{n/m}$ and the image of x agrees with $\overline{\omega}''_0(u^{n/m})$. ◊

Finally, let us return to a general quasi-fine log scheme $(X, \underline{M})$; the theorem will follow from the more precise :

Claim 6.4.86. In the situation of the theorem, suppose moreover that

(a) $\underline{M}$ admits a saturated chart $\alpha : P_X \to \underline{M}$
(b) $\mathscr{I} = J\underline{M}$, where $J \subset P$ is an ideal generated by two elements $a, b \in P$
(c) $\mathscr{I}_\xi$ is not invertible.

Then we have :

(i) There exists a morphism of $(X, \underline{M})$-schemes : $\mathrm{Bl}_{\mathscr{I}}(X, \underline{M}) \to \mathbb{P}^1_{(X,\underline{M})}$ inducing an isomorphism of $\kappa(\xi)$-schemes $\varphi^{-1}(\xi) \xrightarrow{\sim} \mathbb{P}^1_{\kappa(\xi)}$.
(ii) If furthermore, P is fine (and saturated) and α is sharp at ξ, then there exists a morphism of $(X, \underline{M})$-schemes : $\mathrm{sat}.\mathrm{Bl}_{\mathscr{I}}(X, \underline{M}) \to \mathbb{P}^1_{(X,\underline{M})}$ inducing an isomorphism of $\kappa(\xi)$-schemes $\varphi^{-1}_{\mathrm{sat}}(\xi)_{\mathrm{red}} \xrightarrow{\sim} \mathbb{P}^1_{\kappa(\xi)}$.

Proof of the claim. (i): Denote by $\underline{N}$ the log structure of $\mathrm{Bl}_{\mathscr{I}}(X, \underline{M})$; the elements a, b define global sections of the invertible $\underline{N}$-module $\mathscr{I}\underline{N}$, and we claim that the corresponding morphism of $(X, \underline{M})$-schemes $f_{(a,b)} : \mathrm{Bl}_{\mathscr{I}}(X, \underline{M}) \to \mathbb{P}^1_{(X,\underline{M})}$ will do. Indeed, set $(|\xi|, \underline{M}_\xi) := |\xi| \times_X (X, \underline{M})$, and recall that there exists natural isomorphisms

$$|\xi| \times_X \mathbb{P}^1_{(X,\underline{M})} \xrightarrow{\sim} \mathbb{P}^1_{(|\xi|,\underline{M}_\xi)} \qquad |\xi| \times_X \mathrm{Bl}_{\mathscr{I}}(X, \underline{M}) \xrightarrow{\sim} \mathrm{Bl}_{\mathscr{I}\underline{M}_\xi}(|\xi|, \underline{M}_\xi)$$

(example 6.4.59(i) and remark 6.4.62(ii)). Denote by $\underline{N}_\xi$ the log structure of $\mathrm{Bl}_{\mathscr{I}\underline{M}_\xi}(|\xi|, \underline{M}_\xi)$; by example 6.4.59(iii), the base change

$$|\xi| \times_X f : \mathrm{Bl}_{\mathscr{I}\underline{M}_\xi}(|\xi|, \underline{M}_\xi) \to \mathbb{P}^1_{(|\xi|,\underline{M}_\xi)}$$

is the unique morphism $f_{(\overline{a},\overline{b})}$ of $(|\xi|, \underline{M}_\xi)$-schemes corresponding to the pair $(\overline{a}, \overline{b})$ of global sections of $\mathscr{I}\underline{N}_\xi$ obtained by pulling back the pair (a, b). Therefore, in order to check that $|\xi| \times_X f$ is an isomorphism, we may replace from start $(X, \underline{M})$ by $(|\xi|, \underline{M}_\xi)$ (whose log structure is still quasi-fine, by lemma 6.1.16(i), and assume that $X = \mathrm{Spec}\,\kappa$, where κ is a field (resp. a separably closed field, in case $\tau = \text{ét}$), in which case the assertion is just claim 6.4.83.

(ii): Denote by $\underline{N}'$ the log structure of $\mathrm{sat}.\mathrm{Bl}_{\mathscr{I}}(X, \underline{M})$, define a', b' and J' as in the foregoing, and set again $\mathscr{J}' := J'\underline{N}'$. Again, it is easily seen that $\mathscr{J}'$ is an invertible $\underline{N}'$-module, and the pair (a', b') yields a morphism $f_{(a',b')} : \mathrm{sat}.\mathrm{Bl}_{\mathscr{I}}(X, \underline{M}) \to \mathbb{P}^1_{(X,\underline{M})}$ which fulfills the sought condition. Indeed, denote by $\underline{N}'_\xi$ the log structure of $\mathrm{sat}.\mathrm{Bl}_{\mathscr{I}\underline{M}_\xi}(|\xi|, \underline{M}_\xi)$; in light of (6.4.80) and example 6.4.59(iii), the base change

$$|\xi| \times_X f_{(a',b')} : \mathrm{sat}.\mathrm{Bl}_{\mathscr{I}\underline{M}_\xi}(|\xi|, \underline{M}_\xi) \to \mathbb{P}^1_{(|\xi|,\underline{M}_\xi)}$$

is the unique morphism $f_{(\overline{a}',\overline{b}')}$ of $(|\xi|, \underline{M}_\xi)$-schemes corresponding to the pair $(\overline{a}', \overline{b}')$ of global sections of $\mathscr{J}'\underline{N}'_\xi$ obtained by pulling back the pair (a', b'). Thus, the assertion is just claim 6.4.85. $\square$

6.5. **Regular log schemes.** In this section we introduce the logarithmic version of the classical regularity condition for locally noetherian schemes. This theory is essentially due to K.Kato ([53]), and we mainly follow his exposition, except in a few places where his original arguments are slightly flawed, in which cases we supply the necessary corrections.

6.5.1. Let A be a ring, P a monoid. Recall that $\mathfrak{m}_P$ is the maximal (prime) ideal of P (see (3.1.11)). The $\mathfrak{m}_P$*-adic filtration* of P is the descending sequence of ideals :

$$\cdots \subset \mathfrak{m}_P^3 \subset \mathfrak{m}_P^2 \subset \mathfrak{m}_P$$

where $\mathfrak{m}_P^n$ is the n-th power of $\mathfrak{m}$ in the monoid $\mathscr{P}(P)$ (see (3.1.1)). It induces a $\mathfrak{m}_P$*-adic filtration* $\mathrm{Fil}_\bullet M$ on any P-module M and any $A[P]$-algebra B, defined by letting $\mathrm{Fil}_n M := \mathfrak{m}_P^n M$ and $\mathrm{Fil}_n B := A[\mathfrak{m}_P^n] \cdot B$, for every $n \in \mathbb{N}$.

Lemma 6.5.2. *Suppose that P is fine. We have :*

(i) *The $\mathfrak{m}_P$-adic filtration is separated on P.*
(ii) *If P is sharp, $P \setminus \mathfrak{m}_P^n$ is a finite set, for every $n \in \mathbb{N}$.*

Proof. (i): Indeed, choose A to be a noetherian ring, set $J := \bigcap_{n\geq 0} A[\mathfrak{m}_P^n]$ and notice that J is generated by $\mathfrak{m}_P^\infty := \bigcap_{n\in\mathbb{N}} \mathfrak{m}_P^n$. On the other hand, J is annihilated by an element of $1+A[\mathfrak{m}_P]$ ([61, Th.8.9]). Thus, suppose $x \in \mathfrak{m}_P^\infty$, and pick $y \in A[\mathfrak{m}_P]$ such that $(1-y)x = 0$; we may write $y = a_1t_1 + \cdots + a_rt_r$ for certain $a_1,\dots,a_r \in A$ and $t_1,\dots,t_r \in \mathfrak{m}_P$. Therefore $x = a_1xt_1 + \cdots + a_rxt_r$ in $A[P]$, which is absurd, since P is integral.

Assertion (ii) is immediate from the definition. □

6.5.3. Keep the assumptions of lemma 6.5.2. It turns out that P can actually be made into a graded monoid, albeit in a non-canonical manner. We proceed as follows. Let $\varepsilon : P \to P^{\mathrm{sat}}$ the inclusion map, $T \subset P^{\mathrm{sat}}$ the torsion subgroup, set $Q := P^{\mathrm{sat}}/T$, and let $\pi : P^{\mathrm{sat}} \to Q$ be the natural surjection. We may regard $\log Q$ as a submonoid of the polyhedral cone $Q_{\mathbb{R}}$, lying in the vector space $Q_{\mathbb{R}}^{\mathrm{gp}}$, as in (3.4.6). Since $Q_{\mathbb{R}}$ is a rational polyhedral cone, the same holds for $Q_{\mathbb{R}}^\vee$, hence we may find a $\mathbb{Q}$-linear form $\gamma : \log Q^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{Q}$, which is non-negative on $\log Q$, and such that $Q_{\mathbb{R}} \cap \operatorname{Ker} u \otimes_{\mathbb{Q}} \mathbb{R}$ is the minimal face of $Q_{\mathbb{R}}$, *i.e.* the $\mathbb{R}$-vector space spanned by the image of $Q^\times$. If we multiply γ by some large positive integer, we may achieve that $\gamma(\log P) \subset \mathbb{N}$. We set :

$$\mathrm{gr}_n^\gamma P := (\gamma\circ\pi\circ\varepsilon)^{-1}(n) \qquad \text{for every } n \in \mathbb{N}.$$

It is clear that $\mathrm{gr}_n^\gamma P \cdot \mathrm{gr}_m^\gamma P \subset \mathrm{gr}_{n+m}^\gamma P$, hence

$$P = \coprod_{n\in\mathbb{N}} \mathrm{gr}_n^\gamma P$$

is a $\mathbb{N}$-graded monoid, and consequently :

$$A[P] = \bigoplus_{n\in\mathbb{N}} A[\mathrm{gr}_n^\gamma P]$$

is a graded A-algebra. Moreover, it is easily seen that $\mathrm{gr}_0^\gamma P = P^\times$. More generally, for $x \in P$, let $\mu(x)$ be the maximal $n \in \mathbb{N}$ such that $x \in \mathfrak{m}_P^n$; then there exists a constant $C \geq 1$ such that :

$$\gamma(x) \geq \mu(x) \geq C^{-1}\gamma(x) \qquad \text{for every } x \in P$$

so that the $\mathfrak{m}_P$-adic filtration and the filtration defined by $\mathrm{gr}_\bullet^\gamma P$, induce the same topology on P and on $A[P]$. As a corollary of these considerations, we may state the following “regularity criterion” for fine monoids :

Proposition 6.5.4. *Let P be an integral monoid such that $P^\sharp$ is fine. Then we have*

$$\mathrm{rk}^\circ_{P^\times} \mathfrak{m}_P/\mathfrak{m}_P^2 \geq \dim P$$

(notation of example 2.3.18*) and the equality holds if and only if $P^\sharp$ is a free monoid.*

Proof. (Notice that $\mathfrak{m}_P/\mathfrak{m}_P^2$ is a free pointed $P^\times$-module, since $P^\times$ obviously acts freely on $\mathfrak{m}_P\setminus\mathfrak{m}_P^2$.) Since $\mathfrak{m}_{P^\sharp}/\mathfrak{m}_{P^\sharp}^2 = (\mathfrak{m}_P/\mathfrak{m}_P^2)\otimes_P P^\sharp$, we may replace P by $P^\sharp$, and assume from start that P is sharp and fine. Then, the rank of $\mathfrak{m}_P/\mathfrak{m}_P^2$ equals the cardinality of the set $\Sigma := \mathfrak{m}_P \setminus \mathfrak{m}_P^2$, which is finite, by lemma 6.5.2. We have a surjective morphism of monoids $\varphi : \mathbb{N}^{(\Sigma)} \to P$, that sends the basis of $\mathbb{N}^{(\Sigma)}$ bijectively onto $\Sigma \subset P$ (corollary 3.1.10). The sought inequality follows immediately, and it is also clear that we have equality, in case P is free. Conversely, if equality holds, φ^{gp} must be a surjective group homomorphism between free abelian group of the same finite rank (corollary 3.4.10(i)), so it is an isomorphism, and then the same holds for φ. □

Proposition 6.5.5. *Let P be a fine and sharp monoid, A a noetherian local ring. Set $S_P := 1 + A[\mathfrak{m}_P]$; then we have :*

$$\dim S_P^{-1}A[P] = \dim A + \dim P.$$

Proof. To begin with, the assumption that P is sharp implies that $S_P^{-1}A[P]$ is local. Next, notice that $A[P]$ is a free A-module, hence the natural map $A \to S_P^{-1}A[P]$ is a flat and local ring homomorphism. Let k be the residue field of A; in view of [61, Th.15.1(ii)], we deduce :

$$\dim S_P^{-1}A[P] = \dim A + \dim S_P^{-1}k[P].$$

Hence we are reduced to showing the stated identity for $A = k$. However, clearly $S_P^{-1}k[P] = k[P]_\mathfrak{m}$, where $\mathfrak{m}$ is the maximal ideal generated by the image of $\mathfrak{m}_P$, hence it suffices to apply claim 5.9.36(ii) and corollary 3.4.10(i), to conclude. □

6.5.6. Let A be a ring, and P a fine and sharp monoid. We define :

$$A[[P]] := \lim_{n\in\mathbb{N}} A\langle P/\mathfrak{m}_P^n\rangle.$$

Alternatively, this is the completion of $A[P]$ for its $A[\mathfrak{m}_P]$-adic topology. In view of the finiteness properties of the $\mathfrak{m}_P$-adic filtration (lemma 6.5.2(ii)), one may present $A[[P]]$ as the ring of formal infinite sums $\sum_{\sigma\in P} a_\sigma \cdot \sigma$, with arbitrary coefficients $a_\sigma \in A$, where the multiplication and addition are defined in the obvious way. Moreover, we may use a morphism of monoids $\gamma : \log P \to \mathbb{N}$ as in (6.5.3), to see that :

$$A[[P]] = \prod_{n\in\mathbb{N}} A[\mathrm{gr}_n^\gamma P] \qquad (6.5.7)$$

where $A[\mathrm{gr}_n^\gamma P] \cdot A[\mathrm{gr}_m^\gamma P] \subset A[\mathrm{gr}_{n+m}^\gamma P]$ for every $m, n \in \mathbb{N}$. So any element $x \in A[[P]]$ can be decomposed as an infinite sum

$$x = \sum_{n\in\mathbb{N}} \mathrm{gr}_n^\gamma x.$$

The term $\mathrm{gr}_0^\gamma x \in \mathrm{gr}_0^\gamma A = A$ does not depend on the chosen γ : it is the *constant term* of x, *i.e.* the image of x under the natural projection $A[[P]] \to A$.

Corollary 6.5.8. *Let P be a fine and sharp monoid, A a noetherian local ring. Then :*

(i) *For any local morphism $P \to A$ (see* (3.1.11)*), we have the inequality:*

$$\dim A \le \dim A/\mathfrak{m}_P A + \dim P.$$

(ii) $\dim A[P] = \dim A[[P]] = \dim A + \dim P$.

Proof. (i): Set $A_0 := A/\mathfrak{m}_P A$, and $B := S_P^{-1}A_0[P]$, where $S_P \subset A_0[P]$ is the multiplicative subset $1 + A_0[\mathfrak{m}_P]$; if we denote by $\mathrm{gr}_\bullet A$ (resp. $\mathrm{gr}_\bullet B$) the graded A_0-algebra associated to the $\mathfrak{m}_P$-adic filtration on A (resp. on B), we have a natural surjective homomorphism of graded A_0-algebras :

$$\mathrm{gr}_\bullet B \to \mathrm{gr}_\bullet A.$$

Hence $\dim A = \dim \mathrm{gr}_\bullet A \le \dim \mathrm{gr}_\bullet B = \dim B$, by [61, Th.15.7]. Then the assertion follows from proposition 6.5.5.

(ii): Again we consider the ring $B := S_P^{-1}A[P]$, where $S_P := 1 + A[\mathfrak{m}_P]$. One sees easily that the graded ring associated to the $\mathfrak{m}_P$-adic filtration on B is just the ring algebra $A[P]$, whence the first stated identity, taking into account [61, Th.5.7]. The same argument applies as well to the $\mathfrak{m}_P$-adic filtration on $A[[P]]$, and yields the second identity. □

As a first application, we have the following combinatorial version of Kunz's theorem 4.7.30 that characterizes regular rings via their Frobenius endomorphism.

Theorem 6.5.9. *Let P be a monoid such that $P^\sharp$ is fine, $k > 1$ an integer, and suppose that the Frobenius endomorphism $\boldsymbol{k}_P : P \to P$ is flat (see example* 3.5.10(i)*). Then $P^\sharp$ is a free monoid.*

Proof. First, we remark that $\boldsymbol{k}_P^\sharp : P^\sharp \to P^\sharp$ is still flat (corollary 3.1.49(i)). Moreover, $\boldsymbol{k}_P^\sharp$ is injective. Indeed, suppose that $x^k = y^k \cdot u$ for some $x, y \in P$ and $u \in P^\times$; from theorem 3.1.42 we deduce that there exist $b_1, b_2, t \in P$ such that

$$b_1 x = b_2 y \qquad 1 = b_1^k t \qquad u = b_2^k t.$$

Especially, $b_1, b_2 \in P^\times$, so the images of x and y agree in $P^\sharp$. Hence, we may replace P by $P^\sharp$, and assume that $\boldsymbol{k}_P$ is flat and injective, in which case $\mathbb{Z}[\boldsymbol{k}_P] : \mathbb{Z}[P] \to \mathbb{Z}[P]$ is flat (theorem 3.2.3), integral and injective, hence it is faithfully flat. Now, let R be the colimit of the system of rings $(R_n \mid n \in \mathbb{N})$, where $R_n := \mathbb{Z}[P]$, and the transition map $R_n \to R_{n+1}$ is $\mathbb{Z}[\boldsymbol{k}_P]$ for every $n \in \mathbb{N}$. The induced map $j : R_0 \to R$ is still faithfully flat; moreover, let p be any prime divisor of k, and notice that $j \circ \boldsymbol{p}_P = j$ (where $\boldsymbol{p}_P$ is the p-Frobenius map). It follows that $\boldsymbol{p}_P$ is flat and injective as well, so $\mathbb{F}_p[\boldsymbol{p}_P] : \mathbb{F}_p[P] \to \mathbb{F}_p[P]$ is a flat ring homomorphism (again, by theorem 3.2.3), and then the same holds for the induced map $\mathbb{F}_p[[\boldsymbol{p}_P]] : \mathbb{F}_p[[P]] \to \mathbb{F}_p[[P]]$. By Kunz's theorem, we deduce that $\mathbb{F}_p[[P]]$ is a regular local ring, with maximal ideal $\mathfrak{m} := \mathbb{F}_p[[\mathfrak{m}_P]]$; notice that the images of the elements of $\mathfrak{m}_P \setminus \mathfrak{m}_P^2$ yield a basis for the $\mathbb{F}_p$-vector space $\mathfrak{m}/\mathfrak{m}^2$. Say that $\mathfrak{m}_P \setminus \mathfrak{m}_P^2 = \{x_1, \dots, x_s\}$; it follows that the continuous ring homomorphism

$$\mathbb{F}_p[[T_1, \dots, T_s]] \to \mathbb{F}_p[[P]] \qquad T_i \mapsto x_i \qquad \text{for } i = 1, \dots, s$$

is an isomorphism. From the discussion of (6.5.6), we immediately deduce that $P \simeq \mathbb{N}^{\oplus s}$, as required. □

6.5.10. Now we wish to state and prove the combinatorial versions of the Artin-Rees lemma, and of the so-called local flatness criterion (see *e.g.* [61, Th.22.3]). Namely, let P be a *pointed* monoid, such that $P^\sharp$ is finitely generated; let also $(A, \mathfrak{m}_A)$ be a local noetherian ring, N a finitely generated A-module, and :

$$\alpha : P \to (A, \cdot)$$

a morphism of pointed monoids. The following is our version of the Artin-Rees lemma :

Lemma 6.5.11. *In the situation of* (6.5.10)*, let $J \subset P$ be an ideal, M a finitely generated P-module, $M_0 \subset M$ a submodule. Then there exists $c \in \mathbb{N}$ such that :*

(6.5.12) $$J^n M \cap M_0 = J^{n-c}(J^c M \cap M_0) \qquad \text{for every } n > c.$$

Proof. Set $\overline{M} := M/P^\times$, $\overline{M}_0 := M_0/P^\times$ and $\overline{J} := J/P^\times$, the set-theoretic quotients for the respective natural $P^\times$-actions. Notice that $\overline{J}$ is an ideal of $P^\sharp$ and $\overline{M}_0 \subset \overline{M}$ is an inclusion of $\overline{P}$-modules. Moreover, any set of generators of the ideal $\overline{J}$ (resp. of the $P^\sharp$-module $\overline{M}_0$) lifts to a set of generators for J (resp. for the P-module M_0). Furthermore, it is easily seen that (6.5.12) is equivalent to the identity $\overline{J}^n \overline{M} \cap \overline{M}_0 = \overline{J}^{n-c}(\overline{J}^c \overline{M} \cap \overline{M}_0)$. Hence, we are reduced to the case where $P = P^\sharp$ is a finitely generated monoid. Then $\mathbb{Z}[P]$ is noetherian, $\mathbb{Z}[M]$ is a $\mathbb{Z}[P]$-module of finite type, and we notice that :

$$\mathbb{Z}[J^n M \cap M_0] = J^n \mathbb{Z}[M] \cap \mathbb{Z}[M_0] \qquad \mathbb{Z}[J^{n-c}(J^c M \cap M_0)] = J^{n-c}(J^c \mathbb{Z}[M] \cap \mathbb{Z}[M_0]).$$

Thus, the assertion follows from the standard Artin-Rees lemma [61, Th.8.5]. □

Proposition 6.5.13. *In the situation of* (6.5.10)*, suppose moreover that $P^\sharp$ is fine (see remark* 2.3.14(vi)*), and let $\mathfrak{m}_\alpha := \alpha^{-1}\mathfrak{m}_A$. Then the following conditions are equivalent :*

(a) *N is α-flat (see definition* 3.1.34*).*
(b) $\mathrm{Tor}_i^{\mathbb{Z}\langle P\rangle}(N, \mathbb{Z}\langle M\rangle) = 0$ *for every $i > 0$ and every integral pointed P-module M.*
(c) $\mathrm{Tor}_1^{\mathbb{Z}\langle P\rangle}(N, \mathbb{Z}\langle P/\mathfrak{m}_\alpha\rangle) = 0$.
(d) *The natural map :*

$$(\mathfrak{m}_\alpha^n/\mathfrak{m}_\alpha^{n+1}) \otimes_P N \to \mathfrak{m}_\alpha^n N/\mathfrak{m}_\alpha^{n+1} N$$

is an isomorphism of A-modules, for every $n \in \mathbb{N}$ (notation of (3.1.33)*).*

Proof. The assertion (a)⇔(b) is just a restatement of proposition 3.1.40(i), and holds in greater generality, without any assumption on either A or the pointed integral monoid P.

As for the remaining assertions, let $S := \alpha^{-1}(A^\times)$; since the localization $P \to S^{-1}P$ is flat, the natural maps :

$$\mathrm{Tor}_i^{\mathbb{Z}\langle P\rangle}(N, \mathbb{Z}\langle M\rangle) \to \mathrm{Tor}_i^{\mathbb{Z}\langle S^{-1}P\rangle}(N, \mathbb{Z}\langle S^{-1}M\rangle)$$

are isomorphisms, for every $i \in \mathbb{N}$ and every P-module M. Also, notice that the two P-modules appearing in (d) are actually $S^{-1}P$-modules (and the natural map is $\mathbb{Z}\langle S^{-1}P\rangle$-linear). Hence, we can replace everywhere P by $S^{-1}P$, which allows to assume that α is local, *i.e.* $\mathfrak{m}_\alpha = \mathfrak{m}_P$.

Next, obviously (b)⇒(c).

(c)⇒(d): For every $n \in \mathbb{N}$, we have a short exact sequence of pointed P-modules :

$$0 \to \mathfrak{m}_P^n/\mathfrak{m}_P^{n+1} \to P/\mathfrak{m}_P^{n+1} \to P/\mathfrak{m}_P^n \to 0.$$

It is easily seen that $\mathfrak{m}_P^n/\mathfrak{m}_P^{n+1}$ is a free $P/\mathfrak{m}_P$-module (in the category of pointed modules), so the assumption implies that $\mathrm{Tor}_1^{\mathbb{Z}\langle P\rangle}(N, \mathfrak{m}_P^n/\mathfrak{m}_P^{n+1}) = 0$ for every $n \in \mathbb{N}$. By looking at the induced long Tor-sequences, we deduce that the natural map

$$\mathrm{Tor}_1^{\mathbb{Z}\langle P\rangle}(N, P/\mathfrak{m}_P^{n+1}) \to \mathrm{Tor}_1^{\mathbb{Z}\langle P\rangle}(N, P/\mathfrak{m}_P^n)$$

is injective for every $n \in \mathbb{N}$. Then, a simple induction shows that, under assumption (c), all these modules vanish. The latter means that the natural map :

$$\mathfrak{m}_P^n \otimes_P N \to \mathfrak{m}_P^n N$$

is an isomorphism, for every $n \in \mathbb{N}$. We consider the commutative ladder with exact rows :

$$\begin{tikzcd}
\mathfrak{m}_P^{n+1} \otimes_P N \arrow[r] \arrow[d] & \mathfrak{m}_P^n \otimes_P N \arrow[r] \arrow[d] & (\mathfrak{m}_P^n/\mathfrak{m}_P^{n+1}) \otimes_P N \arrow[r] \arrow[d] & 0 \\
\mathfrak{m}_P^{n+1} N \arrow[r] & \mathfrak{m}_P^n N \arrow[r] & \mathfrak{m}_P^n N/\mathfrak{m}_P^{n+1} N \arrow[r] & 0.
\end{tikzcd}$$

By the foregoing, the two left-most vertical arrows are isomorphisms, hence the same holds for the right-most, whence (c).

(d)⇒(c): We have to show that the natural map $u : \mathfrak{m}_P \otimes_P N \to \mathfrak{m}_P N$ is an isomorphism. To this aim, we consider the $\mathfrak{m}_P$-adic filtrations on these two modules; for the associated graded modules one gets :

$$\mathrm{gr}_n(\mathfrak{m}_P N) = \mathfrak{m}_P^n N/\mathfrak{m}_P^{n+1} N \qquad \mathrm{gr}_n(\mathfrak{m}_P \otimes_P N) = (\mathfrak{m}_P^n/\mathfrak{m}_P^{n+1}) \otimes_P N$$

for every $n \in \mathbb{N}$; whence maps of A-modules :

$$\mathrm{gr}_n(\mathfrak{m}_P \otimes_P N) \xrightarrow{\mathrm{gr}_n u} \mathrm{gr}_n(\mathfrak{m}_P N).$$

which are isomorphism by assumption. To conclude, it suffices to show :

Claim 6.5.14. For every ideal $I \subset P$, the $\mathfrak{m}_P A$-adic filtration is separated on the A-module $I \otimes_P N$.

Proof of the claim. Indeed, notice that the ideal $I/P^\times \subset P^\sharp$ is finitely generated (proposition 3.1.9(ii)), hence the same holds for I, so $I \otimes_P N$ is a finitely generated A-module. Then the contention follows from [61, Th.8.10]. ◊

(c)⇒(b): We argue by induction on i. For $i = 1$, suppose first that $M = P/I$ for some ideal $I \subset P$ (notice that any such quotient is an integral pointed P-module); in this case, the assertion to prove is that the natural map $v : I \otimes_P N \to IN$ is an isomorphism. However, consider the $\mathfrak{m}_P$-adic filtration on P/I; for the associated graded module we have :

$$\mathrm{gr}_n(P/I) = (\mathfrak{m}_P^n \cup I)/(\mathfrak{m}^{n+1} \cup I) \qquad \text{for every } n \in \mathbb{N}$$

and it is easily seen that this is a free pointed $P^\times$-module, for every $n \in \mathbb{N}$. Hence, by inspecting the long exact Tor-sequences associated to the exact sequences

$$0 \to \mathrm{gr}_n(P/I) \to P/(\mathfrak{m}_P^{n+1} \cup I) \to P/(\mathfrak{m}_P^n \cup I) \to 0$$

our assumption (c), together with a simple induction yields :

$$\mathrm{Tor}_1^{\mathbb{Z}\langle P\rangle}(N, \mathbb{Z}\langle P/(\mathfrak{m}_P^n \cup I)\rangle) \qquad \text{for every } n \in \mathbb{N}. \tag{6.5.15}$$

Now, fix $n \in \mathbb{N}$; in light of lemma 6.5.11, there exists $k \geq n$ such that $\mathfrak{m}_P^k \cap I \subset \mathfrak{m}_P^n I$. We deduce surjective maps of A-modules :

$$I \otimes_P N \to \frac{I}{\mathfrak{m}_P^k \cap I} \otimes_P N \to \frac{I}{\mathfrak{m}_P^n I} \otimes_P N \xrightarrow{\sim} \frac{I \otimes_P N}{\mathfrak{m}_P^n(I \otimes_P N)}.$$

On the other hand, (6.5.15) says that the natural map $(\mathfrak{m}_P^n \cup I) \otimes_P N \to (\mathfrak{m}_P^n \cup I)N$ is an isomorphism, so the same holds for the induced composed map :

$$\frac{I}{\mathfrak{m}_P^k \cap I} \otimes_P N \xrightarrow{\sim} \frac{\mathfrak{m}_P^k \cup I}{\mathfrak{m}_P^k} \otimes_P N \to \frac{(\mathfrak{m}_P^k \cup I)N}{\mathfrak{m}_P^k N}.$$

Consequently, the kernel of v is contained in $\mathfrak{m}_P^n(I \otimes_P N)$; since n is arbitrary, we are reduced to showing that the $\mathfrak{m}_P$-adic filtration is separated on $I \otimes_P N$, which is claim 6.5.14.

Next, again for $i = 1$, let M be an arbitrary integral P-module. In view of remark 3.1.27(i), we may assume that M is finitely generated; moreover, remark 3.1.27(ii), together with an easy induction further reduces to the case where M is cyclic, in which case, according to remark 3.1.27(iii), M is of the form P/I for some ideal I, so the proof is complete in this case.

Finally, suppose $i > 1$ and assume that the assertion is already known for $i - 1$. We may similarly reduce to the case where $M = P/I$ for some ideal I as in the foregoing; to conclude, we observe that :

$$\mathrm{Tor}_i^{\mathbb{Z}\langle P\rangle}(N, \mathbb{Z}\langle P/I\rangle) \simeq \mathrm{Tor}_{i-1}^{\mathbb{Z}\langle P\rangle}(N, \mathbb{Z}\langle I\rangle).$$

Since obviously I is an integral P-module, the contention follows. $\square$

As a corollary, we have the following combinatorial going-down theorem, which is proved in the same way as its commutative algebra counterpart.

Corollary 6.5.16. *In the situation of* (6.5.10), *assume that $P^\sharp$ is fine, and that A is α-flat. Let $\mathfrak{p} \subset \mathfrak{q}$ be two prime ideals of P, and $\mathfrak{q}' \subset A$ a prime ideal such that $\mathfrak{q} = \alpha^{-1}\mathfrak{q}'$. Then there exists a prime ideal $\mathfrak{p}' \subset \mathfrak{q}'$ such that $\mathfrak{p} = \alpha^{-1}\mathfrak{p}'$.*

Proof. Let $\beta : P_\mathfrak{q} \to A_{\mathfrak{q}'}$ be the morphism induced by α; it is easily seen that $A_{\mathfrak{q}'}$ is β-flat, and moreover $(P_\mathfrak{p})^\sharp = (P^\sharp)^\sharp_\mathfrak{p}$ is still fine (lemma 3.1.20(iv)). Hence we may replace α by β, which (in view of (3.1.11)) allows to assume that $\mathfrak{q}$ (resp. $\mathfrak{q}'$) is the maximal ideal of P (resp. of A). Next, let $P_0 := P/\mathfrak{p}$, $A_0 := A/\mathfrak{p}A$ and denote by $\alpha_0 : P_0 \to A_0$ the morphism induced from α; it is easily seen that A_0 is α_0-flat : for instance, the natural map $(\mathfrak{m}_P^n/\mathfrak{m}_P^{n+1}) \otimes_P A_0 \to \mathfrak{m}_P^n A_0/\mathfrak{m}_P^{n+1}A_0$ is of the type $f \otimes_A A_0$, where f is the map in proposition 6.5.13(c), thus if the latter is bijective, so is the former. Moreover $P_0^\sharp$ is a quotient of $P^\sharp$, hence it is again fine. Therefore we may replace P by P_0 and A by A_0, which allows to further assume that $\mathfrak{p} = \{0\}$, and it suffices to show that there exists a prime ideal $\mathfrak{q}' \subset A$, such that $\alpha^{-1}\mathfrak{q}' = \{0\}$. Set $\Sigma := P \setminus \{0\}$; it is easily seen that the natural morphism $P \to \Sigma^{-1}P$ is injective; moreover, its cokernel C (in the category of pointed P-modules) is integral, so that $\mathrm{Tor}_1^{\mathbb{Z}\langle P\rangle}(A, \mathbb{Z}\langle C\rangle) = 0$ by assumption. It follows that the localization map $A \to \Sigma^{-1}A$ is injective, especially $\Sigma^{-1}A \neq \{0\}$, and therefore it contains a prime ideal $\mathfrak{q}''$. The prime ideal $\mathfrak{q}' := \mathfrak{q}'' \cap A$ will do. $\square$

Lemma 6.5.17. *Let A be a noetherian local ring, N an A-module of finite type, $\alpha : P \to A$ and $\beta : Q \to A$ two morphisms of pointed monoids, with $P^\sharp$ and $Q^\sharp$ both fine. Suppose that α and β induce the same constant log structure on* $\mathrm{Spec}\, A$ *(see* (6.1.13)*). Then N is α-flat if and only if it is β-flat.*

Proof. Let ξ be a τ-point localized at the closed point of $X := \mathrm{Spec}\, A$, set $B := \mathscr{O}_{X,\xi}$ and let $\varphi : A \to B$ be the natural map. Let also M be the push-out of the diagram of monoids $P \leftarrow (\varphi \circ \alpha)^{-1}B^\times \to B^\times$ deduced from α; then $M \simeq P^{\log}_{X,\xi}$, the stalk at the point ξ of the constant log structure on X_τ associated to α. Since φ is faithfully flat, it is easily seen that N is α-flat if and only if $N \otimes_A B$ is $\varphi \circ \alpha$-flat.

Hence we may replace A by B, α by $\varphi \circ \alpha$, N by $N \otimes_A B$, and Q by M, after which we may assume that $Q = P \amalg_{\alpha^{-1}(A^\times)} A^\times$; especially, there exists a morphism of monoids $\gamma : P \to Q$ such that $\alpha = \beta \circ \gamma$, and moreover β is a local morphism.

Next, set $S := \alpha^{-1}(A^\times)$; clearly γ extends to a morphism of monoids $\gamma' : S^{-1}P \to Q$, and α and $\beta \circ \gamma'$ induce the same constant log structure on X_τ. Arguing as in the proof of proposition 6.5.13, we see that N is P-flat if and only if it is $S^{-1}P$-flat. Hence, we may replace P by $S^{-1}P$, which allows to assume that γ induces an isomorphism $P \amalg_{P^\times} A^\times \xrightarrow{\sim} Q$, therefore also an isomorphism $P^\sharp \xrightarrow{\sim} Q^\sharp$. The latter implies that $\mathfrak{m}_Q = \mathfrak{m}_P Q$; moreover, notice that the morphism of monoids $P^\times \to A^\times$ is faithfully flat, so the natural map :

$$\mathrm{Tor}_1^{\mathbb{Z}\langle P\rangle}(N, \mathbb{Z}\langle P/\mathfrak{m}_P\rangle) \to \mathrm{Tor}_1^{\mathbb{Z}\langle Q\rangle}(N, \mathbb{Z}\langle (P/\mathfrak{m}_P) \otimes_P Q\rangle) \to \mathrm{Tor}_1^{\mathbb{Z}\langle Q\rangle}(N, \mathbb{Z}\langle Q/\mathfrak{m}_Q\rangle)$$

is an isomorphism. The assertion follows. ☐

Lemma 6.5.18. *Let M be an integral monoid, A a ring, $\varphi : M \to A$ a morphism of monoids, and set $S := \mathrm{Spec}\, A$. Suppose that A is φ-flat. Then the log structure $(M, \varphi)^{\log}_S$ on S_τ is the subsheaf of monoids of $\mathscr{O}_S$ generated by $\mathscr{O}_S^\times$ and the image of M.*

Proof. To ease notation, set $\underline{M} := (M, \varphi)^{\log}_S$; let ξ be any τ-point of S. Then the stalk $\underline{M}_\xi$ is the push-out of the diagram : $\mathscr{O}^\times_{S,\xi} \leftarrow \varphi_\xi^{-1}(\mathscr{O}^\times_{S,\xi}) \to M$ where $\varphi_\xi : M \to \mathscr{O}_{S,\xi}$ is deduced from φ. Hence $\underline{M}_\xi$ is generated by $\mathscr{O}^\times_{S,\xi}$ and the image of M, and it remains only to show that the structure map $\underline{M}_\xi \to \mathscr{O}_{S,\xi}$ is injective. Therefore, let $a, b \in M$ and $u, v \in \mathscr{O}^\times_{S,\xi}$ such that :

$$(6.5.19) \qquad \varphi_\xi(a) \cdot u = \varphi_\xi(b) \cdot v.$$

We come down to showing :

Claim 6.5.20. There exist $c, d \in M$ such that :

$$\varphi_\xi(c), \varphi_\xi(d) \in \mathscr{O}^\times_{S,\xi} \qquad ac = bd \quad \varphi_\xi(c) \cdot v = \varphi_\xi(d) \cdot u.$$

Proof of the claim. Let $\mathfrak{m}_\xi \subset \mathscr{O}_{S,\xi}$ be the maximal ideal, and set $\mathfrak{p} := \varphi_\xi^{-1}(\mathfrak{m}_\xi)$, so that φ_ξ extends to a local morphism $\varphi_\mathfrak{p} : M_\mathfrak{p} \to \mathscr{O}_{S,\xi}$. Since $\mathscr{O}_{S,\xi}$ is a flat A-algebra, $\mathscr{O}_{S,\xi}$ is φ-flat, and consequently it is faithfully $\varphi_\mathfrak{p}$-flat (lemma 3.1.36). Then, assumption (6.5.19) leads to the identity:

$$aM_\mathfrak{p} \otimes_{M_\mathfrak{p}} \mathscr{O}_{S,\xi} = \varphi_\xi(a) \cdot \mathscr{O}_{S,\xi} = \varphi_\xi(b) \cdot \mathscr{O}_{S,\xi} = bM_\mathfrak{p} \otimes_{M_\mathfrak{p}} \mathscr{O}_{S,\xi}$$

whence $aM_\mathfrak{p} = bM_\mathfrak{p}$, by faithful $\varphi_\mathfrak{p}$-flatness. It follows that there exist $x, y \in M_\mathfrak{p}$ such that $ax = b$ and $by = a$, hence $axy = a$, which implies that $xy = 1$, since $M_\mathfrak{p}$ is an integral module. The latter means that there exist $c, d \in M \setminus \mathfrak{p}$ such that $ac = bd$ in M. We deduce easily that

$$\varphi_\xi(a) \cdot \varphi_\xi(c) \cdot v = \varphi_\xi(a) \cdot \varphi_\xi(d) \cdot u.$$

Thus, to complete the proof, it suffices to show that $\varphi_\xi(a)$ is regular in $\mathscr{O}_{S,\xi}$. However, the morphism of M-modules $\mu_a : M \to M : m \mapsto am$ (for all $m \in M$) is injective, hence the same holds for the map $\mu_a \otimes_M \mathscr{O}_{S,\xi} : \mathscr{O}_{S,\xi} \to \mathscr{O}_{S,\xi}$, which is just multiplication by $\varphi(a)$. ☐

6.5.21. Let $(X, \underline{M})$ be a locally noetherian log scheme, with coherent log structure (on the site X_τ, see (6.2.1)), and let ξ be any τ-point of X. We denote by $I(\xi, \underline{M}) \subset \mathscr{O}_{X,\xi}$ the ideal generated by the image of the maximal ideal of $\underline{M}_\xi$, and we set :

$$d(\xi, \underline{M}) := \dim \mathscr{O}_{X,\xi}/I(\xi, \underline{M}) + \dim \underline{M}_\xi.$$

Lemma 6.5.22. *In the situation of* (6.5.21)*, suppose furthermore that* $(X, \underline{M})$ *is a fs log scheme. Then we have the inequality :*

$$\dim \mathscr{O}_{X,\xi} \leq d(\xi, \underline{M}).$$

Proof. According to corollary 6.1.34(i), there exists a neighborhood $U \to X$ of ξ in X_τ, and a fine and saturated chart $\alpha : P_U \to \underline{M}_{|U}$, which is sharp at the point ξ. Especially, $P \simeq \underline{M}^\sharp_\xi$, therefore $\dim P = \dim \underline{M}_\xi$ (corollary 3.4.10(ii)). Notice that $\mathscr{O}_{X,\xi}$ is a noetherian local ring (this is obvious for $\tau = $ Zar, and follows from [33, Ch.IV, Prop.18.8.8(iv)] for $\tau = $ ét), hence to conclude it suffices to apply corollary 6.5.8(i) to the induced map of monoids $P \to \mathscr{O}_{X,\xi}$. □

Definition 6.5.23. Let $(X, \underline{M})$ be a locally noetherian fs log scheme, ξ a τ-point of X.

(i) We say that $(X, \underline{M})$ is *regular at the point* ξ, if the following holds :
  (a) the inequality of lemma (6.5.22) is actually an equality, and
  (b) the local ring $\mathscr{O}_{X,\xi}/I(\xi, \underline{M})$ is regular.

(ii) We denote by $(X, \underline{M})_{\mathrm{reg}} \subset X$ the set of points x such that $(X, \underline{M})$ is regular at any (hence all) τ-points of X localized at x.

(iii) We say that $(X, \underline{M})$ is *regular*, if $(X, \underline{M})_{\mathrm{reg}} = X$.

(iv) Suppose that K is a field, and X a K-scheme. We say that the K-log scheme $(X, \underline{M})$ is *geometrically regular*, if $E \times_K (X, \underline{M})$ is regular, for every field extension E of K.

Remark 6.5.24. (i) Certain constructions produce log structures $\underline{M} \to \mathscr{O}_X$ that are morphisms of pointed monoids. It is then useful to extend the notion of regularity to such log structures. We shall say that $(X, \underline{M})$ is a *pointed regular* log scheme, if there exists a log structure $\underline{N}$ on X, such that $\underline{M} = \underline{N}_\circ$ (notation of (6.1.9)), and $(X, \underline{N})$ is a regular log scheme.

(ii) Likewise, if K is a field, X a K-scheme, and $\underline{M}$ a log structure on X, we shall say that the K-log scheme $(X, \underline{M})$ is *geometrically pointed regular*, if $\underline{M} = \underline{N}_\circ$ for some log structure $\underline{N}$ on X, such that $(X, \underline{N})$ is geometrically regular.

6.5.25. Let $(X, \underline{M})$ be a locally noetherian fs log scheme, ξ a τ-point of X, and $\mathscr{O}^\wedge_{X,\xi}$ the completion of $\mathscr{O}_{X,\xi}$. The next result is the logarithmic version of the classical characterization of complete regular local rings ([61, Th.29.7 and Th.29.8]).

Theorem 6.5.26. *With the notation of* (6.5.25)*, the log scheme* $(X, \underline{M})$ *is regular at the point* ξ *if and only if there exist :*

(a) *a complete regular local ring* $(R, \mathfrak{m}_R)$*, and a local ring homomorphism* $R \to \mathscr{O}^\wedge_{X,\xi}$*;*

(b) *a fine and saturated chart* $P_{X(\xi)} \to \underline{M}(\xi)$ *which is sharp at the closed point* ξ *of* $X(\xi)$*, such that the induced continuous ring homomorphism*

$$R[[P]] \to \mathscr{O}^\wedge_{X,\xi}$$

is an isomorphism if $\mathscr{O}_{X,\xi}$ *contains a field, and otherwise it is a surjection, with kernel generated by an element* $\vartheta \in R[[P]]$ *whose constant term lies in* $\mathfrak{m}_R \backslash \mathfrak{m}^2_R$.

Proof. Suppose first that (a) and (b) hold. If R contains a field, then it follows that

$$\dim \mathscr{O}_{X,\xi} = \dim \mathscr{O}^\wedge_{X,\xi} = \dim R + \dim P$$

by corollary 6.5.8(ii); moreover, in this case $I(\xi, \underline{M}) = \mathfrak{m}_P \mathscr{O}_{X,\xi}$, hence $\mathscr{O}_{X,\xi}/I(\xi, \underline{M}) = \mathscr{O}_{X,\xi}/\mathfrak{m}_P \mathscr{O}_{X,\xi}$, whose completion is $\mathscr{O}^\wedge_{X,\xi}/\mathfrak{m}_P \mathscr{O}^\wedge_{X,\xi} \simeq R$ so that $\mathscr{O}_{X,\xi}/I(\xi, \underline{M})$ is regular ([30, Ch.0, Prop.17.3.3(i)]). Furthermore,

$$\dim R = \dim \mathscr{O}^\wedge_{X,\xi}/\mathfrak{m}_P \mathscr{O}^\wedge_{X,\xi} = \dim \mathscr{O}_{X,\xi}/I(\xi, \underline{M})$$

([61, Th.15.1]). Hence $(X, \underline{M})$ is regular at the point ξ. If R does not contain a field, we obtain $\dim \mathscr{O}_{X,\xi} = \dim R + \dim P - 1$. On the other hand, let ϑ_0 be the image of ϑ in $\mathfrak{m}_R$; then we have $\mathscr{O}^\wedge_{X,\xi}/\mathfrak{m}_P\mathscr{O}^\wedge_{X,\xi} \simeq R/\vartheta_0 R$, which is regular of dimension $\dim\, R - 1$, and again we invoke [30, Ch.0, Prop.17.3.3(i)] to see that $(X, \underline{M})$ is regular at ξ.

Conversely, suppose that $(X, \underline{M})$ is regular at ξ. Suppose first that $\mathscr{O}_{X,\xi}$ contains a field; then we may find a field $k \subset \mathscr{O}^\wedge_{X,\xi}$ mapping isomorphically to the residue field of $\mathscr{O}^\wedge_{X,\xi}$ ([61, Th.28.3]). Pick a sequence $(t_1, \dots, t_r)$ of elements of $\mathscr{O}_{X,\xi}$ whose image in the regular local ring $\mathscr{O}_{X,\xi}$ forms a regular system of parameters. Let also P a fine saturated monoid for which there exists a neighborhood $U \to X$ of ξ and a chart $P_U \to \underline{M}_{|U}$, sharp at the point ξ. There follows a map of monoids $\alpha : P \to \mathscr{O}_{X,\xi}$, and necessarily the image of $\mathfrak{m}_P$ lies in the maximal ideal of $\mathscr{O}_{X,\xi}$, and generates $I(\xi, \underline{M})$. We deduce a continuous ring homomorphism

$$k[[P \times \mathbb{N}^{\oplus r}]] \to \mathscr{O}^\wedge_{X,\xi}$$

which extends α, and which maps the generators $T_1, \dots, T_r$ of $\mathbb{N}^{\oplus r}$ onto respectively $t_1, \dots, t_r$. This map is clearly surjective, and by comparing dimensions (using corollary 6.5.8(ii)) one sees that it is an isomorphism. Then the theorem holds in this case, with $R := k[[\mathbb{N}^{\oplus r}]]$.

Next, if $\mathscr{O}_{X,\xi}$ does not contain a field, then its residue characteristic is a positive integer p, and we may find a complete discrete valuation ring $V \subset \mathscr{O}^\wedge_{X,\xi}$ whose maximal ideal is pV, and such that V/pV maps isomorphically onto the residue field of $\mathscr{O}^\wedge_{X,\xi}$ ([61, Th.29.3]). Again, we choose a morphism of monoids $\alpha : P \to \mathscr{O}_{X,\xi}$ as in the foregoing, and a sequence $(t_1, \dots, t_r)$ of elements of $\mathscr{O}_{X,\xi}$ lifting a regular system of parameters for $\mathscr{O}_{X,\xi}/I(\xi, \underline{M})$, by means of which we define a continuous ring homomorphism

$$\varphi : V[[P \times \mathbb{N}^{\oplus r}]] \to \mathscr{O}^\wedge_{X,\xi}$$

as in the previous case. Again, it is clear that φ is surjective. The image of the ideal J generated by the maximal ideal of $P \times \mathbb{N}^{\oplus r}$ is the maximal ideal of $\mathscr{O}^\wedge_{X,\xi}$; in particular, there exists $x \in J$ such that $\vartheta := p - x$ lies in $\mathrm{Ker}\,\varphi$. If we let $R := V[[\mathbb{N}^{\oplus r}]]$, it is clear that $\vartheta \in \mathfrak{m}_R \setminus \mathfrak{m}_R^2$.

Claim 6.5.27. Let A be a ring, π a regular element of A such that $A/\pi A$ is an integral domain, P a fine and sharp monoid. Let also ϑ be an element of $A[[P]]$ whose constant term is π (see (6.5.6)). Then $A[[P]]/(\vartheta)$ is an integral domain.

Proof of the claim. To ease notation, set $A_0 := A/\pi A$, $B := A[[P]]$ and $C := B/\vartheta B$. Choose a decomposition (6.5.7), and set $\mathrm{Fil}^\gamma_n B := \prod_{i\geq n} A[[\mathrm{gr}^\gamma_i P]]$ for every $n \in \mathbb{N}$. $\mathrm{Fil}^\gamma_\bullet B$ is a separated filtration by ideals of B, and we may consider the induced filtration $\mathrm{Fil}^\gamma_\bullet C$ on C. First, we remark that $\mathrm{Fil}^\gamma_\bullet C$ is also separated. This comes down to checking that

$$\bigcap_{n\geq 0} \vartheta B + \mathrm{Fil}^\gamma_n B = \vartheta B.$$

To this aim, suppose that, for a given $x \in B$ we have identities of the type $x = \vartheta y_n + z_n$, with $y_n \in B$ and $z_n \in \mathrm{Fil}^\gamma_n B$ for every $n \in \mathbb{N}$. Then, since π is regular, an easy induction shows that $\mathrm{gr}^\gamma_i(y_n) = \mathrm{gr}^\gamma_i(y_m)$ whenever $n, m > i$, and moreover $x = \vartheta \cdot \sum_{i\in\mathbb{N}} \mathrm{gr}^\gamma_i(y_{i+1})$, which shows the contention. It follows that, in order to show that C is a domain, it suffices to show that the same holds for the graded ring $\mathrm{gr}^\gamma_\bullet C$ associated to $\mathrm{Fil}^\gamma_\bullet C$. However, notice that :

$$\mathrm{Fil}^\gamma_n B \cap \vartheta B = \vartheta \cdot \mathrm{Fil}^\gamma_n B \qquad \text{for every } n \in \mathbb{N}.$$

(Indeed, this follows easily from the fact that π is a regular element : the verification shall be left to the reader). Hence, we may compute :

$$\mathrm{gr}^\gamma_n C = \frac{\mathrm{Fil}^\gamma_n B + \vartheta B}{\mathrm{Fil}^\gamma_{n+1} + \vartheta B} \simeq \frac{\mathrm{Fil}^\gamma_n B}{\mathrm{Fil}^\gamma_{n+1} + \vartheta \mathrm{Fil}^\gamma_n} \simeq A[\mathrm{gr}^\gamma_n P]/\vartheta A[\mathrm{gr}^\gamma_n P] \simeq A_0[\mathrm{gr}^\gamma_n P].$$

Thus, $\mathrm{gr}^\gamma_\bullet A \simeq A_0[P]$, which is a domain, since by assumption A_0 is a domain. $\diamondsuit$

From claim 6.5.27(ii) we deduce that $R[[P]]/(\vartheta)$ is an integral domain. Then, again by comparing dimensions, we see that φ factors through an isomorphism $R[[P]]/(\vartheta) \xrightarrow{\sim} \mathscr{O}^\wedge_{X,\xi}$. $\square$

Remark 6.5.28. Resume the notation of (6.5.25), and suppose that $\mathscr{O}_{X,\xi}/I(\xi,\underline{M})$ is a regular local ring. Let $P_{X(\xi)} \to \underline{M}(\xi)$ be a chart as in theorem 6.5.26(b), and $\alpha : P \to \mathscr{O}_{X,\xi}$ the corresponding morphism of monoids. Moreover, if $\mathscr{O}_{X,\xi}$ contains a field, let V denote a coefficient field of $\mathscr{O}^\wedge_{X,\xi}$, and otherwise, let V be a complete discrete valuation ring whose maximal ideal is generated by p, the residue characteristic of $\mathscr{O}_{X,\xi}$. In either case, pick a ring homomorphism $V \to \mathscr{O}^\wedge_{X,\xi}$ inducing an isomorphism of V/pV onto the residue field of $\mathscr{O}_{X,\xi}$. Let as well $t_1,\dots,t_r \in \mathscr{O}_{X,\xi}$ be any sequence of elements whose image in $\mathscr{O}_{X,\xi}/I(\xi,\underline{M})$ forms a regular system of parameters, and extend the map α to a morphism of monoids $P \times \mathbb{N}^{\oplus r} \to \mathscr{O}_{X,\xi}$, by the rule : $e_i \mapsto t_i$, where $e_1,\dots,e_r$ is the natural basis of $\mathbb{N}^{\oplus r}$. Then by inspecting the proof of theorem 6.5.26, we see that the induced continuous ring homomorphism :

$$V[[P \times \mathbb{N}^{\oplus r}]] \to \mathscr{O}^\wedge_{X,\xi}$$

is always surjective, and if V is not a field, its kernel contains an element ϑ whose constant term in V is $\vartheta_0 = p$. If V is a field (resp. if V is a discrete valuation ring) then $(X,\underline{M})$ is regular at the point ξ, if and only if this map is an isomorphism (resp. if and only if the kernel of this map is generated by ϑ).

Corollary 6.5.29. *Let $(X,\underline{M})$ be a regular log scheme. Then the scheme X is normal and Cohen-Macaulay.*

Proof. Let $x \in X$ be any point, and ξ a τ-point localized at x; we have to show that $\mathscr{O}_{X,x}$ is Cohen-Macaulay; in light of [33, Ch.IV, Cor.18.8.13(a)] (when $\tau = \text{ét}$), it suffices to show that the same holds for $\mathscr{O}_{X,\xi}$. Then [61, Th.17.5] further reduces to showing that the completion $\mathscr{O}^\wedge_{X,\xi}$ is Cohen-Macaulay; the latter follows easily from theorems 6.5.26 and 5.9.34(i). Next, in order to prove that X is normal, it suffices to show that $\mathscr{O}_{X,x}$ is regular, whenever x has codimension one in X ([61, Th.23.8]). Again, by [33, Ch.IV, Cor.18.8.13(c)] (when $\tau = \text{ét}$) and [30, Ch.0, Prop.17.1.5], we reduce to showing that $\mathscr{O}^\wedge_{X,\xi}$ is regular for such a point x. However, for a point of codimension one we have $r := \dim \underline{M}_\xi \leq 1$. If $r = 0$, then $I(\xi,\underline{M}) = \{0\}$, hence $\mathscr{O}_{X,\xi}$ is regular. Lastly, if $r = 1$, we see that $\underline{M}^\sharp_\xi \simeq \mathbb{N}$ (theorem 3.4.16(iii)); consequently, there exists a regular local ring R and an isomorphism $\mathscr{O}^\wedge_{X,\xi} \simeq R[[\mathbb{N}]]/(\vartheta)$, where $\vartheta = 0$ if $\mathscr{O}_{X,\xi}$ contains a field, and otherwise the constant term of ϑ lies in $\mathfrak{m}_R \setminus \mathfrak{m}^2_R$. Since $R[[\mathbb{N}]]$ is again regular, the assertion follows in either case. $\square$

Corollary 6.5.30. *Let $i : (X',\underline{M}') \to (X,\underline{M})$ be an exact closed immersion of regular log schemes (see definition* 6.3.22(i)*). Then the underlying morphism of schemes $X' \to X$ is a regular closed immersion.*

Proof. Let ξ be a τ-point of X, and denote by J the kernel of $i^\natural_\xi : \mathscr{O}_{X,\xi} \to \mathscr{O}_{X',\xi}$. To ease notation, let as well $A := \mathscr{O}_{X,\xi}/I(\xi,\underline{M})$ and $A' := \mathscr{O}_{X',\xi}/I(\xi,\underline{M}')$ Since $\log i : i^*\underline{M} \to \underline{M}'$ is an isomorphism, $i^\natural_\xi$ induces an isomorphism :

$$\mathscr{O}_{X,\xi}/(J + I(\xi,\underline{M})) \xrightarrow{\sim} A'.$$

Since A and A' are regular, there exists a sequence of elements $t_1,\dots,t_k \in J$, whose image in A forms the beginning of a regular system of parameters and generate the kernel of the induced map $A \to A'$ ([30, Ch.0, Cor.17.1.9]). Extend this sequence by suitable elements of $\mathscr{O}_{X,\xi}$, to obtain a sequence $(t_1,\dots,t_r)$ whose image in A is a regular system of parameters. We deduce a surjection $\varphi : V[[P \times \mathbb{N}^{\oplus r}]] \to \mathscr{O}^\wedge_{X,\xi}$ as in remark 6.5.28, where V is either a field or a complete discrete valuation ring. Denote by $(e_1,\dots,e_r)$ the natural basis of $\mathbb{N}^{\oplus r}$. Now, suppose first that V is a complete discrete valuation ring; then $\operatorname{Ker}\varphi$ is generated by an element

$\vartheta \in V[[P \times \mathbb{N}^{\oplus r}]]$, whose constant term is a uniformizer in V; we deduce easily from claim 6.5.27 that $(e_1, \dots, e_r, \vartheta)$ is a regular sequence in $V[[P \times \mathbb{N}^{\oplus r}]]$; hence the same holds for the sequence $(\vartheta, e_1, \dots, e_r)$, in view of [61, p.127, Cor.]. This implies that $(t_1, \dots, t_r)$ is a regular sequence in $\mathscr{O}^\wedge_{X,\xi}$, hence $(t_1, \dots, t_k)$ is a regular sequence in $\mathscr{O}_{X,\xi}$, which is the contention. The case where V is a field is analogous, though simpler : the details shall be left to the reader. $\square$

Remark 6.5.31. In the situation of remark 6.5.28, suppose that P is a free monoid of finite rank d, let $\{f_1, \dots, f_d\} \subset \mathscr{O}_{X,\xi}$ be the image of the (unique) system of generators of P, and for every $i \le d$ let $Z_i \subset X$ be the zero locus of f_i; then $\bigcup_{i=1}^d Z_i$ is a strict normal crossing divisor in the sense of example 6.2.11. The proof is analogous to that of corollary 6.5.30 : the sequence $(f_1, \dots, f_d, \vartheta)$ is regular in $V[[P \oplus \mathbb{N}^{\oplus r}]]$, hence also its permutation $(\vartheta, f_1, \dots, f_d)$ is regular, whence the claim (the details shall be left to the reader).

Proposition 6.5.32. *Resume the situation of* (6.5.25). *Then the log scheme $(X, \underline{M})$ is regular at the τ-point ξ if and only if the following two conditions hold :*

(a) *The ring $\mathscr{O}_{X,\xi}/I(\xi, \underline{M})$ is regular.*
(b) *There exists a morphism of monoids $P \to \mathscr{O}_{X,\xi}$ from a fine monoid P, whose associated constant log structure on $X(\xi)$ is the same as $\underline{M}(\xi)$, and such that $\mathscr{O}_{X,\xi}$ is P-flat.*

Proof. Suppose first that $(X, \underline{M})$ is regular at the point ξ; then by definition (a) holds. Next, we may find a fine monoid P and a local morphism $\alpha : P \to A := \mathscr{O}_{X,\xi}$ whose associated constant log structure on $X(\xi)$ is the same as $\underline{M}(\xi)$, and a regular local ring R, with a ring homomorphism $R \to A$ such that the induced continuous map $R[[P]] \to A^\wedge$ fulfills condition (b) of theorem 6.5.26 (where $A^\wedge$ is the completion of A). Since the completion map $\varphi : A \to A^\wedge$ is faithfully flat, A is α-flat if and only if $A^\wedge$ is $\varphi \circ \alpha$-flat; in light of proposition 3.1.39(i), it then suffices to show that, for every ideal $I \subset P$, the natural map

$$A^\wedge \overset{\mathbf{L}}{\otimes}_P P/I \to A^\wedge \otimes_P P/I$$

is an isomorphism in $\mathsf{D}^-(A^\wedge\text{-}\mathbf{Mod})$ (notation of (3.1.33)). To this aim we remark :

Claim 6.5.33. For any ideal $I \subset P$, the natural morphism

$$R[P] \overset{\mathbf{L}}{\otimes}_P P/I \to R\langle P/I \rangle$$

is an isomorphism in $\mathsf{D}^-(R[P]\text{-}\mathbf{Mod})$.

Proof of the claim. We consider the base change spectral sequence for Tor :

$$E^2_{ij} := \mathrm{Tor}_i^{\mathbb{Z}[P]}(\mathrm{Tor}_j^{\mathbb{Z}}(R, \mathbb{Z}[P]), \mathbb{Z}\langle P/I \rangle) \Rightarrow \mathrm{Tor}^{\mathbb{Z}}_{i+j}(R, \mathbb{Z}\langle P/I \rangle)$$

([75, Th.5.6.6]). Clearly $E^2_{ij} = 0$ whenever $j > 0$, whence isomorphisms :

$$\mathrm{Tor}_i^{\mathbb{Z}\langle P \rangle}(R\langle P \rangle, \mathbb{Z}\langle P/I \rangle) \xrightarrow{\sim} \mathrm{Tor}_i^{\mathbb{Z}}(R, \mathbb{Z}\langle P/I \rangle)$$

for every $i \in \mathbb{N}$. The claim follows easily. $\diamondsuit$

In light of claim 6.5.33 we deduce natural isomorphisms :

$$A^\wedge \overset{\mathbf{L}}{\otimes}_P P/I \xrightarrow{\sim} A^\wedge \overset{\mathbf{L}}{\otimes}_{R[P]} (R[P] \overset{\mathbf{L}}{\otimes}_P P/I) \xrightarrow{\sim} A^\wedge \overset{\mathbf{L}}{\otimes}_{R[P]} R\langle P/I \rangle$$

in $\mathsf{D}^-(A^\wedge\text{-}\mathbf{Mod})$. Now, if A contains a field, we have $A^\wedge \simeq R[[P]]$, which is a flat $R[P]$-algebra ([61, Th.8.8]), and the contention follows. If A does not contain a field, the complex

$$0 \to R[[P]] \xrightarrow{\vartheta} R[[P]] \to A^\wedge \to 0$$

is a $R[P]$-flat resolution of $A^\wedge$. Since $R[[P]]/IR[[P]] \simeq \lim\limits_{n \in \mathbb{N}} R\langle P/(I \cup \mathfrak{m}_P^n) \rangle$, we come down to the following :

Claim 6.5.34. Let $\vartheta \in R[P]$ be any element whose constant term ϑ_0 is a regular element in R. Then, for every ideal $I \subset P$, the image of ϑ in $R\langle P/I\rangle$ is a regular element.

Proof of the claim. In view of (6.5.3), P can be regarded as a graded monoid $P = \coprod_{n\in\mathbb{N}} P_n$, so $R[P]$ is a graded algebra, and $I = \coprod_{n\in\mathbb{N}}(I \cap P_n)$ is a graded ideal. Thus, $\langle P/I\rangle$ is a graded R-algebra as well, and the claim follows easily (details left to the reader). ♢

Conversely, suppose that conditions (a) and (b) hold. By virtue of lemma 6.5.17, we may assume that P is fine, sharp and saturated, and that the map $P \to \mathscr{O}_{X,\xi}$ is local. According to remark 6.5.28, we may find a regular local ring R of dimension $\leq 1 + \dim A/\mathfrak{m}_P A$, and a surjective ring homomorphism :

$$\varphi : R[[P]] \to A^\wedge$$

Suppose first that $\dim R = 1 + \dim A/\mathfrak{m}_P A$; then $\mathrm{Ker}\,\varphi$ contains an element ϑ whose constant term ϑ_0 lies in $\mathfrak{m}_R \setminus \mathfrak{m}_R^2$, and φ induces an isomorphism $R_0 := R/\vartheta_0 R \xrightarrow{\sim} A^\wedge/\mathfrak{m}_P A^\wedge$. We have to show that φ induces an isomorphism $\overline{\varphi} : R[[P]]/(\vartheta) \xrightarrow{\sim} A^\wedge$. To this aim, we consider the $\mathfrak{m}_P$-adic filtrations on these rings; for the associated graded rings we have :

$$\mathrm{gr}_n R[[P]]/(\vartheta) \simeq R_0\langle \mathfrak{m}_P^n/\mathfrak{m}_P^{n+1}\rangle \qquad \mathrm{gr}_n A^\wedge = \mathfrak{m}_P^n A^\wedge/\mathfrak{m}_P^{n+1}A^\wedge.$$

Hence $\mathrm{gr}_n\overline{\varphi}$ is the natural map $A^\wedge/\mathfrak{m}_P A^\wedge \otimes_P (\mathfrak{m}_P^n/\mathfrak{m}_P^{n+1}) \to \mathfrak{m}_P^n A^\wedge/\mathfrak{m}_P^{n+1}A^\wedge$, and the latter is an isomorphism, since $A^\wedge$ is P-flat. The assertion follows in this case. The remaining case where $\dim R = \dim A/\mathfrak{m}_P A$ is similar, but easier, so shall be left to the reader, as an exercise. □

Corollary 6.5.35. *Let $(X, \underline{M})$ be a log scheme, ξ a τ-point of X, and suppose that $(X, \underline{M})$ is regular at ξ. Then the following conditions are equivalent :*

(a) *$\mathscr{O}_{X,\xi}$ is a regular local ring.*
(b) *$\underline{M}_\xi^\sharp$ is a free monoid of finite rank.*

Proof. (b)⇒(a) : Indeed, it suffices to show that $\mathscr{O}_{X,\xi}^\wedge$ is regular ([30, Ch.0, Prop.17.3.3(i)]); by theorem 6.5.26, the latter is isomorphic to either $R[[P]]$ or $R[[P]]/\vartheta R[[P]]$, where R is a regular local ring, $P := \underline{M}_\xi$, and the constant term of ϑ lies in $\mathfrak{m}_R \setminus \mathfrak{m}_R^2$. Since, by assumption, P is a free monoid of finite rank, it is easily seen that rings of the latter kind are regular ([30, Ch.0, Cor.17.1.8]).

(a)⇒(b) : By proposition 6.5.32, $R := \mathscr{O}_{X,\xi}$ is P-flat, for a morphism $P := \underline{M}_\xi^\sharp \to R$ that induces the log structure $\underline{M}(\xi)$ on $X(\xi)$. It follows that

$$\mathfrak{m}_P R/\mathfrak{m}_P^2 R = (\mathfrak{m}_P/\mathfrak{m}_P^2) \otimes_P R = R^{\oplus r}$$

where $r := \mathrm{rk}^\circ_{P^\times}\mathfrak{m}_P/\mathfrak{m}_P^2$ is the cardinality of $\mathfrak{m}_P \setminus \mathfrak{m}_P^2$. In view of [33, Ch.IV, Prop.16.9.3, Cor.16.9.4, Cor.19.1.2], we deduce that the image of $\mathfrak{m}_P \setminus \mathfrak{m}_P^2$ is a regular sequence of R of length r, hence $\dim P = \dim R - \dim R/\mathfrak{m}_P R = r$, since $(X, \underline{M})$ is regular at ξ. Then the assertion follows from proposition 6.5.4. □

Corollary 6.5.36. *Let $(X, \underline{M})$ be a regular log scheme, ξ any τ-point of X, and $\mathfrak{p} \subset \underline{M}_\xi$ an ideal. We have :*

(i) *If $\mathfrak{p}$ is a prime ideal, $\mathfrak{p}\mathscr{O}_{X,\xi}$ is a prime ideal of $\mathscr{O}_{X,\xi}$, and $\mathrm{ht}\,\mathfrak{p} = \mathrm{ht}\,\mathfrak{p}\mathscr{O}_{X,\xi}$.*
(ii) *If $\mathfrak{p}$ is a radical ideal, $\mathfrak{p}\mathscr{O}_{X,\xi}$ is a radical ideal of $\mathscr{O}_{X,\xi}$.*

Proof. To ease notation, set $A := \mathscr{O}_{X,\xi}$. Pick a morphism $\beta : P \to \underline{M}_\xi$ as in proposition 6.5.32; by corollary 6.1.34(i) and lemma 6.5.17, we may further assume that P is fine, sharp and saturated. Let $\mathfrak{q} := \beta^{-1}\mathfrak{p} \subset P$; then $\mathfrak{p}A = \mathfrak{q}A$.

(i): Since $\mathfrak{p}$ is a prime ideal, $\mathfrak{q}$ is a prime ideal of P, and in order to prove that $\mathfrak{p}A$ is a prime ideal, it suffices therefore to show that the completion $(A/\mathfrak{q}A)^\wedge$ of $A/\mathfrak{q}A$ is an integral domain. However, remark 6.5.28 implies that $(A/\mathfrak{q}A)^\wedge \simeq B/\vartheta B$, where $B := R[[P\backslash\mathfrak{q}]]$, with $(R, \mathfrak{m}_R)$ a regular local ring, and ϑ is either zero, or else it is an element whose constant term $\vartheta_0 \in R$ lies in $\mathfrak{m}_R \setminus \mathfrak{m}_R^2$. The assertion is obvious when $\vartheta = 0$, and otherwise it follows from claim 6.5.27. Next, by going down (corollary 6.5.16), we see that :

$$\mathrm{ht}\,\mathfrak{p}A \geq \mathrm{ht}\,\mathfrak{p}.$$

On the other hand, notice that we have a natural identification $\mathrm{Spec}\,\underline{M}_\xi \xrightarrow{\sim} \mathrm{Spec}\,P$, especially $\mathrm{ht}\,\mathfrak{p} = \mathrm{ht}\,\mathfrak{q}$, hence $\dim A/\mathfrak{p}A = \dim(A/\mathfrak{q}A)^\wedge = \dim R + \dim(P\setminus\mathfrak{q}) - \varepsilon$, where ε is either 0 or 1 depending on whether R does or does not contain a field (corollary 6.5.8(ii)). Thus :

$$\mathrm{ht}\,\mathfrak{p}A \leq \dim A - \dim A/\mathfrak{p}A = \dim P - \dim(P\setminus\mathfrak{q}) = \mathrm{ht}\,\mathfrak{p}$$

(corollary 3.4.10(iii)), which completes the proof.

(ii): In this case, $\mathfrak{q}$ is a radical ideal of P, so it can be written as a finite intersection of prime ideals of P (lemmata 3.1.15 and 3.1.20(iii)); then the assertion follows from (i) and lemma 3.1.37 (details left to the reader). □

Lemma 6.5.37. *Let X be a scheme, $\underline{M}$ a fs log structure on X_{Zar}, and ξ a geometric point of X. Then $(X,\underline{M})$ is regular at the point $|\xi|$ if and only if $\widetilde{u}^*_X(X,\underline{M})$ is regular at ξ.*

Proof. Set $(Y,\underline{N}) := \widetilde{u}^*_X(X,\underline{M})$ (notation of (6.2.2) : of course $Y = X$, but the sheaf $\mathscr{O}_Y$ is defined on the site $X_{\text{ét}}$, hence $B := \mathscr{O}_{Y,\xi}$ is the strict henselization of $A := \mathscr{O}_{X,|\xi|}$), and let $\alpha : \underline{M}_{|\xi|} \to B$ be the induced morphism of monoids; since α is local, it is easily seen that $\underline{N}_\xi$ is isomorphic to the push-out of the diagram $\underline{M}_{|\xi|} \leftarrow \underline{M}^\times_{|\xi|} \to B^\times$, especially $\dim \underline{M}_{|\xi|} = \dim \underline{N}_\xi$. Moreover, $I(\xi,\underline{N}) = I(|\xi|,\underline{M})B$, so that $B_0 := B/I(\xi,\underline{N})$ is the strict henselization of $A_0 := A/I(|\xi|,\underline{M})$ ([33, Ch.IV, Prop.18.6.8]); hence A_0 is regular if and only if the same holds for B_0 ([33, Ch.IV, Cor.18.8.13]). Finally, $\dim A = \dim B$ and $\dim A_0 = \dim B_0$ ([61, Th.15.1]). The lemma follows. □

6.5.38. Let $(Y,\underline{N})$ be a log scheme, $\overline{y}$ a τ-point of Y, and $\alpha : Q^{\log}_Y \to \underline{N}$ a fine and saturated chart which is sharp at $\overline{y}$. Let also $\varphi : Q \to P$ be an injective morphism of monoids, with P fine and saturated, and such that $P^\times$ is a torsion-free abelian group. Define $(X,\underline{M})$ as the fibre product in the cartesian diagram of log schemes :

$$\begin{array}{ccc} (X,\underline{M}) & \xrightarrow{f} & (Y,\underline{N}) \\ \big\downarrow & & \big\downarrow{\scriptstyle h} \\ \mathrm{Spec}(\mathbb{Z},P) & \xrightarrow{\mathrm{Spec}(\mathbb{Z},\varphi)} & \mathrm{Spec}(\mathbb{Z},Q) \end{array} \tag{6.5.39}$$

where h is induced by α; especially, h is strict.

Lemma 6.5.40. *In the situation of* (6.5.38), *Let $\overline{x}$ be any τ-point of X such that $f(\overline{x}) = \overline{y}$, and suppose that $(Y,\underline{N})$ is regular at $\overline{y}$. Then $(X,\underline{M})$ is regular at $\overline{x}$.*

Proof. To begin with, we show :

Claim 6.5.41. The natural map :

$$\mathscr{O}_{X,\overline{x}} \overset{\mathbf{L}}{\otimes}_P P/I \to \mathscr{O}_{X,\overline{x}} \otimes_P P/I$$

is an isomorphism in $\mathsf{D}^-(\mathscr{O}_{X,\overline{x}}\text{-}\mathbf{Mod})$, for every ideal $I \subset P$ (notation of (3.1.33)).

Proof of the claim. It suffices to show that this map is an isomorphism in $\mathsf{D}^-(\mathscr{O}_{Y,\overline{y}}\text{-}\mathbf{Mod})$, and in the latter category we have a commutative diagram :

$$\begin{tikzcd}
(\mathscr{O}_{Y,\overline{y}} \overset{\mathbf{L}}{\otimes}_Q P) \overset{\mathbf{L}}{\otimes}_P P/I \arrow[rr] \arrow[d] & & \mathscr{O}_{Y,\overline{y}} \overset{\mathbf{L}}{\otimes}_Q P/I \arrow[d] \\
(\mathscr{O}_{Y,\overline{y}} \otimes_Q P) \overset{\mathbf{L}}{\otimes}_P P/I \arrow[r] & (\mathscr{O}_{Y,\overline{y}} \otimes_Q P) \otimes_P P/I \arrow[r, "\sim"] & \mathscr{O}_{Y,\overline{y}} \otimes_Q P/I.
\end{tikzcd} \tag{6.5.42}$$

Since $\mathscr{O}_{X,\overline{x}}$ is a localization of $\mathscr{O}_{Y,\overline{y}} \otimes_Q P$, we are reduced to showing that the bottom arrow of (6.5.42) is an isomorphism. However, the top arrow of (6.5.42) is always an isomorphism. Moreover, on the one hand, since $(Y, \underline{N})$ is regular at $\overline{y}$, the ring $\mathscr{O}_{Y,\overline{y}}$ is Q-flat (proposition 6.5.32), and on the other hand, since φ is injective, P/I is an integral Q-module, so the two vertical arrows are isomorphisms as well, and the claim follows. $\diamondsuit$

Claim 6.5.43. $\mathscr{O}_{X,\overline{x}}/I(\overline{x}, \underline{M})$ is a regular ring.

Proof of the claim. Let $\beta : P \to A := \mathscr{O}_{X,\overline{x}}$ be the morphism deduced from h, and set :

$$S := \beta^{-1}(\mathscr{O}^\times_{X,\overline{x}}) \qquad I(\overline{x}, P) := P \backslash S.$$

Then the $\mathbb{Z}[P]$-algebra A is a localization of the $\mathbb{Z}[P]$-algebra

$$B := S^{-1}(\mathscr{O}_{Y,\overline{y}} \otimes_Q P)$$

and it suffices to show that $B/I(\overline{x}, \underline{M})$ is regular.

It is easily seen that $I(\overline{x}, \underline{M}) = I(\overline{x}, P)A$ and $\varphi(\mathfrak{m}_Q) \subset I(\overline{x}, P)$; on the other hand, since α is sharp at the point $\overline{y}$, we have $I(\overline{y}, \underline{N}) = \mathfrak{m}_Q \mathscr{O}_{Y,\overline{y}}$, and $Q \setminus \mathfrak{m}_Q = \{1\}$. Let p be the residue characteristic of $\mathscr{O}_{Y,\overline{y}}$; there follow isomorphisms of $\mathbb{Z}_{(p)}$-algebras :

$$B/I(\overline{x}, \underline{M})B \simeq S^{-1}\mathscr{O}_{Y,\overline{y}} \otimes_Q P/I(\overline{x}, P) \simeq \mathscr{O}_{Y,\overline{y}}/I(\overline{y}, \underline{N}) \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_{(p)}[S^{\mathrm{gp}}].$$

By assumption, $\mathscr{O}_{Y,\overline{y}}/I(\overline{y}, \underline{N})$ is regular, hence we are reduced to showing that $\mathbb{Z}_{(p)}[S^{\mathrm{gp}}]$ is a smooth $\mathbb{Z}_{(p)}$-algebra ([33, Ch.IV, Prop.17.5.8(iii)]). However, under the current assumptions P^{gp} is a free abelian group of finite rank, hence the same holds for S^{gp}, and the contention follows easily. $\diamondsuit$

In light of proposition 3.1.39(i), claims 6.5.41 and 6.5.43 assert that conditions (a) and (b) of proposition 6.5.32 are satisfied for the τ-point $\overline{x}$ of $(X, \underline{M})$, so the latter is regular at $\overline{x}$, as stated. $\square$

Theorem 6.5.44. *Let $f : (X, \underline{M}) \to (Y, \underline{N})$ be a smooth morphism of locally noetherian fs log schemes, ξ a τ-point of X, and suppose that $(Y, \underline{N})$ is regular at the point $f(\xi)$. Then $(X, \underline{M})$ is regular at the point ξ.*

Proof. In case $\tau = \mathrm{Zar}$, lemma 6.5.37 and corollary 6.3.27(ii) reduce the assertion to the corresponding one for $\widetilde{u}^* f$. Hence, we may assume that $\tau = \text{ét}$. Next, since the assertion is local on X_τ, we may assume that $\underline{N}$ admits a fine and saturated chart $\alpha : Q_Y^{\log} \to \underline{N}$ which is sharp at $f(\xi)$ (corollary 6.1.34(i)); then, by corollary 6.3.42, we may further assume that there exists an injective morphism of fine and saturated monoids $\varphi : Q \to P$, such that $P^\times$ is torsion-free, and a cartesian diagram as in (6.5.39). Then the assertion follows from lemma 6.5.40. $\square$

Corollary 6.5.45. *Let $f : (X, \underline{M}) \to (Y, \underline{N})$ be a smooth morphism of locally noetherian fine log schemes. Then, for every $y \in (Y, \underline{N})_{\mathrm{tr}}$, the log scheme $\operatorname{Spec} \kappa(y) \times_Y (X, \underline{M})$ is geometrically regular.*

Proof. The trivial log structure on $\mathrm{Spec}\,\kappa(y)$ is obviously saturated, so the same holds for the log structure of $\mathrm{Spec}\,\kappa(y) \times_Y (X, \underline{M})$. Hence, the assertion is an immediate consequence of theorem 6.5.44. $\square$

Theorem 6.5.46. *Let $(X, \underline{M})$ be a locally noetherian fs log scheme. Then the subset $(X, \underline{M})_{\mathrm{reg}}$ is closed under generization.*

Proof. Let ξ be a τ-point of X, with support $x \in (X, \underline{M})_{\mathrm{reg}}$, and let η be a generization of ξ, whose support is a generization y of x. We have to show that $(X, \underline{M})$ is regular at the point η. Since the assertion is local on X, we may assume that $(X, \underline{M})$ admits a fine and saturated chart $\beta : P_X \to \underline{M}$, sharp at the point ξ (corollary 6.1.34(i)). Let $\alpha : P \to \mathscr{O}_{X,\eta}$ be the morphism deduced from β_η, and set $\mathfrak{p} := \alpha^{-1}\mathfrak{m}_\eta$, where $\mathfrak{m}_\eta \subset \mathscr{O}_{X,\eta}$ is the maximal ideal. We consider the cartesian diagram of log schemes :

$$\begin{array}{ccc}
(X', \underline{M}')_\circ & \longrightarrow & (X, \underline{M}) \\
\big\downarrow & & \big\downarrow{\scriptstyle g} \\
\mathrm{Spec}\langle \mathbb{Z}, P/\mathfrak{p}\rangle & \longrightarrow & \mathrm{Spec}(\mathbb{Z}, P)
\end{array}$$

where g is induced by β (notation of (6.2.13)). Clearly ξ and η induce τ-points on X', which we denote by the same names. We have natural identifications:

$$\mathscr{O}_{X,\xi}/I(\xi, \underline{M}) \xrightarrow{\sim} \mathscr{O}_{X',\xi}/I(\xi, \underline{M}') \qquad \underline{M}'_\eta = \mathscr{O}^\times_{X',\eta} \qquad \mathscr{O}_{X,\eta}/I(\eta, \underline{M}) \xrightarrow{\sim} \mathscr{O}_{X',\eta}$$

([33, Ch.IV, Prop.18.6.8]). Moreover, from proposition 6.5.32 and lemma 6.5.17 we know that $\mathscr{O}_{X,\xi}$ is P-flat; then corollary 6.5.16 yields the inequality :

$$\dim \mathscr{O}_{X',\xi} = \dim \mathscr{O}_{X,\xi} \otimes_P P/\mathfrak{p} \geq \dim \mathscr{O}_{X,\xi} - \mathrm{ht}(\mathfrak{p}) = \dim \mathscr{O}_{X,\xi}/I(\xi, \underline{M}) + \dim P/\mathfrak{p}$$

in other words : $\dim \mathscr{O}_{X',\xi} = d(\xi, \underline{M}')$ (notation of definition 6.5.23), so $(X', \underline{M}')$ is regular at ξ. By the same token, $\mathscr{O}_{X,\eta}$ is $P_\mathfrak{p}$-flat; from proposition 6.5.32, it follows that $(X, \underline{M})$ is regular at η if and only if the same holds for $(X', \underline{M}')$.

Hence we may replace $(X, \underline{M})$ by $(X', \underline{M}')$, and P by $P \setminus \mathfrak{p}$, after which we may assume that $y \in (X, \underline{M})_{\mathrm{tr}}$. In this case :

$$\alpha(P) \subset \mathscr{O}^\times_{X,\eta} \tag{6.5.47}$$

and we have to show that $\mathscr{O}_{X,\eta}$ is a regular local ring, or equivalently, that $\mathscr{O}_{X,y}$ is regular ([33, Ch.IV, Cor.18.8.13(c)]).

Denote by Y the topological closure of y in X, endowed with its reduced subscheme structure. According to [27, Ch.II, Prop.7.1.7] (and [33, Ch.IV, Prop.18.8.8(iv)] in case $\tau = \text{ét}$), we may find a local injective ring homomorphism $j : \mathscr{O}_{Y,\xi} \to V$, where V is a discrete valuation ring. Let also $\overline{\beta} : P \to \mathscr{O}_{Y,\xi}$ be the morphism deduced from β. Then (6.5.47) implies that $j \circ \overline{\beta}(P) \subset V \setminus \{0\}$, hence $j \circ \overline{\beta}$ extends to a homomorphism of groups $P^{\mathrm{gp}} \to K^\times$, where $K^\times := (V \setminus \{0\})^{\mathrm{gp}}$ is the multiplicative group of the field of fractions of V; after composition with the valuation $K^\times \to \mathbb{Z}$ of V, there follows a group homomorphism

$$\varphi : P \to \mathbb{Z}.$$

Notice also that $j \circ \overline{\beta}$ is a local morphism, since the same holds for j and $\overline{\beta}$; consequently, $\varphi(P) \neq \{0\}$. Set $Q := \varphi^{-1}\mathbb{N}$; then $\varphi(Q)$ is a non-trivial submonoid of $\mathbb{N}$, and $\dim Q = \dim Q/\mathrm{Ker}\,\varphi^{\mathrm{gp}} = \dim \varphi(Q) = 1$. Set

$$T := \mathrm{Spec}\,V \qquad (S, \underline{Q}) := \mathrm{Spec}(\mathbb{Z}, Q).$$

Notice that Q is saturated and fine (corollary 3.4.2), so there exists an isomorphism :

$$\mathbb{Z}^{\oplus r} \times \mathbb{N} \xrightarrow{\sim} Q \qquad \text{for some } r \in \mathbb{N}$$

(theorem 3.4.16(iii)); the latter determines a chart :

$$\mathbb{N}_S \to \underline{Q} \tag{6.5.48}$$

which is sharp at every τ-point of S localized outside the trivial locus $\mathrm{Spec}(\mathbb{Z}, Q)_{\mathrm{tr}}$. We consider the cartesian diagram of log schemes :

$$\begin{tikzcd} (X', g'^*\underline{Q}) \arrow[r, "g'"] \arrow[d, "f'"'] & \mathrm{Spec}(\mathbb{Z}, \underline{Q}) \arrow[d, "f"] \\ (X, \underline{M}) \arrow[r, "g"] & \mathrm{Spec}(\mathbb{Z}, P) \end{tikzcd}$$

where f is the morphism of log schemes induced by the inclusion map $\psi : P \to Q$. Notice that f is a smooth morphism (proposition 6.3.34); moreover, the restriction

$$f_{\mathrm{tr}} : \mathrm{Spec}(\mathbb{Z}, Q)_{\mathrm{tr}} \to \mathrm{Spec}(\mathbb{Z}, P)_{\mathrm{tr}}$$

of f, is just the morphism $\mathrm{Spec}\,\mathbb{Z}[\psi^{\mathrm{gp}}] : \mathrm{Spec}\,\mathbb{Z}[Q^{\mathrm{gp}}] \to \mathrm{Spec}\,\mathbb{Z}[P^{\mathrm{gp}}]$, hence it is an isomorphism of schemes. It follows that f' is a smooth morphism (proposition 6.3.24(ii)), and its restriction f'_{tr} to the trivial loci, is an isomorphism.

The homomorphism j induces a morphism $h : T \to X$, such that the closed point of T maps to x and the generic point maps to y. By construction $g \circ h$ lifts to a morphism of schemes $h' : T \to S$, and the pair (h, h') determines a morphism $T \to X'$. Let $x', y' \in X'$ be the images of respectively the closed point and the generic point of T, and choose τ-points ξ' and η' localized at x' and respectively y'; then the image of x' in X is the point x, therefore $(X', g'^*\underline{Q})$ is regular at the point ξ', by theorem 6.5.44. Furthermore, $g'(\xi')$ lies outside $\mathrm{Spec}(\mathbb{Z}, Q)_{\mathrm{tr}}$, hence (6.5.48) induces a chart $\mathbb{N}_{X'} \to g'^*\underline{Q}$, which is sharp at ξ'. In light of corollary 6.5.35, we deduce that $\mathscr{O}_{X',x'}$ is a regular ring, and then the same holds also for $\mathscr{O}_{X',y'}$ ([30, Ch.0, Cor.17.3.2]). However, y' lies in the trivial locus of $(X', g'^*\underline{Q})$, and its image in X is y, so by the foregoing the natural map $\mathscr{O}_{X,y} \to \mathscr{O}_{X',y'}$ is an isomorphism, and the contention follows. □

6.5.49. Let $(X, \underline{M})$ be any log scheme, ξ a τ-point of X. There follows a continuous map

$$\psi_\xi : X(\xi) \to T_\xi := \mathrm{Spec}\,\underline{M}_\xi$$

that sends the closed point of $X(\xi)$ to the closed point of T_ξ. For every point $\mathfrak{p} \in T_\xi$, let $\overline{\{\mathfrak{p}\}}$ be the topological closure of $\{\mathfrak{p}\}$ in T_ξ; then

$$X(\mathfrak{p}) := \psi_\xi^{-1}\overline{\{\mathfrak{p}\}}$$

is a closed subset of $X(\xi)$, which we endow with its reduced subscheme structure, and we set

$$(X(\mathfrak{p}), \underline{M}(\mathfrak{p})) := ((X(\xi), \underline{M}(\xi)) \times_X X(\mathfrak{p}))_{\mathrm{red}}$$

(notation of example 6.1.10(iv)). Notice that $U_\mathfrak{p} := \psi_\xi^{-1}(\mathfrak{p})$ is an open subset of $X(\mathfrak{p})$, for every $\mathfrak{p} \in T_\xi$. We call the family

$$(U_\mathfrak{p} \mid \mathfrak{p} \in T_\xi)$$

of locally closed subschemes of $X(\xi)$, the *logarithmic stratification* of $(X(\xi), \underline{M}(\xi))$. For instance, $U_\varnothing = (X(\overline{x}), \underline{M}(\overline{x}))_{\mathrm{tr}}$. More generally, it is clear from the definition that

$$U_\mathfrak{p} = (X(\mathfrak{p}), \underline{M}(\mathfrak{p}))_{\mathrm{tr}} \qquad \text{for every } \mathfrak{p} \in T_\xi. \tag{6.5.50}$$

Corollary 6.5.51. *In the situation of* (6.5.49), *suppose that $(X, \underline{M})$ is regular at ξ. We have :*

(i) *The log scheme $(X(\mathfrak{p}), \underline{M}(\mathfrak{p}))$ is pointed regular, for every $\mathfrak{p} \in T_\xi$.*
(ii) *The scheme $X(\mathfrak{p})$ is irreducible, and its codimension in X equals the height of $\mathfrak{p}$ in T_ξ, for every $\mathfrak{p} \in T_\xi$.*
(iii) *The scheme $U_\mathfrak{p}$ is regular and irreducible, for every $\mathfrak{p} \in T_\xi$.*

Proof. (i): By theorem 6.5.46, it suffices to show that $(X(\mathfrak{p}), \underline{M}(\mathfrak{p}))$ is pointed regular at ξ, for every $\mathfrak{p} \in T_\xi$. However, say that $X(\xi) = \mathrm{Spec}\, A$, and let $P \to \underline{M}(\xi)$ be a fine and saturated chart, sharp at the τ-point ξ. Then $X(\mathfrak{p}) = \mathrm{Spec}\, A_0$, where $A_0 := A/\mathfrak{p}A$, and $\underline{M}(\mathfrak{p}) = \underline{N}_\circ$, where $\underline{N}$ is the fs log structure deduced from the induced map $\beta : P\backslash\mathfrak{p} \to A_0$. By proposition 6.5.32 and lemma 6.5.17, the ring A is P-flat; then A_0 is β-flat, and the assertion follows from proposition 6.5.32.

Next, (ii) is a rephrasing of corollary 6.5.36(i), and (iii) follows from (i),(ii) and (6.5.50), by virtue of corollary 6.5.35. □

Proposition 6.5.52. *Let $(X, \beta : \underline{M} \to \mathscr{O}_X)$ be a regular log scheme, set $U := (X, \underline{M})_{\mathrm{tr}}$, and denote by $j : U \to X$ the open immersion. Then the morphism β induces identifications:*

$$\underline{M} \xrightarrow{\sim} j_*\mathscr{O}_U^\times \cap \mathscr{O}_X \qquad \underline{M}^{\mathrm{gp}} \xrightarrow{\sim} j_*\mathscr{O}_U^\times.$$

Proof. Notice that the scheme X is normal (corollary 6.5.29), hence both $j_*\mathscr{O}_U^\times$ and $\mathscr{O}_X$ are subsheaves of the sheaf $i_*\mathscr{O}_{X_0}$, where X_0 is the subscheme of maximal points of X, and $i : X_0 \to X$ is the natural morphism; so we may intersect these two sheaves inside the latter.

In view of lemma 6.5.18 and proposition 6.5.32, we know already that β is injective, and clearly the image of β lands in $j_*\mathscr{O}_U^\times$, so it remains only to show that β (resp. β^{gp}) induces an epimorphism onto $j_*\mathscr{O}_U^\times \cap \mathscr{O}_X$ (resp. onto $j_*\mathscr{O}_U^\times$). The assertions can be checked on the stalks, hence let ξ be any τ-point of X; to begin with, we show :

Claim 6.5.53. The induced map $\beta_\xi^{\mathrm{gp}} : \underline{M}_\xi^{\mathrm{gp}} \to (j_*\mathscr{O}_U^\times)_\xi$ is a surjection.

Proof of the claim. Set $A := \mathscr{O}_{X,\xi}$; notice that $(X(\xi), \underline{M}(\xi))_{\mathrm{tr}}$ is the complement of the union of the finitely many closed subsets of the form $Z_\mathfrak{p} := \mathrm{Spec}\, A/\mathfrak{p}A$, where $\mathfrak{p} \subset \underline{M}_\xi$ runs over the prime ideals of height one. Due to corollary 6.5.36(i), each $Z_\mathfrak{p}$ is an irreducible divisor in $\mathrm{Spec}\, A$. Now, let $s \in (j_*\mathscr{O}_U^\times)_\xi$; then the divisor of s is of the form $\sum_{\mathrm{ht}\mathfrak{p}=1} n_\mathfrak{p}[\mathfrak{p}]$, for some $n_\mathfrak{p} \in \mathbb{Z}$. By lemmata 6.1.16(ii) and 6.2.21(i), $\underline{M}_\xi^\sharp$ is a fine and saturated monoid; then lemma 3.4.22 and proposition 3.4.32 say that the fractional ideal $I := \bigcap_{\mathrm{ht}\,\mathfrak{p}=1} \mathfrak{m}_\mathfrak{p}^{n_\mathfrak{p}}$ of $\underline{M}_\xi$ is reflexive, and therefore the fractional ideal IA of A is reflexive as well (lemma 3.4.27(ii)). Since A is normal, by considering the localizations of $I\mathscr{O}_{X,\xi}$ at the prime ideals of height one of A, we deduce easily that $IA = sA$ ([14, Ch.VII, §4, n.2, Cor. du Th.2]). Thus, we may write $s = \sum_{i=1}^n \beta_\xi^{\mathrm{gp}}(x_i)a_i$ for certain $a_i \in A$ and $x_i \in I$ (for $i = 1, \dots, n$). Then we must have $\beta_\xi^{\mathrm{gp}}(x_i)a_i \notin s \cdot \mathfrak{m}_\xi$ for at least one index $i \leq n$ (where $\mathfrak{m}_\xi \subset A$ is the maximal ideal); for such i, it follows that $s^{-1} \cdot \beta_\xi^{\mathrm{gp}}(x_i) \in A^\times$, whence $s \in \underline{M}_\xi^{\mathrm{gp}}$, as required. ◊

Now, let $s \in (j_*\mathscr{O}_U^\times)_\xi \cap A$; by claim 6.5.53, we may find $x \in \underline{M}_\xi^{\mathrm{gp}}$ such that $\beta_\xi^{\mathrm{gp}}(x) = s$. To conclude, it suffices to show that $x \in \underline{M}_\xi$. By theorem 3.4.16(i), we are reduced to showing that $x \in (\underline{M}_\xi)_\mathfrak{p}$ for every prime ideal $\mathfrak{p} \subset \underline{M}_\xi$ of height one. Set $\mathfrak{q} := \mathfrak{p}A$; then $\mathfrak{q}$ is a prime ideal of height one (corollary 6.5.36(i)), and β_ξ extends to a well defined map of monoids $\beta_\mathfrak{p} : (\underline{M}_\xi)_\mathfrak{p} \to A_\mathfrak{q}$. Furthermore, $A_\mathfrak{q}$ is a discrete valuation ring (corollary 6.5.29), whose valuation we denote $v : A_\mathfrak{q} \setminus \{0\} \to \mathbb{N}$. In view of theorem 3.4.16(ii), we see that $x \in (\underline{M}_\xi)_\mathfrak{p}$ if and only if $(v \circ \beta_\mathfrak{p})^{\mathrm{gp}}(x) \geq 0$. However, clearly $v(s) \geq 0$, whence the contention. □

Remark 6.5.54. Let $\mathbf{reg.log}_\tau$ denote the full subcategory of $\mathbf{log}_\tau$ whose objects are the regular log schemes. As an immediate consequence of proposition 6.5.52 (and of remark 6.2.8(i)) we see that the forgetful functor F of (6.2.1) restricts to a fully faithful functor

$$\mathbf{reg.log}_\tau \to \mathbf{Open} \qquad (X, \underline{M}) \mapsto ((X, \underline{M})_{\mathrm{tr}} \to X)$$

where $\mathbf{Open}$ is full subcategory of $\mathsf{Morph}(\mathbf{Sch})$ whose objects are the open immersions.

6.6. **Resolution of singularities of regular log schemes.** Most of this section concerns results that are special to the class of log schemes over the Zariski topology; these are then applied to étale fs log structures, after we have shown that every such log structure admits a logarithmic blow up which descends to the Zariski topology (proposition 6.6.52). The same statement – with an unnecessary restriction to regular log schemes – can be found in the article [64] by W.Niziol : see theorem 5.6 of *loc.cit.*. We mainly follow her treatment, except for fleshing out some details, and correcting some inaccuracies.

Therefore, *we let here* $\tau = \mathrm{Zar}$*, and all log structures considered in this section until* (6.6.48) *are defined on the Zariski sites of their underlying schemes*.

6.6.1. Let $f : (Y, \underline{N}) \to (X, \underline{M})$ be any morphism of log schemes; we remark that f is a morphism of monoidal spaces, *i.e.* for every $y \in Y$, the map $\underline{M}_{f(y)} \to \underline{N}_y$ induced by $\log f$, is local. Indeed, it has been remarked in (6.1.6) that the natural map $\underline{M}_{f(y)} \to (f^*\underline{M})_y$ is local, and on the other hand, a section $s \in (f^*\underline{M})_y$ is invertible if and only if its image in $\mathscr{O}_{Y,y}$ is invertible, if and only if $\log f_y(s)$ is invertible in $\underline{N}_y$, whence the contention. We shall denote

$$f^\sharp : (Y, \underline{N})^\sharp \to (X, \underline{M})^\sharp$$

the morphism of sharp monoidal spaces induced by f in the obvious way (*i.e.* the underlying continuous map is the same as the continuous map underlying f, and $\log f^\sharp : f^*\underline{M}^\sharp \to \underline{N}^\sharp$ is $(\log f)^\sharp$: see definition 3.5.1).

6.6.2. We let $\mathscr{K}$ be the category whose objects are all data of the form $\underline{X} := ((X, \underline{M}), F, \psi)$, where $(X, \underline{M})$ is a log scheme, F is a fan, and $\psi : (X, \underline{M})^\sharp \to F$ is a morphism of sharp monoidal spaces, such that $\log \psi : \psi^*\mathscr{O}_F \to \underline{M}^\sharp$ is an isomorphism. The morphisms :

$$((Y, \underline{N}), F', \psi') \to \underline{X}$$

in $\mathscr{K}$ are all the pairs (f, φ), where $f : (Y, \underline{N}) \to (X, \underline{M})$ is a morphism of log schemes, and $\varphi : F' \to F$ is a morphism of fans, such that the diagram

$$\begin{tikzcd} (Y, \underline{N})^\sharp \arrow[r, "f^\sharp"] \arrow[d, "\psi'"'] & (X, \underline{M})^\sharp \arrow[d, "\psi"] \\ F' \arrow[r, "\varphi"] & F \end{tikzcd}$$

commutes. Especially, notice that lemma 6.1.4 implies the identity :

$$\mathrm{Str}(f) = \psi'^{-1}\mathrm{Str}(\varphi). \tag{6.6.3}$$

(notation of definitions 3.5.1(ii) and 6.2.7(ii)). We shall say that an object $\underline{X}$ is *locally noetherian*, if the same holds for the scheme X. Likewise, a morphism (f, φ) in $\mathscr{K}$ is *quasi-compact*, (resp. *quasi-separated*, resp. *separated*, resp. *locally of finite type*, resp. *of finite type*) if the same holds for the morphism of schemes underlying f. We say that f is *étale*, if the same holds for the morphism of log schemes underlying f. Also, the *trivial locus* of $\underline{X}$ is defined as the subset $(X, \underline{M})_{\mathrm{tr}}$ of X.

6.6.4. There is an obvious (forgetful) functor :

$$F : \mathscr{K} \to \mathbf{Sch} \qquad ((X, \underline{M}), F, \psi) \mapsto X$$

which is a fibration. More precisely, every morphism of schemes $S' \to S$ induces a *base change* functor (notation of (1.1.16)) :

$$F\mathscr{K}/S \to F\mathscr{K}/S' \qquad \underline{X} \mapsto S' \times_S \underline{X}$$

unique up to natural isomorphism of functors. Namely, for $\underline{X} := ((X, \underline{M}), F, \psi)$ one lets

$$S' \times_S \underline{X} := (S' \times_S (X, \underline{M}), F, \psi')$$

where $\psi' := \psi \circ \pi^\sharp$, and $\pi : S' \times_S (X, \underline{M}) \to (X, \underline{M})$ is the natural projection.

Example 6.6.5. (i) Let P be a monoid, R a ring, and set $(S, P_S^{\log}) := \mathrm{Spec}(R, P)$ (see (6.2.13)). The unit of adjunction $\varepsilon_P : P \to R[P]$ determines a unique morphism of sharp monoidal spaces

$$\psi_P : \mathrm{Spec}(R, P)^\sharp \to T_P := (\mathrm{Spec}\, P)^\sharp$$

(proposition 3.5.6), and we claim that $\underline{S} := (\mathrm{Spec}(R, P), T_P, \psi_P)$ is an object of $\mathscr{K}$.

The assertion can be checked on the stalks, hence let ξ be a point of S; then ξ corresponds to a prime ideal $\mathfrak{p}_\xi \subset R[P]$, and by inspecting the definitions, we see that $\psi_P(\xi) = \varepsilon_P^{-1}(\mathfrak{p}_\xi) = \mathfrak{q}_\xi := P \cap \mathfrak{p}_\xi \in \mathrm{Spec}\, P$. Again, a direct inspection shows that the morphism

$$(\log \psi_P)_\xi : \mathscr{O}_{T_P, \mathfrak{q}_\xi} \to (P_S^{\log})^\sharp_\xi$$

is none else than the natural isomorphism $P^\sharp_{\mathfrak{q}_\xi} \xrightarrow{\sim} P^{\log}_{S,\xi}/\mathscr{O}^\times_{S,\xi}$ deduced from ε_P and the natural identifications

$$P^{\log}_{S,\xi} = \mathscr{O}^\times_{S,\xi} \otimes_{P \setminus \mathfrak{q}_\xi} P = \mathscr{O}^\times_{S,\xi} \otimes_{P^\times_{\mathfrak{q}_\xi}} P_{\mathfrak{q}_\xi}.$$

(ii) The construction of $\underline{S}$ is clearly functorial in P. Namely, say that $\lambda : P \to Q$ is a morphism of monoids, and set $S' := \mathrm{Spec}\, R[Q]$, $T_Q := (\mathrm{Spec}\, Q)^\sharp$. There follows a morphism

$$\underline{S}' := (\mathrm{Spec}(R, Q), T_Q, \psi_Q) \to \underline{S} \tag{6.6.6}$$

in $\mathscr{K}$, whose underlying morphism of log schemes is $\mathrm{Spec}(R, \lambda)$, and whose underlying morphism of fans is just $(\mathrm{Spec}\, \lambda)^\sharp$.

6.6.7. Example 6.6.5 can be globalized to more general log schemes, at least under some additional assumptions. Namely, let $(X, \underline{M})$ be a regular log scheme. For every point x of X, let $\mathfrak{m}_x \subset \mathscr{O}_{X,x}$ be the maximal ideal; we set :

$$F(X) := \{x \in X \mid I(x, \underline{M}) = \mathfrak{m}_x\}$$

(notation of (6.5.21)), and we endow $F(X)$ with the topology induced from X. The *fan of* $(X, \underline{M})$ is the sharp monoidal space

$$F(X, \underline{M}) := (F(X), \underline{M}_{|F(X)})^\sharp.$$

We wish to show that $F(X, \underline{M})$ is indeed a fan. Let $U \subset X$ be any open subset; to begin with, it is clear that $F(U, \underline{M}_{|U})$ is naturally an open subset of $F(X, \underline{M})$; hence the contention is local on X, so we may assume that X is affine, say $X = \mathrm{Spec}\, A$ for a noetherian ring A, and that $\underline{M}$ admits a finite chart $P_X \to \underline{M}$; denote by $\beta : P \to A$ the induced morphism of monoids. Let now ξ be a point of X, and $\mathfrak{p}_\xi \subset A$ the corresponding prime ideal; set $\mathfrak{q}_\xi := \beta^{-1}\mathfrak{p}_\xi \in \mathrm{Spec}\, P$. By inspecting the definitions, it is easily seen that $I(\xi, \underline{M}) = \mathfrak{q}_\xi A_{\mathfrak{p}_\xi}$. Therefore, for any prime ideal $\mathfrak{q} \subset P$, let $V(\mathfrak{q})_{\max}$ be the finite set consisting of all the maximal points of the closed subset $V(\mathfrak{q}) := \mathrm{Spec}\, A/\mathfrak{q}A$ (*i.e.* the minimal prime ideals of $A/\mathfrak{q}A$); in light of corollary 6.5.36(i), it follows easily that

$$F(X) = \bigcup_{\mathfrak{q} \in \mathrm{Spec}\, P} V(\mathfrak{q})_{\max} \tag{6.6.8}$$

especially, $F(X)$ is a finite set (lemma 3.1.20(iii)). Let $t \in F(X)$ be any element, and denote by $U(t) \subset F(X)$ the subset of all generizations of t in $F(X)$; as a corollary, we see that $U(t)$ is an open subset of $F(X)$. Moreover, (6.6.8) also implies that :

$$U(t) = \bigcup_{\mathfrak{q} \in \mathrm{Spec}\, P} (\mathrm{Spec}\, \mathscr{O}_{X,t}/\mathfrak{q}\mathscr{O}_{X,t})_{\max}.$$

However, corollary 6.5.36(i) says that $\mathrm{Spec}\, \mathscr{O}_{X,t}/\mathfrak{q}\mathscr{O}_{X,t}$ is irreducible for every $\mathfrak{q} \in \mathrm{Spec}\, P$, therefore the set $U(t)$ is naturally identified with a subset of $\mathrm{Spec}\, P$. Furthermore, arguing as in example 6.6.5(i) we find a natural isomorphism $P^\sharp_{\mathfrak{q}_t} \xrightarrow{\sim} \mathscr{O}_{F(X,\underline{M}),t}$. Moreover, if $t' \in U(t)$ is any other point, the specialization map $\mathscr{O}_{F(X,\underline{M}),t} \to \mathscr{O}_{F(X,\underline{M}),t'}$ corresponds – under the above isomorphism – to the natural morphism $P^\sharp_{\mathfrak{q}_t} \to P^\sharp_{\mathfrak{q}'_t}$ induced by the localization map $P_{\mathfrak{q}_t} \to P_{\mathfrak{q}'_t}$. This shows that the open monoidal subspace $(U(t), \underline{M}_{|U(t)})$ is naturally isomorphic to $(\mathrm{Spec}\, P_{\mathfrak{q}_t})^\sharp$, hence $F(X, \underline{M})$ is a fan, as stated.

Remark 6.6.9. (i) The discussion in (6.6.7) shows more precisely that, if :

(a) $(U, \underline{M})$ is a log scheme with $U = \mathrm{Spec}\, A$ affine,
(b) there exists a morphism $\beta : P \to A$ from a finitely generated monoid P, inducing the log structure $\underline{M}$ on U, and
(c) $\beta(\mathfrak{q})A$ is a prime ideal for every $\mathfrak{q} \in \mathrm{Spec}\, P$

then $F(U, \underline{M})$ is naturally identified with $(\mathrm{Spec}\, P)^\sharp$. In this situation, denote by

$$f_\beta : (U, \underline{M})^\sharp \to T_P := (\mathrm{Spec}\, P)^\sharp$$

the morphism of sharp monoidal spaces deduced from β (proposition 3.5.6); by inspecting the definitions, we see that the associated map $\log f_\beta : f^*_\beta \mathscr{O}_{T_P} \to \underline{M}^\sharp$ is an isomorphism. Via the foregoing natural identification, there results a morphism $\pi_U : (U, \underline{M})^\sharp \to F(U, \underline{M})$ which can be described without reference to P. Indeed, let $x \in U$ be any point, and $\mathfrak{p} \in \mathrm{Spec}\, A$ the corresponding prime ideal; by inspecting the definitions we find that

$$\pi_U(x) = I(x, \underline{M}) \ \text{ which is the largest prime ideal in } \mathrm{Spec}\, A_{\mathfrak{p}} \cap F(U, \underline{M}).$$

Especially, $\pi_U(x)$ is a generization of x, and the inverse of $(\log \pi_U)_x : \mathscr{O}_{F(U,\underline{M}),\pi_U(x)} \xrightarrow{\sim} \underline{M}^\sharp_x$ is induced by the specialization map $\underline{M}_x \to \underline{M}_{\pi_U(x)}$ (which induces an isomorphism on the associated sharp quotient monoids).

(ii) Finally, it follows easily from corollary 6.5.36(i) and [32, Ch.IV, Cor.8.4.3], that every regular log scheme $(X, \underline{M})$ admits an affine open covering $X = \bigcup_{i \in I} U_i$, such that each $(U_i, \underline{M}_{|U_i})$ fulfills conditions (a)–(c) above, and the intrinsic description in (i) shows that the morphisms π_{U_i} glue to a well defined morphism $\pi_X : (X, \underline{M})^\sharp \to F(X, \underline{M})$ of sharp monoidal spaces, such that the datum

$$\mathscr{K}(X, \underline{M}) := ((X, \underline{M}), F(X, \underline{M}), \pi_X)$$

is an object of $\mathscr{K}$.

(iii) Notice that $\pi^{-1}_X(t)$ is an irreducible locally closed subset of X of codimension equal to the height of t, for every $t \in F(X, \underline{M})$; also $\pi^{-1}_X(t)$ is a regular scheme, for its reduced subscheme structure. Indeed, we have already observed that t is the unique maximal point of $\pi^{-1}_X(t)$, and then the assertion follows immediately from corollary 6.5.51(ii,iii). Furthermore, the inclusion map $j : F(X, \underline{M}) \to X$ is a continuous section of π_X, and notice that $j \circ \pi_X(U) = j^{-1}U$ for every open subset $U \subset X$, since j maps every $t \in F(X, \underline{M})$ to the unique maximal point of $\pi^{-1}_X(t)$. It follows easily that π_X is an open map.

6.6.10. Denote by $\mathscr{K}_{\mathrm{int}}$ the full subcategory of $\mathscr{K}$ whose objects are the data $((X, \underline{M}), F, \psi)$ such that $\underline{M}$ is an integral log structure, and F is an integral fan. There is an obvious functor

$$\mathscr{K}_{\mathrm{int}} \to \mathbf{int.Fan} \quad : \quad ((X, \underline{M}), F, \psi) \mapsto F$$

to the category of integral fans, which shall be used to construct useful morphisms of log schemes, starting from given morphisms of fans. This technique rests on the following three results :

Lemma 6.6.11. *Let $((X, \underline{M}), F, \psi)$ be an object of $\mathscr{K}_{\mathrm{int}}$, with F locally fine. Then, for each point $x \in X$ there exists an open neighborhood $U \subset X$ of x, a fine chart $Q_U \to \underline{M}_{|U}$ (for some fine monoid Q depending on x), and an isomorphism of monoids $Q^\sharp \xrightarrow{\sim} \mathscr{O}_{F,\psi(x)}$.*

Proof. The assertion is local on X, hence we may assume that $F = (\mathrm{Spec}\, P)^\sharp$ for a fine monoid P. In this case, ψ is determined by the corresponding map

$$\overline{\beta} : P_X \to \underline{M}^\sharp.$$

Indeed, for any $x \in X$, the point $\psi(x)$ is the prime ideal $\overline{\beta}_x^{-1}(\mathfrak{m}_x) \subset P$, where $\mathfrak{m}_x \subset \underline{M}_x^\sharp$ is the maximal ideal; moreover :

$$\mathscr{O}_{F,\psi(x)} = P/\overline{S}_x \qquad \text{where} \qquad \overline{S}_x := \overline{\beta}_x^{-1}(1)$$

and – under this identification – the isomorphism $\log \psi_x : \mathscr{O}_{F,\psi(x)} \stackrel{\sim}{\to} \underline{M}_x^\sharp$ is deduced from $\overline{\beta}_x$ in the obvious way.

Now, let $x \in X$ be any point; after replacing X by the open subset $\psi^{-1}U(\psi(x))$, we may assume that $P = \mathscr{O}_{F,\psi(x)}$ (notation of (3.5.16)), and by assumption $\log \psi_x : P \to \underline{M}_x^\sharp$ is an isomorphism. Pick a surjection $\alpha : \mathbb{Z}^{\oplus r} \to P^{\mathrm{gp}}$, and let Q be the pull-back in the cartesian diagram :

$$\begin{tikzcd} Q \arrow[r] \arrow[d] & \mathbb{Z}^{\oplus r} \arrow[d, "(\log \psi_x)^{\mathrm{gp}} \circ \alpha"] \\ \underline{M}_x^\sharp \arrow[r] & \underline{M}_x^{\mathrm{gp}}/\underline{M}_x^\times . \end{tikzcd}$$

By construction, α restricts to a morphism of monoids $\vartheta : Q \to P$, inducing an isomorphism $Q^\sharp \stackrel{\sim}{\to} P$, whence an isomorphisms of fans $(\mathrm{Spec}\, P)^\sharp \stackrel{\sim}{\to} (\mathrm{Spec}\, Q)^\sharp$, and Q is fine, by corollary 3.4.2. Moreover, the choice of a lifting $\mathbb{Z}^{\oplus r} \to \underline{M}_x^{\mathrm{gp}}$ of $(\log \psi_x)^{\mathrm{gp}} \circ \alpha$ determines a morphism

$$\beta_x : Q \to \underline{M}_x^\sharp \times_{\underline{M}_x^{\mathrm{gp}}/\underline{M}_x^\times} \underline{M}_x^{\mathrm{gp}} = \underline{M}_x$$

which lifts $\log \psi_x \circ \vartheta$ (here it is needed that $\underline{M}$ is integral). Next, after replacing X by an open neighborhood of x, we may assume that β_x extends to a map of pre-log structures $\beta : Q_X \to \underline{M}$ lifting $\overline{\beta}$ (lemma 6.1.16(iv.b),(v)). It remains only to show that β is a chart for $\underline{M}$, which can be checked on the stalks. Thus, let $y \in X$ be any point, and set $S_y := \beta_y^{-1}\underline{M}_y^\times$; with the foregoing notation, the stalk $Q_{X,y}^{\log}$ of the induced log structure is naturally isomorphic to $(S_y^{-1}Q \times \mathscr{O}_{X,y}^\times)/S_y^{\mathrm{gp}}$, therefore

$$(Q_{X,y}^{\log})^\sharp \simeq Q/S_y \simeq P/\overline{S}_y$$

and the induced map $P/\overline{S}_y \to \underline{M}_y^\sharp$ is again deduced from $\overline{\beta}_y$, so it is an isomorphism; then the same holds for $\beta_y^{\log} : Q_{X,y}^{\log} \to \underline{M}_y$ (lemma 6.1.4). □

Lemma 6.6.12. *Let $\mu : P \to P'$ be a morphism of integral monoids such that μ^{gp} is surjective, $\varphi : (\mathrm{Spec}\, P')^\sharp \to (\mathrm{Spec}\, P)^\sharp$ the induced morphism of affine fans, and denote by*

$$\lambda : P \to Q := P^{\mathrm{gp}} \times_{P'^{\mathrm{gp}}} P'$$

the map of monoids determined by μ and the unit of adjunction $P \to P^{\mathrm{gp}}$. Then :

- (i) *The natural projection $Q \to P'$ induces an isomorphism $\omega : (\mathrm{Spec}\, P')^\sharp \to (\mathrm{Spec}\, Q)^\sharp$ of affine fans, such that $(\mathrm{Spec}\, \lambda)^\sharp \circ \omega = \varphi$.*
- (ii) *The induced morphism* (6.6.6) *in $\mathscr{K}_{\mathrm{int}}$ (with $R := \mathbb{Z}$) is* $\mathbf{int}.\mathbf{Fan}$*-cartesian.*
- (iii) *If P and P' are fine, the morphism* (6.6.6) *is étale.*

Proof. (See (1.4) for generalities concerning inverse images and cartesian morphisms relative to a functor.) Notice first that, since μ^{gp} is surjective, the projection $Q \to P'$ induces an isomorphism $Q^\sharp \stackrel{\sim}{\to} P'^\sharp$, whence (i).

(ii): Notice that the log structure of $\mathrm{Spec}(\mathbb{Z}, Q)$ is integral, by lemma 6.1.16(iii). Next, set $F' := (\mathrm{Spec}\, P')^\sharp$, define $\underline{S}$, $\underline{S}'$ as in example 6.6.5(ii) (with $R := \mathbb{Z}$), let $g : ((Y, \underline{N}), F'', \psi_Y) \to \underline{S}$ be any morphism of $\mathscr{K}_{\mathrm{int}}$, and $\varphi' : F'' \to F'$ a morphism of integral fans, such that the image of g in $\mathbf{int}.\mathbf{Fan}$ equals $\varphi \circ \varphi'$; we must show that g factors through a unique morphism $h : ((Y, \underline{N}), F'', \psi_Y) \to \underline{S}'$, whose image in $\mathbf{int}.\mathbf{Fan}$ equals φ'. As usual, we may reduce to the case where $Y = \mathrm{Spec}\, A$ is affine, $F'' = \mathrm{Spec}\, P''$ is an affine fan for a sharp integral monoid P'', and ψ_Y is given by a map of sheaves $P''_Y \to \underline{N}^\sharp$. In such situation, the morphism $(Y, \underline{N}) \to \mathrm{Spec}(\mathbb{Z}, P)$ underlying g is determined by a morphism of monoids $P \to \underline{N}(Y)$, or which is the same, a map of sheaves $\gamma_Y : P_Y \to \underline{N}$; likewise, φ' is given by a morphism of monoids $P' \to P''$, and composing with ψ_Y, we get a map of sheaves $\alpha_Y : P'_Y \to \underline{N}^\sharp$. Finally, the condition that g lies over $\varphi \circ \varphi'$ translates as the commutativity of the following diagram of sheaves :

$$\begin{tikzcd} P_Y^{\mathrm{gp}} \arrow[r] \arrow[d, "\gamma_Y^{\mathrm{gp}}"'] & P_Y'^{\mathrm{gp}} \arrow[d] & P'_Y \arrow[l] \arrow[d, "\alpha_Y"] \\ \underline{N}^{\mathrm{gp}} \arrow[r] & \underline{N}^{\mathrm{gp}}/\underline{N}^\times & \underline{N}^\sharp. \arrow[l] \end{tikzcd}$$

There follows a unique morphism of sheaves :

$$Q_Y \to \underline{N}^{\mathrm{gp}} \times_{\underline{N}^{\mathrm{gp}}/\underline{N}^\times} \underline{N}^\sharp = \underline{N} \tag{6.6.13}$$

(here it is needed that $\underline{N}$ is integral) such that the diagram :

$$\begin{tikzcd} P_Y \arrow[r] \arrow[d, "\gamma_Y"'] & Q_Y \arrow[r] \arrow[d] & P'_Y \arrow[d, "\alpha_Y"] \\ \underline{N} \arrow[r, equal] & \underline{N} \arrow[r] & \underline{N}^\sharp \end{tikzcd}$$

commutes. Then (6.6.13) determines a morphism of log schemes $h_Y : (Y, \underline{N}) \to \mathrm{Spec}(\mathbb{Z}, Q)$, such that the pair (h_Y, φ') is the unique morphism h in $\mathscr{K}_{\mathrm{int}}$ with the sought properties.

(iii): If both P and P' are fine, so is Q (corollary 3.4.2), and by construction, λ^{gp} is an isomorphism. Then the assertion follows from theorem 6.3.37. □

Proposition 6.6.14. *Let $\underline{X} := ((X, \underline{M}), F, \psi)$ be an object of $\mathscr{K}_{\mathrm{int}}$. Let also $\varphi : F' \to F$ be an integral partial subdivision, with F locally fine and F' integral. We have :*

(i) *$(X, \underline{M})$ is a fine log scheme.*
(ii) *If F is saturated, $(X, \underline{M})$ is a fs log scheme.*
(iii) *$\underline{X}$ admits an inverse image over φ (relative to the functor of* (6.6.10)*).*
(iv) *If φ is finite, then the cartesian morphism $(f, \varphi) : \varphi^*\underline{X} \to \underline{X}$, is quasi-compact.*
(v) *If F' is locally fine, the morphism (f, φ) is quasi-separated and étale.*

Proof. (i): This is just a restatement of lemma 6.6.11.

(ii): In light of (i), we only have to show that $\underline{M}_x$ is a saturated monoid, for every $x \in X$. Since by assumption $\underline{M}_x^\sharp$ is saturated, the assertion follows from lemma 3.2.9(ii).

Claim 6.6.15. In order to show (iii)–(v), we may assume that :

(a) $F = (\mathrm{Spec}\, P)^\sharp$ for a fine monoid P, and X is an affine scheme.
(b) The map of global sections $P \to \Gamma(X, \underline{M}^\sharp)$ determined by ψ, comes from a morphism of pre-log structures $\beta : P_X \to \underline{M}$ which is a fine chart for $\underline{M}$.

Proof of the claim. To begin with, suppose that $((X', \underline{M}'), F', \psi')$ is the sought preimage of $\underline{X}$; let $U \subset X$ be any open subset, and $V \subset F$ an open subset such that $\psi(U) \subset V$. Then it is easily seen that the object

$$\varphi^*\underline{X} \times_{\underline{X}} (U, V) := ((f^{-1}U, \underline{M}'_{|f^{-1}U}), \varphi^{-1}V, \psi'_{|f^{-1}U})$$

is a preimage of $((U, \underline{M}_{|U}), V, \psi_{|U})$ over the restriction $\varphi^{-1}V \to V$ of φ. Now, suppose that we have found an affine open covering $X = \bigcup_{i \in I} U_i$, and for every $i \in I$ an affine open subset

$V_i \subset F$ with $\psi(U_i) \subset V_i$, such that the object $\underline{U}_i := ((U_i, \underline{M}_{|U_i}), V_i, \psi_{|U_i})$ admits a preimage over the restriction $\varphi_i : \varphi^{-1}V_i \to V_i$ of φ; then the foregoing implies that there are natural isomorphisms:

$$\varphi_i^* \underline{U}_i \times_{\underline{U}_i} (U_{ij}, V_{ij}) \xrightarrow{\sim} \varphi_j^* \underline{U}_j \times_{\underline{U}_j} (U_{ij}, V_{ij})$$

for every $i, j \in I$, where $U_{ij} := U_i \cap U_j$ and $V_{ij} := V_i \cap V_j$. Thus, we may glue all these inverse images along these isomorphisms, to obtain the sought inverse image of $\underline{X}$.

Moreover, if φ is finite, the same will hold for the restrictions φ_i, and if each cartesian morphism $\varphi_i^* \underline{U}_i \to \underline{U}_i$ is quasi-compact, the same will hold also for (f, φ) ([25, Ch.I, §6.1]). Likewise, if each morphism $\varphi_i^* \underline{U}_i \to \underline{U}_i$ is étale and quasi-separated, then the same will hold for (f, φ) ([25, Ch.I, Prop.6.1.11] and proposition 6.3.24(iii)).

Therefore, we may replace X by any U_i, and F by the corresponding V_i, which reduces the proof of (iii)–(v) to the case where condition (a) is fulfilled. Lastly, in light of lemma 6.6.11 we may suppose that the open subsets U_i are small enough, so that also condition (b) is fulfilled. $\diamondsuit$

In view of claim 6.6.15, we shall assume henceforth that conditions (a) and (b) are fulfilled. Let $\underline{S} := (\mathrm{Spec}(\mathbb{Z}, P), T_P, \psi_P)$ be the object of $\mathscr{K}_{\mathrm{int}}$ considered in example 6.6.5(i) (with $R := \mathbb{Z}$); in this situation, β determines a morphism of schemes $f_\beta : X \to S$, and in view of lemma 6.2.14 we have $\underline{X} \simeq X \times_S \underline{S}$ (notation of (6.6.4)). Therefore, if we find an inverse image $\underline{S}'$ for $\underline{S}$ over φ, the object $X \times_S \underline{S}'$ will provide the sought inverse image of $\underline{X}$. Thus, in order to show (iii)–(v), we may further reduce to the case where $\underline{X} = \underline{S}$ ([25, Ch.I, Prop.6.1.5(iii), Prop.6.1.9(iii)] and proposition 6.3.24(ii)).

Claim 6.6.16. In order to prove (iii)–(v), we may assume that F' is affine.

Proof of the claim. Indeed, say that $F' = \bigcup_{i \in I} V_i$ is an open covering, and let $\varphi_i : V_i \to T_P$ be the restriction of φ; suppose that we have found, for each $i \in I$, an inverse image $\varphi_i^* \underline{S}$ of $\underline{S}$ over φ_i; again, it follows easily that an inverse image for $\underline{S}$ over φ can be constructed by gluing the objects $\varphi_i^* \underline{S}$. This already implies that, in order to show (iii), we may assume that F' is affine.

Now, suppose that φ is finite, so we may find an open covering as above, such that furthermore each V_i is affine, and I is a finite set. Suppose that each of the corresponding morphisms $\varphi_i^* \underline{S} \to \underline{S}$ is quasi-compact; by the foregoing, the schemes underlying the objects $\varphi_i^* \underline{S}$ give a finite open covering of the scheme underlying $\varphi^* \underline{S}$, and then it is clear that (iv) holds.

Next, suppose furthermore that each morphism $\varphi_i^* \underline{S} \to \underline{S}$ is étale. It follows that the morphism $\varphi^* \underline{S} \to \underline{S}$ is also étale (proposition 6.3.24(ii)); then, since S is noetherian, the scheme underlying $\varphi^* \underline{S}$ is locally noetherian ([25, Ch.I, Prop.6.2.2]), and therefore the morphism $\varphi^* \underline{S} \to \underline{S}$ is quasi-separated ([25, Ch.I, Cor.6.1.13]). $\diamondsuit$

In view of claim 6.6.16, we shall assume that $F' = (\mathrm{Spec}\, P')^\sharp$ is affine as well, for a sharp and integral P', and that φ is given by a morphism $\lambda : P \to P'$ inducing a surjection on the associated groups (details left to the reader). Then assertions (iii) and (iv) are now straightforward consequences of lemma 6.6.12(i,ii), and (v) follows from lemma 6.6.12(iii), after one remarks that, when F' is fine, one may choose for P' a fine monoid. $\square$

Remark 6.6.17. Keep the assumptions of proposition 6.6.14, and let $t \in F_0$ a point of F of height zero (see (3.5.16)). Notice that the inclusion map $j_t : \{t\} \to F$ is an open immersion, hence the fibre $X_t := \psi^{-1}(t) \subset X$ is open; indeed, it is clear from the definitions, that $\psi^{-1}(F_0)$ is precisely the trivial locus of $\underline{X}$. Moreover, let $t' \in \varphi^{-1}(t)$ since the group homomorphism $\mathscr{O}^{\mathrm{gp}}_{F,t} \to \mathscr{O}^{\mathrm{gp}}_{F',t'}$ is surjective, we see that t' is of height zero in F', and φ restricts to an isomorphism of fans $(\{t'\}, \mathscr{O}_{F,t'}) \xrightarrow{\sim} (\{t\}, \mathscr{O}_{F,t})$. Set $\underline{X}_t := j_t^* \underline{X}$ (whose underlying scheme is X_t), and define likewise $\varphi^* \underline{X}_{t'} := (\varphi \circ j_{t'})^* \underline{X}$ (whose underlying scheme is an open subset of the trivial locus of $\varphi^* \underline{X}$). We deduce that :

- The trivial locus of $\varphi^* \underline{X}$ is the preimage of $(X, \underline{M})_{\mathrm{tr}}$.

- For every $t' \in F'_0$, the restriction of $(f, \varphi) : \varphi^*\underline{X}_{t'} \to \underline{X}_t$ is an isomorphism.

Corollary 6.6.18. *In the situation of proposition* 6.6.14*, let* $((X', \underline{M}'), F', \psi') := \varphi^*\underline{X}$*. The following holds :*

(i) *If* $F' = F^{\mathrm{sat}}$*, and* $\varphi : F^{\mathrm{sat}} \to F$ *is the counit of adjunction, then* $(X', \underline{M}') = (X, \underline{M})^{\mathrm{fs}}$*, and* f *is the counit of adjunction.*

(ii) *If* φ *is the blow up of a coherent ideal* $\mathscr{I}$ *of* $\mathscr{O}_F$*, then* f *is the blow up of* $\mathscr{I}\underline{M}$*, the unique ideal of* $\underline{M}$ *whose image in* $\underline{M}^\sharp$ *equals* $\psi^{-1}\mathscr{I}$*.*

(iii) *Suppose moreover, that* $(X, \underline{M})$ *is regular,* F' *is saturated , and* $\underline{X} = \mathscr{K}(X, \underline{M})$ *(notation of remark* 6.6.9(iii)*). Then* $(X', \underline{M}')$ *is regular, and* $\varphi^*\underline{X}$ *is isomorphic to* $\mathscr{K}(X', \underline{M}')$.

Proof. To start with, we remark that the assertions are local on X. Indeed, this is clear for (i), and for (iii) it follows easily from remark 6.6.9(i,,ii); concerning (ii), suppose that $X = \bigcup_{i\in I} U_i$ is an open covering, such that $\varphi^*(U_i, \underline{M}_{|U_i})$ is the blow up of the ideal $\mathscr{I}\underline{M}_{|U_i}$, for every $i \in I$. For every $i, j \in I$ set $U_{ij} := U_i \cap U_j$; by the universal property of the blow up, there are unique isomorphisms of $(U_{ij}, \underline{M}_{|U_{ij}})$-schemes :

$$U_{ij} \times_{U_i} \varphi^*(U_i, \underline{M}_{|U_i}) \xrightarrow{\sim} U_{ij} \times_{U_j} \varphi^*(U_j, \underline{M}_{|U_j}).$$

Then both $\varphi^*(X, \underline{M})$ and the blow up of $\mathscr{I}\underline{M}$ are necessarily obtained by gluing along these isomorphisms, so they are isomorphic.

Thus, we may assume that $F = (\mathrm{Spec}\, P)^\sharp$ for some fine monoid P, and $(X, \underline{M}) = X \times_S (S, P_S^{\log})$ for some morphism of schemes $X \to S$, where as usual $(S, P_S^{\log}, F, \psi_P)$ is defined as in example 6.6.5(i). In this case, (i) follows from remarks 6.2.36(iii), 3.5.8(ii) and lemma 6.6.12(ii).

Likewise, remark 6.4.62(ii) allows to reduce (ii) to the case where $\underline{X} = (S, P_S^{\log}, F, \psi_P)$, and $\mathscr{I} = I^\sim$ for some ideal $I \subset P$; in which case we conclude by inspecting the explicit description in example 6.4.73.

(iii): From proposition 6.6.14(ii,v) and theorem 6.5.44 we already see that $(X', \underline{M}')$ is regular. Next, we are easily reduced to the case where X is affine, say $X = \mathrm{Spec}\, A$, and $F' = (\mathrm{Spec}\, Q)^\sharp$, for some saturated monoid Q, and by lemma 6.6.12, we may assume that φ is induced by a morphism of monoids $\lambda : P \to Q$ such that λ^{gp} is an isomorphism, and $(X', \underline{M}') = \varphi^*\underline{X} = X \times_S \underline{S}'$, where $\underline{S}' := (\mathrm{Spec}(\mathbb{Z}, Q), F', \psi_Q)$ is defined as in example 6.6.5(ii). Let $\mathfrak{q} \subset Q$ be any prime ideal, and set $\mathfrak{p} := \mathfrak{q} \cap P$; in light of remark 6.6.9(i), it then suffices to show that $Q/\mathfrak{q} \otimes_P A = Q/\mathfrak{q} \otimes_{P/\mathfrak{p}} A/\mathfrak{p}A$ is an integral domain. To this aim, let us remark :

Claim 6.6.19. Let $\lambda : P \to Q$ be a morphism of fine monoids, such that λ^{gp} is an isomorphism, $F \subset Q$ any face, and denote $\lambda_F : F \cap P \to P$ the inclusion map. Then :

(i) The natural map $\mathrm{Coker}\, \lambda_F^{\mathrm{gp}} \to P^{\mathrm{gp}}/(F \cap P)^{\mathrm{gp}}$ is injective.

(ii) If moreover, P is saturated, then $\mathrm{Coker}\, \lambda_F^{\mathrm{gp}}$ is a free abelian group of finite rank.

Proof of the claim. The map of (i) is obtained via the snake lemma, applied to the ladder of abelian groups :

$$\begin{tikzcd} 0 \arrow[r] & (F \cap P)^{\mathrm{gp}} \arrow[r] \arrow[d, "\lambda_F^{\mathrm{gp}}"'] & P^{\mathrm{gp}} \arrow[r] \arrow[d, "\lambda^{\mathrm{gp}}"] & P^{\mathrm{gp}}/(F \cap P)^{\mathrm{gp}} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & F^{\mathrm{gp}} \arrow[r] & Q^{\mathrm{gp}} \arrow[r] & Q^{\mathrm{gp}}/F^{\mathrm{gp}} \arrow[r] & 0. \end{tikzcd}$$

taking into account that both $\mathrm{Ker}\, \lambda^{\mathrm{gp}}$ and $\mathrm{Coker}\, \lambda^{\mathrm{gp}}$ vanish. Then (i) is obvious, and (ii) comes down to checking that $P^{\mathrm{gp}}/(F \cap P)^{\mathrm{gp}}$ is torsion-free, in case P is fine and saturated. But since $F \cap P$ is a face of P, the latter assertion is an easy consequence of proposition 3.4.7. $\Diamond$

Now, set $F := Q \setminus \mathfrak{q}$; we come down to checking that $B := F \otimes_{F\cap P} A/\mathfrak{p}A$ is an integral domain, and remark 6.6.9(i) tells us that $A/\mathfrak{p}A$ is a domain. However, $A/\mathfrak{p}A$ is $(F \cap P)$-flat (proposition 6.5.32 and lemma 6.5.17), hence the natural map

$$B \to C := F^{\mathrm{gp}} \otimes_{(F\cap P)^{\mathrm{gp}}} \mathrm{Frac}(A/\mathfrak{p}A)$$

is injective; on the other hand, claim 6.6.19(ii) implies that $C = A[\mathrm{Coker}\, \lambda_F^{\mathrm{gp}}]$ is a free (polynomial) A-algebra, whence the contention. $\square$

Example 6.6.20. In the situation of example 3.5.27, take $T := \mathrm{Spec}\, P$ for a fine monoid P, and define $\underline{S}$ as in example 6.6.5(i). The k-Frobenius map $\boldsymbol{k}_P$ (example 3.5.10(i)) induces an endomorphism $\boldsymbol{k}_{\underline{S}} := (\mathrm{Spec}(R, \boldsymbol{k}_P), \boldsymbol{k}_T)$ of $\underline{S}$ in $\mathscr{K}$. By proposition 6.6.14(iii) and example 3.5.27, there exists a unique morphism $\underline{g}$ fitting into a commutative diagram of $\mathscr{K}_{\mathrm{int}}$:

$$\begin{tikzcd} \varphi^*\underline{S} \arrow[r] \arrow[d, "\underline{g}"'] & \underline{S} \arrow[d, "\boldsymbol{k}_{\underline{S}}"] \\ \varphi^*\underline{S} \arrow[r] & \underline{S} \end{tikzcd} \tag{6.6.21}$$

(where the horizontal arrows are the cartesian morphisms). Say that $\varphi^*\underline{S} = ((Y, \underline{N}), F, \psi)$; then we have $\underline{g} = (g, \boldsymbol{k}_F)$ for a unique endomorphism $g := (g, \log g)$ of the log scheme $(Y, \underline{N})$. Let $U := \mathrm{Spec}\, P' \subset F$ be any open affine subset ; since $\boldsymbol{k}_F$ is the identity on the underlying topological spaces, g restricts to an endomorphism $g_{|\psi^{-1}U}$ of $\psi^{-1}U \times_Y (Y, \underline{N})$. In view of lemma 6.6.12, the latter log scheme is of the form $\underline{S}'$ as in example 6.6.5(ii), with $Q := P^{\mathrm{gp}} \times_{P'^{\mathrm{gp}}} P'$. Then $g_{|\psi^{-1}U}$ is induced by an endomorphism ν of Q, fitting into a commutative diagram :

$$\begin{tikzcd} P \arrow[r] \arrow[d, "\boldsymbol{k}_P"'] & Q \arrow[d, "\nu"] \\ P \arrow[r] & Q \end{tikzcd}$$

whose horizontal arrows are the natural injections. Since $Q \subset P^{\mathrm{gp}}$, it is clear that $\nu = \boldsymbol{k}_Q$. Especially, $g : Y \to Y$ is a finite morphism of schemes. Furthermore, for every point $y \in Y$, we have a commutative diagram of monoids :

$$\begin{tikzcd} \underline{N}_{g(y)} \arrow[r] \arrow[d, "\log g_y"'] & \underline{N}^\sharp_{g(y)} \arrow[d, "\log g^\sharp_y"'] & \mathscr{O}_{F,\psi(y)} \arrow[l, "\sim"'] \arrow[d, "\boldsymbol{k}_{\psi(y)}:=(\log \boldsymbol{k}_F)_{\psi(y)}"] \\ \underline{N}_y \arrow[r] & \underline{N}^\sharp_y & \mathscr{O}_{F,\psi(y)}. \arrow[l, "\sim"'] \end{tikzcd} \tag{6.6.22}$$

6.6.23. Let $(K, |\cdot|)$ be a valued field, Γ and K^+ respectively the value group and the valuation ring of $|\cdot|$; denote by $1 \in \Gamma$ the neutral element, and by $\Gamma_+ \subset \Gamma$ the submonoid consisting of all elements ≤ 1. Set $S := \mathrm{Spec}\, K^+$; we consider the log scheme $(S, \mathscr{O}_S^*)$, where $\mathscr{O}_S^* \subset \mathscr{O}_S$ is the subsheaf such that $\mathscr{O}_S^*(U) := \mathscr{O}_S(U)\setminus\{0\}$ for every open subset $U \subset S$. To this log scheme we associate the object of $\mathscr{K}_{\mathrm{int}}$:

$$\mathsf{f}(K, |\cdot|) := ((S, \mathscr{O}_S^*), \mathrm{Spec}\, \Gamma_+, \psi_\Gamma)$$

where ψ_Γ is the morphism of monoidal spaces arising from the isomorphism

$$\Gamma_+ \xrightarrow{\sim} (K^+\setminus\{0\})/(K^+)^\times$$

deduced from the valuation $|\cdot|$. It is well known that ψ_Γ is a homeomorphism (see *e.g.* [36, §6.1.26]). We shall use objects of this kind to state some separation and properness criteria for log schemes whose existence is established via proposition 6.6.14. To this aim, we need to digress a little, to prove the following auxiliary results, which are refinements of the standard valuative criteria for morphisms of schemes.

6.6.24. Let $f : X \to Y$ be a morphism of schemes, R an integral ring, and K the field of fractions of R. Let us denote by $X(R)_{\max} \subset X(R)$ the set of morphisms $\operatorname{Spec} R \to X$ which map $\operatorname{Spec} K$ to a maximal point of X. There follows a commutative diagram of sets :

$$
\begin{tikzcd}
X(R)_{\max} \arrow[r] \arrow[d] & X(K)_{\max} \arrow[d] \\
Y(R) \arrow[r] & Y(K).
\end{tikzcd}
\tag{6.6.25}
$$

Proposition 6.6.26. *Let $f : X \to Y$ be a morphism of schemes. The following conditions are equivalent :*

(a) *f is separated.*
(b) *f is quasi-separated, and for every valued field $(K, |\cdot|)$, the map of sets :*

$$X(K^+)_{\max} \to X(K)_{\max} \times_{Y(K)} X(K^+)_{\max} \tag{6.6.27}$$

deduced from diagram (6.6.25) *(with $R := K^+$, the valuation ring of $|\cdot|$), is injective.*

Proof. (a) ⇒ (b) by the valuative criterion of separation ([25, Ch.I, Prop.5.5.4]). Conversely, we shall show that if (b) holds, then the assumptions of the criterion of *loc.cit.* are fulfilled. Indeed, let $(L, |\cdot|_L)$ be any valued field, with valuation ring L^+, and suppose we have two morphisms $\sigma_1, \sigma_2 : \operatorname{Spec} L^+ \to X$ whose restrictions to $\operatorname{Spec} L$ agree, and such that $f \circ \sigma_1 = f \circ \sigma_2$.

Let $s, \eta \in \operatorname{Spec} L^+$ be respectively the closed point and the generic point, set $x := \sigma_1(\eta) = \sigma_2(\eta) \in X$, and denote by $\varphi : \mathscr{O}_{X,x} \to L$ the ring homomorphism corresponding to the restriction of σ_1 (and σ_2); then x admits two specializations $x_i := \sigma_i(s) \in X$ (for $i = 1, 2$) such that φ sends the image $A_i \subset \mathscr{O}_{X,x}$ of the specialization map $\mathscr{O}_{X,x_i} \to \mathscr{O}_{X,x}$, into L^+, and the maximal ideal of A_i into the maximal ideal $\mathfrak{m}_L$ of L^+, for both $i = 1, 2$. Moreover, $y := f(x_1) = f(x_2)$.

Denote by $B \subset \mathscr{O}_{X,x}$ the smallest subring containing A_1 and A_2, and set $\mathfrak{p} := B \cap \varphi^{-1}\mathfrak{m}_L$. Then $\varphi(B_\mathfrak{p}) \subset L^+$ as well, and $B_\mathfrak{p}$ dominates both A_1 and A_2. Now, let $t \in X$ be a maximal point which specializes to x; by [25, Ch.I, Prop.5.5.2] we may find a valued field $(E, |\cdot|_E)$ with a local ring homomorphism $\mathscr{O}_{X,t} \to E$, such that the valuation ring E^+ of $|\cdot|_E$ dominates the image of the specialization map $\mathscr{O}_{X,x} \to \mathscr{O}_{X,t}$. Let $\kappa(E)$ be the residue field of E^+, and $\overline{B}_\mathfrak{p} \subset \kappa(E)$ the image of $B_\mathfrak{p}$. By the same token, we may find a valuation ring $V \subset \kappa(E)$ with fraction field $\kappa(E)$, and dominating $\overline{B}_\mathfrak{p}$; then it is easily seen that the preimage $K^+ \subset E^+$ of V is a valuation ring with field of fractions $K = E$, which dominates the image of $B_\mathfrak{p}$ in $\mathscr{O}_{X,t}$ ([61, Th.10.1(iv)]). Hence, K^+ dominates the images of $\mathscr{O}_{X,x_i}$, for both $i = 1, 2$, and therefore, also the image of $\mathscr{O}_{Y,y}$; in other words, in this way we obtain two elements in $X(K^+)_{\max}$ whose images agree in $Y(K^+)$, and whose restrictions $\operatorname{Spec} K \to X$ coincide. By assumption, these two K^+-points must then coincide, especially $x_1 = x_2$, and therefore $\sigma_1 = \sigma_2$, as required. □

Proposition 6.6.28. *Let $f : X \to Y$ be a quasi-compact morphism of schemes. The following conditions are equivalent :*

(a) *f is universally closed.*
(b) *For every valued field $(K, |\cdot|)$, the corresponding map* (6.6.27) *is surjective.*

Proof. (a) ⇒ (b) by the valuative criterion of [25, Ch.I, Prop.5.5.8]. Conversely, we will show that (b) implies that the conditions of *loc.cit.* are fulfilled. Hence, let $(L, |\cdot|_L)$ be any valued field, with valuation ring L^+, and suppose that we have a morphism $\sigma : \operatorname{Spec} L \to X$, whose composition with f extends to a morphism $\operatorname{Spec} L^+ \to Y$; we have to show that σ extends to a morphism $\operatorname{Spec} L^+ \to X$. Denote by $\eta, s \in \operatorname{Spec} L^+$ respectively the generic and the closed point, let $x \in X$ be the image of η, and $y \in Y$ the image of s. Then σ corresponds to a ring homomorphism $\sigma^\natural : \mathscr{O}_{X,x} \to L$, and L^+ dominates the image of the map $\mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$ determined by f. Moreover, $\sigma^\natural$ factors through the residue field $\kappa(x)$ of $\mathscr{O}_{X,x}$. Then $\kappa(x)^+ := \kappa(x) \cap L^+$ is a valuation ring with fraction field $\kappa(x)$, and we are reduced to showing that there exists a specialization $x' \in X$ of x, such that $f(x') = y$, and such that $\kappa(x)^+$ dominates the image of the specialization map $\mathscr{O}_{X,x'} \to \mathscr{O}_{X,x}$.

Let now $t \in X$ be a maximal point which specializes to x; by [25, Ch.I, Prop.5.5.8] we may find a valued field $(E, |\cdot|_E)$ with a local ring homomorphism $\mathscr{O}_{X,t} \to E$, whose valuation ring E^+ dominates the image of the specialization map $\mathscr{O}_{X,x} \to \mathscr{O}_{X,t}$. Let $\kappa(E)$ be the residue field of E^+; the induced map $\mathscr{O}_{X,x} \to \kappa(E)$ factors through $\kappa(x)$, and by [61, Th.10.2] we may find a valuation ring $V \subset \kappa(E)$ with field of fractions $\kappa(E)$, which dominates $\kappa(x)^+$. The preimage $K^+ \subset E^+$ of V is a valuation ring with field of fractions $K = E$ ([61, Th.10.1]). By construction, K^+ dominates the image of $\mathscr{O}_{X,y}$, in which case assumption (b) says that there exists a specialization x' of x such that $f(x') = y$, and such that K^+ dominates the image of the specialization map $\mathscr{O}_{X,x'} \to \mathscr{O}_{X,t}$. A simple inspection then shows that $\kappa(x)^+$ dominates the image of $\mathscr{O}_{X,x'}$ in $\mathscr{O}_{X,x}$, as required. □

Corollary 6.6.29. *Let $f : X \to Y$ be a morphism of schemes which is quasi-separated and of finite type. Then the following conditions are equivalent :*

(a) *f is proper.*
(b) *For every valued field $(K, |\cdot|)$, the corresponding diagram* (6.6.25) *(with $R := K^+$ the valuation ring of $|\cdot|$) is cartesian.*

Proof. It is immediate from propositions 6.6.26 and 6.6.28. □

6.6.30. We are now ready to return to log schemes. Resume the situation of (6.6.23), let $\underline{X} := ((X, \underline{M}), F, \psi)$ be any object of $\mathscr{K}$, and denote by $\alpha : \underline{M} \to \mathscr{O}_X$ the structure map of $\underline{M}$.

Suppose we are given a morphism $\sigma : S \to X$ of schemes, and we ask whether there exists a morphism of log structures $\beta : \sigma^*\underline{M} \to \mathscr{O}_S^*$, such that the pair (σ, β) is a morphism of log schemes $(S, \mathscr{O}_S^*) \to (X, \underline{M})$. By definition, this holds if and only if the composition

$$\overline{\beta} : \sigma^*\underline{M} \xrightarrow{\sigma^*\alpha} \sigma^*\mathscr{O}_X \xrightarrow{\sigma^\natural} \mathscr{O}_S$$

factors through $\mathscr{O}_S^*$. Moreover, in this case the factorization is unique, so that σ determines β uniquely. Let $\eta \in S$ be the generic point; we claim that the stated condition is fulfilled, if and only if $t := \sigma(\eta)$ lies in $(X, \underline{M})_{\mathrm{tr}}$ (see definition 6.2.7(i)). Indeed, if t lies in the trivial locus, $\underline{M}_t = \mathscr{O}_{X,t}^\times$, so certainly the image of $\overline{\beta}_t : \underline{M}_t \to \mathscr{O}_{S,\eta}$ lies in $\mathscr{O}_{S,\eta}^* = K^\times$. Now, if $s \in S$ is any other point, the composition

$$\underline{M}_{\sigma(s)} \xrightarrow{\overline{\beta}_{\sigma(s)}} \mathscr{O}_{S,s} \to \mathscr{O}_{S,\eta} = K$$

factors through $\underline{M}_t$, hence its image lies in $K^\times \cap \mathscr{O}_{S,s} = \mathscr{O}_{S,s}^*$, whence the contention. Conversely, if $\overline{\beta}$ factors through $\mathscr{O}_S^*$, it follows especially that the image of the stalk of the structure map $\underline{M}_t \to \mathscr{O}_{X,t}$ lies in the preimage of $\mathscr{O}_{S,\eta}^* = \mathscr{O}_{S,\eta}^\times$, and the latter is just $\mathscr{O}_{X,t}^\times$, so t lies in the trivial locus.

Next, let $f : (S, \mathscr{O}_S^*) \to (X, \underline{M})$ be any morphism of log schemes. We claim that there exists a unique morphism of fans $\varphi := (\varphi, \log \varphi) : \mathrm{Spec}\,\Gamma_+ \to F$, such that the pair (f, φ) is a morphism $\mathsf{f}(K, |\cdot|) \to \underline{X}$ in $\mathscr{K}$. Indeed, since ψ_Γ is a homeomorphism, there exists a unique continuous map φ on the underlying topological spaces, such that $\varphi \circ \psi_\Gamma = \psi \circ f$, and for the same reason, the map $\log f^\sharp : f^*\underline{M}/\mathscr{O}_S^\times \to \mathscr{O}_S^*/\mathscr{O}_S^\times$ is of the form $\psi_\Gamma^*(\log \varphi)$ for a unique morphism of sheaves $\log \varphi$ as sought. Then $\log \varphi$ will be a local morphism, since the same holds for $\log f$ (see (6.6.1)).

Summing up, we have shown that the natural map

$$\mathrm{Hom}_{\mathscr{K}}(\mathsf{f}(K,|\cdot|),\underline{X}) \to X(K^+) \quad : \quad (\sigma,\varphi) \mapsto \sigma$$

is injective, and its image is the set of all the morphisms $S \to X$ which map η into $(X,\underline{M})_{\mathrm{tr}}$.

Proposition 6.6.31. *Let $\underline{X} := ((X,\underline{M}),F,\psi)$ be an object of $\mathscr{K}_{\mathrm{int}}$, and $\varphi : F' \to F$ an integral partial subdivision, with F and F' locally fine. We have :*

(i) *If the induced map $F'(\mathbb{N}) \to F(\mathbb{N})$ is injective, the cartesian morphism $\varphi^*\underline{X} \to \underline{X}$ is separated.*

(ii) *If φ is a proper subdivision, the cartesian morphism $\varphi^*\underline{X} \to \underline{X}$ is proper, and induces an isomorphism of schemes $(\varphi^*\underline{X})_{\mathrm{tr}} \xrightarrow{\sim} (X,\underline{M})_{\mathrm{tr}}$.*

Proof. The assertions are local on X (cp. the proof of claim 6.6.15), so we may assume – by lemma 6.6.11 – that $\underline{M}$ admits a chart $P_X \to \underline{M}$, and $F = (\mathrm{Spec}\, P)^\sharp$. In this case, let $S := \mathrm{Spec}\,\mathbb{Z}[P]$, and denote by $\underline{S}$ the object of $\mathscr{K}_{\mathrm{int}}$ attached to P, as in example 6.6.5(i) (with $R := \mathbb{Z}$); then $(X,\underline{M})$ is isomorphic to $X \times_S (S, P_S^{\log})$, and if $f_S : \varphi^*\underline{S} \to \underline{S}$ is the cartesian morphism over φ, then the cartesian morphism $f : \varphi^*\underline{X} \to \underline{X}$ is given by the pair $(\mathbf{1}_X \times_S f_S, \varphi)$ (see the proof of proposition 6.6.14(iii)). Thus, we may replace $\underline{X}$ by $\underline{S}$ ([25, Ch.I, Prop.5.3.1(iv)]), in which case lemma 6.6.12 shows that the log scheme $(X',\underline{M}')$ underlying $\varphi^*\underline{S}$ admits an open covering consisting of affine log schemes of the form $\mathrm{Spec}(\mathbb{Z},Q)$, for a fine monoid Q. Notice that, for such Q we have :

$$\mathrm{Spec}(\mathbb{Z},Q)_{\mathrm{tr}} = \mathbb{Z}[Q^{\mathrm{gp}}]$$

which is a dense open subset of $\mathrm{Spec}\,\mathbb{Z}[Q]$.

(i): According to proposition 6.6.14(v), the morphism f is quasi-separated, so we may apply the criterion of proposition 6.6.26. Indeed, let $(K,|\cdot|)$ be any valued field, and suppose that $\sigma_i \in X'(K^+)_{\max}$ (for $i = 1,2$) are two sections, whose images in $X'(K)_{\max} \times_{S(K)} S(K^+)$ coincide; we have to show that $\sigma_1 = \sigma_2$. However, we have just seen that the maximal points of X' lie in $(X',\underline{M}')_{\mathrm{tr}}$, so the discussion of (6.6.30) shows that both σ_i extend uniquely to morphisms $\sigma'_i : \mathsf{f}(K,|\cdot|) \to \varphi^*\underline{S}$ in $\mathscr{K}$, and it suffices to show that $\sigma'_1 = \sigma'_2$. By definition, the datum of σ'_i is equivalent to the datum of a morphism $\sigma''_i : \mathsf{f}(K,|\cdot|) \to \underline{S}$, and a morphism of fans $\varphi'_i : \mathrm{Spec}\,\Gamma_+ \to F'$. Again by (6.6.30), the morphisms σ''_1 and σ''_2 agree if and only if they induce the same morphisms of schemes; the latter holds by assumption, since σ_1 and σ_2 yield the same element of $S(K^+)$. On the other hand, in view of (b) and proposition 3.5.24, the elements $\varphi_1,\varphi_2 \in F'(\Gamma_+)$ coincide if and only if their images in $F(\Gamma_+)$ coincide; but again, this last condition holds since the images of both σ_i agree in $S(K^+)$.

(ii): In view of (i) and proposition 6.6.14(iv),(v) we know already that f is separated and of finite type, so we may apply the criterion of corollary 6.6.29. Hence, let $\sigma \in S(K^+)$ be a section, and $x \in X'(K)_{\max}$ a K-rational point such that $f(x)$ is the image of σ in $S(K)$; in view of (i), it suffices to show that σ lifts to a section $\widetilde{\sigma} \in X'(K^+)_{\max}$, whose image in $X'(K)$ is x. Since $x \in (X',\underline{M}')_{\mathrm{tr}}$, remark 6.6.17 implies that $f(x) \in (S,P_S^{\log})_{\mathrm{tr}}$, and then the discussion in (6.6.30) says that σ underlies a unique morphism $(\sigma',\beta) : \mathsf{f}(K,|\cdot|) \to \underline{S}$. By proposition 3.5.26, the element $\beta \in F(\Gamma_+)$ lifts to an element $\beta' \in F'(\Gamma_+)$, and finally the pair $((\sigma,\beta),\beta')$ determines a unique morphism $\mathsf{f}(K,|\cdot|) \to \varphi^*\underline{S}$, whose underlying morphism of schemes is the sought $\widetilde{\sigma}$. Lastly, notice that the map $F'(\{1\}) \to F(\{1\})$ induced by φ, is bijective by propositions 3.5.24 and 3.5.26 (where $\{1\}$ is the monoid with one element), which means that φ restricts to a bijection on the points of height zero; then remark 6.6.17 implies the second assertion of (ii). □

Theorem 6.6.32. *Let $(X,\underline{M})$ be a regular log scheme. Then there exists a smooth morphism of log schemes $f : (X',\underline{M}') \to (X',\underline{M})$, whose underlying morphism of schemes is proper and* *birational, and such that X' is a regular scheme. More precisely, f restricts to an isomorphism of schemes $f^{-1}X_{\mathrm{reg}} \to X_{\mathrm{reg}}$ on the preimage of the open locus of regular points of X.*

Proof. We use the object $\underline{X} := ((X, \underline{M}), F(X, \underline{M}), \pi_X)$ attached to $(X, \underline{M})$ as in remark 6.6.9(ii). Indeed, it is clear that $F(X, \underline{M})$ is locally fine and saturated, hence theorem 3.6.31 yields an integral, proper, simplicial subdivision $\varphi : F' \to F(X, \underline{M})$ which restricts to an isomorphism $\varphi^{-1}F(X, \underline{M})_{\mathrm{sim}} \xrightarrow{\sim} F(X, \underline{M})_{\mathrm{sim}}$. Take $(f, \varphi) : \varphi^*\underline{X} \to \underline{X}$ to be the cartesian morphism over φ, and denote by $(X', \underline{M}')$ the log scheme underlying $\varphi^*\underline{X}$; it follows already from proposition 6.6.31(ii) that f is proper on the underlying schemes. Next, corollary 6.5.35 shows that X_{reg} is $\pi^{-1}F(X, \underline{M})_{\mathrm{sim}}$, so f restricts to an isomorphism $f^{-1}X_{\mathrm{reg}} \xrightarrow{\sim} X_{\mathrm{reg}}$. Furthermore, f is étale, by proposition 6.6.14(v), hence the log scheme $(X', \underline{M}')$ is regular (theorem 6.5.44). Finally, again by corollary 6.5.28 we see that X' is regular. □

6.6.33. Let now $(Y, \underline{N})$ be a regular log scheme, such that $F(Y, \underline{N})$ is affine (notation of remark 6.6.9(ii)), say isomorphic to $(\mathrm{Spec}\, P)^\sharp$, for some fine, sharp and saturated monoid P. Let $I \subset P$ be an ideal generated by two elements $a, b \in P$, and denote by $f : (Y', \underline{N}') \to (Y, \underline{N})$ the saturated blow up of the ideal $I\underline{N}$ of $\underline{N}$ (see (6.4.79)). Set $U' := (Y', \underline{N}')_{\mathrm{tr}}$, $U := (Y, \underline{N})_{\mathrm{tr}}$, and denote $j : U \to Y$, $j' : U' \to Y'$ the open immersions. In this situation we have :

Lemma 6.6.34. (i) $H^1(Y', \underline{N}'^{\sharp\mathrm{gp}}) = 0$.

(ii) *Suppose morever, that $R^1j'_*\mathscr{O}^\times_{U'} = 0$. Then $R^1j_*\mathscr{O}^\times_U = 0$.*

Proof. (i): Since $\underline{N}'^{\sharp\mathrm{gp}} = \pi^*_X\mathscr{O}^{\mathrm{gp}}_{F(Y',\underline{N}')}$, claim 6.6.39(ii) and remark 6.6.9(iii) reduce to showing

$$H^1(F(Y', \underline{N}'), \mathscr{O}^{\mathrm{gp}}_{F(Y',\underline{N}')}) = 0. \tag{6.6.35}$$

However, by corollary 6.6.18(iii), the fan $F(Y', \underline{N}')$ is the saturated blow up of the ideal $I\mathscr{O}_{F(Y,\underline{N})}$ of $\mathscr{O}_{F(Y,\underline{N})}$, hence it admits the affine covering

$$F(Y', \underline{N}') = (\mathrm{Spec}\, P[a^{-1}b]^{\mathrm{sat}})^\sharp \cup (\mathrm{Spec}\, P[b^{-1}a]^{\mathrm{sat}})^\sharp.$$

Notice now that every affine fan is a local topological space, hence the left hand-side of (6.6.35) can be computed as the Čech cohomology of $\mathscr{O}^{\mathrm{gp}}_{F(Y',\underline{N}')}$ relative to this covering ([39, II, 5.9.2]). However, the intersection of the two open subsets is $(\mathrm{Spec}\, P[a^{-1}b, b^{-1}a])^\sharp$, and clearly the restriction map

$$H^0(\mathrm{Spec}\, P[a^{-1}b]^{\mathrm{sat}}, \mathscr{O}^{\mathrm{gp}}_{F(Y',\underline{N}')}) \to H^0(\mathrm{Spec}\, P[a^{-1}b, b^{-1}a], \mathscr{O}^{\mathrm{gp}}_{F(Y',\underline{N}')})$$

is surjective. The assertion is an immediate consequence.

(ii): Let $y \in Y$ be any point; according to (6.4.80), the morphism

$$f \times_Y Y(y) : (Y', \underline{N}') \times_Y Y(y) \to (Y(y), \underline{N}(y))$$

is the saturated blow up of the ideal $I\underline{N}(y)$; on the other hand, let $U(y) := U \times_Y Y(y)$, $U'(y) := U' \times_Y Y(y)$ and denote by $j_y : U(y) \to Y(y)$ and $j'_y : U'(y) \to Y' \times_Y Y(y)$ the open immersions; in light of proposition 5.1.15(ii), it suffices to show that $R^1j_{y*}\mathscr{O}_{U(y)} = 0$, and the assumption implies that $R^1j'_{y*}\mathscr{O}_{U'(y)} = 0$. Summing up, we may replace Y be $Y(y)$, and assume from start that Y is local, and y is its closed point. From the assumption we get :

$$H^1(Y', j'_*\mathscr{O}^\times_{U'}) = H^1(U', \mathscr{O}^\times_{Y'}) = \mathrm{Pic}\, U'.$$

On the other hand, recall that $\underline{N}'^{\sharp\mathrm{gp}} = j'_*\mathscr{O}^\times_{U'}/\mathscr{O}^\times_{Y'}$ (proposition 6.5.52); combining with (i), we deduce that the natural map

$$\mathrm{Pic}\, Y' \to \mathrm{Pic}\, U'$$

is surjective. Set $Y'_0 := f^{-1}(y) \subset Y'$, endow Y'_0 with its reduced subscheme structure, and let $i : Y'_0 \to Y'$ be the closed immersion. If I is an invertible ideal of P, then f is an isomorphism, in which case the assertion is obvious. We may then assume that I is not invertible, in which case claim 6.4.86(ii) says that there exists a morphism of $(Y,\underline{N})$-schemes $h:(Y',\underline{N}')\to\mathbb{P}^1_{(Y,\underline{N})}$ inducing an isomorphism of $\kappa(y)$-schemes

$$(h\times_Y \operatorname{Spec}\kappa(y))_{\mathrm{red}} : Y'_0 \xrightarrow{\sim} \mathbb{P}^1_{\kappa(y)}. \tag{6.6.36}$$

Let us remark :

Claim 6.6.37. Let S be a noetherian local scheme, s the closed point of S, and $f:X\to S$ a proper morphism of schemes. Suppose that $\dim X(s)\leq 1$, and $H^1(X(s),\mathscr{O}_{X(s)})=0$. Then the natural map

$$\operatorname{Pic} X\to \operatorname{Pic} X(s)$$

is injective.

Proof of the claim. Say that $S=\operatorname{Spec} A$ for a local ring A, and denote by $\mathfrak{m}_A\subset A$ the maximal ideal. For every $k\in\mathbb{N}$, set $S_n:=\operatorname{Spec} A/\mathfrak{m}_A^{k+1}$, and let $i_n:X_n:=X\times_S S_n\to X$ be the closed immersion. Let $\mathscr{L}$ be any invertible $\mathscr{O}_X$-module, and suppose that $i_0^*\mathscr{L}\simeq\mathscr{O}_{X_0}$; we have to show that $\mathscr{L}\simeq\mathscr{O}_X$. We notice that, for every $k\in\mathbb{N}$, the natural map

$$H^0(X_{k+1},i_{k+1}^*\mathscr{L})\to H^0(X_k,i_k^*\mathscr{L})$$

is surjective : indeed, its cokernel is an A-submodule of $H^1(X_0,\mathfrak{m}_A^k\mathscr{L}/\mathfrak{m}_A^{k+1}\mathscr{L})$, and since $\dim X_0\leq 1$, the natural map

$$(\mathfrak{m}_A^k/\mathfrak{m}_A^{k+1})\otimes_{\kappa(s)} H^1(X_0,i_0^*\mathscr{L})\xrightarrow{\sim} H^1(X_0,(\mathfrak{m}_A^k/\mathfrak{m}_A^{k+1})\otimes_{\kappa(s)}\mathscr{L})\to H^1(X_0,\mathfrak{m}_A^k\mathscr{L}/\mathfrak{m}_A^{k+1}\mathscr{L})$$

is surjective; on the other hand, our assumptions imply that $H^1(X_0,i_0^*\mathscr{L})=0$, whence the contention. Let $A^\wedge$ be the $\mathfrak{m}_A$-adic completion of A; taking into account [28, Ch.III, Th.4.1.5], we deduce that the natural map

$$H^0(X,\mathscr{L})\otimes_A A^\wedge\to H:=H^0(X_0,i^*\mathscr{L})$$

is a continuous surjection, for the $\mathfrak{m}_A$-adic topologies. Since H is a discrete space for this topology, and since the image of $H^0(X,\mathscr{L})$ is dense in the $\mathfrak{m}_A$-adic topology of $H^0(X,\mathscr{L})\otimes_A A^\wedge$, we conclude that the restriction map $H^0(X,\mathscr{L})\to H$ is surjective as well. Let $\overline{s}\in H$ be a global section of $i_0^*\mathscr{L}$ whose image in $i_0^*\mathscr{L}_x$ is a generator of the latter A_0-module, for every $x\in X_0$, and pick $s\in H^0(X,\mathscr{L})$ whose image in H equals $\overline{s}$. It remains only to check that, for every $x\in X$, the A-module $\mathscr{L}_x$ is generated by the image s_x of s. However, since X is proper, every $x\in X$ specializes to a point of X_0, hence we may assume that $x\in X_0$, in which case one concludes easily, by appealing to Nakayama's lemma (details left to the reader). ◊

Combining claim 6.6.37 and [58, Prop.11.1(i)], we see that the induced map

$$i^*:\operatorname{Pic} Y'\to\operatorname{Pic} Y'_0$$

is injective. Let now $\mathscr{L}$ be any invertible $\mathscr{O}_U$-module; we have to show that $\mathscr{L}$ extends to an invertible $\mathscr{O}_Y$-module (which is then isomorphic to $\mathscr{O}_Y$). However, notice that f restricts to an isomorphism $g:U'\xrightarrow{\sim}U$, hence $g^*\mathscr{L}$ is an invertible $\mathscr{O}_{U'}$-module, and by the foregoing there exists an invertible $\mathscr{O}_{Y'}$-module $\mathscr{L}'$ such that $\mathscr{L}'_{|U'}\simeq g^*\mathscr{L}$. In light of the isomorphism (6.6.36), there exists an invertible $\mathscr{O}_{\mathbb{P}^1_Y}$-module $\mathscr{L}''$ such that $i^*\mathscr{L}'\simeq i^*h^*\mathscr{L}''$. Therefore

$$\mathscr{L}'\simeq h^*\mathscr{L}''.$$

Now, on the one hand, claim 6.6.37 implies that $\mathscr{L}''=\mathscr{O}_{\mathbb{P}^1_Y}(n)$ for some $n\in\mathbb{N}$; on the other hand, (6.4.47) implies that h restricts to a morphism of schemes $h_{\mathrm{tr}}:U'\to\mathbb{G}_{m,Y}$, and $\mathscr{L}''_{|\mathbb{G}_{m,Y}}=\mathscr{O}_{\mathbb{P}^1_Y}(n)_{|\mathbb{G}_{m,Y}}=\mathscr{O}_{\mathbb{G}_{m,Y}}$, so finally $g^*\mathscr{L}=\mathscr{O}_{U'}$, hence $\mathscr{L}=\mathscr{O}_U$, whence the contention. □

The following result complements proposition 6.5.52.

Theorem 6.6.38. *Let $(X, \underline{M})$ be any regular log scheme, set $U :=: (X, \underline{M})_{\mathrm{tr}}$ and denote by $j : U \to X$ the open immersion. Then we have :*

$$R^1 j_* \mathscr{O}_U^\times = 0.$$

Proof. We begin with the following general :

Claim 6.6.39. Let $\pi : T_1 \to T_2$ be a continuous open and surjective map of topological spaces, such that $\pi^{-1}(t)$ is an irreducible topological space (with the subspace topology) for every $t \in T_2$. Then, we have :

(i) For every sheaf S on T_2, the natural map $S \to \pi_* \pi^* S$ is an isomorphism.
(ii) For every abelian sheaf S on T_2, the natural map $S[0] \to R\pi_*\pi^* S$ is an isomorphism in $\mathsf{D}(\mathbb{Z}_{T_2}\text{-}\mathbf{Mod})$.

Proof of the claim. (i): Since π is open, $\pi^* S$ is the sheaf associated to the presheaf : $U \mapsto S(\pi U)$, for every open subset U of T_1. We show, more precisely, that this presheaf is already a sheaf; since π is surjective, the claim shall follow immediately. Now, let $U \subset T_1$ be an open subset, and $(U_i \mid i \in I)$ a family of open subsets of X covering U; for every $i, j \in I$, set $U_{ij} := U_i \cap U_j$. It suffices to show that $S(\pi U)$ is the equalizer of the two maps :

$$\prod_{i \in I} S(\pi U_i) \rightrightarrows \prod_{i,j \in I} S(\pi U_{ij}).$$

Since S is a sheaf, the latter will hold, provided we know that $\pi U_i \cap \pi U_j = \pi U_{ij}$ for every $i, j \in I$. Hence, let $t \in \pi U_i \cap \pi U_j$; this means that $\pi^{-1}(t) \cap U_i \neq \varnothing$ and $\pi^{-1}(t) \cap U_j \neq \varnothing$. Since $\pi^{-1}(t)$ is irreducible, we deduce that $\pi^{-1}(t) \cap U_{ij} \neq \varnothing$, as required.

(ii): The proof of (i) also shows that, for every flabby abelian sheaf J on T_2, the abelian sheaf $\pi^* J$ is flabby on T_1. Hence, if S is any abelian sheaf on T_2, we obtain a flabby resolution of $\pi^* S$ of the form $\pi^* J_\bullet$, by taking a flabby resolution $S \to J_\bullet$ of S on T_2. According to lemma 5.1.3, there is a natural isomorphism

$$\pi_* \pi^* J \xrightarrow{\sim} R\pi_* \pi^* S$$

in $\mathsf{D}(\mathbb{Z}_{T_2}\text{-}\mathbf{Mod})$. Then the assertion follows from (i). ◊

After these preliminaries, let us return to the log scheme $(X, \underline{M})$, and its associated object $\underline{X} := ((X, \underline{M}), F(X, \underline{M}), \pi_X)$. The assertion to prove is local on X, hence we may assume that $F(X, \underline{M}) = (\operatorname{Spec} P)^\sharp$, for some sharp, fine and saturated monoid P. Next, by theorem 6.6.32 (and its proof) there exists an integral proper simplicial subdivision $\varphi : F' \to F(X, \underline{M})$, such that the log scheme $(X', \underline{M}')$ underlying $\varphi^* \underline{X}$ is regular, X' is regular, and the morphism $X' \to X$ restricts to an isomorphism $U' := (X', \underline{M}')_{\mathrm{tr}} \to U$. In this situation, we may find a further subdivision $\varphi' : F'' \to F'$ such that both φ' and $\varphi \circ \varphi'$ are compositions of saturated blow up of ideals generated by at most two elements of P (example 3.6.15(iii)). Say that $\varphi \circ \varphi' = \varphi_r \circ \varphi_{r-1} \cdots \circ \varphi_1$, where each φ_i is a saturated blow up of the above type. By proposition 6.6.14(iii), we deduce a sequence of morphisms of log schemes

$$(X_1, \underline{M}_1) \xrightarrow{g_1} \cdots \to (X_{r-1}, \underline{M}_{r-1}) \xrightarrow{g_{r-1}} (X_r, \underline{M}_r) \xrightarrow{g_r} (X_{r+1}, \underline{M}_{r+1}) := (X, \underline{M})$$

each of which is the blow up of a corresponding ideal, and by the same token, φ' induces a morphism $g : (X_1, \underline{M}_1) \to (X', \underline{M}')$ of $(X, \underline{M})$-schemes. For every $i = 1, \cdots, r+1$, set $U_i := (X_i, \underline{M}_i)_{\mathrm{tr}}$, and let $j_i : U_i \to X_i$ be the open immersion; especially, $U_{r+1} = U$. We shall show, by induction on i, that

(6.6.40) $$R^1 j_{i*} \mathscr{O}_{U_i}^\times = 0 \qquad \text{for } i = 1, \dots, r+1.$$

Notice first that the stated vanishing translates the following assertion. For every $x \in X_i$ and every invertible $\mathscr{O}_{U_i}$-module $\mathscr{L}$, there exists an open neighborhood U_x of x in X_i such that

$\mathscr{L}_{|U_x\cap U_i}$ extends to an invertible $\mathscr{O}_{U_x}$-module. However, it follows immediately from propositions 5.6.7 and 5.6.14, that every invertible $\mathscr{O}_{U'}$-module extends to an invertible $\mathscr{O}_{X'}$-module. Since g restricts to an isomorphism $U_1 = g^{-1}U' \xrightarrow{\sim} U'$, we easily deduce that every invertible $\mathscr{O}_{U_1}$-module extends to an invertible $\mathscr{O}_{X_1}$-module (namely : if $\mathscr{L}$ is such a module, extend $g_{|U_1*}\mathscr{L}$ to an invertible $\mathscr{O}_{X'}$-module, and pull the extension back to X_1, via g^*). Summing up, we see that (6.6.40) holds for $i = 1$.

Next, suppose that (6.6.40) has already been shown to hold for a given $i \leq r$; by lemma 6.6.34(ii), it follows that (6.6.40) holds for $i + 1$, so we are done. □

6.6.41. Let X be a scheme, $\underline{M}$ a fine log structure on the Zariski site of X, x a point of X, and notice that every fractional ideal of $\underline{M}_x$ is finitely generated (lemma 3.4.22(iv)). Say that $X(x) = \operatorname{Spec} A$; in view of lemma 3.4.27(ii) there is a natural map of abelian groups

$$\mathrm{Div}(\underline{M}_x) \to \mathrm{Div}(A).$$

Composing with the map $I \mapsto I^{\sim}$ as in (5.6.11), we get, by virtue of *loc.cit.*, a map

$$\mathrm{Div}(\underline{M}_x) \to \mathrm{Div}\, X(x). \qquad (6.6.42)$$

Corollary 6.6.43. *With the notation of* (6.6.41), *suppose furthermore that* $(X, \underline{M})$ *is regular at the point* x. *Then* (6.6.42) *induces an isomorphism*

$$\overline{\mathrm{Div}}(\underline{M}_x) \xrightarrow{\sim} \overline{\mathrm{Div}}(X(x)).$$

Proof. By virtue of proposition 3.4.37(ii) (and remark 5.6.10(i)), the map under investigation is already known to be injective. To show surjectivity, let K be the field of fractions of A, and $L \subset K$ any reflexive fractional ideal of A; we may then regard $\mathscr{L} := L^{\sim}$ as a coherent $\mathscr{O}_X$-submodule of $i_*\mathscr{O}_{X_0}$, where $i : X_0 \to X$ is the inclusion map of the subscheme $X_0 := \operatorname{Spec} K$. By proposition 5.6.14, the $\mathscr{O}_U$-module $\mathscr{L}_{|U}$ is invertible, hence it extends to an invertible $\mathscr{O}_{X(x)}$-module $\mathscr{L}'$, by virtue of theorem 6.6.38. Since $X(x)$ is local, $\mathscr{L}'$ is a free $\mathscr{O}_{X(x)}$-module, and therefore $\mathscr{L}_{|U} \simeq \mathscr{O}_U$. Thus, pick $a \in \mathscr{L}(U) \subset K$ which generates $\mathscr{L}_{|U}$; after replacing L by $a^{-1}L$, we may assume that $\mathscr{L}_{|U} = \mathscr{O}_U$ as subsheaves of $i_*\mathscr{O}_{X_0}$. Let Σ be the set of points of height one of $X(x)$ contained in $X(x) \setminus U$; for each $y \in \Sigma$, the maximal ideal $\mathfrak{m}_y$ of $\mathscr{O}_{X(x),y}$ is generated by a single element a_y, and there exists $k_y \in \mathbb{Z}$ such that $a_y^{k_y}$ generates the $\mathscr{O}_{X(x),y}$-submodule $\mathscr{L}_y$ of K. To ease notation, let $P := \underline{M}_x$, and denote $\psi : X(x) \to \operatorname{Spec} P$ the natural continuous map; also, let $\mathfrak{m}_{\psi(y)}$ be the maximal ideal of the localization $P_{\psi(y)}$ for every $y \in X(x)$, and set

$$I := \bigcap_{y\in\Sigma} \mathfrak{m}_{\psi(y)}^{k_y}.$$

In light of lemma 3.4.22(i,ii) and proposition 3.4.32, it is easily seen that $I/P^{\times}$ is a reflexive fractional ideal of the fine and saturated monoid $P^{\sharp}$, so I is a reflexive fractional ideal of P. Then $IA \subset K$ is a reflexive fractional ideal of A. Set $\mathscr{L}'' := (IA)^{\sim} \subset i_*\mathscr{O}_{X_0}$; then $\mathscr{L}''_{|U} = \mathscr{L}_{|U}$ and $\mathscr{L}''_y = \mathscr{L}_y$ for every $y \in \Sigma$. It follows that, for every $y \in \Sigma$, there exists an open neighborhood U_y of y in $X(x)$ such that $\mathscr{L}''_{|U_y} = \mathscr{L}_{|U_y}$. Let $U' := U \cup \bigcup_{y\in\Sigma} U_y$, and denote by $j' : U' \to X(x)$ the open immersion. Notice that $\delta'(y, \mathscr{O}_X) \geq 2$ for every $y \in X \setminus U'$ (corollary 6.5.29). In light of proposition 5.6.7(ii) (and remark 5.6.10(i)), we deduce

$$\mathscr{L} = j'_*\mathscr{L}_{|U'} = j'_*\mathscr{L}''_{|U'} = \mathscr{L}''$$

whence the contention. □

6.6.44. Let $\underline{M}$ be a log structure on the Zariski site of a local scheme X, such that $(X, \underline{M})$ is a regular log scheme. Let $x \in X$ be the closed point, say that $X = \operatorname{Spec} B$ for some local ring B, and let $\beta : P \to B$ a chart for $\underline{M}$ which is sharp at x. As usual, if M is any B-module, we denote $M^\sim$ the quasi-coherent $\mathscr{O}_X$-module arising from M.

Theorem 6.6.45. *In the situation of* (6.6.44), *suppose as well that* $\dim P = 2$ *and* $\dim X = 2$. *Then every indecomposable reflexive* $\mathscr{O}_X$*-module is isomorphic to* $(IB)^\sim$*, for some reflexive fractional ideal* I *of* P*. (Notation of* (3.4.26)*.)*

Proof. Set $Q := \mathrm{Div}_+(P)$ and define $\varphi : P \to Q$ as in example 3.4.46. The chart β defines a morphism $\psi : (X, \underline{M}) \to \mathrm{Spec}(\mathbb{Z}, P)$, and we let $(X', \underline{M}')$ be the fibre product in the cartesian diagram

$$\begin{tikzcd} (X', \underline{M}') \arrow[r] \arrow[d, "f"'] & \mathrm{Spec}(\mathbb{Z}, Q) \arrow[d, "\mathrm{Spec}(\mathbb{Z},\varphi)"] \\ (X, \underline{M}) \arrow[r, "\psi"] & \mathrm{Spec}(\mathbb{Z}, P). \end{tikzcd}$$

Arguing as in the proof of claim 7.3.36, it is easily seen that X' is a local scheme and $(X', \underline{M}')$ is regular. Moreover, X' is a regular scheme, since Q is a free monoid (corollary 6.5.35), and f is a finite morphism of Kummer type (lemma 7.3.7).

Claim 6.6.46. The $\mathscr{O}_X$-module $f_*\mathscr{O}_{X'}$ is isomorphic to a finite direct sum $(I_1B \oplus \cdots \oplus I_kB)^\sim$, where $I_1, \dots, I_k$ are reflexive fractional ideals of P, and $\mathscr{O}_X$ is a direct summand of $f_*\mathscr{O}_{X'}$.

Proof of the claim. In view of example 3.4.46, we see that $f_*\mathscr{O}_{X'} = (Q \otimes_P B)^\sim$ is the direct sum of the $\mathscr{O}_X$-modules $(\mathrm{gr}_\gamma Q \otimes_P B)^\sim$, where γ ranges over the elements of $Q^{\mathrm{gp}}/P^{\mathrm{gp}}$, and $\mathrm{gr}_\bullet Q$ denotes the φ-grading of Q. But for each such γ, the natural map $\mathrm{gr}_\gamma Q \otimes_P B \to \mathrm{gr}_\gamma Q \cdot B$ is an isomorphism, since B is P-flat (lemma 6.5.17 and proposition 6.5.32). This shows the first assertion, and the second is clear as well, since $\mathrm{gr}_0 Q = P$. ◇

Denote by x' the closed point of X'; from corollary 6.5.36(i) we see that $U := (X, \underline{M})_1$ is the complement of $\{x\}$ and $U' := (X', \underline{M}')_1$ is the complement of $\{x'\}$ (notation of definition 6.2.7(i)). Let also $j : U \to X$ and $j' : U' \to X'$ be the open immersions.

Claim 6.6.47. (i) The restriction $g : U' \to U$ of f is a flat morphism of schemes.

(ii) The restriction functor $j^* : \mathscr{O}_X\text{-}\mathbf{Rflx} \to \mathscr{O}_U\text{-}\mathbf{Rflx}$ is an equivalence (notation of (5.6)).

Proof of the claim. (i): It suffices to check that the restriction $\mathrm{Spec}(\mathbb{Z}, Q)_1 \to \mathrm{Spec}(\mathbb{Z}, P)_1$ of $\mathrm{Spec}(\mathbb{Z}, \varphi)$ is flat. However, we have the affine open covering

$$\mathrm{Spec}(\mathbb{Z}, P)_1 = \operatorname{Spec} \mathbb{Z}[P_{\mathfrak{p}_1}] \cup \operatorname{Spec} \mathbb{Z}[P_{\mathfrak{p}_2}]$$

where $\mathfrak{p}_1, \mathfrak{p}_2 \subset P$ are the two prime ideals of height one (see example 3.4.17(i)). Hence, we are reduced to showing that the morphism of log schemes underlying

$$\mathrm{Spec}(\mathbb{Z}, \varphi_{\mathfrak{p}_i}) : \mathrm{Spec}(\mathbb{Z}, Q_{\mathfrak{p}_i}) \to \mathrm{Spec}(\mathbb{Z}, P_{\mathfrak{p}_i})$$

is flat for $i = 1, 2$. However, it is clear that $Q_{\mathfrak{p}_i}$ is an integral $P_{\mathfrak{p}_i}$-module, hence it suffices to check that $Q_{\mathfrak{p}_i}$ is a flat $P_{\mathfrak{p}_i}$-module, for $i = 1, 2$ (proposition 3.1.52), or equivalently, that $Q^\sharp_{\mathfrak{p}_i}$ is a flat $P^\sharp_{\mathfrak{p}_i}$-module (corollary 3.1.49(ii)). The latter assertion follows immediately from the discussion of (3.4.43).

(ii): From proposition 5.6.7 we see that j^* is full and essentially surjective. Moreover, it follows from remark 5.6.4 that every reflexive $\mathscr{O}_X$-module is S_1, so j^* is also faithful (details left to the reader). ◇

In light of claim 6.6.47(ii), it suffices to show that every indecomposable reflexive $\mathscr{O}_U$-module $\mathscr{F}$ is isomorphic to $(IB)^\sim_{|U}$, for some reflexive fractional ideal I of P. However, for such $\mathscr{F}$, claim 6.6.47(i) and lemma 5.6.6(i) imply that $g^*\mathscr{F}$ is a reflexive $\mathscr{O}_{U'}$-module. From proposition 5.6.7 and corollary 5.4.22 we deduce that $\delta'(x', j'_*g^*\mathscr{F}) \geq 2$, so $j'_*g^*\mathscr{F}$ is a free $\mathscr{O}_{X'}$-module of finite rank ([30, Ch.0, Prop.17.3.4]), and finally, $g^*\mathscr{F}$ is a free $\mathscr{O}_{U'}$-module of finite rank. Taking into account claim 6.6.46, it follows that $g_*g^*\mathscr{F} = \mathscr{F} \otimes_{\mathscr{O}_U} g_*\mathscr{O}_{U'}$ is a direct sum of $\mathscr{O}_U$-modules of the type $(IB)^{\sim}_{|U}$, for various $I \in \mathrm{Div}(P)$; moreover, $\mathscr{F}$ is a direct summand of $g_*g^*\mathscr{F}$. Then, we may find a decomposition $g_*g^*\mathscr{F} = \mathscr{F}_1 \oplus \cdots \oplus \mathscr{F}_t$ such that $\mathscr{F}_i$ is an indecomposable $\mathscr{O}_U$-module for $i = 1, \dots, t$, and $\mathscr{F}_1 = \mathscr{F}$ (details left to the reader : notice that – since reflexive $\mathscr{O}_U$-modules are S_1 – the length t of such a decomposition is bounded by the dimension of the $\kappa(\eta)$-vector space $(g_*g^*\mathscr{F})_\eta$, where η is the maximal point of X). On the other hand, notice that

$$\begin{aligned}
\mathrm{End}_{\mathscr{O}_U}((IB)^{\sim}_{|U}) &= \mathrm{End}_B(IB) && \text{(claim 6.6.47(ii))} \\
&= (IB : IB) && \text{(by (3.4.35))} \\
&= (I : I)B && \text{(lemma 3.4.27(i))} \\
&= B && \text{(proposition 3.4.37(i))}
\end{aligned}$$

for every $I \in \mathrm{Div}(P)$. Now the contention follows from theorem 4.3.3. □

6.6.48. Let now $(X_{\text{ét}}, \underline{M})$ be a quasi-coherent log scheme on the étale site of X. Pick a covering family $(U_\lambda \mid \lambda \in \Lambda)$ of X in $X_{\text{ét}}$, and for every $\lambda \in \Lambda$, a chart $P_{\lambda,U_\lambda} \to \underline{M}_{|U_\lambda}$. The latter induce isomorphisms of log schemes $(U_\lambda, \underline{M}_{|U_\lambda}) \xrightarrow{\sim} \widetilde{u}^*(U_{\lambda,\mathrm{Zar}}, P^{\log}_{U_{\lambda,\mathrm{Zar}}})$ (see (6.2.2)); in other words, every quasi-coherent log structure on $X_{\text{ét}}$ descends, locally on $X_{\text{ét}}$, to a log structure on the Zariski site. However, $(X_{\text{ét}}, \underline{M})$ may fail to descend to a unique log structure on the whole of X_{Zar}. Our present aim is to show that, at least under a few more assumptions on $\underline{M}$, we may find a blow up $(X'_{\text{ét}}, \underline{M}') \to (X_{\text{ét}}, \underline{M})$ such that $(X'_{\text{ét}}, \underline{M}')$ descends to a log structure on X'_{Zar}. To begin with, for every $\lambda \in \Lambda$ let

$$T_\lambda := (\mathrm{Spec}\, P_\lambda)^\sharp \qquad \text{and} \qquad S_\lambda := \mathrm{Spec}\, \mathbb{Z}[P_\lambda].$$

Also, let $\underline{S}_\lambda := (\mathrm{Spec}(\mathbb{Z}, P_\lambda), T_\lambda, \psi_{P_\lambda})$ be the object of $\mathscr{K}$ attached to P_λ, as in example 6.6.5(i). From the isomorphism

$$(U_{\lambda,\mathrm{Zar}}, \widetilde{u}_*\underline{M}_{|U_\lambda}) \xrightarrow{\sim} U_\lambda \times_{S_\lambda} \mathrm{Spec}(\mathbb{Z}, P_\lambda)$$

we deduce an object

$$\underline{U}_\lambda := U_\lambda \times_{S_\lambda} \underline{S}_\lambda = ((U_{\lambda,\mathrm{Zar}}, \widetilde{u}_*\underline{M}_{|U_\lambda}), T_\lambda, \psi_\lambda).$$

Suppose now that $\underline{M}$ is a fs log structure; then we may choose for each P_λ a fine and saturated monoid (lemma 6.1.16(iii)). Next, suppose that X is quasi-compact; in this case we may assume that Λ is a finite set, hence $\mathscr{S} := \{P_\lambda \mid \lambda \in \Lambda\}$ is a finite set of monoids, and consequently we may choose a finite sequence of integers $\underline{c}(\mathscr{S})$ fulfilling the conditions of (3.6.25) relative to the category $\mathscr{S}$-**Fan**. Then, for every $\lambda \in \Lambda$ we have a well defined integral roof $\rho_\lambda : T_\lambda(\mathbb{Q}_+) \to \mathbb{Q}_+$, and we denote by $f_\lambda : T(\rho_\lambda) \to T_\lambda$ the corresponding subdivision. There follows a cartesian morphism

$$f^*_\lambda \underline{U}_\lambda \to \underline{U}_\lambda$$

whose underlying morphism of log schemes is a saturated blow up of the ideal $\mathscr{I}_{\rho_\lambda}\widetilde{u}_*\underline{M}_{|U_\lambda}$ (proposition 3.6.13 and corollary 6.6.18). Next, for $\lambda, \mu \in \Lambda$, set $U_{\lambda\mu} := U_\lambda \times_X U_\mu$ and let

$$\underline{U}_{\lambda\mu} := ((U_{\lambda\mu}, \widetilde{u}_*\underline{M}_{|U_{\lambda\mu}}), T_\lambda, \psi_{\lambda\mu})$$

where $\psi_{\lambda\mu}$ is the composition of ψ_λ and the projection $U_{\lambda\mu} \to U_\lambda$. Denote by $(U^{\sim}_\lambda, \underline{M}^{\sim}_\lambda)$ (resp. $(U^{\sim}_{\lambda\mu}, \underline{M}^{\sim}_{\lambda\mu})$) the log scheme underlying $f^*_\lambda \underline{U}_\lambda$ (resp. $f^*_\lambda \underline{U}_{\lambda\mu}$). Also, for any $\lambda, \mu, \gamma \in \Lambda$, let $\pi_{\lambda\mu\gamma} : U_{\lambda\mu} \times_X U_\gamma \to U_{\lambda\mu}$ be the natural projection.

Lemma 6.6.49. *In the situation of* (6.6.48), *we have :*

(i) *There exists a unique isomorphism of log schemes* $g_{\lambda\mu} : (U^{\sim}_{\lambda\mu}, \underline{M}^{\sim}_{\lambda\mu}) \xrightarrow{\sim} (U^{\sim}_{\mu\lambda}, \underline{M}^{\sim}_{\mu\lambda})$ *fitting into a commutative diagram :*

$$\begin{tikzcd}
(U^{\sim}_{\lambda\mu}, \underline{M}^{\sim}_{\lambda\mu}) \arrow[r, "g_{\lambda\mu}"] \arrow[d] & (U^{\sim}_{\mu\lambda}, \underline{M}^{\sim}_{\mu\lambda}) \arrow[d] \\
U_{\lambda\mu} \arrow[r, equal] & U_{\mu\lambda}
\end{tikzcd}$$

whose vertical arrows are the saturated blow up morphisms.

(ii) *There exist natural isomorphisms of* $\mathscr{O}_{U_{\lambda\mu}}$*-modules*

$$\omega_{\lambda\mu} : g^*_{\lambda\mu}\mathscr{O}_{U_{\mu\lambda}}(1) \xrightarrow{\sim} \mathscr{O}_{U_{\lambda\mu}}(1)$$

such that $(\pi^*_{\mu\gamma\lambda}\omega_{\mu\lambda}) \circ (\pi^*_{\lambda\mu\gamma}\omega_{\lambda\mu}) = \pi^*_{\lambda\gamma\mu}\omega_{\lambda\gamma}$ *for every* $\lambda, \mu, \gamma \in \Lambda$.

Proof. (i): By the universal property of the saturated blow up, it suffices to show that :

(6.6.50) $$\mathscr{I}_{\rho_\lambda}\widetilde{u}_*\underline{M}_{|U_{\lambda\mu}} = \mathscr{I}_{\rho_\mu}\widetilde{u}_*\underline{M}_{|U_{\mu\lambda}} \qquad \text{on } U_{\lambda\mu} = U_{\mu\lambda}.$$

The assertion is local on $U_{\lambda\mu}$, hence let $x \in U_{\lambda\mu}$ be any point; we get an isomorphism of $\widetilde{u}_*\underline{M}_x$-monoids :

$$\mathscr{O}_{T_\lambda,\psi_{\lambda\mu}(x)} \xrightarrow{\sim} \mathscr{O}_{T_\mu,\psi_{\mu\lambda}(x)}$$

whence an isomorphism of fans $U(\psi_{\lambda\mu}(x)) \xrightarrow{\sim} U(\psi_{\mu\lambda}(x))$ (notation of (3.5.16)). Therefore, the subset $U(x) := \psi^{-1}_{\lambda\mu}U(\psi_{\lambda\mu}(x)) \cap \psi^{-1}_{\mu\lambda}U(\psi_{\mu\lambda}(x))$ is an open neighborhood of x in $U_{\lambda\mu}$, and both $\psi_{\lambda\mu}$ and $\psi_{\mu\lambda}$ factor through the same morphism of monoidal spaces :

$$\psi(x) : (U(x), (\widetilde{u}_*\underline{M}^\sharp)_{|U(x)}) \to F(x) := (\mathrm{Spec}\, \widetilde{u}_*\underline{M}_x)^\sharp$$

and open immersions $F(x) \to T_\lambda$ and $F(x) \to T_\mu$. It then follows from (3.6.25) that the preimage of $\mathscr{I}_{\rho_\lambda}$ on $F(x)$ agrees with the preimage of $\mathscr{I}_{\rho_\mu}$, whence the contention.

(ii): By inspecting the definitions, it is easily seen that the epimorphism (6.4.17) identifies naturally $\mathscr{O}_{U^{\sim}_{\lambda\mu}}(1)$ to the ideal $\mathscr{I}_{\rho_{\lambda\mu}}\mathscr{O}_{U^{\sim}_{\lambda\mu}}$ of $\mathscr{O}_{U^{\sim}_{\lambda\mu}}$ generated by the image of $\mathscr{I}_{\rho_\lambda}\widetilde{u}_*\underline{M}_{|U_{\lambda\mu}}$. Hence the assertion follows again from (6.6.50). □

6.6.51. Lemma 6.6.49 implies that

$$((U^{\sim}_\lambda, \mathscr{O}_{U^{\sim}_\lambda}(1)), (g_{\lambda\mu}, \omega_{\lambda\mu}) \mid \lambda, \mu \in \Lambda)$$

is a descent datum – relative to the faithfully flat and quasi-compact morphism $\coprod_{\lambda\in\Lambda} U_\lambda \to X$ – for the fibred category of schemes endowed with an ample invertible sheaf. According to [42, Exp.VIII, Prop.7.8], such a datum is effective, hence it yields a projective morphism $\pi : X^{\sim} \to X$ together with an ample invertible sheaf $\mathscr{O}_{X^{\sim}}(1)$ on $X^{\sim}$, with isomorphisms $g_\lambda : X^{\sim} \times_X U_\lambda \xrightarrow{\sim} U^{\sim}_\lambda$ of U_λ-schemes and $\pi^*_\lambda\mathscr{O}_{X^{\sim}}(1) \xrightarrow{\sim} g^*_\lambda\mathscr{O}_{U^{\sim}_\lambda}(1)$ of invertible modules.

Then the datum $(\widetilde{u}^*\underline{M}^{\sim}_\lambda, \widetilde{u}^*\log g_{\lambda\mu} \mid \lambda, \mu \in \Lambda)$ likewise determines a unique sheaf of monoids $\underline{M}^{\sim}$ on $X^{\sim}_{\text{ét}}$, and the structure maps of the log structures $\underline{M}_\lambda$ glue to a well defined morphism of sheaves of monoids $\underline{M}^{\sim} \to \mathscr{O}_{X^{\sim}_{\text{ét}}}$, so that $(X^{\sim}_{\text{ét}}, \underline{M}^{\sim})$ is a log scheme, and the projection π extends to a morphism of log schemes $(\pi, \log\pi) : (X^{\sim}_{\text{ét}}, \underline{M}^{\sim}) \to (X_{\text{ét}}, \underline{M})$.

Proposition 6.6.52. *In the situation of* (6.6.51)*, the counit of adjunction*

$$\widetilde{u}^*\widetilde{u}_*(X^{\sim}_{\text{ét}}, \underline{M}^{\sim}) \to (X^{\sim}_{\text{ét}}, \underline{M}^{\sim})$$

is an isomorphism.

Proof. (This is the counit of the adjoint pair $(\widetilde{u}^*, \widetilde{u}_*)$ of (6.1.6), relating the categories of log structures on $X^\sim_{\mathrm{Zar}}$ and $X^\sim_{\mathrm{\acute{e}t}}$.) Recall that there exist natural epimorphisms $(\mathbb{Q}_+^{\oplus d})_{T(\rho_\lambda)} \to \mathscr{O}_{T(\rho_\lambda),\mathbb{Q}}$ (see (3.6.27)), which induce epimorphisms of $U^\sim_{\lambda,\mathrm{Zar}}$-monoids

$$\vartheta_\lambda : (\mathbb{Q}_+^{\oplus d})_{U^\sim_{\lambda,\mathrm{Zar}}} \to (\underline{M}^\sim_\lambda)^\sharp_{\mathbb{Q}} \qquad \text{for every } \lambda \in \Lambda.$$

The compatibility with open immersions expressed by (3.6.28) implies that the system of maps $(\widetilde{u}^*\vartheta_\lambda \mid \lambda \in \Lambda)$ glues to a well defined epimorphism of $X^\sim_{\mathrm{\acute{e}t}}$-monoids :

$$\vartheta : (\mathbb{Q}_+^{\oplus d})_{X^\sim_{\mathrm{\acute{e}t}}} \to (\underline{M}^\sim)^\sharp_{\mathbb{Q}}.$$

In view of lemma 2.4.26(ii), it follows that the counit of adjunction :

$$\widetilde{u}^*\widetilde{u}_*(\underline{M}^\sim)^\sharp_{\mathbb{Q}} \to (\underline{M}^\sim)^\sharp_{\mathbb{Q}}$$

is an isomorphism. By applying again lemma 2.4.26(ii) to the monomorphism $(\underline{M}^\sim)^\sharp \to (\underline{M}^\sim)^\sharp_{\mathbb{Q}}$, we deduce that also the counit

$$\widetilde{u}^*\widetilde{u}_*(\underline{M}^\sim)^\sharp \to (\underline{M}^\sim)^\sharp$$

is an isomorphism. Then the assertion follows from proposition 6.2.3(iii). $\square$

Corollary 6.6.53. *Let $(X_{\mathrm{\acute{e}t}}, \underline{M})$ be a quasi-compact regular log scheme. Then there exists a smooth morphism of log schemes $(X'_{\mathrm{\acute{e}t}}, \underline{M}') \to (X_{\mathrm{\acute{e}t}}, \underline{M})$ whose underlying morphism of schemes is proper and birational, and such that X' is a regular scheme. More precisely, f restricts to an isomorphism $(X'_{\mathrm{\acute{e}t}}, \underline{M}')_{\mathrm{tr}} \to (X_{\mathrm{\acute{e}t}}, \underline{M})_{\mathrm{tr}}$ on the trivial loci.*

Proof. Given such $(X_{\mathrm{\acute{e}t}}, \underline{M})$, we construct first the morphism $\pi : (X^\sim_{\mathrm{\acute{e}t}}, \underline{M}^\sim) \to (X_{\mathrm{\acute{e}t}}, \underline{M})$ as in (6.6.51). Since $\pi \times_X \mathbf{1}_{U_\lambda}$ is proper for every $\lambda \in \Lambda$, if follows that π is proper ([31, Ch.IV, Prop.2.7.1]). Likewise, notice that each morphism $U^\sim_\lambda \to U_\lambda$ induces an isomorphism $(U^\sim_\lambda, \underline{M}^\sim_\lambda)_{\mathrm{tr}} \xrightarrow{\sim} (U_\lambda, \widetilde{u}_*\underline{M}_{|U_\lambda})_{\mathrm{tr}}$ (remark 6.6.17). It follows easily that π restricts an isomorphism on the trivial loci. Hence, we may replace $(X_{\mathrm{\acute{e}t}}, \underline{M})$ by $(X^\sim_{\mathrm{\acute{e}t}}, \underline{M}^\sim_{\mathrm{\acute{e}t}})$. Then, by corollary 6.3.27(ii) and proposition 6.6.52 we are further reduced to showing that there exists a proper morphism $\pi' : (X'_{\mathrm{Zar}}, \underline{M}') \to \widetilde{u}_*(X^\sim_{\mathrm{\acute{e}t}}, \underline{M}^\sim)$ with X' regular, restricting to an isomorphism on the trivial loci. However, in light of lemma 6.5.37 (and again, proposition 6.6.52), the sought π' is provided by the more precise theorem 6.6.32. $\square$

6.7. **Local properties of the fibres of a smooth morphism.** Resume the situation of example 6.6.5(ii), and to ease notation set $\varphi := (\operatorname{Spec}\lambda)^\sharp$, and $(f, \log f) := \operatorname{Spec}(R, \lambda)$. Suppose now, that $\lambda : P \to Q$ is an integral, local and injective morphism of fine monoids. Then $f : S' \to S$ is flat and finitely presented (theorem 3.2.3). Moreover :

Lemma 6.7.1. *In the situation of* (6.7), *for every point $s \in S$, the fibre $f^{-1}(s)$ is either empty, or else it is pure-dimensional, of dimension*

$$\dim f^{-1}(s) = \dim Q - \dim P = \mathrm{rk}_{\mathbb{Z}} \operatorname{Coker} \lambda^{\mathrm{gp}}.$$

Proof. To begin with, notice that $\lambda^{-1}Q^\times = P^\times$, whence the second stated identity, in view of corollary 3.4.10(i). To prove the first stated identity, we easily reduce to the case where R is a field. Notice that the image of f is an open subset $U \subset S$ ([31, Ch.IV, Th.2.4.6]), especially U (resp. S') is pure-dimensional of dimension $\mathrm{rk}_{\mathbb{Z}}P^{\mathrm{gp}}$ (resp. $\mathrm{rk}_{\mathbb{Z}}Q^{\mathrm{gp}}$) by claim 5.9.36(ii). Hence, fix any closed point $s \in S$, and set $X := f^{-1}(s)$. From [31, Ch.IV, Cor.6.1.2] we deduce that, for every closed point $s' \in X$, the Krull dimension of $\mathscr{O}_{X,s'}$ equals $r := \mathrm{rk}_{\mathbb{Z}}\mathrm{Coker}\,\lambda^{\mathrm{gp}}$. More precisely, say that $Z \subset X$ is any irreducible component; we may find a closed point $s' \in Z$ which does not lie on any other irreducible component of X, and then the foregoing implies that the dimension of Z equals r, as stated.

Next, let $s \in U$ be any point, and denote K the residue field of $\mathscr{O}_{U,s}$, and $\pi : R[P] \to K$ the natural map. Let $y \in \mathrm{Spec}\, K[P]$ be the K-rational closed point such that $a(y) = \pi(a)$ for every $a \in P$; then the image of y in S equals s, and if we let $f_K := \mathrm{Spec}\, K[\lambda]$, we have an isomorphism of K-schemes $f_K^{-1}(y) \xrightarrow{\sim} f^{-1}(s)$. The foregoing shows that $f_K^{-1}(y)$ is pure-dimensional of dimension r, hence the same holds for $f^{-1}(s)$. $\square$

Now, let us fix $s \in S$, such that $f^{-1}(s) \neq \varnothing$, and suppose that $\psi_P(s) = \mathfrak{m}_P$ is the closed point of T_P. For every $\mathfrak{q} \in \varphi^{-1}(\mathfrak{m}_P)$, the closure $\overline{\{\mathfrak{q}\}}$ of $\{\mathfrak{q}\}$ in T_Q is the image of the natural map $\mathrm{Spec}\, Q/\mathfrak{q} \to T_Q$. We deduce natural isomorphisms of schemes :

$$S_0 := \psi_P^{-1}\{\mathfrak{m}_\mathfrak{p}\} \xrightarrow{\sim} \mathrm{Spec}\, R\langle P/\mathfrak{m}_P\rangle \qquad S'_\mathfrak{q} := \psi_Q^{-1}\overline{\{\mathfrak{q}\}} \xrightarrow{\sim} \mathrm{Spec}\, R\langle Q/\mathfrak{q}\rangle$$

under which, the restriction $f_\mathfrak{q} : S'_\mathfrak{q} \to S_0$ of f is identified with $\mathrm{Spec}\, R\langle\lambda_\mathfrak{q}\rangle$, where $\lambda_\mathfrak{q} : P/\mathfrak{m}_P \to Q/\mathfrak{q}$ is induced by λ. The latter is an integral and injective morphism as well (corollary 3.1.51). Explicitly, set $F := Q\setminus\mathfrak{q}$, and let $\lambda_F : P^\times \to F$ be the restriction of λ; then $\lambda_\mathfrak{q} = (\lambda_F)_\circ$, and lemma 6.7.1 yields the identity :

(6.7.2) $$\dim f_\mathfrak{q}^{-1}(s) = \dim Q/\mathfrak{q} = \dim Q - \mathrm{ht}\,\mathfrak{q}.$$

Also, notice that T_Q is a finite set under the current assumptions (lemma 3.1.20(iii)), and clearly

$$f^{-1}(s) = \bigcup_{\mathfrak{q}\in\mathrm{Max}(\varphi^{-1}\mathfrak{m}_P)} f_\mathfrak{q}^{-1}(s)$$

where, for a topological space T, we denote by $\mathrm{Max}(T)$ the set of maximal points of T. Therefore, for every irreducible component Z of $f^{-1}(s)$ there must exist $\mathfrak{q} \in \mathrm{Max}(\varphi^{-1}\mathfrak{m}_P)$ such that $Z \subset f_\mathfrak{q}^{-1}(s)$, and especially,

(6.7.3) $$\dim f^{-1}(s) = \dim f_\mathfrak{q}^{-1}(s).$$

Conversely, we claim that (6.7.3) holds for every $\mathfrak{q} \in \mathrm{Max}(\varphi^{-1}\mathfrak{m}_P)$. Indeed, suppose that $\dim f_\mathfrak{q}^{-1}(s)$ is strictly less than $\dim f^{-1}(s)$ for one such $\mathfrak{q}$, and let Z be an irreducible component of $f_\mathfrak{q}^{-1}(s)$; let also Z' be an irreducible component of $f^{-1}(s)$ containing Z. By the foregoing, there exists $\mathfrak{q}' \in \mathrm{Max}(\varphi^{-1}\mathfrak{m}_P)$ with $Z' \subset f_{\mathfrak{q}'}^{-1}(s)$. Set $\mathfrak{q}'' := \mathfrak{q}\cup\mathfrak{q}'$; then clearly $\mathfrak{q}'' \in \varphi^{-1}(\mathfrak{m}_P)$, and

$$\overline{\{\mathfrak{q}''\}} = \overline{\{\mathfrak{q}\}} \cap \overline{\{\mathfrak{q}'\}}.$$

Especially, $Z \subset f_{\mathfrak{q}''}^{-1}(s)$; however, it follows from (6.7.2) that $\dim f_{\mathfrak{q}''}^{-1}(s) < \dim f_\mathfrak{q}^{-1}(s)$ (since $\mathrm{ht}\,\mathfrak{q} < \mathrm{ht}\,\mathfrak{q}''$); but this is absurd, since $\dim Z = \dim f_\mathfrak{q}^{-1}(s)$ (lemma 6.7.1). The same counting argument also shows that every maximal point of $f^{-1}(s)$ gets mapped necessarily to a maximal point of $\varphi^{-1}(\mathfrak{m}_P)$; in other words, we have shown that ψ_Q restricts to a surjective map :

$$\mathrm{Max}(f^{-1}s) \to \mathrm{Max}(\varphi^{-1}\mathfrak{m}_P).$$

More precisely, let $s' \in f^{-1}(s)$ be a point such that $\psi_Q(s') = \mathfrak{m}_Q$, the closed point of T_Q. Then clearly $s' \in f_\mathfrak{q}^{-1}(s)$ for every $\mathfrak{q} \in \mathrm{Max}(\varphi^{-1}\mathfrak{m}_P)$, and it follows that the foregoing surjection restricts further to a surjective map :

(6.7.4) $$f_{s'}^{-1}(s) \to \mathrm{Max}(\varphi^{-1}\mathfrak{m}_P).$$

On the other hand, since $\log\psi_Q$ is an isomorphism, we have

$$(Q_{S'}^{\log})^\sharp_{\overline{s}'} = \mathscr{O}_{T_Q,\mathfrak{q}} = Q^\sharp_\mathfrak{q} \qquad \text{for every } s' \in \psi_Q^{-1}(\mathfrak{q})$$

(where $\overline{s}'$ denotes any τ-point of S' localized at s'); explicitly, if $F = Q\setminus\mathfrak{q}$, then $Q^\sharp_\mathfrak{q} = Q/F$; likewise, we get $(P_S^{\log})^\sharp_{\overline{s}} = P/\varphi^{-1}F$. Taking into account (6.7.2), we deduce :

$$\dim f^{-1}(f(s')) - \dim f_\mathfrak{q}^{-1}(f(s')) = \mathrm{rk}_\mathbb{Z}\, \mathrm{Coker}\,(\log f)^{\mathrm{gp}}_{\overline{s}'} \qquad \text{for every } s' \in \psi_Q^{-1}(\mathfrak{q}).$$

Thus, for every $n \in \mathbb{N}$, let

$$U_n := \{s' \in S' \mid \mathrm{rk}_{\mathbb{Z}}\, \mathrm{Coker}\, (\log f)^{\mathrm{gp}}_{\overline{s}'} = n\}.$$

The foregoing implies that for every $s \in S$, the subset $U_0 \cap f^{-1}(s)$ is open and dense in $f^{-1}(s)$, and $U_n \cap f^{-1}(s)$ is either empty, or else it is a subset of pure codimension n in $f^{-1}(s)$.

6.7.5. In the situation of (6.7), suppose moreover that $\log f$ is saturated; notice that this condition holds if and only if $\log \varphi$ is saturated, if and only if the same holds for λ (lemma 3.2.12(iii)). Then, corollary 3.2.32(ii) says that $\mathrm{Coker}(\log f^\sharp)^{\mathrm{gp}}_{\overline{s}'}$ is torsion-free for every $s' \in S'$; especially, $\mathrm{Coker}(\log f^\sharp)^{\mathrm{gp}}_{\overline{s}'}$ vanishes for every $s' \in U_0$. Then corollary 3.2.32(i) implies that $\log f^\sharp_{\overline{s}'}$ is an isomorphism for every $s' \in U_0$, in which case the same holds for $\log f_{\overline{s}'}$ (lemma 6.1.4); in other words, U_0 is the strict locus of f (see definition 6.2.7(ii)).

Lemma 6.7.6. *Let K be an algebraically closed field of characteristic p, and $\chi : P \to (K, \cdot)$ a local morphism of monoids. Let also $\lambda : P \to Q$ be as in* (6.7.5). *We have :*

(i) *The K-algebra $Q \otimes_P K$ is Cohen-Macaulay.*
(ii) *Suppose moreover, that either $p = 0$, or else the order of the torsion subgroup of $\Gamma := \mathrm{Coker}\, \lambda^{\mathrm{gp}}$ is not divisible by p. Then $Q \otimes_P K$ is reduced (*i.e. *its nilradical is trivial).*

Proof. (i): Since χ is a local morphism, we have $\chi^{-1}(0) = \mathfrak{m}_P$, and χ is determined by its restriction $P^\times \to K^\times$, which is a homomorphism of abelian groups. Notice that $K^\times$ is divisible, hence it is injective in the category of abelian groups; since the unit of adjunction $P \to P^{\mathrm{sat}}$ is local (lemma 3.2.9(iii)), it follows that χ extends to a local morphism $\chi' : P^{\mathrm{sat}} \to K$. Notice that $Q \otimes_P K = Q^{\mathrm{sat}} \otimes_{P^{\mathrm{sat}}} K$ (lemma 3.2.12(iv)), hence we may replace P by P^{sat} and Q by Q^{sat}, which allows to assume from start that Q is saturated. In this case, by theorem 5.9.34(i), both $K[P]$ and $K[Q]$ are Cohen-Macaulay; since $K[\lambda]$ is flat, theorem 5.4.36 and [31, Ch.IV, Cor.6.1.2] imply that $Q \otimes_P K$ is Cohen-Macaulay as well.

(ii): Let $Q = \bigoplus_{\gamma \in \Gamma} Q_\gamma$ be the λ-grading of Q (see remark 3.2.5(iii)). Under the current assumptions, λ is exact (lemma 3.2.30(ii)), consequently $Q_0 = P$ (remark 3.2.5(v)). Moreover, Q_γ is a finitely generated P-module, for every $\gamma \in \Gamma$ (corollary 3.4.3), hence either $Q_\gamma = \varnothing$, or else Q_γ is a free P-module of rank one, say generated by an element u_γ (remark 3.2.5(iv)). Furthermore, $Q_\gamma^k = Q_{k\gamma}$ for every integer $k > 0$ and every $\gamma \in \Gamma$ (proposition 3.2.31). Thus, whenever $Q_\gamma \neq \varnothing$, the element $u_\gamma \otimes 1$ is a basis of the K-vector space $Q_\gamma \otimes_P K$, and $(u_\gamma \otimes 1)^k$ is a basis of $Q_{k\gamma} \otimes_P K$, for every $k > 0$; especially, $u_\gamma \otimes 1$ is not nilpotent. In view of proposition 4.4.12(ii), the contention follows. $\square$

6.7.7. Let $f : (X, \underline{M}) \to (Y, \underline{N})$ be a smooth and log flat morphism of fine log schemes. For every $n \in \mathbb{N}$, let $U(f, n) \subset X$ be the subset of all $x \in X$ such that $\mathrm{rk}_{\mathbb{Z}}\mathrm{Coker}\, (\log f^\sharp_{\overline{x}})^{\mathrm{gp}} = n$, for any τ-point $\overline{x}$ localized at x. By lemma 6.2.21(iii), $U(f, n)$ is a locally closed subset (resp. an open subset) of X, for every $n > 0$ (resp. for $n = 0$).

Theorem 6.7.8. *In the situation of* (6.7.7)*, we have :*

(i) *f is a flat morphism of schemes.*
(ii) *For all $y \in Y$, every connected component of $f^{-1}(y)$, is pure dimensional, and $f^{-1}(y) \cap U_n$ is either empty, or else it has pure codimension n in $f^{-1}(y)$, for every $n \in \mathbb{N}$.*
(iii) *Moreover, if f is saturated, we have :*
  (a) *The strict locus $\mathrm{Str}(f)$ of f is open in X, and $\mathrm{Str}(f) \cap f^{-1}(y)$ is a dense subset of $f^{-1}(y)$, for every $y \in Y$.*
  (b) *$f^{-1}(y)$ is geometrically reduced and Cohen-Macaulay, for every $y \in Y$.*
  (c) *$\mathrm{Coker}(\log f^\sharp_{\overline{x}})^{\mathrm{gp}}$ is a free abelian group of finite rank, for every τ-point $\overline{x}$ of X.*

Proof. Let ξ be any τ-point of X; according to corollary 6.1.33, there exists a neighborhood V of ξ, and a fine chart $P_V \to \underline{N}_{|V}$, such that P^{gp} is a free abelian group of finite rank, and the induced morphism of monoids $P \to \mathscr{O}_{Y,\xi}$ is local.

By lemma 6.1.12, [31, Ch.IV, Th.2.4.6, Prop.2.5.1] and [33, Ch.IV, Prop.17.5.7], it suffices to prove the theorem for the morphism $f \times_Y V : (X, \underline{M}) \times_Y V \to (V, \underline{N}_{|V})$. Hence we may assume from start that $\underline{N}$ admits a fine chart $\beta : P_Y \to \underline{N}$, such that P^{gp} is torsion-free and the induced map $P \to \mathscr{O}_{Y,\xi}$ is local.

Next, by theorem 6.3.37, remark 6.1.21(i), [31, Ch.IV, Th.2.4.6] and [33, Ch.IV, Prop.17.5.7] (and again lemma 6.1.12) we may further assume that f admits a fine chart $(\beta, \omega_Q : Q_X \to \underline{M}, \lambda)$, such that λ is injective, the torsion subgroup of $\mathrm{Coker}\, \lambda^{\mathrm{gp}}$ is a finite group whose order is invertible in $\mathscr{O}_X$, and the induced morphism of schemes $X \to Y \times_{\mathrm{Spec}\, \mathbb{Z}[Q]} \mathrm{Spec}\, \mathbb{Z}[P]$ is étale. Moreover, by theorem 6.1.35(iii), we may assume – after replacing Q by a localization, and X by a neighborhood of ξ in X_τ – that the morphism $\lambda : P \to Q$ is integral (resp. saturated, if f is saturated), and the morphism $Q \to \mathscr{O}_{X,\xi}$ induced by $\omega_{P,\xi}$ is local, so λ is local as well.

In this case, the same sort of reduction as in the foregoing shows that, in order to prove (i) and (ii), it suffices to consider a morphism f as in (6.7), for which these assertions have already been established. Likewise, in order to show (iii), it suffices to consider a morphism f as in (6.7.5). For such a morphism, assertions (iii.a) and (iii.c) are already known, and (iii.b) is an immediate consequence of lemma 6.7.6. $\square$

Theorem 6.7.8(iii) admits the following partial converse :

Proposition 6.7.9. *Let $f : (X, \underline{M}) \to (Y, \underline{N})$ be a smooth and log flat morphism of fs log schemes, such that $f^{-1}(y)$ is geometrically reduced, for every $y \in Y$. Then f is saturated.*

Proof. Fix a τ-point ξ of X; the assertion can be checked on stalks, hence we may assume that $\underline{N}$ admits a fine and saturated chart $\beta : P_Y \to \underline{N}$ (lemma 6.1.16(iii)), such that the induced morphism $\alpha_P : P \to \mathscr{O}_{Y,f(\xi)}$ is local (claim 6.1.29). Then, by corollary 6.3.42, we may find an étale morphism $g : U \to X$ and a τ-point ξ' of U with $g(\xi') = \xi$, such that the induced morphism of log schemes $f_U : (U, g^*\underline{M}) \to (Y, \underline{N})$ admits a fine and saturated chart $(\beta, \omega_Q : Q_U \to g^*\underline{M}, \lambda)$, where λ is injective, and the induced ring homomorphism

$$Q \otimes_P \mathscr{O}_{Y,f(\xi)} \to \mathscr{O}_{U,\xi'} \tag{6.7.10}$$

is étale. By [33, Ch.IV, Prop.17.5.7], the fibres of f_U are still geometrically reduced, hence we are reduced to the case where $U = X$ and $\xi = \xi'$. Furthermore, after replacing Q by a localization, and X by a neighborhood of ξ, we may assume that the map $\alpha_Q : Q \to \mathscr{O}_{X,\xi}$ induced by $\omega_{Q,\xi}$ is local and λ is integral (theorem 6.1.35(iii) and lemma 3.2.9(i)). Lastly, let K be the residue field of $\mathscr{O}_{Y,f(\xi)}$; our assumption on $f^{-1}(y)$ means that the ring $A := \mathscr{O}_{X,\xi} \otimes_{\mathscr{O}_{Y,f(\xi)}} K$ is reduced.

We shall apply the criterion of proposition 3.2.31. Thus, let $Q = \bigoplus_{\gamma \in \Gamma} Q_\gamma$ be the λ-grading of Q and notice as well that λ is a local morphism (since the same holds for α_P and α_Q), therefore it is exact (lemma 3.2.30(ii)); consequently $Q_0 = P$ (remark 3.2.5(v)). Moreover, Q_γ is a finitely generated P-module, for every $\gamma \in \Gamma$ (corollary 3.4.3), hence either $Q_\gamma = \varnothing$, or else Q_γ is a free P-module of rank one.

We have to prove that $Q_\gamma^k = Q_{k\gamma}$ for every integer $k > 0$ and every $\gamma \in \Gamma$. In case $Q_\gamma = \varnothing$, this is the assertion that $Q_{k\gamma} = \varnothing$ as well. However, since Q is saturated, the same holds for $\Gamma' := (\lambda P)^{-1}Q/(\lambda P)^{\mathrm{gp}}$ (lemma 3.2.9(i,ii)), so it suffices to remark that $\Gamma' \subset \Gamma$ is precisely the submonoid consisting of all those $\gamma \in \Gamma$ such that $Q_\gamma \neq \varnothing$.

Therefore, fix a generator u_γ for every P-module $Q_\gamma \neq \varnothing$, and by way of contradiction, suppose that there exist $\gamma \in \Gamma'$ and $k > 0$ such that u_γ^k does not generate the P-module $Q_{k\gamma}$; this means that there exists $a \in \mathfrak{m}_P$ such that $u_\gamma^k = a \cdot u_{k\gamma}$. Now, notice that the induced morphism of monoids $\beta : P \to K$ is local, especially $\beta(a) = 0$, therefore $(u_\gamma \otimes 1)^k = 0$ in the K-algebra $Q \otimes_P K$. Denote by $I \subset Q \otimes_P K$ the annihilator of $u_\gamma \otimes 1$, and notice that, since (6.7.10) is flat, IA is the annihilator of the image u' of $u_\gamma \otimes 1$ in A.

On the other hand, it is easily seen that I is the graded ideal generated by $(u_\mu \otimes 1 \mid \mu \in \Gamma'')$ where $\Gamma'' \subset \Gamma'$ is the subset of all μ such that $u_\gamma \cdot u_\mu$ is not a generator of the P-module $Q_{\gamma+\mu}$. Clearly $u_\mu \notin Q^\times$ for every $\mu \in \Gamma''$, therefore the image of $u_\mu \otimes 1$ in A lies in the maximal ideal. Therefore $IA \neq A$, *i.e.* u' is a non-zero nilpotent element, a contradiction. $\square$

6.7.11. Let $(X, \underline{M})$ be any log scheme, and $\overline{x}$ any geometric point, localized at a point $x \in X$. Suppose that $y \in X$ is a generization of x, and $\overline{y}$ a geometric point localized at y; then, arguing as in (2.4.22), we may extend uniquely any strict specialization morphism $X(\overline{y}) \to X(\overline{x})$ to a morphism of log schemes $(X(\overline{y}), \underline{M}(\overline{y})) \to (X(\overline{x}), \underline{M}(\overline{x}))$ fitting into a commutative diagram

$$\begin{tikzcd}
(X(\overline{y}), \underline{M}(\overline{y})) \arrow[r] \arrow[d] & (X(y), \underline{M}(y)) \arrow[d] \\
(X(\overline{x}), \underline{M}(\overline{x})) \arrow[r] & (X(x), \underline{M}(x))
\end{tikzcd}$$

whose right vertical arrow is induced by the natural isomorphism

$$(X(y), \underline{M}(y)) \xrightarrow{\sim} (X(x), \underline{M}(x)) \times_{X(x)} X(y).$$

A simple inspection shows that the induced morphism

$$\Gamma(X(\overline{x}), \underline{M}(\overline{x}))^\sharp \to \Gamma(X(\overline{y}), \underline{M}(\overline{y}))^\sharp$$

is naturally identified with the morphism $\underline{M}^\sharp_{\overline{x}} \to \underline{M}^\sharp_{\overline{y}}$ obtained from the specialization map $\underline{M}_{\overline{x}} \to \underline{M}_{\overline{y}}$.

6.7.12. Let $g : (X, \underline{M}) \to (Y, \underline{N})$ be a morphism of log schemes, $\overline{x}$ any geometric point of X, and set $\overline{y} := g(\overline{x})$. The log structures $\underline{M} \to \mathscr{O}_X$ and $\underline{N} \to \mathscr{O}_Y$, and the map $\log g_{\overline{x}}$ induce a commutative diagram of continuous maps

$$\begin{tikzcd}
X(\overline{x}) \arrow[r, "g_{\overline{x}}"] \arrow[d, "\psi_{\overline{x}}"'] & Y(\overline{y}) \arrow[d, "\psi_{\overline{y}}"] \\
\mathrm{Spec}\, \underline{M}_{\overline{x}} \arrow[r, "\varphi_{\overline{x}}"] & \mathrm{Spec}\, \underline{N}_{\overline{y}}
\end{tikzcd} \tag{6.7.13}$$

(notation of 2.4.19), and notice that $\psi_{\overline{x}}$ (resp. $\psi_{\overline{y}}$) maps the closed point of $X(\overline{x})$ (resp. of $Y(\overline{y})$) to the closed point $t_{\overline{x}} \in \mathrm{Spec}\, \underline{M}_{\overline{x}}$ (resp. $t_{\overline{y}} \in \mathrm{Spec}\, \underline{N}_{\overline{y}}$).

Proposition 6.7.14. *In the situation of* (6.7.12)*, suppose that g is a smooth morphism of fine log schemes, and moreover :*

(a) *either g is a saturated morphism*
(b) *or $(X, \underline{M})$ is a fs log scheme.*

Then the following holds :

(i) *The map $\psi_{\overline{x}}$ restricts to a bijection :*

$$\mathrm{Max}(g_{\overline{x}}^{-1}(\overline{y})) \xrightarrow{\sim} \mathrm{Max}(\varphi_{\overline{x}}^{-1}(t_{\overline{y}})).$$

(ii) *For every irreducible component Z of $g_{\overline{x}}^{-1}(\overline{y})$, set*

$$(Z, \underline{M}(Z)) := (X(\overline{x}), \underline{M}(\overline{x})) \times_{X(\overline{x})} Z.$$

Then the $\kappa(\overline{y})$-log scheme $(Z, \underline{M}(Z)_{\mathrm{red}})$ is geometrically pointed regular. (Notation of example 6.1.10(iv) *and remark* 6.5.24(ii)*.)*

Proof. By corollary 6.1.33, there exists a neighborhood $U \to Y$ of $\overline{y}$, and a fine chart $\beta : P_U \to \underline{N}_{|U}$ such that $\beta_{\overline{y}}$ is a local morphism, and P^{gp} is torsion-free. Now, let

$$g_{\overline{x}} := (g_{\overline{x}}, \log g_{\overline{x}}) : (X(\overline{x}), \underline{M}(\overline{x})) \to (Y(\overline{y}), \underline{N}(\overline{y}))$$

be the morphism of log schemes induced by g (notation of 6.7.11); by theorem 6.3.37, we may find a fine chart for $g_{\overline{x}}$ of the type $(i^*_{\overline{y}}\beta, \omega, \lambda : P \to Q)$, where λ is injective, and the order of the torsion subgroup of $\mathrm{Coker}\,\lambda^{\mathrm{gp}}$ is invertible in $\mathscr{O}_{X,x}$. Moreover, set $R := \mathscr{O}_{Y(\overline{y}),\overline{y}}$; then the induced map $X(\overline{x}) \to \mathrm{Spec}\, Q \otimes_P R$ is ind-étale, and – after replacing Q by some localization – we may assume that $\omega_{\overline{x}} : Q \to \mathscr{O}_{X(\overline{x}),\overline{x}}$ is local (claim 6.1.29), hence the same holds for λ. Furthermore, under assumption (a) (resp. (b)), we may also suppose that Q is saturated, by lemmata 3.2.9(ii) and 6.1.16(ii) (resp. that λ is saturated, by theorem 6.1.35(iii)).

Let us now define $f : S' \to S$ as in (6.7); notice that $\omega_{\overline{x}}$ induces a closed immersion $Y(\overline{y}) \to S$, and we have a natural identification of $Y(\overline{y})$-schemes :

$$\mathrm{Spec}\, Q \otimes_P R = Y(\overline{y}) \times_S S'.$$

Denote by $\overline{s}$ the image of $\overline{y}$ in S, and $\overline{s}'$ the image of $\overline{x}$ in $Y(\overline{y}) \times_S S' \subset S'$; there follows an isomorphism of $Y(\overline{y})$-schemes :

$$X(\overline{x}) \xrightarrow{\sim} Y(\overline{y}) \times_{S(\overline{s})} S'(\overline{s}')$$

([33, Ch.IV, Prop.18.8.10]). Moreover, our chart induces isomorphisms :

$$\mathrm{Spec}\, \underline{M}_{\overline{x}} \xrightarrow{\sim} T_Q \qquad \mathrm{Spec}\, \underline{N}_{\overline{y}} \xrightarrow{\sim} T_P$$

which identify $\varphi_{\overline{x}}$ to the map $\varphi : T_Q \to T_P$ of (6.7). In view of these identifications, we see that (6.7.13) is the restriction to the closed subset $X(\overline{x})$, of the analogous diagram :

$$\begin{tikzcd} S'(\overline{s}') \arrow[r, "f_{\overline{s}'}"] \arrow[d, "\psi_{\overline{s}'}"'] & S(\overline{s}) \arrow[d, "\psi_{\overline{s}}"] \\ T_Q \arrow[r, "\varphi"] & T_P. \end{tikzcd}$$

We are thus reduced to the case where $X = S'$, $Y = S$ and $g = f$. Moreover, let s (resp. s') be the support of $\overline{s}$ (resp. of $\overline{s}'$); the morphism $\pi : S'(\overline{s}') \to S'(s')$ is flat, hence it restricts to a surjective map

$$\mathrm{Max}(f^{-1}_{\overline{s}'}\overline{s}) \to \mathrm{Max}(f^{-1}_{s'}s).$$

In order to prove (i), it suffices then to show that the map $\mathrm{Max}(f^{-1}_{\overline{s}'}\overline{s}) \to \mathrm{Max}(\varphi^{-1}\mathfrak{m}_P)$, defined as the composition of the foregoing map and the surjection (6.7.4), is injective. This boils down to the assertion that, for every $\mathfrak{q} \in \mathrm{Max}(\varphi^{-1}\mathfrak{m}_P)$ the point s' lies on a unique irreducible component of the fibre $\pi^{-1}(f^{-1}_{\mathfrak{q},s'}s)$. However, let $\overline{\beta} : P \to \kappa(s)$ be the composition of the chart $P \to R$ and the projection $R \to \kappa(s)$; then $f^{-1}_{\mathfrak{q}}(s) = \mathrm{Spec}\, Q/\mathfrak{q} \otimes_P \kappa(s)$. Since π is ind-étale, the assertion will follow from [33, Ch.IV, Prop.17.5.7] and corollary 6.5.29, together with :

Claim 6.7.15. The log scheme $W_{\mathfrak{q}} := \mathrm{Spec}\langle \mathbb{Z}, Q/\mathfrak{q}\rangle \times_{\mathrm{Spec}\,\mathbb{Z}[P]} \mathrm{Spec}\, \kappa(s)$ is geometrically pointed regular.

Proof of the claim. To ease notation, set $F := Q \setminus \mathfrak{q}$; by assumption $\lambda^{-1}F = P^\times$, so that

$$W_{\mathfrak{q}} = (W'_{\mathfrak{q}})_\circ \qquad \text{where} \qquad W'_{\mathfrak{q}} := \mathrm{Spec}(\mathbb{Z}, F) \times_{\mathrm{Spec}(\mathbb{Z},P^\times)} \mathrm{Spec}\, \kappa(s)$$

(notation of (6.2.13); notice that the log structures of $\mathrm{Spec}(\mathbb{Z}, P^\times)$ and $\mathrm{Spec}\, \kappa(s)$ are trivial). Moreover, let $\lambda_F : P^\times \to F$ be the restriction of λ; then λ_F^{gp} is injective, and

$$\mathrm{Coker}\,\lambda_F^{\mathrm{gp}} \subset \mathrm{Coker}\,\lambda^{\mathrm{gp}}$$

(corollary 3.2.33(i)), hence the order of the torsion subgroup of $\mathrm{Coker}\lambda_F^{\mathrm{gp}}$ is invertible in $\mathscr{O}_{S,s}$. Moreover, if λ is saturated, then the same holds for λ_F (corollary 3.2.33(ii)), and it is easily seen that, if Q is saturated, the same holds for F. Consequently, the morphism $\mathrm{Spec}(\mathbb{Z}, \lambda_F)$ is smooth (proposition 6.3.34). The same then holds for the morphism $W'_{\mathfrak{q}} \to \mathrm{Spec}\,\kappa(s)$ obtained after base change of $\mathrm{Spec}(\mathbb{Z}, \lambda_F)$ along the morphism $h : \mathrm{Spec}(\mathbb{Z}, P^\times) \to \mathrm{Spec}\,\kappa(s)$ induced by $\overline{\beta}$ (proposition 6.3.24(ii)).

Now we notice that under either of the assumptions (a) or (b), $W'_{\mathfrak{q}}$ is a fs log scheme. Indeed, under assumption (b), this follows by remarking that $\mathrm{Spec}\,\kappa(s)$ is trivially a fs log scheme, and $\mathrm{Spec}(\mathbb{Z}, \lambda_F)$ is saturated. Under assumption (a), $\mathrm{Spec}(\mathbb{Z}, F)$ is a fs log scheme, and it suffices to observe that h is a strict morphism. Lastly, since the morphism $W'_{\mathfrak{q}} \to \mathrm{Spec}\,\kappa(s)$ is obviously saturated, we apply corollary 6.5.45 to conclude. ◊

(ii): In light of the foregoing, we see that, for any irreducible component Z of $g_{\overline{x}}^{-1}(\overline{y})$, there exists a unique $\mathfrak{q}(Z) \in \mathrm{Max}(\varphi^{-1}\mathfrak{m}_P)$ such that Z is isomorphic to the strict henselization of $f_{\mathfrak{q}(Z)}^{-1}(s)$, at the geometric point $\overline{s}'$. Notice now that the log structure of $W_{\mathfrak{q}(Z)}$ is reduced, by virtue of claim 6.7.15 and proposition 6.5.52; then, a simple inspection of the definitions shows that $(Z, \underline{M}(Z)_{\mathrm{red}})$ is isomorphic to the strict henselization $W_{\mathfrak{q}(Z)}(\overline{s}')$. Invoking again claim 6.7.15, we deduce the contention. □

6.7.16. In the situation of (6.7.12), suppose moreover that Y is a normal scheme, and $(Y, \underline{N})_{\mathrm{tr}}$ is a dense subset of Y. Let $\overline{\eta}$ be a geometric point of $Y(\overline{y})$, localized at the generic point η. Let

$$(U_{\mathfrak{q}} \mid \mathfrak{q} \in \mathrm{Spec}\,\underline{M}_{\overline{x}})$$

be the logarithmic stratification of $(X(\overline{x}), \underline{M}(\xi))$ (see (6.5.49)). Notice that $\psi_{\overline{x}}(g^{-1}(\eta))$ lies in the preimage $\Sigma \subset \mathrm{Spec}\,\underline{M}_{\overline{x}}$ of the maximal point $\varnothing$ of $\mathrm{Spec}\,\underline{N}_{\overline{y}}$; therefore, $g_{\overline{x}}^{-1}(\eta)$ is the union of the subsets $U_{\mathfrak{q}} \times_Y |\eta|$, for all $\mathfrak{q} \in \Sigma$.

Proposition 6.7.17. *In the situation of* (6.7.16), *suppose that g is a smooth morphism of fine log schemes. Then the following holds :*

(i) *X is a normal scheme.*
(ii) *The scheme $g_{\overline{x}}^{-1}(\overline{\eta})$ is normal and irreducible.*
(iii) *For every $\mathfrak{q} \in \Sigma$, the $\kappa(\eta)$-scheme $U_{\mathfrak{q}} \times_Y |\eta|$ is non-empty, geometrically normal and geometrically irreducible, of pure dimension* $\dim g_{\overline{x}}^{-1}(\eta) - \mathrm{ht}\,\mathfrak{q}$.
(iv) *Especially, set $W := (X(\overline{x})\backslash U_\varnothing) \times_Y |\overline{\eta}|$; then $\psi_{\overline{x}}$ induces a bijection :*

$$\mathrm{Max}(W) \xrightarrow{\sim} \{\mathfrak{q} \in \Sigma \mid \mathrm{ht}\mathfrak{q} = 1\}.$$

(v) *For every $w \in \mathrm{Max}(W)$, the stalk $\mathscr{O}_{g_{\overline{x}}^{-1}(\overline{\eta}),w}$ is a discrete valuation ring.*

Proof. Set $R := \mathscr{O}_{Y(\overline{y}),\overline{y}}$; arguing as in the proof of proposition 6.7.14, we may find :

- a local, flat and saturated morphism $\lambda : P \to Q$ of fine monoids, such that the order of the torsion subgroup of $\mathrm{Coker}\,\lambda^{\mathrm{gp}}$ is invertible in R;
- local morphisms of monoids $P \to R$, $Q \to \mathscr{O}_{X(\overline{x}),\overline{x}}$ which are charts for the log structures deduced from $\underline{N}$ and respectively $\underline{M}$, and such that the induced morphism of $Y(\overline{y})$-schemes $X(\overline{x}) \to \mathrm{Spec}\,Q \otimes_P R$ is ind-étale.

By [33, Ch.IV, Prop.17.5.7], we may then assume that $(X, \underline{M})$ (resp. $(Y, \underline{N})$) is the scheme $\mathrm{Spec}\,Q \otimes_P R$ (resp. $\mathrm{Spec}\,R$), endowed with the log structure deduced from the natural map $Q \to Q \otimes_P R$ (resp. the chart $P \to R$), and g is the natural projection. Suppose first that R is excellent, and let R' be the normalization of R in a finite extension K' of $\mathrm{Frac}(R)$; then R' is also strictly local and noetherian, and if y' denotes the closed point of $Y' := \mathrm{Spec}\,R'$, then the residue field extension $\kappa(y) \subset \kappa(y')$ is purely inseparable. Set $(X', \underline{M}') := (X, \underline{M}) \times_Y Y'$, $(Y', \underline{N}') := (Y, \underline{N}) \times_Y Y'$, and let $g' : (X', \underline{M}') \to (Y', \underline{N}')$ be the induced morphism of log schemes; it follows especially that the restriction $g'^{-1}(y') \to g^{-1}(y)$ is a homeomorphism on the underlying topological spaces. Hence, there is a geometric point $\overline{x}'$ of X', unique up to isomorphism, whose image in X agrees with $\overline{x}$, and we easily deduce an isomorphism of Y-schemes ([33, Ch.IV, Prop.18.8.10])

$$X'(\overline{x}') \xrightarrow{\sim} X(\overline{x}) \times_Y Y'. \tag{6.7.18}$$

Let η' be the generic point of Y'; by assumption, $\underline{N}'$ is trivial in a Zariski neighborhood of η', hence $(X', \underline{N}') \times_{Y'} |\eta'|$ is a fs log schemes (since g is saturated), and then the same log scheme is also regular (theorem 6.5.44), therefore $g'^{-1}(\eta')$ is a normal scheme (corollary 6.5.29). On the other hand, R' is a Krull domain (see remark 7.1.27), and g' is flat with reduced fibres (theorem 6.7.8(iii.b)), so X' is a noetherian normal scheme (claim 7.1.32), and consequently the same holds for $X'(\overline{x}')$ ([33, Ch.IV, Prop.18.8.12(i)]); especially, the latter is irreducible, so the same holds for $X'(\overline{x}') \times_{Y'} |\eta'|$. In view of (6.7.18), it follows that $X(\overline{x}) \times_Y |\eta'|$ is also normal and irreducible. Since K' is arbitrary, this completes the proof of (i) and (ii), in this case.

Next, if R is any normal ring, we may write R as the union of a filtered family $(R_i \mid i \in I)$ of excellent normal local subrings ([31, Ch.IV, (7.8.3)(ii,vi)]). For each $i \in I$, denote by $\overline{y}_i$ the image of $\overline{y}$ in $\mathrm{Spec}\, R_i$; then the strict henselization R_i^{sh} of R_i at $\overline{y}_i$ is also a subring of R, so we may replace R_i by R_i^{sh}, which allows to assume that each R_i is strictly local, normal and noetherian ([33, Ch.IV, Prop.18.8.8(iv), Prop.18.8.12(i)]). Up to replacing I by a cofinal subset, we may assume that the image of P lies in R_i, for every $i \in I$. For each $i \in I$, set $X_i := \mathrm{Spec}\, Q \otimes_P R_i$, $Y_i := \mathrm{Spec}\, R_i$, and endow X_i (resp. Y_i) with the log structure $\underline{M}_i$ (resp. $\underline{N}_i$) deduced from the natural map $Q \to Q \otimes_P R_i$ (resp. $P \to R_i$). There follows a system of morphisms of log schemes $g_i : (X_i, \underline{M}_i) \to (Y_i, \underline{N}_i)$ for every $i \in I$, whose limit is the morphism g. Moreover, since $(Y, \underline{N})_{\mathrm{tr}}$ is dense in Y, the image of P lies in $R \backslash \{0\}$, hence it lies in $R_i \setminus \{0\}$ for every $i \in I$, and the latter means that $(Y_i, \underline{N}_i)_{\mathrm{tr}}$ is dense in Y_i, for every $i \in I$. Let $\overline{\eta}_i$ (resp. $\overline{x}_i$) be the image of $\overline{\eta}$ (resp. of $\overline{x}$) in Y_i (resp. in X_i); moreover, for each $i \in I$, let $x_i \in X_i$ be the support of $\overline{x}_i$. By the previous case, we know that $g^{-1}_{i,\overline{x}_i}(\overline{\eta}_i)$ is normal and irreducible. However, X (resp. $g^{-1}_{\overline{x}}(\overline{\eta})$) is the limit of the system of schemes $(X_i \mid i \in I)$ (resp. $(g^{-1}_{i,\overline{x}_i}(\overline{\eta}_i) \mid i \in I)$), so (i) and (ii) follow. (Notice that the colimit of a filtered system of integral (resp. normal) domains, is an integral (resp. normal) domain : exercise for the reader.)

(iii): For every $\mathfrak{q} \in \mathrm{Spec}\, Q = \mathrm{Spec}\, \underline{M}_{\overline{x}}$, set $X_\mathfrak{q} := \mathrm{Spec}\, Q/\mathfrak{q} \otimes_P R$; since the chart $Q \to \mathscr{O}_{X(\overline{x}),\overline{x}}$ is local, it is clear that $x \in X_\mathfrak{q}$ for every such $\mathfrak{q}$, and :

$$X_\mathfrak{q}(\overline{x}) = \bigcup_{\mathfrak{p} \in \mathrm{Spec}\, Q/\mathfrak{q}} U_\mathfrak{p}.$$

If $\varphi(\mathfrak{q}) = \varnothing$, the induced map $P \to Q/\mathfrak{q}$ is still flat (corollary 3.1.51), hence the projection $X_\mathfrak{q}(\overline{x}) \to Y$ is a flat morphism of schemes, especially $g^{-1}_{\overline{x}}(\eta) \cap X_\mathfrak{q}(\overline{x}) \neq \varnothing$. Furthermore, the subset $X_\mathfrak{q}(\overline{x})$ is pure-dimensional, of codimension $\mathrm{ht}\, \mathfrak{q}$ in $g^{-1}_{\overline{x}}(\eta)$, by (6.7.2) and [31, Ch.IV, Cor.6.1.4]. It follows that $U_\mathfrak{q}$ is a dense open subset of $X_\mathfrak{q}(\overline{x})$. To conclude, it remains only to show that each $X_\mathfrak{q}(\overline{x})$ is geometrically normal and geometrically irreducible; however, let $j_\mathfrak{q} : X_\mathfrak{q} \to X$ be the closed immersion; the induced morphism of log schemes $g_\mathfrak{q} : (X_\mathfrak{q}, j_\mathfrak{q}^* \underline{M}) \to (Y, \underline{N})$ is also smooth, hence the assertion follows from (ii).

(iv) is a straightforward consequence of (iii).

(v): Notice that $A := \mathscr{O}_{g^{-1}_{\overline{x}}(\overline{\eta}),w}$ is ind-étale over the noetherian ring $Q \otimes_P \kappa(\overline{\eta})$, hence its strict henselization is noetherian, and then A itself is noetherian ([33, Ch.IV, Prop.18.8.8(iv)]). Since X is normal, and w is a point of height one in $g^{-1}_{\overline{x}}(\overline{\eta})$, we conclude that A is a discrete valuation ring. $\square$

7. Étale coverings of schemes and log schemes

7.1. **Acyclic morphisms of schemes.** For any scheme X, we shall denote by :

$$\mathbf{Cov}(X)$$

the category whose objects are the finite étale morphisms $E \to X$; the morphisms $(E \to X) \to (E' \to X)$ are the X-morphisms of schemes $E \to E'$. By faithfully flat descent, $\mathbf{Cov}(X)$ is naturally equivalent to the subcategory of $X^{\sim}_{\text{ét}}$ consisting of all locally constant constructible sheaves. If $f : X \to Y$ is any morphism of schemes, and $\varphi : E \to Y$ is an object $\mathbf{Cov}(Y)$, then $f^*\varphi := \varphi \times_Y X : E \times_Y X \to X$ is an object of $\mathbf{Cov}(X)$; more precisely, we have a fibration :

$$\mathbf{Cov} \to \mathbf{Sch} \tag{7.1.1}$$

over the category of schemes, whose fibre, over any scheme X, is the category $\mathbf{Cov}(X)$.

Lemma 7.1.2. *Let f be a morphism of schemes, and suppose that either one of the following conditions holds :*

(a) *f is integral and surjective.*
(b) *f is faithfully flat.*
(c) *f is proper and surjective.*

Then f is of universal 2-descent for the fibred category (7.1.1).

Proof. This is [4, Exp.VIII, Th.9.4]. □

Lemma 7.1.3. *In the situation of definition* 5.5.40*, suppose that* $\mathrm{Lef}(X,Y)$ *holds. Let $U \subset X$ be any open subset such that $Y \subset U$, and denote by $j : Y \to U$ the closed immersion. Then the induced functor*

$$j^* : \mathbf{Cov}(U) \to \mathbf{Cov}(Y)$$

is fully faithful.

Proof. Let $\mathbf{Cov}(\mathfrak{X})$ be the full subcategory of $\mathscr{O}_{\mathfrak{X}}\text{-}\mathbf{Alg}_{\mathrm{lfft}}$ consisting of all finite étale $\mathscr{O}_{\mathfrak{X}}$-algebras (notation of lemma 5.5.41(ii)); the category $\mathbf{Cov}(U)$ is a full subcategory of the category $\mathscr{O}_U\text{-}\mathbf{Alg}_{\mathrm{lfft}}$, so lemma 5.5.41(ii) already implies that the functor $\mathbf{Cov}(U) \to \mathbf{Cov}(\mathfrak{X})$ is fully faithful, hence we are reduced to showing that the functor :

$$\mathbf{Cov}(\mathfrak{X}) \to \mathbf{Cov}(Y) \qquad \mathscr{A} \mapsto \mathrm{Spec}\,\mathscr{A} \otimes_{\mathscr{O}_{\mathfrak{X}}} \mathscr{O}_Y$$

is fully faithful. To this aim, let $\mathscr{I} \subset \mathscr{O}_X$ be the ideal defining the closed immersion $Y \subset X$; consider the direct system of schemes

$$(Y_n := \mathrm{Spec}\,\mathscr{O}_X/\mathscr{I}^{n+1} \mid n \in \mathbb{N})$$

and let $\mathbf{Cov}(Y_\bullet)$ be the category consisting of all direct systems $(E_n \mid n \in \mathbb{N})$ of schemes, such that E_n is finite étale over Y_n, and such that the transition maps $E_n \to E_{n+1}$ induce isomorphisms $E_n \xrightarrow{\sim} E_{n+1} \times_{Y_{n+1}} Y_n$ for every $n \in \mathbb{N}$. The morphisms in $\mathbf{Cov}(Y_\bullet)$ are the morphisms of direct systems of schemes. We have a natural functor :

$$\mathbf{Cov}(\mathfrak{X}) \to \mathbf{Cov}(Y_\bullet) \qquad \mathscr{A} \mapsto (\mathrm{Spec}\,\mathscr{A}/\mathscr{I}^{n+1}\mathscr{A} \mid n \in \mathbb{N})$$

which is fully faithful, by [26, Ch.I, Cor.10.6.10(ii)]. Finally, the functor $\mathbf{Cov}(Y_\bullet) \to \mathbf{Cov}(Y)$ given by the rule $(E_n \mid n \in \mathbb{N}) \to E_0$ is an equivalence, by [33, Ch.IV, Th.18.1.2]. The claim follows. □

7.1.4. Consider now a cofiltered family $\mathscr{S} := (S_\lambda \mid \lambda \in \Lambda)$ of affine schemes. Denote by S the limit of $\mathscr{S}$, and suppose moreover that Λ admits a final element $0 \in \Lambda$. Let $f_0 : X_0 \to S_0$ be a finitely presented morphism of schemes, and set :

$$X_\lambda := X_0 \times_{S_0} S_\lambda \qquad f_\lambda := f_0 \times_{S_0} S_\lambda : X_\lambda \to S_\lambda \qquad \text{for every } \lambda \in \Lambda.$$

Set as well $X := X_0 \times_{S_0} S$ and $f := f_0 \times_{S_0} S : X \to S$. For every $\lambda \in \Lambda$, let $p_\lambda : S \to S_\lambda$ (resp. $p'_\lambda : X \to X_\lambda$) be the natural morphism.

The functors $p'^*_\lambda : \mathbf{Cov}(X_\lambda) \to \mathbf{Cov}(X)$ define a pseudo-cocone in the 2-category $\mathbf{Cat}$, whence a functor :

$$2\text{-}\operatorname*{colim}_{\lambda\in\Lambda} \mathbf{Cov}(X_\lambda) \to \mathbf{Cov}(X). \tag{7.1.5}$$

Lemma 7.1.6. *In the situation of* (7.1.4)*, the functor* (7.1.5) *is an equivalence.*

Proof. It is a rephrasing of [32, Th.8.8.2, Th.8.10.5] and [33, Ch.IV, Prop.17.7.8(ii)]. □

Lemma 7.1.7. *Let X be a scheme, $j : U \to X$ an open immersion with dense image, and $f : X' \to X$ an integral surjective and radicial morphism. The following holds :*

(i) *The morphism f induces an equivalence of sites*
$$f^* : X'_{\text{ét}} \to X_{\text{ét}}$$
and the functor $f^* : \mathbf{Cov}(X) \to \mathbf{Cov}(X')$ is an equivalence.

(ii) *Suppose that X is separated. Then :*
  (a) *The functor $j^* : \mathbf{Cov}(X) \to \mathbf{Cov}(U)$ is faithful.*
  (b) *If moreover, X is reduced and normal, and has finitely many maximal points, j^* is fully faithful, and its essential image consists of all the objects φ of $\mathbf{Cov}(U)$ such that $\varphi \times_X X(\overline{x})$ lies in the essential image of the pull-back functor :*
  $$\mathbf{Cov}(X(\overline{x})) \to \mathbf{Cov}(X(\overline{x}) \times_X U)$$
  for every geometric point $\overline{x}$ of X. (Notation of definition 2.4.17(ii).*)*

(iii) *Furthermore, if X is locally noetherian and regular, and $X \setminus U$ has codimension ≥ 2 in X, then j^* is an equivalence.*

Proof. (i) follows from [4, Exp.VIII, Th.1.1].

(ii.a): Indeed, let $\varphi : E \to X$ be any finite étale morphism; it is easily seen that E is a separated scheme, and since φ is flat and U is dense in X, we see that $E \times_X U$ is a dense subscheme of E, so the claim is clear.

(ii.b): Choose a covering $X = \bigcup_{i\in I} V_i$ consisting of affine open subsets, and let
$$g : X' := \coprod_{i\in I} V_i \to X$$
be the induced morphism; set also $j'_i := j \times_X V_i$ for every $i \in I$. By lemma 7.1.2, g is of universal 2-descent for the fibred category $\mathbf{Cov}$. On the other hand, $X'' := X' \times_X X'$ is separated, and the induced open immersion $j'' : U \times_X X'' \to X''$ has dense image, hence the corresponding functor j''^* is faithful, by (i). By corollary 1.5.35(ii), the full faithfulness of j^* follows from the full faithfulness of the pull-back functor j'^* corresponding to the open immersion $j' := j \times_X X'$. The latter holds if and only if the same holds for all the pull-back functors j'^*_i. Hence, we may replace X by V_i, and assume from start that X is affine, say $X = \operatorname{Spec} A$. Let $\eta_1, \dots, \eta_s \in X$ be the maximal points. Under the current assumptions, A is the product of s domains, and its total ring of fractions $\operatorname{Frac} A$ is the product of fields $\kappa(\eta_1) \times \cdots \times \kappa(\eta_s)$. Let $E \to X$ and $E' \to X$ be two objects of $\mathbf{Cov}(X)$, and $h : E \times_X U \to E' \times_X U$ a morphism; we may write $E = \operatorname{Spec} B$, $E' = \operatorname{Spec} B'$ for finite étale A-algebras B and B', and the restrictions $h_{\eta_i} := h \times_U X(\eta_i) : E(\eta_i) \to E'(\eta_i)$ induce a map of $\operatorname{Frac} A$-algebras
$$h^\natural_\eta := \prod_{i=1}^s h^\natural_{\eta_i} : B' \otimes_A \operatorname{Frac} A \to B \otimes_A \operatorname{Frac} A.$$
On the other hand, by [33, Ch.IV, Prop.17.5.7], B (resp. B') is the normalization of A in $B \otimes_A \operatorname{Frac} A$ (resp. in $B' \otimes_A \operatorname{Frac} A$). It follows that $h^\natural_\eta$ restricts to a map $B' \to B$, and the corresponding morphism $E \to E'$ is necessarily an extension of h. This shows that j^* is fully faithful. To proceed, we make the following general remark.

Claim 7.1.8. Let Z be a scheme, $V_0 \subset Z$ an open subset, and $\varphi : E \to V_0$ an object of $\mathbf{Cov}(V_0)$. Suppose that, for every open subset $V \subset Z$ containing V_0, the pull-back functor $\mathbf{Cov}(V) \to \mathbf{Cov}(V_0)$ is fully faithful. Let $\mathscr{F}$ be the family consisting of all the data (V, ψ, α) where $V \subset X$ is any open subset with $V_0 \subset V$, $\psi : E_V \to V$ is a finite étale morphism, and $\alpha : \psi^{-1}V_0 \xrightarrow{\sim} E$ is an isomorphism of V_0-schemes. $\mathscr{F}$ is partially ordered by the relation such that $(V, \psi, \alpha) \geq (V', \psi', \alpha')$ if and only if $V' \subset V$ and there is an isomorphism $\beta : \psi^{-1}V' \xrightarrow{\sim} E_{V'}$ of V'-schemes, such that $\alpha' \circ \beta_{|\psi^{-1}V'} = \alpha_{|\psi^{-1}V'}$. Then $\mathscr{F}$ admits a supremum.

Proof of the claim. Using Zorn lemma, it is easily seen that $\mathscr{F}$ admits maximal elements, and it remains to show that any two maximal elements (V, ψ, α) and (V', α', ψ) are isomorphic; to see this, set $V'' := V \cap V'$: by assumption, the isomorphism $\alpha'^{-1} \circ \alpha : \psi^{-1}V_0 \xrightarrow{\sim} \psi'^{-1}V_0$ extends to an isomorphism of V''-schemes $\psi^{-1}V'' \xrightarrow{\sim} \psi'^{-1}V''$, using which one can glue E_V and $E_{V'}$ to obtain a datum $(V \cup V', \psi'', \alpha'')$ which is larger than both our maximal elements, hence it is isomorphic to both. $\diamondsuit$

Claim 7.1.9. In the situation of claim 7.1.8, suppose that Z is reduced and normal, and has finitely many maximal points. Let $(U_{\max}, \psi, \alpha)$ be a supremum for $\mathscr{F}$, and $\overline{z}$ a geometric point of Z, such that the morphism $\varphi \times_Z Z(\overline{z})$ extends to a finite étale covering of $Z(\overline{z})$; then the support of $\overline{z}$ lies in $U_{\max}$.

Proof of the claim. By lemma 7.1.6, there exists an étale neighborhood $g : Y \to Z$ of $\overline{z}$, with Y affine, a finite étale morphism $\varphi_Y : E_Y \to Y$, and an isomorphism $h : \varphi \times_U Y \simeq \varphi_Y \times_Z U$. We have a natural essentially commutative diagram :

$$\begin{array}{ccccc}
\mathbf{Cov}(gY) & \xrightarrow{\alpha} & \mathsf{Desc}(\mathbf{Cov}, g) & \longrightarrow & \mathbf{Cov}(Y) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbf{Cov}(U \cap gY) & \xrightarrow{\beta} & \mathsf{Desc}(\mathbf{Cov}, g \times_Z U) & \longrightarrow & \mathbf{Cov}(Y \times_Z U)
\end{array}$$

where α and β are equivalences, by lemma 7.1.2. Moreover, let $Y' := Y \times_Z Y$ and $Y'' := Y' \times_Z Y$; by the foregoing, the functors

$$\mathbf{Cov}(Y') \to \mathbf{Cov}(Y' \times_Z U) \qquad \text{and} \qquad \mathbf{Cov}(Y'') \to \mathbf{Cov}(Y'' \times_Z U)$$

are fully faithful, hence the right square subdiagram is 2-cartesian (corollary 1.5.35(iii)). Thus, the datum $(\varphi \times_Z gY, \varphi_Y, h)$ determines an object φ' of $\mathbf{Cov}(gY)$, together with an isomorphism $\varphi' \times_Z U \simeq \varphi \times_Z gY$, which we may use to glue φ and φ' to a single object φ'' of $\mathbf{Cov}(U \cup gY)$. The claim follows. $\diamondsuit$

The foregoing shows that the assumptions of claim 7.1.8 are fulfilled, with $Z := X$, $V_0 := U$ and any object φ of $\mathbf{Cov}(U)$, hence there exists a largest open subset $U_{\max} \subset X$ over which φ extends. However, claim 7.1.9 shows that $U_{\max} = X$, so the proof of (ii.b) is complete.

(iii): In view of (ii.b) and [33, Ch.IV, Cor.18.8.13], we are reduced to the case where X is a regular local scheme, and it suffices to show that j^* is essentially surjective. We argue by induction on the dimension n of $X \setminus U$. If $n = 0$, then $X \setminus U$ is the closed point, in which case it suffices to invoke the Zariski-Nagata purity theorem ([44, Exp.X, Th.3.4(i)]). Suppose $n > 0$ and that the assertion is already known for smaller dimensions. Let φ be a given finite étale covering of U, and x a maximal point of $X \setminus U$; then $X(x) \setminus U = \{x\}$, so $\varphi_{|U \cap X(x)}$ extends to a finite étale morphism φ_x over $X(x)$. In turns, φ_x extends to an affine open neighborhood $V \subset X$ of X, and up to shrinking V, this extension φ' agrees with φ on $U \cap V$, by lemma 7.1.6. Hence we can glue φ and φ', and replace U by $U \cup V$. Repeating the procedure for every maximal point of $X \setminus U$, we reduce the dimension of $X \setminus U$; then we conclude by the inductive assumption. $\square$

Definition 7.1.10. Let $f : X \to S$ be a morphism of schemes, $\overline{x}$ a geometric point of X localized at $x \in X$; and set $s := f(x)$, $\overline{s} := f(\overline{x})$. Let also $n \in \mathbb{N}$ be any integer, and $\mathbb{L} \subset \mathbb{N}$ any non-empty set of prime numbers.

(i) We say that $\overline{x}$ is *strict* if $\kappa(\overline{x})$ is a separable closure of $\kappa(x)$. (Notation of (2.4.16).)

(ii) We associate to $\overline{x}$ a strict geometric point $\overline{x}^{\mathrm{st}}$, constructed as follows. Let $\kappa(\overline{x}^{\mathrm{st}}) := \kappa(x)^{\mathrm{s}}$, the separable closure of $\kappa(x)$ inside $\kappa(\overline{x})$; then the inclusion $\kappa(\overline{x}^{\mathrm{st}}) \subset \kappa(\overline{x})$ defines a morphism of schemes :

$$|\overline{x}| \to |\overline{x}^{\mathrm{st}}| := \mathrm{Spec}\,\kappa(\overline{x}^{\mathrm{st}}) \tag{7.1.11}$$

and $\overline{x}$ is the composition of (7.1.11) and a unique strict geometric point

$$\overline{x}^{\mathrm{st}} : |\overline{x}^{\mathrm{st}}| \to X$$

localized at x.

(iii) Let $f : X \to Y$ be a morphism of schemes; we call $f(\overline{x})^{\mathrm{st}}$ the *strict image* of $\overline{x}$ in Y. Notice that the natural identification $|\overline{x}| = |f(\overline{x})|$ induces a morphism of schemes :

$$|\overline{x}^{\mathrm{st}}| \to |f(\overline{x})^{\mathrm{st}}|.$$

(iv) Denote by $f_{\overline{x}} : X(\overline{x}) \to S(\overline{s})$ the morphism of strictly local schemes induced by f. We say that f is *locally* (-1)*-acyclic at the point* x, if the scheme $f_{\overline{x}}^{-1}(\xi)$ is non-empty for every strict geometric point ξ of $S(\overline{s})$.

(v) We say that f is *locally* 0*-acyclic at the point* x, if the scheme $f_{\overline{x}}^{-1}(\xi)$ is non-empty and connected for every strict geometric point ξ of $S(\overline{s})$.

(vi) We say that a group G is a *finite* $\mathbb{L}$*-group* if G is finite and all the primes dividing the order of G lie in $\mathbb{L}$. We say that G is an $\mathbb{L}$*-group* if it is a filtered union of finite $\mathbb{L}$-groups.

(vii) We say that f is *locally* 1*-aspherical for* $\mathbb{L}$ *at the point* x, if we have :

$$H^1(f_{\overline{x}}^{-1}(\xi)_{\mathrm{ét}}, G) = \{1\}$$

for every strict geometric point ξ of $S(\overline{s})$, and every $\mathbb{L}$-group G (where 1 denotes the trivial G-torsor).

(viii) We say that f is *locally* (-1)*-acyclic* (resp. *locally* 0*-acyclic*, resp. *locally* 1*-aspherical for* $\mathbb{L}$), if f is locally (-1)-acyclic (resp. locally 0-acyclic, resp. locally 1-aspherical for $\mathbb{L}$) at every point of X.

(ix) We say that f is (-1)*-acyclic* (resp. 0*-acyclic*) if the unit of adjunction : $\mathscr{F} \to f_*f^*\mathscr{F}$ is a monomorphism (resp. an isomorphism) for every sheaf $\mathscr{F}$ on $S_{\mathrm{ét}}$.

See section 2.1 for generalities about G-torsors for a group object G on a topos T. Here we shall be mainly concerned with the case where T is the étale topos $X_{\mathrm{ét}}^{\sim}$ of a scheme X, and G is representable by a group scheme, finite and étale over X. In this case, using faithfully flat descent one can show that any G-torsor is representable by a *principal* G*-homogeneous space*, *i.e.* a finite, surjective, étale morphism $E \to X$ with a G-action $G \to \mathrm{Aut}_X(E)$ such that the induced morphism of X-schemes

$$G \times E \to E \times_X E$$

is an isomorphism.

7.1.12. If G_X is the constant $X_{\mathrm{ét}}^{\sim}$-group arising from a finite group G and X is non-empty and connected, right G_X-torsors are also understood as G-valued characters of the étale fundamental group of X. Indeed, let ξ be a geometric point of X; recall ([42, Exp.V, §7]) that $\pi := \pi_1(X_{\mathrm{ét}}, \xi)$ is defined as the automorphism group of the fibre functor

$$F_\xi : \mathbf{Cov}(X) \to \mathbf{Set} \qquad (E \xrightarrow{f} X) \mapsto f^{-1}\xi.$$

We endow π with its natural profinite topology, as in (1.6.7), so that F_ξ can be viewed as an equivalence of categories

$$F_\xi : \mathbf{Cov}(X) \to \pi\text{-}\mathbf{Set}$$

(see (1.6.7)).

Lemma 7.1.13. *In the situation of* (7.1.12)*, there exists a natural bijection of pointed sets :*

$$H^1(X_{\text{ét}}, G_X) \xrightarrow{\sim} H^1_{\text{cont}}(\pi, G)$$

from the pointed set of right G_X-torsors, to the first non-abelian continuous cohomology group of P with coefficients in G (see (1.6.1)*).*

Proof. Let $f : E \to X$ be a right G_X-torsor, and fix a geometric point $s \in F_\xi(E)$; given any $\sigma \in \pi$, there exists a unique $g_{s,\sigma} \in G$ such that

$$s \cdot g_{s,\sigma} = \sigma_E(s).$$

Any $g \in G$ determines a X-automorphism $g_E : E \to E$, and by definition, the automorphism $F_\xi(g_E)$ on $F_\xi(E)$ commutes with the left action of any element $\tau \in \pi$; however $F_\xi(g_E)$ is just the right action of g on $F_\xi(E)$, hence we may compute :

$$s \cdot g_{s,\tau} \cdot g_{s,\sigma} = (\tau_E(s)) \cdot g_{s,\sigma} = \tau_E(s \cdot g_{s,\tau}) = \tau_E \cdot \sigma_E(s)$$

so the rule $\sigma \mapsto g_{s,\sigma}$ defines a group homomorphism $\rho_{s,f} : \pi \to G$ which is clearly continuous. We claim that the conjugacy class of $\rho_{s,f}$ does not depend on the choice of s. Indeed, if $s' \in F_\xi(E)$ is another choice, there exists a (unique) element $h \in G$ such that $h(s) = s'$; arguing as in the foregoing we see that σ_E commutes with the right action of h on $F_\xi(E)$. In other words, $\sigma_E(s) = h^{-1} \circ \sigma_E(s')$, so that $g_{s',\sigma} = h \circ g_{s,\sigma} \circ h^{-1}$.

Therefore, denote by ρ_f the conjugacy class of $\rho_{s,f}$; we claim that ρ_f depends only on the isomorphism class of the G_X-torsor E. Indeed, any isomorphism $t : E \xrightarrow{\sim} E'$ of right G_X-torsors induces a bijection $F_\xi(t) : F_\xi(E) \xrightarrow{\sim} F_\xi(E')$, equivariant for the action of G, and for any $\sigma \in \pi$ we have $F_\xi(t) \circ \sigma_E = \sigma_{E'} \circ F_\xi(t)$, whence the assertion.

Conversely, given a continuous group homomorphism $\rho : \pi \to G$, let us endow G with the induced left π-action, and right G-action. Then G is an object of π-$\mathbf{Set}$, to which there corresponds a finite étale morphism $E_\rho \to X$, with an isomorphism $E_\rho \times_X |\xi| \xrightarrow{\sim} G$ of sets with left π-action. Since the right action of G is π-equivariant, we have a corresponding G-action by X-automorphisms on E_ρ, so E_ρ is G-torsor, and its image under the map of the lemma is clearly the conjugacy class of ρ.

Finally, in order to show that the map of the lemma is injective, it suffices to prove that, for any right G_X-torsor $(f : E \to X, \varphi : E \times G \to E)$ and any $s \in F_\xi(E)$, there exists an isomorphism of right G_X-torsors $E_{\rho_{s,f}} \xrightarrow{\sim} E$. However, s and $\rho_{s,f}$ determine an identification of sets with left π-action :

$$G \xrightarrow{\sim} F_\xi(E) \tag{7.1.14}$$

whence an isomorphism $t : E_{\rho_{s,f}} \xrightarrow{\sim} E$ in $\mathbf{Cov}(X)$. Moreover, (7.1.14) also identifies the right G-action on $F_\xi(E)$ to the natural right G-action on G; the latter is π-equivariant, hence it induces a right G-action $\varphi' : E \times G \to E$, such that t is G-equivariant. To conclude, it suffices to show that $\varphi = \varphi'$. In view of [33, Ch.IV, Cor.17.4.8], the latter assertion can be checked on the stalks over the geometric point ξ, where it holds by construction. $\square$

Remark 7.1.15. (i) In the situation of (7.1.12), let $E \to X$ be any right G_X-torsor, and $\rho_E : \pi \to G$ the corresponding representation. Then the proof of lemma 7.1.13 shows that the left π-action on $F_\xi(E)$ is isomorphic to the left π-action on G induced by ρ_E; especially, ρ_E is surjective if and only if π acts transitively on $F_\xi(E)$, if and only if the scheme E is connected (since a decomposition of E into connected components corresponds to a decomposition of $F_\xi(E)$ into orbits for the π-action).

(ii) Let $\varphi : G' \to G$ be a homomorphism of finite groups, and

$$H^1_{\mathrm{cont}}(\pi, \varphi) : H^1_{\mathrm{cont}}(\pi, G') \to H^1_{\mathrm{cont}}(\pi, G)$$

the induced map. Denote by r (resp. l) the right (resp. left) translation action of G on itself. Let also $E' \to X$ be a principal G'-homogeneous space, given by a map $\rho : G' \to \mathrm{Aut}_X(E')$, and denote by $c' \in H^1_{\mathrm{cont}}(\pi, G')$ the class of E'. Then the class $c := H^1_{\mathrm{cont}}(\pi, \varphi)(c)$ can be described geometrically as follows. The scheme $E' \times G$ admits an obvious right G-action, induced by r. Moreover, it admits as well a right G'-action : namely, to any element $g \in G'$, we assign the X-automorphism $\rho_g \times l_{\varphi(g^{-1})} : E' \times G \xrightarrow{\sim} E' \times G$. Set $E := (E \times G)/G'$; it is easily seen that E is a principal G-homogeneous space, and its class is precisely c (the detailed verification shall be left as an exercise for the reader).

(iii) Consider a commutative diagram of schemes

$$\begin{tikzcd} E' \arrow[rr, "g"] \arrow[dr, "f'"'] & & E \arrow[dl, "f"] \\ & X & \end{tikzcd}$$

where f (resp. f') is a G-torsor (resp. a G'-torsor) for a given finite group G (resp. G'). Denote by $\rho : G \to \mathrm{Aut}_X(E)$ and $\rho' : G' \to \mathrm{Aut}_X(E')$ the respective actions. We have :

(a) Suppose that there exists a group homomorphism $\varphi : G' \to G$, such that $\rho \circ \varphi = g \circ \rho'$. Then g induces a G'-equivariant map

$$F_\xi(E') \to \mathrm{Res}(\varphi) F_\xi(E)$$

whence an isomorphism $(E' \times G)/G' \xrightarrow{\sim} G$ of G-torsors. Hence, let c (resp. c') denote any representative of the equivalence class of E (resp. E') in $H^1_{\mathrm{cont}}(\pi, G)$ (resp. $H^1_{\mathrm{cont}}(\pi, G')$); in view of (ii), it follows that

$$H^1(\pi, \varphi)(c') = c.$$

In other words, the induced diagram of continuous group homomorphisms

$$\begin{tikzcd} & \pi_1(X_{\text{ét}}, \xi) \arrow[dl, "c'"'] \arrow[dr, "c"] & \\ G' \arrow[rr, "\varphi"] & & G \end{tikzcd}$$

commutes, up to composition with an inner automorphism of G. (Details left to ther reader.)

(b) If E is connected, a group homomorphism $\varphi : G \to G'$ fulfilling the condition of (a) exists and is unique up to composition with an inner automorphism of G. Indeed, fix any $e' \in F_\xi(E')$ and let $e := f(e')$; if $g' \in G'$, define $\varphi(g')$ as the unique $g \in G$ such that $f(e' \cdot g') = e \cdot g$; also, in view of (i) we may pick $\sigma_{g'} \in \pi_1(X, \xi)$ such that $\sigma_{g'} \cdot e' = e' \cdot g'$, and notice that $\sigma_{g'} \cdot e = e \cdot \varphi(g')$ for every $g' \in G'$. Now, if $h' \in G'$ is any other element, we may compute :

$$f(e' \cdot g'h') = f(\sigma_{g'} \cdot e' \cdot h') = \sigma_{g'} \cdot f(e' \cdot h') = \sigma_{g'} \cdot e \cdot \varphi(h') = e \cdot \varphi(g') \cdot \varphi(h')$$

whence $\varphi(g'h') = \varphi(g') \cdot \varphi(h')$, as required.

Lemma 7.1.16. *Let $f : X \to Y$ be a morphism of schemes, $\mathscr{F}, \mathscr{G}$ two sheaves on $Y_{\text{ét}}$. Then :*

(i) *If $\mathscr{F}$ is locally constant and constructible, the natural map :*

$$\vartheta_f : f^*\mathscr{H}om_{Y^\sim_{\text{ét}}}(\mathscr{F},\mathscr{G}) \to \mathscr{H}om_{X^\sim_{\text{ét}}}(f^*\mathscr{F},f^*\mathscr{G})$$

is an isomorphism.

(ii) *If f is 0-acyclic, the functor*

$$f^* : \mathbf{Cov}(Y) \to \mathbf{Cov}(X) \quad : \quad (E \to Y) \mapsto (E \times_Y X \to X)$$

is fully faithful.

Proof. (i): Suppose we have a cartesian diagram of schemes :

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y. \end{tikzcd}$$

Then, according to (2.3.2), we have a natural isomorphism :

$$\vartheta_{g'} \circ g'^*\vartheta_f \Rightarrow \vartheta_{f'} \circ f'^*\vartheta_g$$

(an invertible 2-cell, in the terminology of (1.3.2)). Now, if g – and therefore g' – is a covering morphism, ϑ_g and $\vartheta_{g'}$ are isomorphisms, and $g'^*\vartheta_f$ is an isomorphism if and only if the same holds for ϑ_f. Summing up, in this case ϑ_f is an isomorphism if and only if the same holds for $\vartheta_{f'}$. Thus, we may choose g such that g^* is a constant sheaf, and after replacing f by f', we may assume that $\mathscr{F} = S_Y$ is the constant sheaf associated to a finite set S. Since the functors

$$\mathscr{H}om_{Y^\sim_{\text{ét}}}(-,\mathscr{G}) : (Y^\sim_{\text{ét}})^o \to Y^\sim_{\text{ét}} \quad \text{and} \quad f^* : Y^\sim_{\text{ét}} \to X^\sim_{\text{ét}}$$

are left exact, we may further reduce to the case where $S = \{1\}$ is the set with one element, in which case $\mathscr{F} = 1_Y$ is the final object of $Y^\sim_{\text{ét}}$, and $f^*\mathscr{F} = 1_X$ is the final object of $X^\sim_{\text{ét}}$. Moreover, we have a natural identification :

$$\mathscr{H}om_{Y^\sim_{\text{ét}}}(1_Y,\mathscr{G}) \xrightarrow{\sim} \mathscr{G} \quad : \quad \sigma \mapsto \sigma(1)$$

and likewise for $\mathscr{H}om_{X^\sim_{\text{ét}}}(1_Y, f^*\mathscr{G})$. Using the foregoing characterization, it is easily checked that, under these identifications, ϑ_f is the identity map of $f^*\mathscr{G}$, whence the claim.

(ii): It has already been remarked that $\mathbf{Cov}(Y)$ is equivalent to the category of locally constant constructible sheaves on $Y_{\text{ét}}$, and likewise for $\mathbf{Cov}(X)$. Let E and F be two objects of $\mathbf{Cov}(Y)$; we have natural bijections :

$$\begin{aligned} \mathrm{Hom}_{\mathbf{Cov}(X)}(f^*E, f^*F) &\xrightarrow{\sim} \Gamma(X, \mathscr{H}om_{X^\sim_{\text{ét}}}(f^*E, f^*F)) \\ &\xrightarrow{\sim} \Gamma(Y, f_*f^*\mathscr{H}om_{Y^\sim_{\text{ét}}}(E,F)) && \text{by (i)} \\ &\xrightarrow{\sim} \Gamma(Y, \mathscr{H}om_{Y^\sim_{\text{ét}}}(E,F)) && \text{since } f \text{ is } 0\text{-acyclic} \\ &\xrightarrow{\sim} \mathrm{Hom}_{\mathbf{Cov}(Y)}(E,F) \end{aligned}$$

as stated. ☐

Lemma 7.1.17. *Let $f : X \to S$ be a quasi-compact morphism of schemes, and suppose that :*

(a) *f is locally (-1)-acyclic.*

(b) *For every strict geometric point ξ of S, the induced morphism $f_\xi : f^{-1}(\xi) \to |\xi|$ is 0-acyclic (*i.e. *f has non-empty geometrically connected fibres).*

Then f is 0-acyclic.

Proof. Let $\mathscr{F}$ be a sheaf on $S_{\text{ét}}$. For every strict geometric point ξ of S, we have a commutative diagram :

$$\begin{tikzcd} \mathscr{F}_\xi \arrow[r, "\varepsilon_\xi"] \arrow[d, "\alpha"'] & (f_*f^*\mathscr{F})_\xi \arrow[d] \\ \Gamma(|\xi|, \xi^*\mathscr{F}) \arrow[r, "f_\xi^*"] & \Gamma(f^{-1}(\xi), f_\xi^* \circ \xi^*\mathscr{F}) \end{tikzcd} \tag{7.1.18}$$

where $\varepsilon : \mathscr{F} \to f_*f^*\mathscr{F}$ is the unit of adjunction. The map α is an isomorphism, and the same holds for f_ξ^*, since f_ξ is 0-acyclic. Hence ε_ξ is injective, which shows already that f is (-1)-acyclic. It remains to show that ε_ξ is surjective. Hence, let $t \in (f_*f^*\mathscr{F})_\xi$ be any section. From (7.1.18) we see that there exists a section $t' \in \mathscr{F}_\xi$ such that the images of t and $\varepsilon_\xi(t')$ agree on $\Gamma(f^{-1}(\xi), f_\xi^* \circ \xi^*\mathscr{F})$. We may find an étale neighborhood $g : U \to S$ of ξ, such that t' (resp. t) extends to a section $t'_U \in \mathscr{F}(U)$ (resp. $t_U \in \Gamma(X \times_S U, f^*\mathscr{F})$). Let $X_U := X \times_S U$, $f_U := f \times_S U : X_U \to U$, and for every geometric point $\overline{x}$ of X_U, denote by $f_{\overline{x}}^* : \mathscr{F}_{f_U(\overline{x})} \to f_U^*\mathscr{F}_{\overline{x}}$ the natural isomorphism (2.4.21). We set

$$V := \{x \in X_U \mid t_{U,\overline{x}} = f_{\overline{x}}^*(t'_{U,f(\overline{x})})\}$$

where $\overline{x}$ is any geometric point of X localized at x, and $t_{U,\overline{x}} \in f^*\mathscr{F}_{\overline{x}}$ (resp. $t'_{U,f(\overline{x})} \in \mathscr{F}_{f_U(\overline{x})}$) denotes the image of t_U (resp. of t'_U). Clearly V is an open subset of X_U, and we have :

Claim 7.1.19. (i) $V = f_U^{-1}f_U(V)$.

(ii) $f_U(V) \subset U$ is an open subset.

Proof of the claim. (i): Given a point $u \in U$, choose a strict geometric point $\overline{u}$ localized at u, and set $\overline{s} := g(\overline{u})^{\text{st}}$; by assumption, the morphism $f_{\overline{s}} : f^{-1}(\overline{s}) \to |\overline{s}|$ is 0-acyclic, hence the image of t_U in $\Gamma(f^{-1}(\overline{s}), f_{\overline{s}}^* \circ \overline{s}^*\mathscr{F})$ is of the form $f_{\overline{s}}^*t''$, for some $t'' \in \mathscr{F}_{\overline{s}}$. It follows that $V \cap f_U^{-1}(u)$ is either the whole of $f_U^{-1}(u)$ or the empty set, according to whether t'' agrees or not with the image of t'_U in $\mathscr{F}_{\overline{s}} = g^*\mathscr{F}_{\overline{u}}$.

(ii): The subset $X \setminus V$ is closed, especially pro-constructible; since f is quasi-compact, we deduce that $f_U(X \setminus V)$ is a pro-constructible subset of U ([30, Ch.IV, Prop.1.9.5(vii)]). It then follows from (i) that $f_U(V)$ is ind-constructible, hence we are reduced to showing that $f_U(V)$ is closed under generizations ([30, Ch.IV, Th.1.10.1]). To this aim, since V is open, it suffices to show that f_U is *generizing*, *i.e.* that the induced maps $X_U(x) \to U(u)$ are surjective, for every $u \in U$ and every $x \in f_U^{-1}(u)$. However, choose a geometric point $\overline{x}$ localized at x, and let $\overline{u} := f_U(\overline{x})$; since the natural maps $X_U(\overline{x}) \to X_U(x)$ and $U(\overline{u}) \to U(u)$ are surjective, it suffices to show that the same holds for the map $f_{U,\overline{x}} : X_U(\overline{x}) \to U(\overline{u})$. The image of $\overline{x}$ (resp. $\overline{U}$) in X (resp. in S) is a geometric point which we denote by the same name; since the natural maps $X_U(\overline{x}) \to X(\overline{x})$ and $U(\overline{u}) \to S(\overline{u})$ are isomorphisms, we are reduced to showing that $f_{\overline{x}} : X(\overline{x}) \to S(\overline{u})$ is surjective, which holds, since f is locally (-1)-acyclic. ◊

Set $W := f_U(V)$; in view of claim 7.1.19, W is an étale neighborhood of ξ, and the natural map $\mathscr{F}(W) \to f^*\mathscr{F}(U)$ sends the restriction $t'_{U|W}$ of t'_U to the restriction $t_{U|V}$ of t_U, whence the claim. □

7.1.20. Consider now a cartesian diagram of schemes :

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd} \tag{7.1.21}$$

where g is a local morphism of strictly local schemes, and denote by s (resp. by s') the closed point of S (resp. of S'). Let $x' \in f'^{-1}(s')$ be any point, $\overline{x}'$ a geometric point of X' localized at x', and set $x := g'(x')$, $\overline{x} := g'(\overline{x}')$. Then g' induces a morphism of S'-schemes :

$$X'(\overline{x}') \to X(\overline{x}) \times_S S'. \tag{7.1.22}$$

Lemma 7.1.23. *In the situation of* (7.1.20)*, suppose that g is an integral morphism. Then :*

(i) *The induced morphism $f'^{-1}(s') \to f^{-1}(s)$ induces a homeomorphism on the underlying topological spaces.*
(ii) (7.1.22) *is an isomorphism.*

Proof. If g is integral, $\kappa(s')$ is a purely inseparable algebraic extension of $\kappa(s)$, hence the morphism $T' := \operatorname{Spec} \kappa(s') \to T := \operatorname{Spec} \kappa(s)$ is radicial, and the same holds for the induced morphisms :

$$f'^{-1}(s') \xrightarrow{\sim} f^{-1}(s) \times_T T' \to f^{-1}(s) \qquad f_x^{-1}(s) \times_T T' \to f_x^{-1}(s)$$

([26, Ch.I, Prop.3.5.7(ii)]). Especially (i) holds, and therefore the natural map $X'(x') \to X(x) \times_S S'$ is an isomorphism; we see as well that $f_x^{-1}(s) \times_T T'$ is a local scheme. Then the assertion follows from [33, Ch.IV, Rem.18.8.11]. $\square$

Proposition 7.1.24. *Let $f : X \to S$ and $g : S' \to S$ be morphisms of schemes, with g quasi-finite, and set $X' := X \times_S S'$. Suppose that f is locally (-1)-acyclic (resp. locally 0-acyclic); then the same holds for $f' := f \times_S S' : X' \to S'$.*

Proof. Let $\overline{s}'$ be any geometric point of S', and set $\overline{s} := g(\overline{s}')$. Denote by $s \in S$ (resp. $s' \in S'$) the support of $\overline{s}$ (resp. $\overline{s}'$); then f is locally (-1)-acyclic at the points of $f^{-1}(s)$, if and only if $f_{\overline{s}} := f \times_S S(\overline{s}) : X \times_S S(\overline{s}) \to S(\overline{s})$ enjoys the same property at the points of $f_{\overline{s}}^{-1}(\overline{s})$. Likewise, f' is locally (-1)-acyclic (resp. locally 0-acyclic) at the points of $f'^{-1}(s')$, if and only if $f'_{\overline{s}'} : X' \times_{S'} S'(\overline{s}') \to S'(\overline{s}')$ enjoys the same property at the points of $f_{\overline{s}}^{-1}(\overline{s})$. Hence, we may replace g by $g_{\overline{s}'} : S'(\overline{s}') \to S(\overline{s})$, and f by the induced morphism $X \times_S S(\overline{s}) \to S(\overline{s})$, which allows to assume that g is finite ([33, Ch.IV, Th.18.5.11]), hence integral. Let ξ' (resp. $\overline{x}'$) be any strict geometric point of S' (resp. of $f'^{-1}(s')$), and let $\xi := g(\xi')^{\mathrm{st}}$, (resp. let $\overline{x}$ be the image of $\overline{x}'$ in X); we have natural morphisms :

$$f'^{-1}_{\overline{x}'}(\xi') \xrightarrow{\alpha} X(\overline{x}) \times_S \xi' \xrightarrow{\beta} f_{\overline{x}}^{-1}(\xi).$$

However, α is an isomorphism, by lemma 7.1.23(ii), and β is a radicial morphism, since the field extension $\kappa(\xi) \subset \kappa(\xi')$ is purely inseparable ([26, Ch.I, Prop.3.5.7(ii)]). The claim follows. $\square$

Lemma 7.1.25. *Let S be a strictly local scheme, $s \in S$ the closed point, $f : X \to S$ a morphism of schemes, $\overline{x}$ (resp. ξ) a strict geometric point of $f^{-1}(s)$ (resp. of S). Then we may find :*

(a) *A cartesian diagram* (7.1.21)*, with S' strictly local, irreducible and normal.*
(b) *A strict geometric point $\overline{x}'$ of $f'^{-1}(s')$ with $g'(\overline{x}')^{\mathrm{st}} = \overline{x}$.*
(c) *A strict geometric point ξ' of S' localized at the generic point of S', with $g(\xi')^{\mathrm{st}} = \xi$, and such that* (7.1.22) *induces an isomorphism :*

$$f'^{-1}_{\overline{x}'}(\xi') \xrightarrow{\sim} f_{\overline{x}}^{-1}(\xi). \tag{7.1.26}$$

Proof. Denote by $Z \subset S$ the closure of the image of ξ, endow Z with its reduced subscheme structure, set $Y := X \times_S Z \subset X$, and let $h_{\overline{x}} : Y(\overline{x}) \to Z$ be the natural morphism. Then Z is a strictly local scheme ([33, Ch.IV, Prop.18.5.6(i)]). Moreover, the closed immersion $Y \to X$ induces an isomorphism of Z-schemes : $Y(\overline{x}) \xrightarrow{\sim} X(\overline{x}) \times_S Z$ (lemma 7.1.23(ii)). By construction, ξ factors through a strict geometric point ξ' of Z, and we deduce an isomorphism : $h_{\overline{x}}^{-1}(\xi') \xrightarrow{\sim} f_{\overline{x}}^{-1}(\xi)$ of Z-schemes. Thus, we may replace (S, X, ξ) by (Z, Y, ξ'), and assume that S is the spectrum of a strictly local domain, and ξ is localized at the generic point of S. Say that

$S = \mathrm{Spec}\, A$, and denote by A^ν the normalization of A in its field of fractions F. Then A^ν is the union of a filtered family $(A_\lambda \mid \lambda \in \Lambda)$ of finite A-subalgebras of F; since A is henselian, each A_λ is a product of henselian local rings, hence it is a local henselian ring, so the same holds for A^ν. Moreover, the residue field $\kappa(s')$ of A^ν is an algebraic extension of the residue field $\kappa(s)$ of A, which is separably closed, hence $\kappa(s')$ is separably closed, *i.e.* A^ν is strictly henselian, so we may fulfill condition (a) by taking $S' := \mathrm{Spec}\, A^\nu$. Condition (b) holds as well, due to lemma 7.1.23(i). Finally, it is clear that ξ lifts to a unique strict geometric point ξ' of S', and it follows from lemma 7.1.23(ii) that (7.1.26) is an isomorphism, as required. □

Remark 7.1.27. In the situation of lemma 7.1.25, suppose furthermore that S is noetherian. A direct inspection reveals that the scheme S' exhibited in the proof of the lemma, is the spectrum of the normalization of a noetherian domain. Quite generally, the normalization of a noetherian domain is a *Krull domain* ([63, Th.33.10]).

7.1.28. In the situation of (7.1.4), let $x \in X$ be any point, and $s := f(x)$. Let also ξ be a strict geometric point of S. We deduce a compatible system of points $x_\lambda := p'_\lambda(x) \in X_\lambda$, whence a cofiltered system of local schemes

$$\mathscr{X} := (X_\lambda(x_\lambda) \mid \lambda \in \Lambda).$$

For every $\lambda \in \Lambda$, set $\xi_\lambda := p_\lambda(\xi)^{\mathrm{st}}$. This yields a compatible system of strict geometric points $(\xi_\lambda \mid \lambda \in \Lambda)$, such that :

$$\xi \xrightarrow{\sim} \lim_{\lambda\in\Lambda} \xi_\lambda.$$

Choose a geometric point $\overline{x}$ of X localized at x, and set likewise $\overline{x}_\lambda := p'_\lambda(\overline{x})$; then the system $\mathscr{X}$ lifts to a system $\mathscr{X}^{\mathrm{sh}} := (X_\lambda(\overline{x}_\lambda) \mid \lambda \in \Lambda)$, whose limit is naturally isomorphic to $X(\overline{x})$ ([33, Ch.IV, Prop.18.8.18(ii)]). Furthermore, $\mathscr{X}^{\mathrm{sh}}$ induces a natural isomorphism of $\kappa(\xi)$-schemes :

$$f_{\overline{x}}^{-1}(\xi) \xrightarrow{\sim} \lim_{\lambda\in\Lambda} f^{-1}_{\lambda,\overline{x}_\lambda}(\xi_\lambda) \qquad (7.1.29)$$

where, as usual, $f_{\overline{x}} : X(\overline{x}) \to S$ (resp. $f_{\lambda,\overline{x}_\lambda} : X_\lambda(\overline{x}_\lambda) \to S_\lambda$) is deduced from f (resp. from f_λ). These remarks, together with the following lemma 7.1.30, and the previous lemma 7.1.25, will allow in many cases, to reduce the study of the fibres of $f_{\overline{x}}$, to the case where the base S is strictly local, excellent and normal.

Lemma 7.1.30. *Let S be a strictly local normal scheme. Then there exists a cofiltered family $\mathscr{S} := (S_\lambda \mid \lambda \in \Lambda)$ consisting of strictly local normal excellent schemes, such that :*

(a) *S is isomorphic to the limit of $\mathscr{S}$.*
(b) *The natural morphism $S \to S_\lambda$ is dominant for every $\lambda \in \Lambda$.*

Proof. Say that $S = \mathrm{Spec}\, A$, and write A as the union of a filtered family $\mathscr{A} := (A_\lambda \mid \lambda \in \Lambda)$ of excellent noetherian local subrings, which we may assume to be normal, by [31, Ch.IV, (7.8.3)(ii),(vi)]. Proceeding as in (7.1.28), we choose a compatible family of geometric points $\overline{s}_\lambda$ localized at the closed points of $\mathrm{Spec}\, A_\lambda$, for every $\lambda \in \Lambda$; using these geometric points, we lift $\mathscr{A}$ to a filtered family $(A_\lambda^{\mathrm{sh}} \mid \lambda \in \Lambda)$ of strict henselizations, whose colimit is naturally isomorphic to A. Moreover, each A_λ^{sh} is noetherian, normal and excellent ([33, Ch.IV, Prop.18.8.8(iv), Prop.18.8.12(i)] and proposition 4.8.35(ii)). Let η be the generic point of S, $h_\lambda : S \to S_\lambda := \mathrm{Spec}\, A_\lambda^{\mathrm{sh}}$ the natural morphism, and $\eta_\lambda^{\mathrm{sh}} := h_\lambda(\eta)$ for every $\lambda \in \Lambda$. The cofiltered system $(S_\lambda \mid \lambda \in \Lambda)$ fulfills condition (a). Moreover, by construction, the image of $\eta_\lambda^{\mathrm{sh}}$ in $\mathrm{Spec}\, A_\lambda$ is the generic point η_λ; then $\eta_\lambda^{\mathrm{sh}}$ is the generic point of S_λ, since the latter is the only point of S_λ lying over η_λ. Hence (b) holds as well. □

Proposition 7.1.31. *Let $f : X \to S$ be a flat morphism of schemes. We have :*

(i) *f is locally (-1)-acyclic.*

(ii) *Suppose moreover, that f has geometrically reduced fibres, and :*
 (a) *either f is locally finitely presented,*
 (b) *or else, S is locally noetherian.*
Then f is locally 0-acyclic.

Proof. Let $x \in X$ be any point, set $s := f(x)$, choose a geometric point $\overline{x}$ of X localized at x, set $\overline{s} := f(\overline{x})$, and let ξ be any strict geometric point of $S(\overline{s})$.

(i): If f is flat, the induced morphism $f_{\overline{x}} : X(\overline{x}) \to S(\overline{s})$ is faithfully flat; especially, $f_{\overline{x}}$ is surjective ([61, Th.7.3(i)]).

(ii): We shall use the following :

Claim 7.1.32. Let A be a Krull domain, F the field of fractions of A, B a flat A-algebra, and suppose that $B \otimes_A \kappa(\mathfrak{p})$ is reduced, for every prime ideal $\mathfrak{p} \in \mathrm{Spec}\, A$ of height one. Then B is integrally closed in $B \otimes_A F$.

Proof of the claim. See [61, §12] for the basic generalities on Krull domains; especially, [61, Th.12.6] asserts that, if A is a Krull domain, the natural sequence of A-modules :

$$\mathscr{E} \quad : \quad 0 \to A \to F \to \bigoplus_{\mathrm{ht}\,\mathfrak{p}=1} F/A_{\mathfrak{p}} \to 0$$

is short exact, where $\mathfrak{p} \in \mathrm{Spec}\, A$ ranges over the prime ideals of height one. By flatness, $\mathscr{E} \otimes_A B$ is still exact, hence $B = \bigcap_{\mathrm{ht}\,\mathfrak{p}=1} B \otimes_A A_{\mathfrak{p}}$ (where the intersection takes place in $B \otimes_A F$). We are therefore reduced to the case where A is a discrete valuation ring. Let t denote a chosen generator of the maximal ideal of A, and suppose that $x \in B \otimes_A F$ is integral over B, so that $x^n + b_1 x^{n-1} + \cdots + b_n = 0$ in $B \otimes_A F$, for some $b_1, \dots, b_n \in B$; let also $r \in \mathbb{N}$ be the minimal integer such that we have $x = t^{-r} b$ for some $b \in B$. We have to show that $r = 0$; to this aim, notice that $b^n + t^r b_1 b^{n-1} + \cdots + t^{rn} b_n = 0$ in B; if $r > 0$, it follows that the image $\overline{b}$ of b in B/tB satisfies the identity : $\overline{b}^n = 0$, therefore $\overline{b} = 0$, since by assumption, B/tB is reduced. Thus $b = tb'$ for some $b' \in B$, and $x = t^{1-r} b'$, contradicting the minimality of r. ◊

Now, in the general situation of (ii), set $X' := X \times_S S(\overline{s})$. The natural morphism $X(\overline{x}) \to X$ factors uniquely through a morphism of $S(\overline{s})$-schemes $j : X(\overline{x}) \to X'$, and if we denote by $\overline{x}'$ the image in X' of $\overline{x}$, then j induces an isomorphism of $S(\overline{s})$-schemes : $X(\overline{x}) \xrightarrow{\sim} X'(\overline{x}')$. Hence, f is locally 0-acyclic at the point x, if and only if the induced morphism $X' \to S(\overline{s})$ is locally 0-acyclic at the support x' of $\overline{x}'$, so we may replace S by $S(\overline{s})$, and assume that S is strictly local, when (a) holds, and even strictly local and noetherian, when (b) holds.

We have to show that $f_{\overline{x}}^{-1}(\xi)$ is connected, and by lemma 7.1.25 and remark 7.1.27, we are further reduced to the case where $S = S(\overline{s}) = \mathrm{Spec}\, A$ is strictly local and normal, ξ is localized at the generic point of S, and moreover :

(a') either f is finitely presented,
(b') or else, A is a (not necessarily noetherian) Krull domain.

Claim 7.1.33. In case (b') holds, $f_{\overline{x}}^{-1}(\xi)$ is connected.

Proof of the claim. Indeed, let F be the field of fractions of A; the field $\kappa(\xi)$ is algebraic over F, hence it suffices to show that $X(\overline{x}) \times_S \mathrm{Spec}\, K$ is connected for every finite field extension $F \subset K$ ([32, Ch.IV, Prop.8.4.1(ii)]). Let A_K be the normalization of A in K; then A_K is again a Krull domain ([14, Ch.VII, §1, n.8, Prop.12]), and $B := \mathscr{O}_{X,x}^{\mathrm{sh}} \otimes_A A_K$ is a flat A_K-algebra. Notice that the geometric fibres of the induced morphism $f_{\overline{x},K} : \mathrm{Spec}\, B \to \mathrm{Spec}\, A_K$ are cofiltered limits of schemes that are étale over the fibres of f; since the fibres of f are geometrically reduced, it follows that the same holds for the fibres of $f_{\overline{x},K}$. Hence B is integrally closed in $B \otimes_A F$ (claim 7.1.32); especially these two rings have the same idempotents, whence the contention. ◊

Finally, suppose that (a') holds. By lemma 7.1.30, the scheme S is the limit of a cofiltered family $(S_\lambda \mid \lambda \in \Lambda)$ of strictly local excellent and normal schemes, such that the natural maps $S \to S_\lambda$ are dominant. By [32, Ch.IV, Th.8.8.2(ii)], we may find $\lambda \in \Lambda$, a finitely presented morphism $f_\lambda : X_\lambda \to S_\lambda$, and an isomorphism of S-schemes : $X \xrightarrow{\sim} X_\lambda \times_{S_\lambda} S$. We set $X_\mu := X_\lambda \times_{S_\lambda} S_\mu$, for every $\mu \in \Lambda$ with $\mu \geq \lambda$; then, up to replacing Λ by a cofinal subset, we may assume that $f_\mu : X_\mu \to S_\mu$ is defined for every $\mu \in \Lambda$. By [32, Ch.IV, Cor.11.2.6.1], after replacing Λ by a cofinal system, we may assume that f_λ is flat for every $\lambda \in \Lambda$. For every $\mu \in \Lambda$, let $Z_\mu \subset S_\mu$ be the subset consisting of all points $s \in S_\mu$ such that the fibre $f_\mu^{-1}(s)$ is not geometrically reduced; by [32, Ch.IV, Th.9.7.7], Z_μ is a constructible subset of S_μ. By assumption, we have

$$\bigcap_{\mu \in \Lambda} p_\mu^{-1} Z_\mu = \varnothing$$

and it is clear that $p_\mu^{-1} Z_\mu \subset p_\lambda^{-1} Z_\lambda$ whenever $\mu \geq \lambda$. Then, $Z_\mu = \varnothing$ for some $\mu \in \Lambda$ ([30, Ch.IV, Prop.1.8.2, Cor.1.9.8]), hence we may replace Λ by a still smaller cofinal subset, and achieve that all the f_λ have geometrically reduced fibres. For every $\lambda \in \Lambda$, let $x_\lambda \in X_\lambda$ be the image of the point x. Arguing as in (7.1.28), we obtain a compatible system of strict geometric points ξ_λ of S_λ (resp. $\overline{x}_\lambda$ of X_λ), such that $p_\lambda(\xi)$ factors through ξ_λ; whence an isomorphism (7.1.29). Thus, $f_{\overline{x}}^{-1}(\xi)$ is reduced if and only if $f_{\lambda,\overline{x}_\lambda}^{-1}(\xi_\lambda)$ is reduced for every sufficiently large $\lambda \in \Lambda$ ([32, Ch.IV, Prop.8.7.2]). Furthermore, since p_λ is dominant, ξ_λ is localized at the generic point of S_λ, for every $\lambda \in \Lambda$. Thus, we are reduced to the case where $S = S(s)$ is the spectrum of a strictly local noetherian normal domain A, and ξ is localized at the generic point of S; since such A is a Krull domain ([61, Th.12.4(i)]), this is covered by claim 7.1.33. ☐

Example 7.1.34. (i) Let A be an excellent local ring, and $A^\wedge$ the completion of A. Then the natural morphism :

$$f : \mathrm{Spec}\, A^\wedge \to \mathrm{Spec}\, A$$

is locally 0-acyclic. Indeed, this follows from proposition 7.1.31(ii) (and from the excellence assumption, which includes the geometric regularity of the formal fibres of A).

(ii) Suppose additionally, that A is strictly local. Then f is 0-acyclic. To see this, we apply the criterion of lemma 7.1.17 : indeed, since f is flat, it is (-1)-acyclic (proposition 7.1.31(i)); it remains to show that f has geometrically connected fibres, and since $A^\wedge$ is strictly local ([33, Ch.IV, Prop.18.5.14]), this is the same as showing that f is locally 0-acyclic at the closed point of $\mathrm{Spec}\, A^\wedge$, which has already been remarked in (i).

(iii) More generally, f is 0-acyclic whenever A is excellent and henselian. Indeed, in this case the argument of (ii) again reduces to showing that f has geometrically connected fibres. However, consider the natural commutative diagram :

(7.1.35)
$$\begin{tikzcd} \mathrm{Spec}\,(A^\wedge)^{\mathrm{sh}} \arrow[r] \arrow[d, "f^{\mathrm{sh}}"'] & \mathrm{Spec}\, A^\wedge \arrow[d, "f"] \\ \mathrm{Spec}\, A^{\mathrm{sh}} \arrow[r] & \mathrm{Spec}\, A. \end{tikzcd}$$

Since A is henselian, A^{sh} is the colimit of a filtered family of finite étale and local A-algebras. Since A and $A^\wedge$ have the same residue field, it follows easily that $A^\wedge \otimes_A A^{\mathrm{sh}}$ is the colimit of a filtered family of finite étale and local $A^\wedge$-algebras, hence it is strictly henselian, and therefore (7.1.35) is cartesian, especially the geometric fibres of f are connected if and only if the same holds for the geometric fibres of f^{sh}, and the latter are reduced (even regular), since A is excellent. Hence, we come down to showing that f^{sh} is locally 0-acyclic at the closed point of $\mathrm{Spec}\,(A^\wedge)^{\mathrm{sh}}$, which holds again by proposition 7.1.31(ii).

For future use, we point out the following

Proposition 7.1.36. *Let $g : X \to Y$ be a flat morphism of excellent noetherian schemes, with X strictly local, and Y normal. Let $U \subset X$ be an open subset, and $Z \subset Y$ a closed subscheme. Suppose that :*

(i) *$g^{-1}(z) \subset U$ for every maximal point z of Z.*
(ii) *$U \cap g^{-1}(z)$ is a dense open subset of $g^{-1}(z)$, for every $z \in Z$.*
(iii) *The fibres $g^{-1}(z)$ are reduced, for every $z \in Z$.*

Then the induced functor $\mathbf{Cov}(U) \to \mathbf{Cov}(U \times_Y Z)$ is fully faithful.

Proof. Indeed, say that $X = \operatorname{Spec} B$, $Y = \operatorname{Spec} A$, $Z = V(I)$ for some ideal $I \subset A$, and denote by $B^\wedge$ the $\mathfrak{m}_B$-adic completion of the local ring B (where $\mathfrak{m}_B \subset B$ denotes the maximal ideal). Let also $f : \operatorname{Spec} B^\wedge \to Y$ be the induced morphism, and $U^\wedge \subset \operatorname{Spec} B^\wedge$ the preimage of U. In light of example 7.1.34(ii) and lemma 7.1.16(ii), it suffices to show that the induced functor $\mathbf{Cov}(U^\wedge) \to \mathbf{Cov}(U^\wedge \times_Y Z)$ is fully faithful. In view of lemma 7.1.3, we are further reduced to checking that conditions (a)–(c) of proposition 5.5.44 hold for the induced ring homomorphism $\varphi : A \to B^\wedge$, the open subset $U^\wedge$, and the ideal I. However, by example 5.5.39, we have $\operatorname{Ass}_A(I, A) = \operatorname{Max}(Z)$, hence (c) follows trivially from our assumption (i). Next, since $B^\wedge$ is a faithfully flat B-algebra, assumption (ii) implies that $U^\wedge \cap f^{-1}(z)$ is a dense open subset of $f^{-1}(z)$, for every $z \in Z$. Moreover, since B is excellent, the natural morphism $\operatorname{Spec} B^\wedge \to X$ is regular, so the same holds for the induced morphism $f^{-1}(z) \to g^{-1}(z)$, and then our assumption (iii) implies – together with [61, Th.32.3(i)] – that $f^{-1}(z)$ is reduced, for every $z \in Z$, whence condition (b). Lastly, we check condition (a), *i.e.* we show that $B^\wedge$ is I-adically complete. Indeed, let C be the I-adic completion of $B^\wedge$; the natural map $B^\wedge \to C$ is injective, and it admits a left inverse, constructed as follows. Let $\underline{a} := (a_n \mid n \in \mathbb{N})$ be a given sequence of elements of $B^\wedge$, which is Cauchy for the I-adic topology; then $\underline{a}$ is also Cauchy for the $\mathfrak{m}_B$-adic topology, and it is easily seen that the limit l of $\underline{a}$ in the $\mathfrak{m}_B$-adic topology depends only on the class $[\underline{a}]$ of $\underline{a}$ in C, so we get a well defined ring homomorphism $\lambda : C \to B^\wedge$ by the rule : $[\underline{a}] \mapsto l$, and clearly λ is the sought left inverse. It remains to check that λ is injective; thus, suppose that $l = 0$, and that $[\underline{a}] \neq 0$; this means that there exists $N \in \mathbb{N}$ such that $a_n \notin I^N$, for every $n \in \mathbb{N}$. Now, the induced sequence $(\overline{a}_n \mid n \in \mathbb{N})$ of elements of $B^\wedge/I^N$ is stationary, and on the other hand, it converges $\mathfrak{m}_B$-adically to 0; therefore $\overline{a}_n = 0$ for every sufficiently large $n \in \mathbb{N}$, a contradiction. $\square$

7.1.37. Let A be a noetherian normal ring, and endow the A-algebra $A[[t]]$ with its t-adic topology. Let

$$\varphi : \mathfrak{X} := \operatorname{Spf} A[[t]] \to X := \operatorname{Spec} A[[t]] \qquad \pi : X \to S := \operatorname{Spec} A \qquad i : S \to X$$

be respectively the natural morphism of locally ringed spaces, the natural projection, and the closed immersion determined by the ring homomorphism $A[[t]] \to A$ given by the rule : $f(t) \mapsto f(0)$, for every $f(t) \in A[[t]]$. Let also $U_0 \subset \operatorname{Spec} A$ be an open subset, $U := \pi^{-1}U_0$ and $\mathfrak{U} := \varphi^{-1}U$. Finally, denote by $\mathscr{E}$ a locally free $\mathscr{O}_U$-module of finite rank, and set $\mathscr{E}^\wedge := \varphi^*_{|\mathfrak{U}}\mathscr{E}$, which is a locally free $\mathscr{O}_\mathfrak{U}$-module of finite rank.

Lemma 7.1.38. *In the situation of* (7.1.37)*, suppose that $S \setminus U_0$ has codimension ≥ 2 in S. Then:*

(i) *The natural map*
$$\Gamma(U, \mathscr{E}) \to \Gamma(\mathfrak{U}, \mathscr{E}^\wedge)$$
is an isomorphism of $A[[t]]$-modules.

(ii) *The pull-back functor :*
$$i^*_{|U_0} : \mathbf{Cov}(U) \to \mathbf{Cov}(U_0) \quad : \quad (E \to U) \mapsto (E \times_U U_0 \to U_0)$$
is an equivalence.

Proof. (i): To begin with, set $Z := S\setminus U_0$; since the morphism π is flat, hence generizing ([61, Th.9.5]), the closed subset $X\setminus U = \pi^{-1}Z$ has codimension ≥ 2 in X. Since A and $A[[t]]$ are both normal, we deduce :

$$\text{(7.1.39)} \qquad \mathrm{depth}_{X\setminus V}\mathscr{O}_X \geq 2 \qquad \mathrm{depth}_Z\mathscr{O}_S \geq 2$$

(theorem 5.4.20(ii) and [61, Th.23.8]); therefore (corollary 5.4.22) :

$$\text{(7.1.40)} \qquad \Gamma(U,\mathscr{O}_X) = \Gamma(X,\mathscr{O}_X) = A[[t]].$$

Next, the short exact sequences of $\mathscr{O}_X$-modules :

$$0 \to i_*\mathscr{O}_S \to \mathscr{O}_X/t^{n+1}\mathscr{O}_X \to \mathscr{O}_X/t^n\mathscr{O}_X \to 0 \qquad \text{for every } n \in \mathbb{N}$$

induce exact sequences

$$\text{(7.1.41)} \qquad R^j\Gamma_Z i_*\mathscr{O}_S \to R^j\Gamma_Z\mathscr{O}_X/t^{n+1}\mathscr{O}_X \to R^j\Gamma_Z\mathscr{O}_X/t^n\mathscr{O}_X \qquad \text{for every } n,j \in \mathbb{N}.$$

Then (7.1.39) and (7.1.41) yield inductively :

$$\mathrm{depth}_Z\mathscr{O}_X/t^n\mathscr{O}_X \geq 2 \qquad \text{for every } n \in \mathbb{N}$$

and again corollary 5.4.22 implies :

$$\text{(7.1.42)} \qquad \Gamma(U,\mathscr{O}_X/t^n\mathscr{O}_X) = A[t]/t^nA[t] \qquad \text{for every } n \in \mathbb{N}.$$

Since U is quasi-compact, we may find a left exact sequence $P := (0 \to \mathscr{E} \to \mathscr{O}_U^{\oplus m} \to \mathscr{O}_U^{\oplus n})$ of $\mathscr{O}_U$-modules (corollary 5.2.17). Since φ is a flat morphism of locally ringed spaces, the sequence φ^*P is still left exact. Since the global section functors are left exact, we are then reduced to the case where $\mathscr{E} = \mathscr{O}_U$. Then we may write :

$$\mathscr{E}^\wedge = \mathscr{O}_{\mathfrak{U}} = \lim_{n\in\mathbb{N}} \mathscr{O}_U/t^n\mathscr{O}_U$$

where, for each $n \in \mathbb{N}$, we regard $\mathscr{O}_U/t^n\mathscr{O}_U$ as a sheaf of (pseudo-discrete) rings on $\mathfrak{U} = V(t) \subset U$. The functor $\Gamma(\mathfrak{U},-)$ is a right adjoint, hence commutes with limits, and we deduce an isomorphism :

$$\Gamma(\mathfrak{U},\mathscr{E}^\wedge) \xrightarrow{\sim} \lim_{n\in\mathbb{N}} \Gamma(U,\mathscr{O}_U/t^n\mathscr{O}_U).$$

(This is even a homeomorphism, provided we view the target as a limit of rings with the discrete topology.) Taking (7.1.42) into account, we obtain $\Gamma(\mathfrak{U},\mathscr{E}^\wedge) = A[[t]]$ which, together with (7.1.40), implies the contention.

(ii): Notice first that (i) and lemma 5.5.42 imply that $\mathrm{Lef}(U, i(U_0))$ holds (see definition 5.5.40). Since the pull-back functor $\pi^*_{|U} : \mathbf{Cov}(U_0) \to \mathbf{Cov}(U)$ is a right quasi-inverse to $i_{|U_0}$, it is clear that the latter is essentially surjective. The full faithfulness is a special case of lemma 7.1.3. □

7.2. **Local asphericity of smooth morphisms of schemes.** Let S be a strictly local scheme, $s \in S$ the closed point, $f : X \to S$ a smooth morphism, $\overline{x}$ any geometric point of $f^{-1}(s)$, and denote by $f_{\overline{x}} : X(\overline{x}) \to S$ the induced morphism of strictly local schemes. For any open subset $U \subset S$ we have a base change functor :

$$\text{(7.2.1)} \qquad f^*_{\overline{x}} : \mathbf{Cov}(U) \to \mathbf{Cov}(f^{-1}_{\overline{x}}U) \qquad (E \to U) \mapsto (E \times_U f^{-1}_{\overline{x}}U).$$

Theorem 7.2.2. *In the situation of* (7.2), *we have :*

(i) *The functor* (7.2.1) *is fully faithful.*
(ii) *Suppose moreover that S is excellent and normal, and that $S\setminus U$ has codimension ≥ 2 in S. Then* (7.2.1) *is an equivalence of categories.*

Proof. (i): In view of lemma 7.1.16(ii) it suffices to show that $f_{\overline{x}}$ is 0-acyclic (since in that case, the same will obviously hold also for its restriction $f_{\overline{x}}^{-1}U \to U$). To begin with, $f_{\overline{x}}$ is locally (-1)-acyclic, by proposition 7.1.31(i), hence it remains only to show that f is locally 0-acyclic at the point x (lemma 7.1.17). The latter assertion follows from proposition 7.1.31(ii) and [33, Ch.IV, Th.17.5.1].

(ii): In light of (i), it suffices to show that (7.2.1) is essentially surjective, under the assumptions of (ii). We argue by induction on the relative dimension n of f. Let $x \in X$ be the support of $\overline{x}$. We may find an open neighborhood $U \subset X$ of x, and an étale morphism of S-schemes $\varphi : U \to \mathbb{A}^n_S$ ([33, Ch.IV, Cor.17.11.4]). Let $\overline{x}' := \varphi(\overline{x})$; there follows an isomorphism of S-schemes : $X(\overline{x}) \stackrel{\sim}{\to} \mathbb{A}^n_S(\overline{x}')$, hence we may assume from start that $X = \mathbb{A}^n_S$, and f is the natural projection. Especially, the theorem holds for $n = 0$. Suppose then, that $n > 0$, and that the theorem is already known when the relative dimension is $< n$. Write f as the composition $f = h \circ g$, where

$$g : X \simeq \mathbb{A}^{n-1}_S \times_S \mathbb{A}^1_S \to \mathbb{A}^1_S \qquad \text{and} \qquad h : \mathbb{A}^1_S \to S$$

are the natural projections; set $\overline{x}_1 := g(\overline{x})$, and $U_1 := h_{\overline{x}_1}^{-1}U$, (where $h_{\overline{x}_1} : S_1 := \mathbb{A}^1_S(\overline{x}_1) \to S$ is the morphism induced by h). We have $S_1 \backslash U_1 = h_{\overline{x}_1}^{-1}(S \backslash U)$, and since flat maps are generizing ([61, Th.9.5]) we easily see that the codimension of $S_1 \backslash U_1$ in S_1 equals the codimension of $S \backslash U$ in S. From our inductive assumption, we deduce that the base change functor $\mathbf{Cov}(U_1) \to \mathbf{Cov}(f_{\overline{x}}^{-1}U)$ is essentially surjective, and hence it suffices to show that the same holds for the functor $\mathbf{Cov}(U) \to \mathbf{Cov}(U_1)$. Thus, we are reduced to the case where $X = \mathbb{A}^1_S$. Suppose now, that $E \to f_{\overline{x}}^{-1}U$ is a finite étale morphism; we can write $f_{\overline{x}}$ as the limit of a cofiltered family of smooth morphisms $(f_\lambda : Y_\lambda \to S \mid \lambda \in \Lambda)$, where each Y_λ is an affine étale $\mathbb{A}^1_S$-scheme. Then $f_{\overline{x}}^{-1}U$ is the limit of the family $(Y_\lambda \times_S U \mid \lambda \in \Lambda)$. By [32, Ch.IV, Th.8.8.2(ii), Th.8.10.5] and [33, Ch.IV, Prop.17.7.8], we may find a $\lambda \in \Lambda$, a finite étale morphism $E_\lambda \to Y_\lambda \times_S U$ and an isomorphism of $f_{\overline{x}}^{-1}U$-schemes : $E_\lambda \times_{Y_\lambda} \mathbb{A}^1_S(\overline{x}) \stackrel{\sim}{\to} E$. Denote by $y \in Y_\lambda$ the image of the closed point of $\mathbb{A}^1_S(\overline{x})$, and by $\overline{y}$ the geometric point of Y_λ obtained as the image of $\overline{x}$ (the latter is viewed naturally as a geometric point of $\mathbb{A}^1_S(\overline{x})$); by construction, y lies in the closed fibre $Y_0 := Y_\lambda \times_S \operatorname{Spec} \kappa(s)$, which is an étale $\mathbb{A}^1_{\kappa(s)}$-scheme, and we may therefore find a specialization $z \in Y_0$ of y, with z a closed point. Pick a geometric point $\overline{z}$ of Y_0 localized at z, and a strict specialization map $Y_\lambda(\overline{z}) \to Y_\lambda(\overline{y})$ as in (2.4.22); there follows a commutative diagram :

$$\begin{tikzcd}
\mathbb{A}^1_S(\overline{x}) \simeq Y_\lambda(\overline{y}) \arrow[r] \arrow[d] & Y_\lambda(\overline{z}) \arrow[d] \\
Y_\lambda(y) \arrow[r] & Y_\lambda(z).
\end{tikzcd}$$

The finite étale covering $E_\lambda \times_{Y_\lambda} Y_\lambda(y) \to Y_\lambda(y) \times_S U$ lies in the essential image of the functor

$$\mathbf{Cov}(Y_\lambda(z) \times_S U) \to \mathbf{Cov}(Y_\lambda(y) \times_S U) \qquad C \mapsto C \times_{Y_\lambda(z)} Y_\lambda(y).$$

It follows that $E \to f_{\overline{x}}^{-1}U$ lies in the essential image of the functor

$$\mathbf{Cov}(f_{\lambda,\overline{z}}^{-1}U) \to \mathbf{Cov}(f_{\overline{x}}^{-1}U) \qquad C \mapsto C \times_{Y(\overline{z})} Y(\overline{y}) \simeq C \times_{Y(\overline{z})} \mathbb{A}^1_S(\overline{x})$$

and therefore it suffices to show that the pull-back functor $\mathbf{Cov}(U) \to \mathbf{Cov}(f_{\overline{z}}^{-1}U)$ is essentially surjective. In other words, we may replace x by z, and assume throughout that x is a closed point of $\mathbb{A}^1_S$.

Claim 7.2.3. Under the current assumptions, we may find a strictly local normal scheme T, with closed point t, a finite surjective morphism $g : T \to S$, and a finite morphism of $\operatorname{Spec} \kappa(s)$-schemes :

$$\operatorname{Spec} \kappa(t) \to \operatorname{Spec} \kappa(x).$$

Proof of the claim. Since $\kappa(x)$ is a finite extension of $\kappa(s)$, it is generated by finitely many algebraic elements $u_1, \dots u_n$, and an easy induction allows to assume that $n = 1$. In this case, one constructs first a scheme T' by taking any lifting of the minimal polynomial of u_1 : for the details, see e.g. [28, Ch.0, (10.3.1.2.)], which shows that the resulting T is local, finite and flat over S, so T' maps surjectively onto S. Next, we may replace T' by its maximal reduced subscheme, which is still strictly local and finite over S. Next, since S is excellent, the normalization $(T')^\nu$ of T' is finite over S ([31, Ch.IV, Scholie 7.8.3(vi)]); let T be any irreducible component of $(T')^\nu$; by [33, Ch.IV, Prop.18.8.10], T fulfills all the sought conditions. $\diamondsuit$

Choose $g : T \to S$ as in claim 7.2.3; since the residue field extension $\kappa(s) \to \kappa(t)$ is algebraic and purely inseparable, there exists a unique point $x' \in \mathbb{A}^1_S(\overline{x}) \times_S T$ lying over t, and we may find a unique strict geometric point $\overline{x}'$ of $\mathbb{A}^1_S(\overline{x}) \times_S T$ localized at x', and lying over $\overline{x}$. In view of [33, Ch.IV, Prop.18.8.10], there follows a natural isomorphism of T-schemes :

$$\mathbb{A}^1_S(\overline{x}) \times_S T \xrightarrow{\sim} \mathbb{A}^1_T(\overline{x}').$$

Denote by $f_{\overline{x}'} := f_{\overline{x}} \times_S T : \mathbb{A}^1_T(\overline{x}') \to T$ the natural projection, and set $U_T := g^{-1}U$; since the morphism $g : T \to S$ is generizing ([61, Th.9.4(ii)]), it is easily seen that $T \setminus U_T$ has codimension ≥ 2 in T.

Let $F : \mathbf{Cov} \to \mathbf{Sch}$ be the fibred category (7.1.1). We have a natural essentially commutative diagram of categories :

$$\begin{tikzcd}
\mathbf{Cov}(U) \arrow[r] \arrow[d] & \mathsf{Desc}(F, g \times_S U) \arrow[d, "\delta"] \\
\mathbf{Cov}(f_{\overline{x}}^{-1}U) \arrow[r] & \mathsf{Desc}(F, g \times_S f_{\overline{x}}^{-1}U)
\end{tikzcd} \tag{7.2.4}$$

where, for any morphism of schemes h, we have denoted by $\mathsf{Desc}(F, h)$ the category of descent data for the fibred category F, relative to the morphism h.

According to lemma 7.1.2, the morphism g is of universal 2-descent for the fibred category F, so the horizontal arrows in (7.2.4) are equivalences. Hence, the theorem will follow, once we know that δ is essentially surjective. However, we have :

Claim 7.2.5. (i) Set $U'_T := U_T \times_S T$ and $U''_T := U'_T \times_S T$. The pull-back functors :

$$\mathbf{Cov}(U'_T) \to \mathbf{Cov}(\mathbb{A}^1_T(\overline{x}') \times_T U'_T) \qquad \mathbf{Cov}(U''_T) \to \mathbf{Cov}(\mathbb{A}^1_T(\overline{x}') \times_T U''_T)$$

are fully faithful.

(ii) Suppose that the pull-back functor

$$\mathbf{Cov}(U_T) \to \mathbf{Cov}(f_{\overline{x}'}^{-1}U_T)$$

is essentially surjective. Then the same holds for the functor δ.

Proof of the claim. (i): Let $\overline{z}''$ be any geometric point of $X'' := \mathbb{A}^1_T \times_S T \times_S T$ whose strict image in $\mathbb{A}^1_T$ is $\overline{x}'$, and let $\overline{z}'$ be the image of $\overline{z}''$ in $X' := \mathbb{A}^1_T \times_S T$; by lemma 7.1.23(ii), the natural morphisms :

$$X'(\overline{z}') \to \mathbb{A}^1_T(\overline{x}') \times_S T \qquad X''(\overline{z}'') \to \mathbb{A}^1(\overline{x}') \times_S T \times_S T$$

are isomorphisms (notice that $T \times_S T$ is also strictly local). Then the claim follows from assertion (i) of the theorem, applied to the projections $X' \to T \times_S T$ and $X'' \to T \times_S T \times_S T$.

(ii): Recall that an object of $\mathsf{Desc}(F, g \times_S f_{\overline{x}'}^{-1}U)$ consists of a finite étale morphism $E'_T \to f_{\overline{x}'}^{-1}U_T$ and a X'-isomorphism $\beta' : E'_T \times_S T \xrightarrow{\sim} T \times_S E'_T$ fulfilling a cocycle condition on $E \times_S T \times_S T$. By assumption, E_T descends to a finite étale morphism $E_T \to U_T$; then (i) implies that β' descends to a U'_T-isomorphism $\beta : E_T \times_S T \xrightarrow{\sim} T \times_S E_T$, and the cocycle identity for β' descends to a cocycle identity for β. $\diamondsuit$

In view of claim 7.2.5, we may replace (S, U, x) by (T, U_T, x'), and therefore assume that x is a $\kappa(s)$-rational point of $\mathbb{A}^1_{\kappa(s)}$. In this case, any choice of coordinate t on $\mathbb{A}^1_S$ yields a section $\sigma_{\overline{x}} : S \to \mathbb{A}^1_S(\overline{x})$ of the natural projection, such that $\sigma_{\overline{x}}(s) = x$. To conclude the proof of the theorem, it suffices to show that the pull-back functor :

$$\mathbf{Cov}(f_{\overline{x}}^{-1}U) \xrightarrow{\sigma_{\overline{x}}^*} \mathbf{Cov}(U)$$

is fully faithful.

Say that $S = \operatorname{Spec} A$; then the scheme $\mathbb{A}^1_S(\overline{x})$ is the spectrum of $A\{t\}$, the henselization of $A[t]$ along the ideal $\mathfrak{m}\{t\}$ generated by t and the maximal ideal $\mathfrak{m}$ of A. Let $A^\wedge$ (resp. $A\{t\}^\wedge$) be the $\mathfrak{m}$-adic (resp. $\mathfrak{m}\{t\}$-adic) completion of A (resp. of $A\{t\}$), and notice the natural isomorphism:

$$A\{t\}^\wedge / tA\{t\}^\wedge \xrightarrow{\sim} A^\wedge$$

(indeed, it is easy to check that $A\{t\}^\wedge \simeq A^\wedge[[t]]$), whence a natural diagram of schemes :

$$\begin{tikzcd} X^\wedge := \operatorname{Spec} A\{t\}^\wedge \arrow[r, "g'"] \arrow[d, shift right, "\pi"'] & \operatorname{Spec} A\{t\} \arrow[d, shift right, "f_{\overline{x}}"'] \\ S^\wedge := \operatorname{Spec} A^\wedge \arrow[r, "g"] \arrow[u, shift right, "\sigma"'] & \operatorname{Spec} A \arrow[u, shift right, "\sigma_{\overline{x}}"'] \end{tikzcd}$$

(where π is the natural projection) whose horizontal arrows commute with both the downward arrows and the upward ones. Set $U^\wedge := g^{-1}U$; by example 7.1.34(ii) and lemma 7.1.16(ii), the pull-back functors

$$g^* : \mathbf{Cov}(U) \to \mathbf{Cov}(U^\wedge) \qquad g'^* : \mathbf{Cov}(f_{\overline{x}}^{-1}U) \to \mathbf{Cov}(\pi^{-1}U^\wedge)$$

are fully faithful. Consequently, we are easily reduced to showing that the pull-back functor : $\mathbf{Cov}(\pi^{-1}U^\wedge) \xrightarrow{\sigma^*} \mathbf{Cov}(U^\wedge)$ is an equivalence. The latter holds by lemma 7.1.38(ii). ☐

Example 7.2.6. As an application of theorem 7.2.2, suppose that $K \subset E$ is an extension of separably closed fields, V_K a geometrically normal and strictly local K-scheme, $U \subset V_K$ an open subset, and ξ a geometric point of $V_E := V_K \times_K E$, whose image in V_K is supported on the closed point. Then the induced functor

$$\mathbf{Cov}(U) \to \mathbf{Cov}(U \times_{V_K} V_E(\xi))$$

is fully faithful, and it is an equivalence in case $V_K \backslash U$ has codimension ≥ 2 in V_K.

Indeed, let K^{a} (resp. E^{a}) be an algebraic closure of K (resp. E), and choose a homomorphism $K^{\mathrm{a}} \to E^{\mathrm{a}}$ extending the inclusion of K into E. Then both $V_{K^{\mathrm{a}}} := V_K \times_K K^{\mathrm{a}}$ and $V_{E^{\mathrm{a}}}(\xi) := V_E(\xi) \times_E E^{\mathrm{a}}$ are still normal and strictly local (lemma 7.1.23(ii)), and the induced functors

$$\mathbf{Cov}(U) \to \mathbf{Cov}(U \times_K K^{\mathrm{a}}) \qquad \mathbf{Cov}(U \times_{V_K} V_E(\xi)) \to \mathbf{Cov}(U \times_{V_K} V_{E^{\mathrm{a}}}(\xi))$$

are equivalences (lemma 7.1.7(i)). It then suffices to show that the induced functor

$$\mathbf{Cov}(U \times_K K^{\mathrm{a}}) \to \mathbf{Cov}(U \times_{V_K} V_{E^{\mathrm{a}}}(\xi))$$

has the asserted properties. Hence, we may replace K by K^{a} and E by E^{a}, and assume from start that $K \subset E$ is an extension of algebraically closed fields. In this case, E can be written as the colimit of a filtered family $(R_\lambda \mid \lambda \in \Lambda)$ of smooth K-algebras; correspondingly, V_E is the limit of a cofiltered system $(V_\lambda \mid \lambda \in \Lambda)$ of smooth V_K-schemes, and – by lemma 7.1.6 – $\mathbf{Cov}(U \times_{V_K} V_E(\xi))$ is the 2-colimit of the system of categories

$$\mathbf{Cov}(U \times_{V_K} V_\lambda(\xi_\lambda)) \qquad (\lambda \in \Lambda)$$

(where, for each $\lambda \in \Lambda$, we denote by ξ_λ the image of ξ in V_λ). Now the contention follows directly from theorem 7.2.2.

Theorem 7.2.7. *Let $f : X \to S$ is a smooth morphism of schemes, $\mathbb{L} \subset \mathbb{N}$ be a set of primes, and suppose that all the elements of $\mathbb{L}$ are invertible in $\mathscr{O}_S$. Then f is 1-aspherical for $\mathbb{L}$.*

Proof. Let $\overline{x}$ be any geometric point of X, $\overline{s} := f(\overline{x})$, and $\overline{\eta}$ a strict geometric point of $S(\overline{s})$. To ease notation, set $T := X(\overline{x})$, let $f_{\overline{x}} : T \to S$ be the natural map, and $T_{\overline{\eta}} := f_{\overline{x}}^{-1}(\overline{\eta})$; we have to show that $H^1(T_{\overline{\eta},\text{ét}}, G) = \{1\}$ for every $\mathbb{L}$-group G. Arguing as in the proof of proposition 7.1.31, we reduce to the case where $S = S(\overline{s})$. Then, by lemma 7.1.25, we can further assume that S is normal and $\overline{\eta}$ is localized at the generic point η of S. By lemma 7.1.30, S is the limit of a cofiltered system $(S_\lambda \mid \lambda \in \Lambda)$ of strictly local, normal and excellent schemes, and as usual, after replacing Λ by a cofinal subset, we may assume that f (resp. $\overline{\eta}$) descends to a compatible system of morphisms $(f_\lambda : X_\lambda \to S_\lambda \mid \lambda \in \Lambda)$, (resp. of strict geometric points $\overline{\eta}_\lambda$ localized at the generic point of S_λ). By [33, Ch.IV, Prop.17.7.8(ii)], there exists $\lambda \in \Lambda$ such that f_μ is smooth for every $\mu \geq \lambda$. Then, in view of [4, Exp.VII, Rem.5.14] and the isomorphism (7.1.29), we may replace f by f_λ, and $\overline{\eta}$ by $\overline{\eta}_\lambda$, and assume from start that S is strictly local, normal and excellent, and G is a finite $\mathbb{L}$-group.

Let $\varphi : E_{\overline{\eta}} \to T_{\overline{\eta}}$ be a principal G-homogeneous space; we come down to showing that $E_{\overline{\eta}}$ admits a section $T_{\overline{\eta}} \to E_{\overline{\eta}}$. By [33, Ch.IV, Prop.17.7.8(ii)] and [32, Ch.IV, Th.8.8.2(ii), Th.8.10.5], we may find a finite separable extension $\kappa(\eta) \subset L$, and a principal G-homogeneous space

$$\varphi_L : E_L \to T_L := T \times_S \operatorname{Spec} L \qquad \rho_L : G \to \operatorname{Aut}_{T_L}(E_L)$$

such that

$$\varphi = \varphi_L \times_{\operatorname{Spec} L} \operatorname{Spec} \kappa(\overline{\eta}) \qquad \rho = \rho_L \times_{\operatorname{Spec} L} \operatorname{Spec} \kappa(\overline{\eta}).$$

Say that $S = \operatorname{Spec} A$, denote by A_L the normalization of A in L, and set $S_L := \operatorname{Spec} A_L$. Then S_L is again normal and excellent ([31, Ch.IV, (7.8.3)(ii),(vi)]), and the residue field of A_L is an algebraic extension of the residue field of A, hence it is separably closed, so S_L is strictly local as well. Thus, we may replace S by S_L, and assume that $E_{\overline{\eta}}$ descends to a principal G-homogeneous space $E_\eta \to T_\eta := f_{\overline{x}}^{-1}(\eta)$ on T_η. Next, we may write η as the limit of the filtered system of affine open subsets of S, so that – by the same arguments – we find an affine open subset $U \subset S$ and a principal G-homogeneous space $E_U \to T_U := f_{\overline{x}}^{-1}U$, with a G-equivariant isomorphism of T_η-schemes : $E_U \times_{T_U} T_\eta \xrightarrow{\sim} E_\eta$. Denote by $D_1, \dots, D_n$ the irreducible components of $S \setminus U$ which have codimension one in S, and for every $i \leq n$, set $D'_i := f_{\overline{x}}^{-1} D_i$. Let also η_T be the generic point of T.

Claim 7.2.8. For given $i \leq n$, let y be the generic point of D_i, and z a maximal point of D'_i. We have:

(i) T and E_U are normal schemes, and $T(y)$ is regular.
(ii) D'_i is a closed subset of pure codimension one in T.
(iii) Let $\mathfrak{m}_y$ (resp. $\mathfrak{m}_z$) be the maximal ideal of $\mathscr{O}_{S,y}$ (resp. of $\mathscr{O}_{T,z}$); then $\mathfrak{m}_y \cdot \mathscr{O}_{T,z} = \mathfrak{m}_z$.
(iv) Let $t \in A$ be any element such that $t \cdot \mathscr{O}_{S,y} = \mathfrak{m}_y$. Then there exist an integer $m > 0$ such that $(m, \operatorname{char} \kappa(s)) = 1$, a finite étale covering

$$E_y \to T(y)[t^{1/m}] := T(y) \times_S \operatorname{Spec} A[T]/(T^m - t)$$

and an isomorphism of $T(y)[t^{1/m}] \times_T T_U$-schemes :

$$E_y \times_T T_U \xrightarrow{\sim} E_U \times_T T(y)[t^{1/m}].$$

Proof of the claim. (i): Since S is normal by assumption, the assertion for T and E_U follows from [33, Ch.IV, Prop.17.5.7, Prop.18.8.12(i)]. Next, set $W := X(y)$; since $\mathscr{O}_{S,y}$ is a discrete valuation ring, W is a regular scheme ([33, Ch.IV, Prop.17.5.8(iii)]). For any $w \in T(y) \subset W$, the natural map $\mathscr{O}_{T(y),w} \to W(w)^{\text{sh}}$ is faithfully flat, and $W(w)^{\text{sh}}$ is regular ([33, Ch.IV, Cor.18.8.13]), therefore $\mathscr{O}_{T(y),w}$ is regular, by [30, Ch.0, Prop.17.3.3(i)].

(ii): Say that $\mathfrak{p} \subset A$ is the prime ideal of height one such that $V(\mathfrak{p}) = D_i$; to ease notation, let also $B := \mathscr{O}^{\mathrm{sh}}_{X,x}$. Let $\{\mathfrak{q}_1, \dots, \mathfrak{q}_k\} \subset \operatorname{Spec} B$ be the set of maximal points of D'_i. Using the fact that flat morphisms are generizing ([61, Th.9.5]), one verifies easily that $A \cap \mathfrak{q}_j = \mathfrak{p}$ for every $j \leq k$. Fix $j \leq k$, and set $\mathfrak{q} := \mathfrak{q}_j$. Since A is normal, $A_\mathfrak{p}$ is a discrete valuation ring, hence $\mathfrak{p}A_\mathfrak{p}$ is a principal ideal, say generated by $t \in A_\mathfrak{p}$; therefore $\mathfrak{q}B_\mathfrak{q}$ is the minimal prime ideal of $B_\mathfrak{q}$ containing t, so $\mathfrak{q}B_\mathfrak{q}$ has height at most one, by Krull's Hauptidealsatz ([61, Th.13.5]). However, a second application of [61, Th.9.5] shows that the height of $\mathfrak{q}$ in B cannot be lower than one, hence $\mathfrak{q}$ has height one, which is the contention.

(iii): From (i) and (ii) we see that $\mathscr{O}_{S,y}$ and $\mathscr{O}_{T,z}$ are discrete valuation rings; then the assertion follows easily from [33, Ch.IV, Th.17.5.1].

(iv): To begin with, since $T(y)$ is regular, it decomposes as a disjoint union of connected components, in natural bijection with the set of maximal points of D'_i. Let $Z \subset T(y)$ be the connected open subscheme containing z; it suffices to show that there exists a finite étale covering :

$$E_Z \to Z[t^{1/m}] := Z \times_{T(y)} T(y)[t^{1/m}]$$

with an isomorphism of $Z[t^{1/m}] \times_T T_U$-schemes : $E_Z \times_T T_U \xrightarrow{\sim} E_U \times_T Z[t^{1/m}]$. By (iii) we have $t \cdot \mathscr{O}_{T,z} = \mathfrak{m}_z$. Notice that $T(z) \times_T T_U = T(\eta_T)$, and $E_{\eta_T} := E_U \times_T T(z)$ is a disjoint union of spectra of finite separable extensions $L_1, \dots, L_k$ of $\kappa(\eta_T)$. Moreover, E_{η_T} is a principal G-homogeneous space over $T(\eta_T)$, *i.e.* every L_j is a Galois extension of $\kappa(\eta_T)$, with Galois group $G_j := \mathrm{Gal}(L_j/\kappa(\eta_T)) \subset G$. Since G is an $\mathbb{L}$-group, the same holds for G_j, hence $E_U \times_T Z$ is tamely ramified along the divisor $\overline{\{z\}}$ (the topological closure of $\{z\} \subset Z$), and the assertion follows from Abhyankar's lemma [42, Exp.XIII, Prop.5.2]. $\Diamond$

Claim 7.2.9. There exist :

(a) a finite dominant morphism $S' \to S$, such that both S' and $T' := T \times_S S'$ are strictly local and normal;
(b) an open subset $U' \subset S'$, such that $S' \setminus U'$ has codimension ≥ 2 in S';
(c) a finite étale morphism $E' \to T'_{U'} := T \times_S U'$, with an isomorphism of $T'_{U'}$-schemes :

$$E' \times_T T_\eta \simeq E_\eta \times_T T'_{U'}.$$

Proof of the claim. For every $i \leq n$, let y_i be the maximal point of D_i, and choose $t_i \in A$ whose image in $\mathscr{O}_{S,y_i}$ generates the maximal ideal. Choose also $m_i \in \mathbb{N}$ with $(m_i, \operatorname{char} \kappa(s)) = 1$ and such that there exists a finite étale covering $E_i \to T(y_i)[t_i^{1/m_i}]$ extending the étale covering $E_U \times_T T(y_i)[t_i^{1/m_i}]$ (claim 7.2.8(ii.d)). Let S' be the normalization of $\operatorname{Spec} A[t_1^{1/m_1}, \dots, t_s^{1/m_n}]$. Then S' is finite over S, hence it is excellent ([31, Ch.IV, (7.8.3)(ii,vi)]), and strictly local (cp. the proof of lemma 7.1.25). Set $E'_\eta := E_\eta \times_S S'$, $T' := T \times_S S'$; since the geometric fibres of $f_{\overline{x}}$ are connected (proposition 7.1.31(ii)), the same holds for the geometric fibres of the induced morphism $T' \to S'$, therefore T' is connected, and then it is also strictly local, by the usual arguments. Notice also that T' is the limit of a cofiltered family of smooth S'-schemes, hence it is reduced and normal ([33, Ch.IV, Prop.17.5.7]). Say that $E'_\eta = \operatorname{Spec} C$, $T = \operatorname{Spec} B$, $T' = \operatorname{Spec} B'$, and let C' be the integral closure of B' in C. Notice that $C \otimes_B \kappa(\eta_T)$ is a finite product of finite separable extensions of the field $B' \otimes_B \kappa(\eta_T)$, and consequently the natural morphism $\varphi' : E_{T'} := \operatorname{Spec} C' \to T'$ is finite ([61, §33, Lemma 1]). Define :

$$U' := \{y \in S' \mid \varphi' \times_{S'} S'(y) : E_{T'}(y) \to T'(y) \text{ is étale}\}.$$

Let now $y \in U'$ any point; then $S'(y)$ is the limit of the cofiltered family $(U_\lambda \mid \lambda \in \Lambda)$ of affine open neighborhoods of y in S', and $\varphi' \times_{S'} S'(y)$ the limit of the system of morphisms $(\varphi'_\lambda := \varphi' \times_{S'} U_\lambda \mid \lambda \in \Lambda)$; we may then find $\lambda \in \Lambda$ such that φ'_λ is étale ([33, Ch.IV, Prop.17.7.8(ii)]), hence $U_\lambda \subset U'$, which shows that U' is open. Furthermore, from [33, Ch.IV, Prop.17.5.7] it follows that $E_U \times_T T'$ is normal, whence an isomorphism of T'-schemes :

$$E_{T'} \times_S U \simeq E_U \times_T T'$$

(cp. the proof of lemma 7.1.7(ii.b)) especially, $U \times_S S' \subset U'$. Likewise, by construction we have natural morphisms : $T'(y_i) \to T(y_i)[t_i^{1/m_i}]$, and using the fact that all the schemes in view are normal we deduce isomorphisms of $T'(y_i)$-schemes :

$$E_{T'}(y_i) \xrightarrow{\sim} E_i \times_{T(y_i)[t_i^{1/m_i}]} T'(y_i).$$

Thus, U' contains all the points of S' of codimension ≤ 1, since the image in S of any such point lies in $U \cup \{y_1, \dots, y_n\}$. The morphism $E' := E_{T'} \times_{S'} U' \to T'_{U'}$ fulfills conditions (a)-(c). ◊

Now, choose $S' \to S$, $U' \subset S'$, and $E' \to T'_{U'}$ as in claim 7.2.9; since the corresponding T' is local, there exists a unique point $x' \in X' := X \times_S S'$ lying over x; pick a geometric point $\overline{x}'$ of X' localized at x', and lying over $\overline{x}$; it then follows from [33, Ch.IV, Prop.18.8.10] that the natural morphism $X'(\overline{x}') \to T'$ is an isomorphism. In such situation, theorem 7.2.2 says that there exists a finite étale covering $E \to U'$ with an isomorphism of $T'_{U'}$-schemes : $E \times_{U'} T'_{U'} \xrightarrow{\sim} E'$, whence an isomorphism of $T_{\overline{\eta}}$-schemes :

$$E_{\overline{\eta}} \simeq E(\overline{\eta}) \times_{\operatorname{Spec} \kappa(\overline{\eta})} T_{\overline{\eta}}.$$

Since $\kappa(\overline{\eta})$ is separably closed, the étale morphism $E(\overline{\eta}) \to \operatorname{Spec} \kappa(\overline{\eta})$ admits a section, hence the same holds for φ, as claimed. □

7.2.10. Let $f : X \to S$ be a morphism of schemes, and $j : U \subset X$ an open immersion such that $U_\eta := U \cap f^{-1}(\eta) \neq \varnothing$ for every $\eta \in S_{\max}$, where $S_{\max} \subset S$ denotes the subset of all maximal points of S. We deduce a natural essentially commutative diagram of functors :

$$\mathscr{D}(S,f,U) \quad : \quad \begin{tikzcd} \mathbf{Cov}(X) \arrow[r, "j^*"] \arrow[d, "\prod_\eta \iota_\eta^*"'] & \mathbf{Cov}(U) \arrow[d, "\prod_\eta \iota_{\eta|U}^*"] \\ \prod_{\eta \in S_{\max}} \mathbf{Cov}(f^{-1}\eta) \arrow[r, "\prod_\eta j_\eta^*"] & \prod_{\eta \in S_{\max}} \mathbf{Cov}(U_\eta) \end{tikzcd}$$

where $j_\eta : U_\eta \to f^{-1}(\eta)$ is the restriction of j and $\iota_\eta : f^{-1}(\eta) \to X$ is the natural immersion. Let us say that $U \subset X$ is *fibrewise dense*, if $f^{-1}(s) \cap U$ is dense in $f^{-1}(s)$, for every $s \in S$. Then we have :

Theorem 7.2.11. *In the situation of* (7.2.10)*, suppose that f is smooth, and U is fibrewise dense. The following holds :*

(i) *The restriction functor j^* is fully faithful.*
(ii) *The diagram $\mathscr{D}(S,f,U)$ is 2-cartesian.*
(iii) *If furthermore, $f^{-1}S_{\max} \subset U$, then j^* is an equivalence.*

Proof. Assertion (ii) means that the functors j^* and ι_η^* induce an equivalence $(j, \iota_\bullet)^*$ from $\mathbf{Cov}(X)$ to the category $\mathscr{C}(X,U)$ of data

$$\underline{E} := (\varphi, (\psi_\eta, \alpha_\eta \mid \eta \in S_{\max})) \qquad (7.2.12)$$

where φ (resp. ψ_η) is an object of $\mathbf{Cov}(U)$ (resp. of $\mathbf{Cov}(f^{-1}\eta)$, for every $\eta \in S_{\max}$), and $\alpha_\eta : \varphi \times_U \operatorname{Spec} \kappa(\eta) \xrightarrow{\sim} \psi_\eta \times_{f^{-1}\eta} U_\eta$ is an isomorphism of U-schemes, for every $\eta \in S_{\max}$ (see (1.3.16)). On the basis of this description, it is easily seen that (i),(ii)⇒(iii). Furthermore, we remark :

Claim 7.2.13. (i) If j^* is fully faithful, then the same holds for $(j, \iota_\bullet)^*$.

(ii) For every open subset $U' \subset X$ containing U, suppose that :
(a) The pull-back functor $\mathbf{Cov}(U') \to \mathbf{Cov}(U)$ is fully faithful.
(b) If $f^{-1}S_{\max} \subset U'$, the pull-back functor $\mathbf{Cov}(X) \to \mathbf{Cov}(U')$ is an equivalence.
Then assertion (ii) holds.

Proof of the claim. (i): Since $f^{-1}\eta$ is a normal (even regular) scheme ([33, Ch.IV, Prop.17.5.7]), the pull-back functors ι_η^* are fully faithful (lemma 7.1.7(ii.b)); the assertion is an immediate consequence.

(ii): In light of (i), it remains only to check that $(j, \iota_\bullet)^*$ is essentially surjective. Thus, let $\varphi : E \to U$ be a finite étale morphism, such that $i_\eta^*\varphi$ extends to a finite étale morphism $\varphi'_\eta : E'_\eta \to f^{-1}(\eta)$, for every maximal point $\eta \in S$. By claim 7.1.8, there is a largest open subset $U_{\max}$ containing U, over which φ extends to a finite étale morphism $\varphi_{\max}$. To conclude, we have to show that $U_{\max} = X$. However, for any maximal point η, let $i_\eta : f^{-1}(\eta) \to X(\eta)$ be the natural closed immersion. By lemma 7.1.7(i), i_η^* is an equivalence, hence we may find a finite étale morphism $\varphi'_{(\eta)} : E'(\eta) \to X(\eta)$ such that $i_\eta^*\varphi'_{(\eta)} \simeq \varphi'_\eta$. By the same token, we also see that $E'(\eta) \times_{X(\eta)} U(\eta)$ is $U(\eta)$-isomorphic to $E \times_U U(\eta)$.

Next, $S(\eta)$ is the limit of the filtered system $\mathscr{V}$ of all open subsets $V \subset S$ with $\eta \in V$, hence lemma 7.1.6 ensures that we may find $V \in \mathscr{V}$ and an object $\varphi'_V : E'_V \to f^{-1}V$ of $\mathbf{Cov}(f^{-1}V)$ such that $\varphi'_V \times_V S(\eta) \simeq \varphi'_{(\eta)}$, and after shrinking V, we may also assume (again by lemma 7.1.6) that $E'_V \times_X U$ is U-isomorphic to $E \times_S V$. Hence we may glue E' and E along the common intersection, to deduce a finite étale morphism $E' \to U' := U \cup f^{-1}V$ that extends φ. It follows that $f^{-1}(\eta) \subset f^{-1}V \subset U_{\max}$. Since η is arbitrary, (b) implies that the pull-back functor $\mathbf{Cov}(X) \to \mathbf{Cov}(U_{\max})$ is an equivalence, especially φ lies in the essential image of j^*, as claimed. ◊

Claim 7.2.14. (i) Suppose that S is noetherian and normal, and X is separated. Then (i) holds.
(ii) If furthermore, S is also excellent, then (ii) holds as well.

Proof of the claim. (i): Under the assumptions of the claim, X is normal and noetherian ([33, Ch.IV, Prop.17.5.7]), so (i) follows from lemma 7.1.7(ii.b), which also says – more generally – that assumption (a) of claim 7.2.13(ii) holds in this case, hence in order to show (ii) it suffices to check that assumption (b) of claim 7.2.13(ii) holds whenever $U \cup f^{-1}S_{\max} \subset U' \subset X$, especially $X\backslash U'$ has codimension ≥ 2 in X. Suppose first that S is regular; then the same holds for X ([33, Ch.IV, Prop.17.5.8]), and the contention follows from lemma 7.1.7(iii).

In the general case, let $S_{\rm reg} \subset S$ be the regular locus, which is open since S is excellent, and contains all the points of codimension ≤ 1, by Serre's normality criterion ([31, Ch.IV, Th.5.8.6]). Consider the restriction $f^{-1}S_{\rm reg} \to S_{\rm reg}$ of f, and the fibrewise dense open immersion $j_{\rm reg} : U' \cap f^{-1}S_{\rm reg} \subset f^{-1}S_{\rm reg}$; by the foregoing, the functor $j_{\rm reg}^*$ is an equivalence, hence we are easily reduced to showing that the functor $\mathbf{Cov}(X) \to \mathbf{Cov}(U' \cup f^{-1}S_{\rm reg})$ is an equivalence, *i.e.* we may assume that $V := f^{-1}S_{\rm reg} \subset U'$. Moreover, since the full faithfulness of j^* is already known, we only need to show that any finite étale morphism $\varphi : E \to U'$ extends to an object of $\mathbf{Cov}(X)$. To this aim, by lemma 7.1.7(ii.b), it suffices to prove that $\varphi \times_{U'} (X(\overline{x}) \times_X U')$ extends to an object of $\mathbf{Cov}(X(\overline{x}))$, for every geometric point $\overline{x}$ of X. Let $\overline{s} := f(\overline{x})$, and denote by $s \in S$ the support of $\overline{s}$; by assumption, we may find a geometric point ξ of $f^{-1}(s)$, whose support lies in $U' \cap f^{-1}(s)$, and a strict specialization morphism $X(\xi) \to X(\overline{x})$. There follows an essentially commutative diagram :

$$\begin{tikzcd}
\mathbf{Cov}(X(\overline{x})) \arrow[r, "\rho"] \arrow[d, "\delta"'] & \mathbf{Cov}(X(\overline{x}) \times_X V) \arrow[d, "\gamma"] & \mathbf{Cov}(S(\overline{s}) \times_S S_{\rm reg}) \arrow[l, "\alpha"'] \arrow[dl, "\beta"] \\
\mathbf{Cov}(X(\xi)) \arrow[r, "\tau"] & \mathbf{Cov}(X(\xi) \times_X V)
\end{tikzcd}$$

where α and β are both equivalences, by theorem 7.2.2(ii); hence γ is an equivalence as well. Moreover, both $\mathbf{Cov}(X(\overline{x}))$ and $\mathbf{Cov}(X(\xi))$ are equivalent to the category of finite sets, and δ is obviously an equivalence. By construction, $\gamma(\varphi \times_{U'} (X(\xi) \times_X V))$ lies in the essential image of τ, hence $\varphi \times_{U'} (X(\xi) \times_X V)$ lies in the essential image of ρ, so say it is isomorphic to $\rho(\varphi')$ for some object φ' of $\mathbf{Cov}(X(\overline{x}))$. Using (i) (and [33, Ch.IV, Prop.18.8.12]) one checks easily that $\varphi' \times_X U' \simeq \varphi \times_{U'} (X(\xi) \times_X U')$, whence the contention. $\Diamond$

Claim 7.2.15. Let $m \in \mathbb{N}$ be any integer. Assertions (i) and (ii) hold if S and X are affine schemes of finite type over $\operatorname{Spec}\mathbb{Z}$, the fibres of f have pure dimension m, and furthermore :

(7.2.16) $$\dim f^{-1}(s)\backslash U < m \qquad \text{for every } s \in S.$$

Proof of the claim. Indeed, in this situation, S admits finitely many maximal points, hence the normalization morphism $S^\nu \to S$ is integral and surjective. Set :

$$S_2 := S^\nu \times_S S^\nu \qquad U_1 := U \times_S S^\nu \qquad U_2 := U \times_S S_2.$$

Let $\beta : X_1 := X \times_S S^\nu \to X$, $f_1 : X_1 \to S^\nu$ and $j_2 : U_2 \to X_2 := X \times_S S_2$ be the induced morphisms; clearly $f_1^{-1}(s')$ has pure dimension m for every $s' \in S^\nu$, and from (7.2.16) we deduce that $\dim f_1^{-1}(s') \setminus U_1 < m$, especially, U_1 is dense in every fibre of f_1; by the same token, U_2 is dense in X_2. Then j_2^* is faithful (lemma 7.1.7(ii.a)), and lemma 7.1.2 and corollary 1.5.35(ii) imply that j^* is fully faithful, provided the same holds for the functor $j_1^* : \mathbf{Cov}(X_1) \to \mathbf{Cov}(U_1)$. In other words, in order to prove assertion (i), we may replace (f, U) by (f_1, U_1), which allows to assume that S is an affine normal scheme, and then it suffices to invoke claim 7.2.14, to conclude.

Concerning assertion (ii) : by the foregoing, we already know that j^* is fully faithful, hence the same holds for $(j, \iota_\bullet)^*$ (claim 7.2.13(i)). To show that $(j, \iota_\bullet)^*$ is essentially surjective, let $\underline{E}$ be an object as in (7.2.12) of the category $\mathscr{C}(X, U)$; the normalization morphism induces a bijection $S^\nu_{\max} \xrightarrow{\sim} S_{\max} : \eta^\nu \mapsto \eta$, and clearly $\kappa(\eta^\nu) = \kappa(\eta)$ for every $\eta \in S_{\max}$, whence a datum

$$\underline{E}^\nu := (\varphi_1 := \varphi \times_U U_1, (\psi_\eta, \alpha_\eta \mid \eta^\nu \in S^\nu_{\max}))$$

of the analogous category $\mathscr{C}(X_1, U_1)$; by claim 7.2.14(ii), we may find an object φ'_1 of $\mathbf{Cov}(X_1)$ and an isomorphism $\alpha : \varphi'_1 \times_{X_1} U_1 \xrightarrow{\sim} \varphi_1$. Let $U_3 := U_2 \times_U U_1$, and denote by $j_3 : U_3 \to X_3 := X_2 \times_X X_1$ the natural open immersion; by the foregoing, we know already that both j_2^* and j_3^* are fully faithful; then corollary 1.5.35(iii) says that the natural essentially commutative diagram :

$$\begin{array}{ccc}
\mathsf{Desc}(\mathbf{Cov}, \beta) & \longrightarrow & \mathsf{Desc}(\mathbf{Cov}, \beta \times_X U) \\
\downarrow & & \downarrow \\
\mathbf{Cov}(X_1) & \longrightarrow & \mathbf{Cov}(U_1)
\end{array}$$

is 2-cartesian. Thus, let $\rho : \mathbf{Cov}(U) \to \mathsf{Desc}(\mathbf{Cov}, \beta \times_X U)$ be the functor defined in (1.5.27); it follows that the datum $(\varphi'_1, \rho(\varphi), \alpha)$ comes from a descent datum (φ'_1, ω) in $\mathsf{Desc}(\mathbf{Cov}, \beta)$. By lemma 7.1.2, the latter descends to an object φ' of $\mathbf{Cov}(X)$, and by construction we have $(j, \iota_\bullet)^*\varphi' = \underline{E}$, as required. $\Diamond$

Next, we consider assertions (i) and (ii) in case where both X and S are affine. We may find an affine open covering $X = V_0 \cup \cdots \cup V_n$ such that the fibres of $f_{|V_i} : V_i \to fV_i$ are of pure dimension i, for every $i = 0, \dots, n$ ([33, Ch.IV, Prop.17.10.2]). For $i = 0, \dots, n$, let $j_i : V_i \cap U \to V_i$ be the induced open immersion; we have natural equivalences of categories :

$$\mathbf{Cov}(X) \xrightarrow{\sim} \prod_{i=0}^n \mathbf{Cov}(V_i) \qquad \mathbf{Cov}(U) \xrightarrow{\sim} \prod_{i=0}^n \mathbf{Cov}(V_i \cap U)$$

which induce a natural identification : $j^* = j_0^* \times \cdots \times j_n^*$. It follows j^* is fully faithful if and only if the same holds for every j_i^*, and moreover $\mathscr{D}(S, f, U)$ decomposes as a product of n diagrams $\mathscr{D}(S, f_{|V_i}, U \cap V_i)$. Hence we may replace f by $f_{|V_m}$, for any $m \leq n$, after which we may also assume that all the fibres f have the same pure dimension m. In that case, notice that the assumption on U is equivalent to (7.2.16). Next, say that $X = \operatorname{Spec} A$, and let $I \subset A$ be an ideal such that $V(I) = X \backslash U$; we may write I as the union of the filtered family $(I_\lambda \mid \lambda \in \Lambda)$ of its finitely generated subideals. Set $U_\lambda := X \backslash V(I_\lambda)$ for every $\lambda \in \Lambda$; it follows that $U = \bigcup_{\lambda \in \Lambda} U_\lambda$. For every $\lambda \in \Lambda$, set

$$Z_\lambda := \{s \in S \mid \dim f^{-1}(s) \backslash U_\lambda < m\}.$$

Claim 7.2.17. (i) Z_λ is a constructible subset of S, for every $\lambda \in \Lambda$.

(ii) We have $Z_\lambda \subset Z_\mu$ whenever $\mu \geq \lambda$, and moreover $S = \bigcup_{\lambda \in \Lambda} Z_\lambda$.

Proof of the claim. (i): Let $V_\lambda := \{x \in X \mid \dim_x f^{-1}(f(x)) \setminus U_\lambda = m\}$; according to [32, Ch.IV, Prop.9.9.1], every V_λ is a constructible subset of X, hence $f(V_\lambda)$ is a constructible subset of S ([30, Ch.IV, Th.1.8.4]), so the same holds for $Z_\lambda = S \backslash f(V_\lambda)$.

(ii): Let $\mu, \lambda \in \Lambda$, such that $\mu \geq \lambda$; then it is clear that $f^{-1}(s) \backslash U_\lambda \subset f^{-1}(s) \backslash U_\mu$ for every $s \in S$; using (7.2.16), the claim follows easily. $\diamondsuit$

Claim 7.2.17 and [30, Ch.IV, Cor.1.9.9] imply that $Z_\lambda = S$ for every sufficiently large $\lambda \in \Lambda$. Hence, after replacing Λ by a cofinal subset, we may assume that all the open subsets U_λ are fibrewise dense. We have a natural essentially commutative diagram :

$$\begin{tikzcd}
\mathbf{Cov}(U) \arrow[r] \arrow[d] & \underset{\lambda \in \Lambda}{2\text{-}\lim}\ \mathbf{Cov}(U_\lambda) \arrow[d] \\
\prod_{\eta \in S_{\max}} \mathbf{Cov}(U_\eta) \arrow[r] & \prod_{\eta \in S_{\max}} \underset{\lambda \in \Lambda}{2\text{-}\lim}\ \mathbf{Cov}(f^{-1}(\eta) \cap U_\lambda)
\end{tikzcd}$$

whose horizontal arrows are equivalences (notation of definition 1.3.12(i)); it follows formally that j^* is fully faithful, provided the same holds for all the pull-back functors $\mathbf{Cov}(X) \to \mathbf{Cov}(U_\lambda)$, and likewise, $\mathscr{D}(S, f, U)$ is 2-cartesian, provided the same holds for all the diagrams $\mathscr{D}(S, f, U_\lambda)$. Hence, we may replace U by U_λ, and assume that U is constructible, and (7.2.16) still holds.

Next, we may write S as the limit of a cofiltered family $(S_\lambda \mid \lambda \in \Lambda)$ of affine schemes of finite type over $\operatorname{Spec} \mathbb{Z}$, and f as the limit of a cofiltered family $f_\bullet := (f_\lambda : X_\lambda \to S_\lambda \in \Lambda)$ of affine finitely presented morphisms, such that :

- The natural morphism $g_\lambda : S \to S_\lambda$ is dominant for every $\lambda \in \Lambda$.
- f_λ is smooth for every $\lambda \in \Lambda$ ([33, Ch.IV, Prop.17.7.8(ii)]), and $f_\mu = f_\lambda \times_{S_\lambda} S_\mu$ whenever $\mu \geq \lambda$.

Furthermore, we may find $\lambda \in \Lambda$ such that $U = U_\lambda \times_{S_\lambda} S$ ([32, Ch.IV, Cor.8.2.11]), so that j is the limit of the cofiltered system of open immersions $(j_\mu : U_\mu := U_\lambda \times_{S_\lambda} S_\mu \to X_\lambda \mid \mu \geq \lambda)$, and after replacing Λ by a cofinal subset, we may assume that j_μ is defined for every $\mu \in \Lambda$. For every $\lambda \in \Lambda$ and $n \in \mathbb{N}$, let $X_{\lambda,n} \subset X_\lambda$ be the open and closed subset consisting of all $x \in X_\lambda$ such that $\dim_x f^{-1} f(x) = n$; clearly $f_\bullet$ restricts to a cofiltered family $f_{\bullet,m} := (f_{\lambda|X_{\lambda,m}} : X_{\lambda,m} \to S_\lambda \mid \lambda \in \Lambda)$, whose limit is again f. Hence we may replace X_λ by $X_{\lambda,m}$, and assume that the fibres of f_λ have pure dimension m, for every $\lambda \in \Lambda$. For every $\lambda \in \Lambda$, let :

$$Z'_\lambda := \{s \in S_\lambda \mid \dim f_\lambda^{-1}(s) \backslash U_\lambda = m\}.$$

and endow Z'_λ with its constructible topology $\mathscr{T}_\lambda$; since Z'_λ is a constructible subset of S_λ ([32, Ch.IV, Prop.9.9.1]), $(Z'_\lambda, \mathscr{T}_\lambda)$ is a compact topological space, and due to (7.2.16), we have :

$$\lim_{\lambda\in\Lambda} Z'_\lambda = \varnothing.$$

Then [20, Ch.I, §9, n.6, Prop.8] implies that $Z'_\lambda = \varnothing$ for every sufficiently large $\lambda \in \Lambda$. Set :

$$\mathbf{Cov}(X_\bullet) := 2\text{-}\operatorname*{colim}_{\mu\geq\lambda} \mathbf{Cov}(X_\mu) \qquad \mathbf{Cov}(U_\bullet) := 2\text{-}\operatorname*{colim}_{\mu\geq\lambda} \mathbf{Cov}(U_\mu).$$

(See definition 1.3.12(ii).) There follows an essentially commutative diagram of categories:

$$\begin{tikzcd} \mathbf{Cov}(X) \arrow[r] \arrow[d, "j^*"'] & \mathbf{Cov}(X_\bullet) \arrow[d, "j^*_\bullet"] \\ \mathbf{Cov}(U) \arrow[r] & \mathbf{Cov}(U_\bullet) \end{tikzcd} \tag{7.2.18}$$

where $j^*_\bullet$ is the 2-colimit of the system of pull-back functors $j^*_\mu : \mathbf{Cov}(X_\mu) \to \mathbf{Cov}(U_\mu)$. In light of lemma 7.1.6, the horizontal arrows of (7.2.18) are equivalences, so j^* will be fully faithful, provided the same holds for the functors j^*_μ, for every large enough $\mu \in \Lambda$.

Hence, in order to prove assertion (i) when X and S are affine, we may assume that S is of finite type over $\operatorname{Spec}\mathbb{Z}$, the fibres of f have pure dimension m, and (7.2.16) holds, which is the case covered by claim 7.2.15.

Concerning assertion (ii), since the morphism g_μ is dominant, for every $\eta' \in (S_\mu)_{\max}$ we may find $\eta \in S_{\max}$ such that $g_\mu(\eta) = \eta'$. Denote by :

$$h : f^{-1}\eta \to f_\mu^{-1}\eta' \qquad \text{and} \qquad j'_\mu : (U_\mu)_{\eta'} := U_\mu \cap f_\mu^{-1}\eta' \to f_\mu^{-1}\eta'$$

the natural morphisms. With this notation, we have the following :

Claim 7.2.19. The induced essentially commutative diagram :

$$\begin{tikzcd} \mathbf{Cov}(f_\mu^{-1}\eta') \arrow[r, "j'^*_\mu"] \arrow[d, "h^*"'] & \mathbf{Cov}((U_\mu)_{\eta'}) \arrow[d, "h^*_U"] \\ \mathbf{Cov}(f^{-1}\eta) \arrow[r, "j^*_\eta"] & \mathbf{Cov}(U_\eta) \end{tikzcd}$$

is 2-cartesian.

Proof of the claim. The pair (h^*, j'^*_μ) induces a functor $(h, j'_\mu)^*$ from $\mathbf{Cov}(f_\mu^{-1}\eta')$ to the category of data of the form $(\varphi, \varphi', \alpha)$, where φ' (resp φ) is a finite étale covering of $(U_\mu)_{\eta'}$ (resp. of $f^{-1}\eta$) and $\alpha : \varphi \times_{f^{-1}\eta} U_\eta \xrightarrow{\sim} \varphi' \times_{\eta'} \eta$ is an isomorphism in $\mathbf{Cov}(U_\eta)$, and the contention is that $(h, j'_\mu)^*$ is an equivalence. The full faithfulness of the functors j'^*_μ and j^*_η (lemma 7.1.7(ii.b)) easily implies the full faithfulness of $(h, j'_\mu)^*$. To prove that $(h, j'_\mu)^*$ is essentially surjective, amounts to showing that if $\varphi' : E' \to (U_\mu)_{\eta'}$ is a finite étale morphism and

$$\varphi'' := \varphi' \times_{\eta'} \eta : E'' := E' \times_{\eta'} \eta \to U_\eta$$

extends to a finite étale morphism $\varphi : E \to f^{-1}\eta$, then φ' extends to a finite étale covering of $f_\mu^{-1}\eta'$. Now, let L be the maximal purely inseparable extension of $\kappa(\eta')$ contained in $\kappa(\eta)$. Since the induced morphism $\eta'' := \operatorname{Spec} L \to \operatorname{Spec}\kappa(\eta')$ is radicial, the base change functors

$$\mathbf{Cov}(f_\mu^{-1}\eta') \to \mathbf{Cov}((f_\mu^{-1}\eta') \times_{\eta'} \eta'') \qquad \mathbf{Cov}((U_\mu)_{\eta'}) \to \mathbf{Cov}((U_\mu)_{\eta'} \times_{\eta'} \eta'')$$

are equivalences (lemma 7.1.7(i)). Thus, we may replace η' by η'', and assume that the field extension $\kappa(\eta') \subset \kappa(\eta)$ is separable, hence the induced morphism $\operatorname{Spec}\kappa(\eta) \to \operatorname{Spec}\kappa(\eta')$ is regular ([14, Ch.VIII, §7, no.3, Cor.1]), and then the same holds for the morphism h ([31, Ch.IV, Prop.6.8.3(iii)]). Given φ' as above, set $\mathscr{A} := (j'_\mu \circ \varphi')_* \mathscr{O}_{E'}$; then $\mathscr{A}$ is a quasi-coherent

$\mathscr{O}_{f_\mu^{-1}\eta'}$-algebra, and we may define the quasi-coherent $\mathscr{O}_{f_\mu^{-1}\eta'}$-algebra $\mathscr{B}$ as the integral closure of $\mathscr{O}_{f_\mu^{-1}\eta'}$ in $\mathscr{A}$. By [31, Ch.IV, Prop.6.14.1], $h^*\mathscr{B}$ is the integral closure of $\mathscr{O}_{f^{-1}\eta}$ in $h^*\mathscr{A} = j_{\eta*} \circ h_U^*(\varphi''_*\mathscr{O}_{E''}) = j_{\eta*}(\varphi_*\mathscr{O}_E)$. By [33, Ch.IV, Prop.17.5.7], it then follows that $h^*\mathscr{B} = \varphi^*\mathscr{O}_E$, therefore $\mathscr{B}$ is a finite étale $\mathscr{O}_{f_\mu^{-1}\eta'}$-algebra ([33, Ch.IV, Prop.17.7.3(ii)] and [31, Ch.IV, Prop.2.7.1]). The claim follows. $\diamondsuit$

By the foregoing, we already know that j^* is fully faithful, hence the same holds for $(j, \iota_\bullet)^*$ (claim 7.2.13(i)). To show that $(j, \iota_\bullet)^*$ is essentially surjective, consider any $\underline{E}$ as in (7.2.12); we may find $\mu \in \Lambda$, and a finite étale morphism $\varphi' : E_\mu \to U_\mu$ such that $\varphi = \varphi' \times_{U_\mu} U$, whence objects $\varphi'_{\eta'} := \varphi' \times_{U_\mu} (U_\mu)_{\eta'}$ in $\mathbf{Cov}((U_\mu)_{\eta'})$, for every $\eta' \in (S_\mu)_{\max}$. By construction, we have $\varphi'_{\eta'} \times_{\eta'} \eta \simeq \psi_\eta \times_{f^{-1}\eta} U_\eta$ for every $\eta' \in (S_\mu)_{\max}$ and every $\eta \in S_{\max}$ such that $g_\mu(\eta) = \eta'$. Then claim 7.2.19 shows that, for every $\eta' \in (S_\mu)_{\max}$ there exists an object $\psi'_{\eta'}$ of $\mathbf{Cov}(f_\mu^{-1}\eta')$ with isomorphisms :

$$\psi'_{\eta'} \times_{f_\mu^{-1}\eta'} f^{-1}\eta \simeq \psi_\eta \qquad \alpha_{\eta'} : \psi'_{\eta'} \times_{f_\mu^{-1}\eta'} (U_\mu)_{\eta'} \xrightarrow{\sim} \varphi'_{\eta'}.$$

Therefore, the datum $\underline{E}_\mu := (\varphi', (\psi'_{\eta'}, \alpha'_{\eta'} \mid \eta' \in (S_\mu)_{\max}))$ is an object of the 2-limit of the diagram of categories

$$\mathbf{Cov}(U_\mu) \xleftarrow{j_\mu^*} \mathbf{Cov}(X_\mu) \xrightarrow{\prod_{\eta'} \iota_{\eta'}^*} \prod_{\eta' \in (S_\mu)_{\max}} \mathbf{Cov}(f_\mu^{-1}\eta')$$

(where $\iota_{\eta'} : f_\mu^{-1}\eta' \to X_\mu$ is the natural immersion, for every $\eta' \in (S_\mu)_{\max}$). By claim 7.2.15, the datum $\underline{E}_\mu$ comes from an object φ_μ of $\mathbf{Cov}(X_\mu)$. Let φ'' be the image of φ'_μ in $\mathbf{Cov}(X)$; by construction we have $(j, \iota_\bullet)^*\varphi'' = \underline{E}$, as required.

This conclude the proof of (i) and (ii), in case X and S are affine. To deal with the general case, let $X = \bigcup_{i \in I} V_i$ be a covering consisting of affine open subschemes, and for every $i \in I$, let $fV_i = \bigcup_{\lambda \in \Lambda_i} S_{i\lambda}$ be an affine open covering of the open subscheme $fV_i \subset S$; set also $V_{i\lambda} := V_i \cap f^{-1}S_{i\lambda}$ for every $i \in I$ and $\lambda \in \Lambda_i$. The restrictions $f_{|V_{i\lambda}} : V_{i\lambda} \to S_{i\lambda}$ are smooth morphisms; moreover, the image of the open immersion

$$j_{i\lambda} := j_{|U \cap V_{i\lambda}} : U \cap V_{i\lambda} \to V_{i\lambda}$$

is dense in every fibre of $f_{|V_i}$. The induced morphism

$$\beta : X' := \coprod_{i \in I} \coprod_{\lambda \in \Lambda_i} V_{i\lambda} \to X \tag{7.2.20}$$

is faithfully flat, hence of universal 2-descent for (7.1.1); moreover, it is easily seen that $X'' := X' \times_X X'$ is separated, and $j'' := j \times_X X''$ is a dense open immersion, hence j''^* is faithful (lemma 7.1.7(ii.a)). Then, by corollary 1.5.35(ii), j^* is fully faithful, provided the pull-back functor $\mathbf{Cov}(X') \to \mathbf{Cov}(X' \times_X U)$ is fully faithful, *i.e.* provided the same holds for the functors $j_{i\lambda}^* : \mathbf{Cov}(V_{i\lambda}) \to \mathbf{Cov}(V_{i\lambda} \cap U)$. However, each $V_{i\lambda}$ is affine ([26, Ch.I, Prop.5.5.10]), hence assertion (i) is already known for the morphisms $f_{|V_{i\lambda}}$ and the open subsets $U \cap V_{i\lambda}$; this concludes the proof of (i).

To show (ii), we use the criterion of claim 7.2.13(ii) : indeed, assumption (a) is already known, hence we are reduced to showing that assertion (iii) holds. To this aim, we consider again the morphism β of (7.2.20), and denote by $f'' : X'' \to S$ the induced morphism. Clearly f'' is smooth, and j'' is an open immersion, such that $f''^{-1}(s) \times_X U$ is dense in $f''^{-1}(s)$, for every $s \in S$; then assertion (i) implies that j''^* is fully faithful. Moreover it is easily seen that $X''' := X'' \times_X X'$ is separated, and $j \times_X X'''$ is a dense open immersion, so j'''^* is faithful (lemma 7.1.7(ii.a)), and therefore corollary 1.5.35(ii) reduces to showing that the pull-back functor $\mathbf{Cov}(X') \to \mathbf{Cov}(X' \times_X U)$ is an equivalence, or – what is the same – that this holds for the pull-back functors $j_{i\lambda}^*$, which is already known. $\square$

7.2.21. We consider now the local counterpart of theorem 7.2.11. Namely, suppose that $f : X \to S$ is a smooth morphism, let $\overline{x}$ be any geometric point of X, set $\overline{s} := f(\overline{x})$, and let $s \in S$ be the support of $\overline{s}$. Then f induces the morphism $f_{\overline{x}} : X(\overline{x}) \to S(\overline{s})$, and for every open immersion $j : U \to X(\overline{x})$, we may then consider the diagram $\mathscr{D}(S(\overline{x}), f_{\overline{x}}, U)$ as in (7.2.10). Notice as well that, for every geometric point ξ of S, the fibre $f_{\overline{x}}^{-1}(\xi)$ is normal, since it is a cofiltered limit of smooth $|\xi|$-schemes; on the other hand, $f_{\overline{x}}^{-1}(\xi)$ is also connected, by proposition 7.1.31(ii), hence $f_{\overline{x}}$ has geometrically irreducible fibres.

Theorem 7.2.22. *In the situation of* (7.2.21)*, suppose that U contains the generic point of $f_{\overline{x}}^{-1}(s)$. Then :*

(i) $j^* : \mathbf{Cov}(X(\overline{x})) \to \mathbf{Cov}(U)$ *is fully faithful.*
(ii) *The diagram $\mathscr{D}(S(\overline{x}), f_{\overline{x}}, U)$ is 2-cartesian.*

Proof. (i): To begin with, since $f_{\overline{x}}$ is generizing ([61, Th.9.5]), and S is local, every fibre of $f_{\overline{x}}$ has a point that specializes to the generic point η_s of $f_{\overline{x}}^{-1}(s)$; since the fibres are irreducible, it follows that the generic point of every fibre specializes to η_s. Therefore U is fibrewise dense in $X(\overline{x})$, and moreover it is connected. Now, the category $\mathbf{Cov}(X)$ is equivalent to the category of finite sets, hence every object in the essential image of j^* is (isomorphic to) a finite disjoint union of copies of U; since U is connected, the morphisms of U-schemes between two such objects E and E' are in natural bijection with the set-theoretic mappings $\pi_0(E) \to \pi_0(E')$ of their sets of connected components, whence the assertion.

(ii): Let us write U as the union of a filtered family $(U_\lambda \mid \lambda \in \Lambda)$ of constructible open subsets of $X(\overline{x})$; up to replacing Λ by a cofinal subset, we may assume that $\eta_s \in U_\lambda$ for every $\lambda \in \Lambda$. Arguing as in the proof of theorem 7.2.11, we see that $\mathscr{D}(S(\overline{x}), f_{\overline{x}}, U)$ is the 2-limit of the system of diagrams $\mathscr{D}(S(\overline{x}), f_{\overline{x}}, U_\lambda)$, hence it suffices to show the assertion with $U = U_\lambda$, for every $\lambda \in \Lambda$, which allows to assume that U is quasi-compact. Next, arguing as in the proof of proposition 7.1.31, we are reduced to the case where $S = S(\overline{s})$. We may write $X(\overline{x})$ as the limit of a cofiltered system $(X_\lambda \mid \lambda \in \Lambda)$ of affine schemes, étale over X, and for $\lambda \in \Lambda$ large enough, we may find an open subset $U_\lambda \subset X_\lambda$ such that $U = U_\lambda \times_{X_\lambda} X(\overline{x})$ ([32, Ch.IV, Cor.8.2.11]). For every $\mu \geq \lambda$, set $U_\mu := U_\lambda \times_{X_\lambda} X_\mu$, and denote by $f_\mu : X_\mu \to S$ the natural morphism. Suppose first that S is irreducible; then, from lemma 7.1.6 it is easily seen that $\mathscr{D}(S, f_{\overline{x}}, U)$ is the 2-colimit of the system of diagrams $\mathscr{D}(S, f_\mu, U_\mu)$, so the assertion follows from theorem 7.2.11(ii) (more generally, this argument works whenever $S_{\max}$ is a finite set, since filtered 2-colimits of categories commute with finite products).

In the general case, let $\varphi : E \to U$ be a finite étale morphism, and suppose that $\varphi_\eta := \varphi \times_{X(\overline{x})} f_{\overline{x}}^{-1}(\eta)$ extends to an object ψ_η of $\mathbf{Cov}(f_{\overline{x}}^{-1}\eta)$, for every $\eta \in S_{\max}$. The assertion boils down to showing that φ extends to an object φ' of $\mathbf{Cov}(X(\overline{x}))$. To this aim, for every $\eta \in S_{\max}$, let $Z_\eta \to S$ be the closed immersion of the topological closure of η in S (which we endow with its reduced scheme structure); set also $Y_\eta := X \times_S Z_\eta$. Then Z_η is a strictly local scheme ([33, Ch.IV, Prop.18.5.6(i)]), and $\overline{x}$ factors through the closed immersion $Y \to X$, which induces an isomorphism of Z-schemes :

$$Y_\eta(\overline{x}) \xrightarrow{\sim} X(\overline{x}) \times_S Z_\eta$$

(lemma 7.1.23(ii)). By the foregoing case, $\varphi \times_S Z_\eta$ extends to an object $\overline{\psi}_\eta$ of $\mathbf{Cov}(Y_\eta(\overline{x}))$. However, Z_η is the limit of the cofiltered system $(Z_{\eta,i} \mid i \in I(\eta))$ consisting of the constructible closed subschemes of S that contain Z_η. By lemma 7.1.6, it follows that we may find $i \in I(\eta)$ and an object $\overline{\psi}_{\eta,i}$ of $\mathbf{Cov}(X(\overline{x}) \times_S Z_{\eta,i})$ whose image in $\mathbf{Cov}(Y_\eta(\overline{x}))$ is isomorphic to $\overline{\psi}_\eta$, and if i is large enough, $\overline{\psi}_{\eta,i} \times_{X(\overline{x})} U$ agrees with $\varphi \times_S Z_{\eta,i}$ in $\mathbf{Cov}(U \times_S Z_{\eta,i})$. For each $\eta, \eta' \in S_{\max}$, choose $i \in I(\eta)$, $i' \in I(\eta')$ with these properties, and to ease notation, set :

$$X'_\eta := X \times_S Z_{\eta,i} \qquad X''_{\eta\eta'} := X'_\eta \times_S Z_{\eta',i'} \qquad \varphi'_\eta := \overline{\psi}_{\eta,i}$$

and denote by $\alpha_\eta : \varphi'_\eta \times_{X(\overline{x})} U \xrightarrow{\sim} \varphi \times_S Z_{\eta,i}$ the given isomorphism. As in the foregoing, we notice that $\overline{x}$ factors through X'_η, and the closed immersion $X'_\eta \to X$ induces an isomorphism $X'_\eta(\overline{x}) \xrightarrow{\sim} X(\overline{x}) \times_S Z_{\eta,i}$ of $Z_{\eta,i}$-schemes. According to [30, Ch.IV, Cor.1.9.9], we may then find a finite subset $T \subset S_{\max}$ such that the induced morphism :

$$\beta : X_1 := \coprod_{\eta \in T} X'_\eta(\overline{x}) \to X(\overline{x})$$

is surjective. Set $X_2 := X_1 \times_{X(\overline{x})} X_1$ and $X_3 := X_2 \times_{X(\overline{x})} X_1$; notice that X_2 is the disjoint union of schemes of the form $X(\overline{x}) \times_S Z_{\eta,i} \times_S Z_{\eta',i'}$, for $\eta, \eta' \in T$, and again, the latter is naturally isomorphic to $X''_{\eta\eta'}(\overline{x})$, for a unique lifting of the geometric point $\overline{x}$ to a geometric point of $X''_{\eta\eta'}$. Similar considerations can be repeated for X_3, and in light of (i), we deduce that the pull-back functors :

$$\mathbf{Cov}(X_i \times_{X(\overline{x})} U) \to \mathbf{Cov}(X_i) \qquad i = 1, 2, 3$$

are fully faithful, in which case corollary 1.5.35(iii) says that the essentially commutative diagram of categories :

$$\begin{tikzcd}
\mathsf{Desc}(\mathbf{Cov}, \beta) \arrow[r] \arrow[d] & \mathsf{Desc}(\mathbf{Cov}, \beta \times_{X(\overline{x})} U) \arrow[d] \\
\prod_{\eta \in T} \mathbf{Cov}(X'_\eta(\overline{x})) \arrow[r] & \prod_{\eta \in T} \mathbf{Cov}(X'_\eta(\overline{x}) \times_{X(\overline{x})} U)
\end{tikzcd}$$

is 2-cartesian. Let $\rho : \mathbf{Cov}(U) \to \mathsf{Desc}(\mathbf{Cov}, \beta \times_{X(\overline{x})} U)$ be the functor defined in (1.5.27); it follows that the datum $((\varphi'_\eta, \alpha_\eta \mid \eta \in T), \rho(\varphi))$ comes from a descent datum (φ'_1, ω) in $\mathsf{Desc}(\mathbf{Cov}, \beta)$. By lemma 7.1.2, the latter descends to an object φ' of $\mathbf{Cov}(X(\overline{x}))$, and by construction we have $j^*\varphi' = \varphi$, as required. $\square$

7.3. **Étale coverings of log schemes.** We resume the general notation of (6.2.1), especially, we choose implicitly τ to be either the Zariski or étale topology, and we shall omit further mention of this choice, unless the omission might be a source of ambiguities.

7.3.1. Let $\underline{Y} := ((Y, \underline{N}), T, \psi)$ be an object of the category $\mathscr{K}_{\mathrm{int}}$ (see (6.6.10) : especially $\tau = \mathrm{Zar}$ here), $\varphi : T' \to T$ an integral proper subdivision of the fan T (definition 3.5.22(ii),(iii)), and suppose that both T and T' are locally fine and saturated. Set $((Y', \underline{N}'), T', \psi') := \varphi^*\underline{Y}$ (proposition 6.6.14(iii)), and let $f : Y' \to Y$ be the morphism of schemes underlying the cartesian morphism $\varphi^*\underline{Y} \to \underline{Y}$.

Proposition 7.3.2. *In the situation of* (7.3.1)*, the following holds :*

(i) *For every geometric point ξ of Y, the fibre $f^{-1}(\xi)$ is non-empty and connected.*
(ii) *If Y is connected, the functor $f^* : \mathbf{Cov}(Y) \to \mathbf{Cov}(Y')$ is an equivalence.*

Proof. (i): The assertion is local on Y, hence we may assume that $T = (\mathrm{Spec}\, P)^\sharp$ for a fine, sharp and saturated monoid P. In this case, we may find a subdivision $\varphi' : T'' \to T'$ such that both φ' and $\varphi \circ \varphi'$ are compositions of saturated blow up of ideals generated by at most two elements of P (example 3.6.15(iii)). We are reduced to showing the assertion for the morphisms φ' and $\varphi \circ \varphi'$, after which, we may further assume that φ is the saturated blow up of an ideal generated by two elements of $\Gamma(T, \mathscr{O}_T)$, in which case the assertion follows from the more precise theorem 6.4.81.

(ii): First, we claim that the assertion is local on the Zariski topology of Y. Indeed, let $Y = \bigcup_{i \in I} U_i$ be a Zariski open covering, and set $U_{ij} := U_i \cap U_j$ for every $i, j \in I$; according to corollary 1.5.35(ii) and lemma 7.1.2, it suffices to prove the contention for the objects $(U_i \times_Y$

$(Y, \underline{N}), T, \psi_{|U_i})$ and $(U_{ij} \times_Y (Y, \underline{N}), T, \psi_{|U_{ij}})$, and the morphism φ. Hence, we may again suppose that $T = \operatorname{Spec} P$, for a monoid P as in (i).

Arguing as in the proof of (i), we may next reduce to the case where φ is the saturated blow up of an ideal generated by two elements of $\Gamma(T, \mathscr{O}_T)$. We remark now that Y' is connected as well. Indeed, suppose that Y' is the disjoint union of two open and closed subsets Z_1 and Z_2; it follows easily from (i) that $f(Z_1) \cap f(Z_2) = \varnothing$, and clearly $Y = f(Z_1) \cup f(Z_2)$. On the other hand, proposition 6.6.31(ii) implies that $f(Z_1)$ and $f(Z_2)$ are closed in Y, which is impossible, since Y is connected.

Pick a geometric point ξ of Y'; in this situation, assertion (ii) is equivalent to :

Claim 7.3.3. The continuous group homomorphism :

$$\pi_1(Y'_{\text{ét}}, \xi) \to \pi_1(Y_{\text{ét}}, f(\xi))$$

induced by f, is an isomorphism of topological groups (see (7.1.12)).

Proof of the claim. In view of lemmata 7.1.13 and 1.6.2, it suffices to show that the induced map :

$$H^1(Y_{\text{ét}}, G_X) \to H^1(Y'_{\text{ét}}, G_{X'}) \tag{7.3.4}$$

is a bijection for every finite group G. However, the morphism of schemes f induces a morphism of topoi $Y'^{\sim}_{\text{ét}} \to Y^{\sim}_{\text{ét}}$ which we denote again by f; let also $g : Y^{\sim}_{\text{ét}} \to \mathbf{Set}$ be the unique morphism of topoi (given by the global section functor on $Y^{\sim}_{\text{ét}}$). With this notation, theorem 2.4.9 specializes to the exact sequence of pointed sets :

$$\{1\} \to H^1(Y_{\text{ét}}, f_*G_{Y'}) \xrightarrow{u} H^1(Y'_{\text{ét}}, G_{Y'}) \to H^0(Y_{\text{ét}}, R^1 f_* G_{Y'}).$$

On the other hand, assertion (i) and the proper base change theorem [5, Exp.XII, Th.5.1(i)] imply that the unit of adjunction $G_Y \to f_*f^*G_Y = f_*G_{Y'}$ is an isomorphism, hence u is naturally identified to (7.3.4), and we are reduced to showing that the natural morphism $\tau_{f,G_{Y'}} : 1_{Y^{\sim}_{\text{ét}}} \to R^1 f_* G_{Y'}$ is an isomorphism (notation of (2.4.3)). The latter can be checked on the stalks, and in view of [5, Exp.XII, Cor.5.2(ii)] we are reduced to showing that $H^1(f^{-1}(\xi)_{\text{ét}}, G) = \{1\}$ for every finite group G, and every geometric point ξ of Y. However, according to theorem 6.4.81, the reduced geometric fibre $f^{-1}(\xi)_{\text{red}}$ is either isomorphic to $|\xi|$ (in which case the contention is trivial), or else it is isomorphic to the projective line $\mathbb{P}^1_{\kappa(\xi)}$, in which case – in view of lemma 7.1.7(i) – it suffices to show that every finite étale morphism $E \to \mathbb{P}^1_{\kappa(\xi)}$ admits a section, which is well known. $\square$

The class of étale morphisms of log schemes was introduced in section 6.3 : its definition and its main properties parallel those of the corresponding notion for schemes, found in [30, Ch.IV, §17]. In the present section this theme is further advanced : we will consider the logarithmic analogue of the classical notion of *étale covering* of a scheme. To begin with, we make the following :

Definition 7.3.5. (i) Let $\varphi : T \to S$ be a morphism of fans. We say that φ is *of Kummer type*, if the map $(\log \varphi)_t : \mathscr{O}_{S,\varphi(t)} \to \mathscr{O}_{T,t}$ is of Kummer type, for every $t \in T$ (see definition 3.4.40).

(ii) Let $f : (Y, \underline{N}) \to (X, \underline{M})$ be a morphism of log schemes.

- (a) We say that f is *of Kummer type*, if for every τ-point ξ of Y, the morphism of monoids $(\log f)_\xi : f^*\underline{M}_\xi \to \underline{N}_\xi$ is of Kummer type.
- (b) A *Kummer chart* for f is the datum of charts

$$\omega_P : P_Y \to \underline{N} \qquad \omega_Q : Q_X \to \underline{M}$$

and a morphism of monoids $\vartheta : Q \to P$ such that $(\omega_P, \omega_Q, \vartheta)$ is a chart for f (see definition 6.1.15(iii)), and ϑ is of Kummer type.

Remark 7.3.6. (i) Let $f : P \to Q$ be a morphism of monoids of Kummer type, with P integral and saturated. It follows easily from lemma 3.4.41(iii,v), that the induced morphism of fans $(\mathrm{Spec}\, f)^\sharp : (\mathrm{Spec}\, Q)^\sharp \to (\mathrm{Spec}\, P)^\sharp$ is of Kummer type.

(ii) In the situation of definition 7.3.5(ii), it is easily seen that f is of Kummer type, if and only if the morphism of Y-monoids $f^*\underline{M}^\sharp_\xi \to \underline{N}^\sharp_\xi$ deduced from $\log f$, is of Kummer type for every τ-point ξ of Y.

(iii) In the situation of definition 7.3.5(ii.b), suppose that the chart $(\omega_P, \omega_Q, \vartheta)$ is of Kummer type, and Q is integral and saturated. It then follows easily from (i), (ii) and example 6.6.5(ii), that f is of Kummer type. In the same vein, we have the following :

Lemma 7.3.7. *Let $f : (Y, \underline{N}) \to (X, \underline{M})$ be a morphism of log schemes with coherent log structures, ξ a τ-point of Y, and suppose that :*

(a) $\underline{M}_{f(\xi)}$ *is fine and saturated.*

(b) *The morphism $(\log f)_\xi : \underline{M}_{f(\xi)} \to \underline{N}_\xi$ is of Kummer type.*

Then there exists a (Zariski) open neighborhood U of $|\xi|$ in Y, such that the restriction $f_{|U} : (U, \underline{N}_{|U}) \to (X, \underline{M})$ of f is of Kummer type.

Proof. By corollary 6.1.34(i) and theorem 6.1.35(ii), we may find a neighborhood $U' \to Y$ of ξ in Y_τ, and a finite chart $(\omega_P, \omega_Q, \vartheta)$ for the restriction $f_{|U'}$, with Q fine and saturated. Set

$$S_P := \omega^{-1}_{P,\xi}\underline{N}^\times_\xi \quad S_Q := \omega^{-1}_{Q,f(\xi)}\underline{M}^\times_{f(\xi)} \quad P' := S_P^{-1}P \quad Q' := S_Q^{-1}Q.$$

According to claim 6.1.29(iii) we may find neighborhoods $U'' \to U'$ of ξ in Y_τ, and $V \to X$ of $f(\xi)$ in X_τ, such that the charts $\omega_{P|U''}$ and $\omega_{Q|V}$ extend to charts

$$\omega_{P'} : P'_{U''} \to \underline{N}_{|U''} \qquad \omega_{Q'} : Q'_V \to \underline{M}_{|V}.$$

Clearly ϑ extends as well to a unique morphism $\vartheta' : Q' \to P'$, and after shrinking U'' we may assume that the restriction $f_{|U''} : (U'', \underline{N}_{|U''}) \to (X, \underline{M})$ factors through a morphism $f' : (U'', \underline{N}_{|U''}) \to (V, \underline{M}_{|V})$, in which case it is easily seen that the datum $(\omega_{P'}, \omega_{Q'}, \vartheta')$ is a chart for f'. Notice as well that Q' is still fine and saturated (lemma 3.2.9(i)), and by claim 6.1.29(iv) the maps $\omega_{P'}$ and $\omega_{Q'}$ induce isomorphisms :

$$P'^\sharp \xrightarrow{\sim} \underline{N}^\sharp_\xi \qquad Q'^\sharp \xrightarrow{\sim} \underline{M}^\sharp_{f(\xi)}.$$

Our assumption (b) then implies that the map $Q'^\sharp \to P'^\sharp$ deduced from ϑ' is of Kummer type, and then the morphism of fans $\mathrm{Spec}\,\vartheta' : \mathrm{Spec}\, P' \to \mathrm{Spec}\, Q'$ is of Kummer type as well (remark 7.3.6(i)). However, let

$$\overline{\omega}_{P'} : (U'', \underline{N}^\sharp_{|U''}) \to (\mathrm{Spec}\, P')^\sharp \qquad \overline{\omega}_{Q'} : (V, \underline{M}^\sharp_{|V}) \to (\mathrm{Spec}\, Q')^\sharp$$

be the morphisms of monoidal spaces deduced from $\omega_{P'}$ and $\omega_{Q'}$; we obtain a morphism

$$(f', \mathrm{Spec}\,\vartheta') : (U'', \underline{N}_{|U''}, (\mathrm{Spec}\, P')^\sharp, \overline{\omega}_{P'}) \to (V, \underline{M}_{|V}, (\mathrm{Spec}\, Q')^\sharp, \overline{\omega}_{Q'})$$

in the category $\mathscr{K}$ of (6.6.2), which – in view of remark 7.3.6(ii) – shows that f' is of Kummer type. This already concludes the proof in case $\tau = \mathrm{Zar}$, and for $\tau = \text{ét}$ it suffices to remark that the image of U'' in Y is a Zariski open neighborhood U of ξ, such that the restriction $f_{|U}$ is of Kummer type. $\square$

We shall use the following criterion :

Proposition 7.3.8. *Let k be a field, $f : P \to Q$ an injective morphism of fine monoids, with P sharp. Suppose that the scheme $\mathrm{Spec}\, k\langle Q/\mathfrak{m}_P Q\rangle$ admits an irreducible component of Krull dimension 0. We have :*

(i) *f is of Kummer type, and $k\langle Q/\mathfrak{m}_P Q\rangle$ is a finite k-algebra.*

(ii) *If moreover, $Q^\times$ is a torsion-free abelian group, then Q is sharp, and $k\langle Q/\mathfrak{m}_P Q\rangle$ is a local k-algebra with k as residue field.*

Proof. Notice that the assumption means especially that $Q/\mathfrak{m}_P Q \neq \{1\}$, so f is a local morphism. Set $I := \mathrm{rad}(\mathfrak{m}_P Q)$ (notation of definition 3.1.8(ii)), and let $\mathfrak{p}_1, \dots, \mathfrak{p}_n \subset Q$ be the minimal prime ideals containing I; then $I = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_n$, by lemma 3.1.15. Clearly the natural closed immersion $\mathrm{Spec}\, k\langle Q/I\rangle \to \mathrm{Spec}\, k\langle Q/\mathfrak{m}_P Q\rangle$ is a homeomorphism; on the other hand, say that $\mathfrak{q} \subset k\langle Q\rangle$ is a prime ideal containing I; then $\mathfrak{q} \cap Q$ is a prime ideal of Q containing I, hence $\mathfrak{p}_i \subset \mathfrak{q}$ for some $i \leq n$, *i.e.* $\mathrm{Spec}\, k\langle Q/I\rangle$ is the union of its closed subsets $\mathrm{Spec}\, k\langle Q/\mathfrak{p}_i Q\rangle$, for $i = 1, \dots, n$. The Krull dimension of each irreducible component of $\mathrm{Spec}\, k\langle Q/\mathfrak{p}_i\rangle = \mathrm{Spec}\, k[Q\setminus\mathfrak{p}_i]$ equals

$$\mathrm{rk}_{\mathbb{Z}}(Q\setminus\mathfrak{p})^\times + \dim Q\setminus\mathfrak{p} = \mathrm{rk}_{\mathbb{Z}} Q^\times + d - \mathrm{ht}(\mathfrak{p}_i) \qquad \text{where } d := \dim Q$$

(claim 5.9.36(ii) and corollary 3.4.10(i,ii)). Our assumption then implies that $Q^\times$ is a finite group, and $\mathrm{ht}(\mathfrak{p}_i) = d$, *i.e.* $\mathfrak{p}_i = \mathfrak{m}_Q$, for at least an index $i \leq n$, and therefore $I = \mathfrak{m}_Q$. Furthermore, since $\mathfrak{m}_Q$ is finitely generated, we have $\mathfrak{m}_Q^n \subset \mathfrak{m}_P Q$ for a sufficiently large integer $n > 0$, and then it follows easily that $k\langle Q/\mathfrak{m}_P Q\rangle$ is a finite k-algebra. If $Q^\times$ is torsion-free, then we also deduce that Q is sharp, and moreover the maximal ideal of $k\langle Q/\mathfrak{m}_P Q\rangle$ generated by $\mathfrak{m}_Q$ is nilpotent, hence the latter k-algebra is local.

Let now $\mathfrak{q} \subset Q$ be any prime ideal, and pick $x \in \mathfrak{m}_Q \setminus \mathfrak{q}$; the foregoing implies that there exists an integer $r > 0$ such that $x^r = f(p_\mathfrak{q})q$ for some $p_\mathfrak{q} \in \mathfrak{m}_P$ and $q \in Q$, hence $f(p_\mathfrak{q}) \notin \mathfrak{q}$, and $f(p_\mathfrak{q})$ is not invertible in Q, since f is local. If now $\mathfrak{q}$ has height $d-1$, it follows that $f(p_\mathfrak{q})$ is a generator of $(Q\setminus\mathfrak{q})_\mathbb{R}$, which is an extremal ray of the polyhedral cone $Q_\mathbb{R}$, and every such extremal ray is of this form (proposition 3.4.7); furthermore, the latter cone is strongly convex, since $Q^\times$ is finite. Hence the set $S := (f(p_\mathfrak{q}) \mid \mathrm{ht}(\mathfrak{q}) = d-1)$ is a system of generators of the polyhedral cone $Q_\mathbb{R}$ (see (3.3.15)). Let $Q' \subset Q$ be the submonoid generated by S; then $S_\mathbb{Q} = S_\mathbb{R} \cap Q_\mathbb{Q} = Q_\mathbb{Q}$ (proposition 3.3.22(iii)), *i.e.* f is of Kummer type. □

7.3.9. Let $f : (Y_{\mathrm{Zar}}, \underline{N}) \to (X_{\mathrm{Zar}}, \underline{M})$ be a morphism of log schemes with Zariski log structures; it follows easily from the isomorphism (6.1.8) and remark 7.3.6(ii) that f is of Kummer type if and only if $\widetilde{u}^* f : (X_{\text{ét}}, \widetilde{u}^*\underline{M}) \to (Y_{\text{ét}}, \widetilde{u}^*\underline{N})$ is a morphism of Kummer type between schemes with étale log structures (notation of (6.2.2)). Suppose now that $\underline{M}$ is an integral and saturated log structure on X_{Zar}, and denote by $\mathbf{s.Kum}(X_{\mathrm{Zar}}, \underline{M})$ (resp. $\mathbf{s.Kum}(X_{\text{ét}}, \widetilde{u}^*\underline{M})$) the full subcategory of $\mathbf{sat.log}/(X_{\mathrm{Zar}}, \underline{M})$ (resp. of $\mathbf{sat.log}/(X_{\text{ét}}, \widetilde{u}^*\underline{M})$ whose objects are all the morphisms of Kummer type. In view of the foregoing (and of lemma 6.1.16(i)), we see that $\widetilde{u}^*$ restricts to a functor :

(7.3.10) $$\mathbf{s.Kum}(X_{\mathrm{Zar}}, \underline{M}) \to \mathbf{s.Kum}(X_{\text{ét}}, \widetilde{u}^*\underline{M}).$$

Lemma 7.3.11. *The functor* (7.3.10) *is an equivalence.*

Proof. By virtue of proposition 6.2.3(ii) we know already that (7.3.10) is fully faithful, hence we only need to show its essential surjectivity. Thus, let $f : (Y_{\text{ét}}, \underline{N}) \to (X_{\text{ét}}, \widetilde{u}^*\underline{M})$ be a morphism of Kummer type. By remark 7.3.6(ii), we know that $\log f$ induces an isomorphism

$$\widetilde{u}^* f^* \underline{M}^\sharp_\mathbb{Q} \xrightarrow{\sim} f^* \widetilde{u}^* \underline{M}^\sharp_\mathbb{Q} \xrightarrow{\sim} \underline{N}^\sharp_\mathbb{Q}$$

(notation of (3.3.20)). Since $\underline{N}$ is integral and saturated, the natural map $\underline{N}^\sharp \to \underline{N}^\sharp_\mathbb{Q}$ is a monomorphism, so that the counit of adjunction $\widetilde{u}^*\widetilde{u}_*\underline{N}^\sharp \to \underline{N}^\sharp$ is an isomorphism (lemma 2.4.26(ii)), and then the same holds for the counit of adjunction $\widetilde{u}^*\widetilde{u}_*(Y, \underline{N}) \to (Y, \underline{N})$ (proposition 6.2.3(iii)). □

Proposition 7.3.12. *Let $f : (Y, \underline{N}) \to (X, \underline{M})$ be a morphism of fs log schemes. The following conditions are equivalent:*

(a) *Every geometric point of X admits an étale neighborhood $U \to X$ such that $Y_U := U \times_X Y$ decomposes as a disjoint union $Y_U = \bigcup_{i=1}^n Y_i$ of open and closed subschemes (for some $n \in \mathbb{N}$), and we have :*
   (i) *Each restriction $Y_i \times_Y (Y, \underline{N}) \to U \times_X (X, \underline{M})$ of $f \times_X \mathbf{1}_U$ admits a fine, saturated Kummer chart $(\omega_{P_i}, \omega_{Q_i}, \vartheta_i)$ such that P_i and Q_i are sharp, and the order of $\mathrm{Coker}\, \vartheta_i^{\mathrm{gp}}$ is invertible in $\mathscr{O}_U$.*
   (ii) *The induced morphism of U-schemes $Y_i \to U \times_{\mathrm{Spec}\, \mathbb{Z}[P_i]} \mathrm{Spec}\, \mathbb{Z}[Q_i]$ is an isomorphism, for every $i = 1, \dots, n$.*

(b) *f is étale, and the morphism of schemes underlying f is finite.*

Proof. (a) ⇒ (b): Indeed, it is easily seen that the morphism $\mathrm{Spec}\, \mathbb{Z}[\vartheta_i]$ is finite, hence the same holds for the restriction $Y_i \to U$ of f, in view of (a.ii), and then the same holds for $f \times_X \mathbf{1}_U$, so finally f is finite on the underlying schemes ([31, Ch.IV, Prop.2.7.1]), and it is étale by the criterion of theorem 6.3.37.

(b) ⇒ (a): Arguing as in the proof of theorem 6.3.37, we may reduce to the case where $\tau = \mathrm{\acute{e}t}$. Suppose first that both X and Y are strictly local. In this case, $\underline{M}$ admits a fine and saturated chart $\omega_P : P_X \to \underline{M}$, sharp at the closed point (corollary 6.1.34(i)). Moreover, f admits a chart $(\omega_P, \omega_Q : Q_Y \to \underline{N}, \vartheta : P \to Q)$, for some fine monoid Q such that $Q^\times$ is torsion-free; also ϑ is injective, the induced morphism of X-schemes

$$g : Y \to X' := X \times_{\mathrm{Spec}\, \mathbb{Z}[P]} \mathrm{Spec}\, \mathbb{Z}[Q]$$

is étale, and the order of $\mathrm{Coker}\, \vartheta^{\mathrm{gp}}$ is invertible in $\mathscr{O}_X$ (corollary 6.3.42). Since f is finite, g is also closed ([27, Ch.II, Prop.6.1.10]), hence its image Z is an open and closed local subscheme, finite over X. Let k be the residue field of the closed point of X; it follows that $X' \times_X \mathrm{Spec}\, k \simeq \mathrm{Spec}\, k\langle Q/\mathfrak{m}_P Q\rangle$ admits an irreducible component of Krull dimension zero (namely, the intersection of Z with the fibre of X' over the closed point of X), in which case the criterion of proposition 7.3.8(ii) ensures that ϑ is of Kummer type and Q is sharp, hence X' is finite over X, and moreover $X' \times_X \mathrm{Spec}\, k$ is a local scheme with k as residue field. Since X is strictly henselian, it follows that X' itself is strictly local, and therefore g is an isomorphism, so the proposition is proved in this case.

Let now X be a general scheme, and ξ a geometric point of X; denote by $X(\xi)$ the strict henselization of X at the point ξ, and set $Y(\xi) := X(\xi) \times_X Y$. Since f is finite, $Y(\xi)$ decomposes as the disjoint union of finitely many open and closed strictly local subschemes $Y_1(\xi), \dots, Y_n(\xi)$. Then we may find an étale neighborhood $U \to X$ of ξ, and open and closed subschemes $Y_1, \dots, Y_n$ of $Y \times_X U$, with isomorphisms of $X(\xi)$-schemes $Y_i(\xi) \xrightarrow{\sim} Y_i \times_U X(\xi)$, for every $i = 1, \dots, n$ ([32, Ch.IV, Cor.8.3.12]). We may then replace X by U, and we reduce to proving the proposition for each of the restrictions $Y_i \to U$ of $f \times_X \mathbf{1}_U$; hence we may assume that $Y(\xi)$ is strictly local. By the foregoing case, we may find a chart

$$P_{X(\xi)} \to \underline{M}(\xi) \qquad Q_{Y(\xi)} \to \underline{N}(\xi) \qquad \vartheta : P \to Q \tag{7.3.13}$$

of $f \times_X \mathbf{1}_{X(\xi)}$, with ϑ of Kummer type, such that P and Q are sharp, the order d of $\mathrm{Coker}\, \vartheta^{\mathrm{gp}}$ is invertible in $\mathscr{O}_{X(\xi)}$, and the induced morphism of $X(\xi)$-schemes $Y(\xi) \to X(\xi) \times_{\mathrm{Spec}\, \mathbb{Z}[P]} \mathrm{Spec}\, \mathbb{Z}[Q]$ is an isomorphism. By corollary 6.2.34 we may find an étale neighborhood $U \to X$ of ξ such that (7.3.13) extends to a chart for $f \times_X \mathbf{1}_U$. After shrinking U, we may assume that d is invertible in $\mathscr{O}_U$. Lastly, after further shrinking of U, we may ensure that the induced morphism of U-schemes $U \times_X Y \to U \times_{\mathrm{Spec}\, \mathbb{Z}[P]} \mathrm{Spec}\, \mathbb{Z}[Q]$ is an isomorphism ([32, Ch.IV, Cor.8.8.2.4]). □

Definition 7.3.14. Let $f : (Y, \underline{N}) \to (X, \underline{M})$ be a morphism of fs log schemes. We say that f is an *étale covering* of $(X, \underline{M})$, if f fulfills the equivalent conditions (a) and (b) of proposition 7.3.12. We denote by

$$\mathbf{Cov}(X, \underline{M})$$

the full subcategory of the category of $(X, \underline{M})$-schemes, whose objects are the étale coverings of $(X, \underline{M})$.

Remark 7.3.15. (i) Notice that all morphisms in $\mathbf{Cov}(X, \underline{M})$ are étale coverings, in light of corollary 6.3.25(ii).

(ii) Moreover, $\mathbf{Cov}(X, \underline{M})$ is a Galois category (see [42, Exp.V, Déf.5.1]), and if ξ is any geometric point of $(X, \underline{M})_{\mathrm{tr}}$, we obtain a fibre functor for this category, by the rule : $f \mapsto f^{-1}(\xi)$, for every étale covering f of $(X, \underline{M})$. (Details left to the reader.) We shall denote by

$$\pi_1((X, \underline{M})_{\text{ét}}, \xi)$$

the corresponding fundamental group.

Example 7.3.16. Let $(f, \log f) : (Y, \underline{N}) \to (X, \underline{M})$ be an étale covering of a regular log scheme $(X, \underline{M})$, and suppose that X is strictly local of dimension 1 and Y is connected (hence strictly local as well). Let $x \in X$ (resp. $y \in Y$) be the closed point; it follows that $\mathscr{O}_{X,x}$ is a strictly henselian discrete valuation ring (corollary 6.5.29). The same holds for $\mathscr{O}_{Y,y}$ in view of theorem 6.5.44 and [33, Ch.IV, Prop.18.5.10]. In case $\dim \underline{M}_x = 0$, the log structures $\underline{M}$ and $\underline{N}$ are trivial, so $f : Y \to X$ is an étale morphism of schemes. Otherwise we have $\dim \underline{M}_x = 1 = \dim \underline{N}_y$ (lemma 3.4.41(i)), and then $\underline{M}_x^\sharp \simeq \mathbb{N} \simeq \underline{N}_y^\sharp$ (theorem 3.4.16(ii)); also, the choice of a chart for f as in proposition 7.3.12, induces an isomorphism

$$\mathscr{O}_{X,x} \otimes_{\underline{M}_x^\sharp} \underline{N}_y^\sharp \xrightarrow{\sim} \mathscr{O}_{Y,y}$$

where the map $\underline{M}_x^\sharp \to \underline{N}_y^\sharp$ is the N-Frobenius map of $\mathbb{N}$, where $N > 0$ is an integer invertible in $\mathscr{O}_{X,x}$. Moreover, notice that the image of the maximal ideal of $\underline{M}_x$ generates the maximal ideal of $\mathscr{O}_{X,x}$ (and likewise for the image of $\underline{N}_y$ in $\mathscr{O}_{Y,y}$), so the structure map of $\underline{M}$ induces an isomorphism $\underline{M}_x^\sharp \xrightarrow{\sim} \Gamma_+$, onto the submonoid of the value group $(\Gamma, \leq)$ of $\mathscr{O}_{X,x}$ consisting of all elements ≤ 1 (and likewise for $\underline{N}_y^\sharp$). In other words, $\mathscr{O}_{Y,y}$ is obtained from $\mathscr{O}_{X,x}$ by adding the N-th root of a uniformizer. It then follows that the ring homomorphism $\mathscr{O}_{X,x} \to \mathscr{O}_{Y,y}$ is an algebraic *tamely ramified extension* of discrete valuation rings (see *e.g.* [36, Cor.6.2.14]).

Definition 7.3.17. Let X be a normal scheme, and $Z \to X$ a closed immersion such that :

(a) Z is a union of irreducible closed subsets of codimension one in X.
(b) For every maximal point $z \in Z$, the stalk $\mathscr{O}_{X,z}$ is a discrete valuation ring.

Set $U := X \setminus Z$, let $f : U' \to U$ be an étale covering, and $g : X' \to X$ the normalization of X in U'. Notice that $g^{-1}Z$ is a union of irreducible closed subsets of codimension one in X' (by the going down theorem [61, Th.9.4(ii)]).

(i) We say that f is *tamely ramified along* Z if, for every geometric point $\xi \in X'$ localized at a maximal point of $g^{-1}Z$, the induced finite extension

$$\mathscr{O}^{\mathrm{sh}}_{X,g(\xi)} \to \mathscr{O}^{\mathrm{sh}}_{X',\xi}$$

of strictly henselian discrete valuation rings ([33, Ch.IV, Cor.18.8.13]), is tamely ramified. Recall that the latter condition means the following : let $\Gamma_{g(\xi)} \to \Gamma_\xi$ be the map of valuation groups induced by the above extension; then the *ramification index of* f *at* ξ

$$e_\xi(f) := (\Gamma_\xi : \Gamma_{f(\xi)})$$

is invertible in the residue field $\kappa(\xi)$, and g induces an isomorphism $\kappa(f(\xi)) \xrightarrow{\sim} \kappa(\xi)$.

(ii) Suppose that f is tamely ramified along Z, and that the set of maximal points of Z is finite. Then the *ramification index of* f *along* Z is the least common multiple $e_Z(f)$ of the ramification indices $e_\xi(f)$, where ξ ranges over the set of all geometric points of X' supported at a maximal point of $g^{-1}Z$.

(iii) We denote by $\mathbf{Tame}(X, U)$ the full subcategory of $\mathbf{Cov}(U)$ whose objects are the étale coverings of U that are tamely ramified along Z.

Lemma 7.3.18. *Let $\varphi : X' \to X$ be a dominant morphism of normal schemes, $Z \subset X$ a closed subset, and set $Z' := \varphi^{-1}Z$. Suppose that :*

(a) *The closed immersions $Z \to X$ and $Z' \to X'$ satisfy conditions* (a) *and* (b) *of definition* 7.3.17.
(b) *φ restricts to a map $\operatorname{Max} Z' \to \operatorname{Max} Z$ on the subsets of maximal points of Z and Z'.*

We have :

(i) *$\varphi^* : \mathbf{Cov}(U) \to \mathbf{Cov}(U')$ restricts to a functor*
$$\mathbf{Tame}(X, U) \to \mathbf{Tame}(X', U').$$
(ii) *Suppose that $\operatorname{Max} Z$ and $\operatorname{Max} Z'$ are finite sets. Then, for any object $f : V \to U$ of $\mathbf{Tame}(X, U)$, the index $e_{Z'}(\varphi^* f)$ divides $e_Z(f)$.*
(iii) *Suppose that X is regular, and let $j : U \to X$ be the open immersion. Then the essential image of the pull-back functor $j^* : \mathbf{Cov}(X) \to \mathbf{Tame}(X, U)$ consists of the objects $f : V \to U$ such that $e_Z(f) = 1$.*

Proof. Let $V \to U$ be an object of $\mathbf{Tame}(X, U)$, and denote by $g : W \to X$ (resp. $g' : W' \to X'$) the normalization of X in V (resp. of X' in $V \times_U U'$). Let w' be a geometric point of W' localized at a maximal point of $g'^{-1}Z'$, and denote by w (resp. x', resp. x) the image of w' in W (resp. in X', resp. in X); in order to check (i) and (ii), we have to show that the ramification index of g' at the geometric point w' is invertible in $\kappa(w')$, and the residue field extension $\kappa(x') \subset \kappa(w')$ is trivial. To this aim, in view of [33, Ch.IV, Prop.17.5.7], we may replace X' by $X'(x')$, X by $X(x)$, W by $W(w)$, and assume from start that X, X' and W are strictly local, say $X = \operatorname{Spec} A$, $X' = \operatorname{Spec} B$ and $C = \operatorname{Spec} W$ for some strictly henselian local rings A, B, C. In this case, $D := B \otimes_A C$ is a direct product of finite and local B-algebras, and x' is localized at a closed point of X', so w' is localized at a closed point of W', therefore $\mathscr{O}_{W',w'}$ is a direct factor of D, and $\kappa(w')$ is a quotient of $\kappa(x') \otimes_{\kappa(x)} \kappa(w)$; but we have $\kappa(w) = \kappa(x)$ by assumption, so this already yields $\kappa(w') = \kappa(x')$. Moreover, by construction, $\mathscr{O}_{W',w'} \otimes_A \operatorname{Frac} A$ is a finite separable extension of $\operatorname{Frac} B$, and the diagram

$$\begin{array}{ccc} \operatorname{Frac} A & \longrightarrow & \operatorname{Frac} C \\ \downarrow & & \downarrow \\ \operatorname{Frac} B & \longrightarrow & \operatorname{Frac} \mathscr{O}_{W',w'} \end{array} \tag{7.3.19}$$

is cocartesian (in the category of fields). Since V is tamely ramified along Z, the top horizontal arrow of (7.3.19) is a finite (abelian) extension whose degree e_w is invertible in $\kappa(x)$ ([36, Cor.6.2.14]); it follows that the bottom horizontal arrows is a Galois extension as well, and its degree divides e_w, as required.

(iii): Let $f : V \to U$ be an object of $\mathbf{Tame}(X, U)$. By claim 7.1.8, there exists a largest open subset $U_{\max} \subset X$ containing U, and such that f is the restriction of an étale covering $f_{\max} : V' \to U_{\max}$; now, if $e_Z(f) = 1$, every point of codimension one of $X \setminus U$ lies in $U_{\max}$, by claim 7.1.9. Hence $X \setminus U$ has codimension ≥ 2 in X, in which case lemma 7.1.7(iii) implies that $X = U_{\max}$, as required. $\square$

Remark 7.3.20. The assumptions (a) and (b) of lemma 7.3.18 are fulfilled, notably, when f is finite and dominant, or when f is flat ([61, Th.9.4(ii), Th.9.5]).

7.3.21. Let now $\underline{X} := (X_i \mid i \in I)$ be a cofiltered system of normal schemes, such that, for every morphism $\varphi : i \to j$ in the indexing category I, the corresponding transition morphism $f_\varphi : X_i \to X_j$ is dominant and affine. Suppose also, that for every $i \in I$, there exists a closed immersion $Z_i \to X_i$, fulfilling conditions (a) and (b) of definition 7.3.17, such that $\operatorname{Max} Z_i$ is a finite set, and for every morphism $\varphi : i \to j$ of I, we have :

- $Z_i = f_\varphi^{-1} Z_j$.
- The corresponding morphism f_φ restricts to a map $\mathrm{Max}\, Z_i \to \mathrm{Max}\, Z_j$.

Let also X be the limit of $\underline{X}$, and Z the limit of the system $(Z_i \mid i \in I)$, and suppose furthermore that the closed immersion $Z \to X$ fulfills as well conditions (a) and (b) of definition 7.3.17.

Proposition 7.3.22. *In the situation of* (7.3.21)*, we have :*

(i) *The induced morphism* $\pi_i : X \to X_i$ *restricts to a map*

$$\mathrm{Max}\, Z \to \mathrm{Max}\, Z_i \qquad \textit{for every } i \in I.$$

(ii) *The morphisms* π_i *induce an equivalence of categories :*

$$2\text{-}\operatorname*{colim}_I \mathbf{Tame}(X_i, X_i \setminus Z_i) \to \mathbf{Tame}(X, X \setminus Z).$$

Proof. (i): More precisely, we shall prove that there is a natural homeomorphism :

$$\mathrm{Max}\, Z \xrightarrow{\sim} \lim_{i \in I} \mathrm{Max}\, Z_i.$$

(Notice the each $\mathrm{Max}\, Z_i$ is a discrete finite set, hence this will show that $\mathrm{Max}\, Z$ is a profinite topological space.) Indeed, suppose that $z := (z_i \mid i \in I)$ is a maximal point of Z. For every $i \in I$, let $T_i \subset \mathrm{Max}\, Z_i$ be the subset of maximal generizations of z_i in Z_i. It is easily seen that $f_\varphi T_i \subset T_j$ for every morphism $\varphi : i \to j$ in I. Clearly T_i is a finite non-empty set for every $i \in I$, hence the limit T of the cofiltered system $(T_i \mid i \in I)$ is non-empty. However, any point of T is a generization of z in Z, hence it must coincide with z. The assertion follows easily.

(ii): To begin with, lemma 7.3.18(i) shows that, for every morphism $\varphi : i \to j$ in I, the transition morphism f_φ induces a pull-back functor $\mathbf{Tame}(X_j, X_j \setminus Z_j) \to \mathbf{Tame}(X_i, X_i \setminus Z_i)$, so the 2-colimit in (ii) is well-defined, and combining (i) with lemma 7.3.18(i) we obtain indeed a well-defined functor from this 2-colimit to $\mathbf{Tame}(X, X \setminus Z)$.

The full faithfulness of the functor of (ii) follows from lemma 7.1.6. Next, let $g : V \to X \setminus Z$ be an object of $\mathbf{Tame}(X, X \setminus Z)$; invoking again lemma 7.1.6, we may descend g to an étale covering $g_j : V_j \to X_j \setminus Z_j$, for some $j \in I$; after replacing I by I/j, we may assume that j is the final object of I and we may define $g_i := f_\varphi^*(g_j)$ for every $\varphi : i \to j$ in I. To conclude the proof, it suffices to show that there exists $i \in I$ such that g_i is tamely ramified along Z_i.

Now, let $\overline{g}_i : \overline{V}_i \to X_i$ be the normalization of X_i in $V_i \times_{X_j} X_i$, for every $i \in I$, and $\overline{g} : \overline{V} \to X$ the normalization of X in V. Given a geometric point $\overline{v}$ localized at a maximal point of $\overline{g}^{-1} Z$, let $\overline{v}_i$ (resp. $\overline{z}_i$) be the image of $\overline{v}$ in $\overline{V}_i$ (resp. in Z_i), for every $i \in I$. Let also $\overline{z}$ be the image of $\overline{v}$ in Z. Then

$$\mathscr{O}^{\mathrm{sh}}_{X,\overline{z}} = \operatorname*{colim}_{i \in I} \mathscr{O}^{\mathrm{sh}}_{X_i,\overline{z}_i} \qquad \mathscr{O}^{\mathrm{sh}}_{\overline{V},\overline{v}} = \operatorname*{colim}_{i \in I} \mathscr{O}^{\mathrm{sh}}_{\overline{\mathbb{V}}_i,\overline{v}_i}.$$

([33, Ch.IV, Prop.18.8.18(ii)]), and it follows easily that there exists $i \in I$ such that the finite extension $\mathscr{O}^{\mathrm{sh}}_{X_i,\overline{z}_i} \to \mathscr{O}^{\mathrm{sh}}_{\overline{\mathbb{V}}_i,\overline{v}_i}$ is already tamely ramified. Since only finitely many points of $\overline{V}_i$ lie over the support of $\overline{z}_i$, and since I is cofiltered, it follows that there exists $i \in I$ such that the induced morphism $\overline{V}_i \times_{X_i} X_i(\overline{z}_i) \to X_i(\overline{z}_i)$ is already tamely ramified. However, notice that $\pi_i^{-1}(z_i)$ is open in $\mathrm{Max}\, Z$, for every $i \in I$, and every $z_i \in \mathrm{Max}\, Z_i$. Therefore, we may find a finite subset $I_0 \subset I$, and for every $i \in I_0$ a subset $T_i \subset \mathrm{Max}\, Z_i$, such that :

- $\mathrm{Max}\, Z = \bigcup_{i \in I_0} \pi_i^{-1}(T_i)$.
- For every geometric point $\overline{z}_i$ localized in T_i, the morphism $\overline{V}_i \times_{X_i} X_i(\overline{z}_i) \to X_i(\overline{z}_i)$ is tamely ramified.

Since I is cofiltered, we may find $k \in I$ with morphisms $\varphi_i : k \to i$ for every $i \in I_0$; after replacing T_i by $f_{\varphi_i}^{-1}(T_i)$ for every $i \in I_0$, we may then assume that $I_0 = \{k\}$, so that $\mathrm{Max}\, Z = \pi_k^{-1}(T_k)$. It follows that $\mathrm{Max}\, Z_i = f_\varphi^{-1} T_k$ for some $i \in I$ and some $\varphi : i \to k$. Then it is clear that g_i is tamely ramified along Z_i, as required. $\square$

7.3.23. Let X be a scheme, $U \subset X$ a connected open subset, ξ any geometric point of U, and $N > 0$ an integer which is invertible in $\mathscr{O}_U$; then the N-Frobenius map $\boldsymbol{N}$ of $\mathscr{O}_U^\times$ gives a *Kummer exact sequence* of abelian sheaves on $X_{\text{ét}}$:

$$0 \to \boldsymbol{\mu}_{N,U} \to \mathscr{O}_U^\times \xrightarrow{\boldsymbol{N}} \mathscr{O}_U^\times \to 0$$

(where $\boldsymbol{\mu}_{N,U}$ is the N-torsion subsheaf of $\mathscr{O}_{U_{\text{ét}}}^\times$: see [5, Exp.IX, §3.2]). Suppose now that $\boldsymbol{\mu}_{N,U}$ is a constant sheaf on $U_{\text{ét}}$; this means especially that $\boldsymbol{\mu}_N := (\boldsymbol{\mu}_{N,U})_\xi$ is a cyclic group of order N. Indeed, $\boldsymbol{\mu}_N$ certainly contains such a subgroup (since N is invertible in $\mathscr{O}_U$), so denote by ζ one of its generators; then every $u \in \boldsymbol{\mu}_N$ satisfies the identity $0 = u^N - 1 = \prod_{i=1}^N (u - \zeta^i)$, and each factor u_ζ^i of this decomposition vanishes on a closed subset U_i of $U(\xi)$; clearly $U_i \cap U_j = \varnothing$ for $i \neq j$, so we get a decomposition of $U(\xi)$ as a disjoint union of open and closed subsets $U(\xi) = U_1 \cup \cdots \cup U_N$ such that $u = \zeta^i$ on U_i, for every $i \leq N$; since $U(\xi)$ is connected, it follows that $U(\xi) = U_i$ for some i, *i.e.* ζ generates $\boldsymbol{\mu}_N$. There follows a natural map :

$$\partial_N : \Gamma(U_{\text{ét}}, \mathscr{O}_U^\times) \to H^1(U_{\text{ét}}, \boldsymbol{\mu}_{N,U}) \xrightarrow{\sim} \mathrm{Hom}_{\text{cont}}(\pi_1(U_{\text{ét}}, \xi), \boldsymbol{\mu}_N)$$

(lemma 7.1.13). Recall the geometric interpretation of ∂_N : a given $u \in \mathscr{O}_U^\times(U)$ is viewed as a morphism of schemes $u : U \to \mathbb{G}_m$, where $\mathbb{G}_m := \operatorname{Spec} \mathbb{Z}[\mathbb{Z}]$ is the standard multiplicative group scheme; we let U' be the fibre product in the cartesian diagram of schemes :

(7.3.24)
$$\begin{tikzcd} U' \arrow[r] \arrow[d, "\varphi_u"'] & \mathbb{G}_m \arrow[d, "\operatorname{Spec} \mathbb{Z}[\boldsymbol{N}_{\mathbb{Z}}]"] \\ U \arrow[r, "u"] & \mathbb{G}_m. \end{tikzcd}$$

Then φ_u is a torsor under the $U_{\text{ét}}$-group $\boldsymbol{\mu}_{N,U}$, and to such torsor, lemma 7.1.13 associates a well defined continuous group homomorphism as required.

If $M > 0$ is any integer dividing N, a simple inspection yields a commutative diagram :

(7.3.25)
$$\begin{tikzcd} \Gamma(U_{\text{ét}}, \mathscr{O}_X^\times) \arrow[r, "\partial_N"] \arrow[dr, "\partial_M"'] & \mathrm{Hom}_{\text{cont}}(\pi_1(U_{\text{ét}}, \xi), \boldsymbol{\mu}_N) \arrow[d, "\pi_{N,M}"] \\ & \mathrm{Hom}_{\text{cont}}(\pi_1(U_{\text{ét}}, \xi), \boldsymbol{\mu}_M) \end{tikzcd}$$

where $\pi_{N,M}$ is induced by the map $\boldsymbol{\mu}_N \to \boldsymbol{\mu}_M$ given by the rule : $x \mapsto x^{N/M}$ for every $x \in \boldsymbol{\mu}_N$.

7.3.26. Let now X be a strictly local scheme, x the closed point of X, and denote by p the characteristic exponent of the residue field $\kappa(x)$ (so p is either 1 or a positive prime integer). Let also $\beta : \underline{M} \to \mathscr{O}_X$ be a log structure on $X_{\text{ét}}$, take $U := (X, \underline{M})_{\text{tr}}$, suppose that $U \neq \varnothing$, and fix a geometric point ξ of U; for every integer $N > 0$ with $(N, p) = 1$, we get a morphism of monoids :

(7.3.27)
$$\underline{M}_x \xrightarrow{\beta_x} \Gamma(U_{\text{ét}}, \mathscr{O}_X^\times) \xrightarrow{\partial_N} \mathrm{Hom}_{\text{cont}}(\pi_1(U_{\text{ét}}, \xi), \boldsymbol{\mu}_N)$$

whose kernel contains $\underline{M}_x^N$, the image of the N-Frobenius endomorphism of $\underline{M}_x$. Notice that the N-Frobenius map of $\mathscr{O}_{X,x}^\times$ is surjective : indeed, since the residue field $\kappa(x)$ is separably closed, and $(N, p) = 1$, all polynomials in $\kappa(x)[T]$ of the form $T^N - u$ (for $u \neq 0$) split as a product of distinct monic polynomials of degree 1; since $\mathscr{O}_{X,x}$ is henselian, the same holds for all polynomials in $\mathscr{O}_{X,x}[T]$ of the form $T^N - u$, with $u \in \mathscr{O}_{X,x}^\times$. It follows that (7.3.27) factors through a natural map:

$$\underline{M}_x^{\sharp\text{gp}} \to \mathrm{Hom}_{\text{cont}}(\pi_1(U_{\text{ét}}, \xi), \boldsymbol{\mu}_N)$$

which is the same as a group homomorphism :

(7.3.28)
$$\underline{M}_x^{\sharp\text{gp}} \times \pi_1(U_{\text{ét}}, \xi) \to \boldsymbol{\mu}_N.$$

In view of the commutative diagram (7.3.25), it is easily seen that the pairings (7.3.28), for N ranging over all the positive integers with $(N,p)=1$, assemble into a single pairing :

$$\underline{M}_x^{\mathrm{gp}} \times \pi_1((X,\underline{M})_{\mathrm{tr,\acute{e}t}},\xi) \to \prod_{\ell\neq p} \mathbb{Z}_\ell(1)$$

(where ℓ ranges over the prime numbers different from p, and $\mathbb{Z}_\ell(1) := \lim_{n\in\mathbb{N}} \boldsymbol{\mu}_{\ell^n}$). The latter is the same as a group homomorphism :

(7.3.29)
$$\pi_1((X,\underline{M})_{\mathrm{tr,\acute{e}t}},\xi) \to \underline{M}_x^{\mathrm{gp}\vee} \otimes_{\mathbb{Z}} \prod_{\ell\neq p} \mathbb{Z}_\ell(1).$$

7.3.30. Let $f : (X,\underline{M}) \to (Y,\underline{N})$ be a morphism of log schemes, such that both X and Y are strictly local, and f maps the closed point x of X to the closed point y of Y. Then f restricts to a morphism of schemes $f_{\mathrm{tr}} : (X,\underline{M})_{\mathrm{tr}} \to (Y,\underline{N})_{\mathrm{tr}}$ (remark 6.2.8(i)). Fix again a geometric point ξ of $(X,\underline{M})_{\mathrm{tr}}$; we get a diagram of group homomorphisms :

$$\begin{tikzcd}
\pi_1((X,\underline{M})_{\mathrm{tr,\acute{e}t}},\xi) \arrow[r, "\pi_1(f_{\mathrm{tr}},\xi)"] \arrow[d] & \pi_1((Y,\underline{N})_{\mathrm{tr,\acute{e}t}},f(\xi)) \arrow[d] \\
\underline{N}_y^{\mathrm{gp}\vee} \otimes_{\mathbb{Z}} \prod_{\ell\neq p} \mathbb{Z}_\ell(1) \arrow[r] & \underline{M}_x^{\mathrm{gp}\vee} \otimes_{\mathbb{Z}} \prod_{\ell\neq p} \mathbb{Z}_\ell(1)
\end{tikzcd}$$

whose vertical arrows (resp. bottom horizontal arrow) are the maps (7.3.29) (resp. is induced by $(\log f_x)^{\mathrm{gp}\vee}$), and by inspecting the constructions, it is easily seen that this diagram commutes.

7.3.31. Suppose now that $(X,\underline{M})$ as in (7.3.26) is a fs log scheme, so that $P := \underline{M}_x^\sharp$ is fine and saturated; in this case, we wish to give a second construction of the pairing (7.3.28).

- Namely, for every integer $N > 0$ set $S_P := \operatorname{Spec} \mathbb{Z}[P]$ and consider the finite morphism of schemes $g_N : S_P \to S_P$ induced by the N-Frobenius endomorphism of P. Notice that S_P contains the dense open subset $U_P := \operatorname{Spec} \mathbb{Z}[1/N, P^{\mathrm{gp}}]$, and it is easily seen that the restriction $g_{N|U_P} : U_P \to U_P$ of g_N is an étale morphism. Let $\tau \in U_P$ be any geometric point; then τ corresponds to a ring homomorphism $\mathbb{Z}[1/N, P^{\mathrm{gp}}] \to \kappa := \kappa(\tau)$, which is the same as a character $\chi_\tau : P^{\mathrm{gp}} \to \kappa^\times$. Likewise, any $\tau' \in g_N^{-1}(\tau)$ is determined by a character $\chi_{\tau'} : P^{\mathrm{gp}} \to \kappa^\times$ extending χ_τ, *i.e.* such that $\chi_{\tau'}(x^N) = \chi_\tau(x)$ for every $x \in P^{\mathrm{gp}}$. Notice that every character χ_τ as above admits at least one such extension $\chi_{\tau'}$, since κ is separably closed, and N is invertible in κ. Let C_N be the cokernel of the N-Frobenius endomorphism of P^{gp}; there follows a short exact sequence of finite abelian groups :

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(C_N,\kappa^\times) \to \mathrm{Hom}_{\mathbb{Z}}(P^{\mathrm{gp}},\kappa^\times) \xrightarrow{N} \mathrm{Hom}_{\mathbb{Z}}(P^{\mathrm{gp}},\kappa^\times) \to 0.$$

Especially, we see that the fibre $g_N^{-1}(\tau)$ is naturally a torsor under the group $\mathrm{Hom}_{\mathbb{Z}}(C_N,\kappa^\times) = \mathrm{Hom}_{\mathbb{Z}}(P^{\mathrm{gp}},\boldsymbol{\mu}_N(\kappa))$, where $\boldsymbol{\mu}_N(\kappa) \subset \kappa^\times$ is the N-torsion subgroup. Hence $g_{N|U_P}$ is an étale Galois covering of U_P, whose Galois group is naturally isomorphic to $\mathrm{Hom}_{\mathbb{Z}}(P^{\mathrm{gp}},\boldsymbol{\mu}_N(\kappa))$; therefore, $g_{N|U_P}$ is classified by a well defined continuous representation :

(7.3.32)
$$\pi_1(U_{P,\mathrm{\acute{e}t}},\tau) \to \mathrm{Hom}_{\mathbb{Z}}(P^{\mathrm{gp}},\boldsymbol{\mu}_N(\kappa))$$

(lemma 7.1.13). The corresponding action of $\mathrm{Hom}_{\mathbb{Z}}(P^{\mathrm{gp}},\boldsymbol{\mu}_N(\kappa))$ on U_P can be extracted from the construction : namely, let $\chi : P^{\mathrm{gp}} \to \boldsymbol{\mu}_N(\kappa)$ be a given character; by definition, the action of χ sends the geometric point τ to the geometric point τ' whose character $\chi_{\tau'} : P^{\mathrm{gp}} \to \kappa^\times$ is given by the rule : $x \mapsto \chi(x)\cdot\chi_\tau$ for every $x \in P^{\mathrm{gp}}$. Consider the automorphism ρ_χ of the $\mathbb{Z}$-algebra $\mathbb{Z}[P^{\mathrm{gp}}]$ given by the rule $x \mapsto \chi(x)\cdot x$ for every $x \in P^{\mathrm{gp}}$, and notice that $\rho(\tau) = \tau'$; since the fibre functors are faithful on étale coverings, we conclude that χ acts as $\operatorname{Spec}\rho_\chi$ on U_P.

• Moreover, if $\lambda : Q \to P$ is any morphism of fine and saturated monoids, clearly we have a commutative diagram

$$\begin{tikzcd} U_P \arrow[r, "\mathrm{Spec}\,\mathbb{Z}[\lambda]_{|U_P}"] \arrow[d, "g_{N|U_P}"'] & U_Q \arrow[d, "g_{N|U_Q}"] \\ U_P \arrow[r, "\mathrm{Spec}\,\mathbb{Z}[\lambda]_{|U_P}"] & U_Q \end{tikzcd}$$

whence, by remark 7.1.15(iii.a), a well defined group homomorphism

$$\mathrm{Hom}_{\mathbb{Z}}(P^{\mathrm{gp}}, \boldsymbol{\mu}_N) \to \mathrm{Hom}_{\mathbb{Z}}(Q^{\mathrm{gp}}, \boldsymbol{\mu}_N) \tag{7.3.33}$$

that makes commute the induced diagram :

$$\begin{tikzcd} \pi_1(U_{P,\mathrm{\acute{e}t}}, \xi) \arrow[r] \arrow[d] & \pi_1(U_Q, \xi') \arrow[d] \\ \mathrm{Hom}_{\mathbb{Z}}(P^{\mathrm{gp}}, \boldsymbol{\mu}_N) \arrow[r] & \mathrm{Hom}_{\mathbb{Z}}(Q^{\mathrm{gp}}, \boldsymbol{\mu}_N) \end{tikzcd}$$

whose vertical arrows are the maps (7.3.32). By inspecting the constructions, it is easily seen that (7.3.33) is just the map $\mathrm{Hom}_{\mathbb{Z}}(\lambda^{\mathrm{gp}}, \boldsymbol{\mu}_N)$.

• Next, by corollary 6.1.34(i) the projection $\underline{M}_x \to P$ admits a splitting

$$\alpha : P \to \underline{M}_x$$

(which defines a sharp chart for $\underline{M}$); if N is invertible in $\mathscr{O}_X$, this splitting induces a morphism of schemes $X \to S_P$, which restricts to a morphism $U \to U_P$. If we let τ be the image of ξ, we deduce a continuous group homomorphism $\pi_1(U_{\mathrm{\acute{e}t}}, \xi) \to \pi_1(U_{P,\mathrm{\acute{e}t}}, \tau)$ ([42, Exp.V, §7]), whose composition with (7.3.32) yields a continuous map :

$$\pi_1(U_{\mathrm{\acute{e}t}}, \xi) \to \mathrm{Hom}_{\mathbb{Z}}(P^{\mathrm{gp}}, \boldsymbol{\mu}_N(\kappa)). \tag{7.3.34}$$

We claim that the pairing $P^{\mathrm{gp}} \times \pi_1(U_{\mathrm{\acute{e}t}}, \xi) \to \boldsymbol{\mu}_N(\kappa)$ arising from (7.3.34) agrees with (7.3.28), under the natural identification $\boldsymbol{\mu}_N(\kappa) \stackrel{\sim}{\to} \boldsymbol{\mu}_N$. Indeed, for given $s \in P$, let $j_s : \mathbb{Z} \to P^{\mathrm{gp}}$ be the map such that $j_s(n) := s^n$ for every $n \in \mathbb{Z}$; by composing (7.3.34) with $\mathrm{Hom}_{\mathbb{Z}}(j_s, \boldsymbol{\mu}_N(\kappa))$, we obtain a map $\pi_1(U_{\mathrm{\acute{e}t}}, \xi) \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \boldsymbol{\mu}_N(\kappa)) \stackrel{\sim}{\to} \boldsymbol{\mu}_N(\kappa)$; in view of the foregoing, it is easily seen that the corresponding $\boldsymbol{\mu}_N(\kappa)$-torsor is precisely $\varphi_{\alpha(s)}$, in the notation of (7.3.24), whence the contention.

Proposition 7.3.35. *In the situation of* (7.3.26), *suppose that* $(X, \underline{M})$ *is a regular log scheme. Then* (7.3.29) *is a surjection.*

Proof. From the discussion of (7.3.31), we are reduced to showing that the morphism (7.3.34) is surjective, for every $N > 0$ such that $(N, p) = 1$. The latter comes down to showing that the corresponding torsor $g_{N|U_P} \times_{U_P} U : U' \to U$ is connected (remark 7.1.15(i)). However, the map $\alpha : P \to \underline{M}_x$ induces a morphism of log schemes $\psi : (X, \underline{M}) \to \mathrm{Spec}(\mathbb{Z}, P)$ (see (6.2.13)); we remark the following :

Claim 7.3.36. Slightly more generally, for any Kummer morphism $\nu : P \to Q$ of monoids, with Q fine, sharp and saturated, define $(X_\nu, \underline{M}_\nu)$ as the fibre product in the cartesian diagram:

$$\begin{tikzcd} (X_\nu, \underline{M}_\nu) \arrow[r] \arrow[d, "f_\nu"'] & \mathrm{Spec}(\mathbb{Z}, Q) \arrow[d, "\mathrm{Spec}(\mathbb{Z},\nu)"] \\ (X, \underline{M}) \arrow[r, "\psi"] & \mathrm{Spec}(\mathbb{Z}, P) \end{tikzcd} \tag{7.3.37}$$

Then $(X_\nu, \underline{M}_\nu)$ is regular, and X_ν is strictly local.

Proof of the claim. By lemma 6.5.40, $(X_\nu, \underline{M}_\nu)$ is regular at every point of the closed fibre $X_\nu \times_X \operatorname{Spec} \kappa(x)$, and since f_ν is a finite morphism, every point of X_ν specializes to a point of the closed fibre, therefore $(X_\nu, \underline{M}_\nu)$ is regular (theorem 6.5.46). Moreover, X_ν is connected, provided the same holds for the closed fibre; in turn, this follows immediately from proposition 7.3.8(ii). Then X_ν is strictly local, by [33, Ch.IV, Prop.18.5.10]. ◊

From claim 7.3.36 we see that X_ν is normal (corollary 6.5.29) for every such ν, especially this holds for the N-Frobenius endomorphism of P, in which case U' is an open subset of X_ν; since the latter is connected, the same then holds for U'. □

Example 7.3.38. (i) In the situation of example 6.6.20, let $X \to S$ be any morphism of schemes, set $\underline{X} := X \times_S \underline{S}$, denote by $(X, \underline{M})$ the log scheme underlying $\underline{X}$, and define $(X_{(k)}, \underline{M}_{(k)})$ as the fibre product in the cartesian diagram of log schemes :

$$\begin{array}{ccc} (X_{(k)}, \underline{M}_{(k)}) & \xrightarrow{\pi_{(k)}} & \operatorname{Spec}(R, P) \\ {\scriptstyle \boldsymbol{k}_X}\downarrow & & \downarrow{\scriptstyle \operatorname{Spec}(R, \boldsymbol{k}_P)} \\ (X, \underline{M}) & \xrightarrow{\pi} & \operatorname{Spec}(R, P) \end{array}$$

where π is the natural projection. Set as well

$$\underline{X}_{(k)} := ((X_{(k)}, \underline{M}_{(k)}), T_P, \psi_P \circ \pi^\sharp_{(k)})$$

which is an object of $\mathscr{K}_{\mathrm{int}}$; by proposition 6.6.14(iii), diagram (3.5.28) (with $T := T_P$) underlies a commutative diagram in $\mathscr{K}_{\mathrm{int}}$:

$$\begin{array}{ccc} \varphi^* \underline{X}_{(k)} & \longrightarrow & \underline{X}_{(k)} \\ {\scriptstyle \underline{g}_X}\downarrow & & \downarrow{\scriptstyle (\boldsymbol{k}_X, \boldsymbol{k}_{T_P})} \\ \varphi^* \underline{X} & \longrightarrow & \underline{X} \end{array}$$

whose horizontal arrows are cartesian. Clearly this diagram is isomorphic to $X \times_S$ (6.6.21); we deduce that the morphism g_X of log schemes underlying $\underline{g}_X$ is finite, and of Kummer type; by proposition 7.3.12, g_X is even an étale covering, if k is invertible in $\mathscr{O}_X$.

(ii) Suppose furthermore, that $(X, \underline{M})$ is regular; by claim 7.3.36 it follows that $(X_{(k)}, \underline{M}_{(k)})$ is regular as well, and then the same holds for the log schemes $(X_\varphi, \underline{M}_\varphi)$ and $(Y, \underline{N})$ underlying respectively $\varphi^* \underline{X}$ and $\varphi^* \underline{X}_{(k)}$ (proposition 6.6.14(v) and theorem 6.5.44). Let now $y \in Y$ be a point of height one in Y, lying in the closed subset $Y \setminus (Y, \underline{N})_{\mathrm{tr}}$, and set $x := g_X(y)$; arguing as in example 7.3.16, we see that $\underline{N}^\sharp_y$ and $\underline{M}^\sharp_{\varphi,x}$ are both isomorphic to $\mathbb{N}$, and the induced map $\mathscr{O}_{X_\varphi,x} \to \mathscr{O}_{Y,y}$ is an extension of discrete valuation rings, whose corresponding extension of valued fields is finite. Denote by $\Gamma_x \to \Gamma_y$ the associated extension of valuation groups; in view of (6.6.22) we see that the ramification index $(\Gamma_y : \Gamma_x)$ equals k.

7.3.39. Resume the situation of (7.3.31). Every (finite) discrete quotient map

$$\rho : \underline{M}^{\mathrm{gp}\vee}_x \otimes_{\mathbb{Z}} \prod_{\ell \neq p} \mathbb{Z}_\ell(1) \to G$$

corresponds, via composition with (7.3.29), to a G_U-torsor, which can be explicitly constructed as follows. Set $P := \underline{M}^\sharp_x$, pick a splitting α as in (7.3.31), choose an integer $N > 0$ large enough, so that $(N, p) = 1$, and ρ factors through a group homomorphism

$$\overline{\rho} : \operatorname{Hom}_{\mathbb{Z}}(P^{\mathrm{gp}}, \boldsymbol{\mu}_N(\kappa)) \to G$$

and set $H := \operatorname{Ker} \overline{\rho}$. Now, via (7.3.29), the quotient $G' := \operatorname{Hom}_{\mathbb{Z}}(P^{\mathrm{gp}}, \boldsymbol{\mu}_N(\kappa))$ corresponds to the G'_U-torsor on U obtained from $g_N : U_P \to U_P$ via pull-back along the morphism $U \to U_P$ given by the chart α (notation of (7.3.31)). According to remark 7.1.15(ii), the sought G_U is therefore isomorphic to the one obtained from the quotient $\overline{g}_N : U_P/H \to U_P$, via pull-back along the same morphism. To exhibit such quotient, consider the exact sequence:

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(G, \kappa^\times) \to P^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{Z}/N\mathbb{Z} \to \mathrm{Hom}_{\mathbb{Z}}(H, \kappa^\times) \to 0.$$

Define $Q^{\mathrm{gp}} \subset P^{\mathrm{gp}}$ as the kernel of the induced map $P^{\mathrm{gp}} \to \mathrm{Hom}_{\mathbb{Z}}(H, \kappa^\times)$, and set $Q := Q^{\mathrm{gp}} \cap P$. By construction, the N-Frobenius endomorphism of P factors through an injective map $\nu : P \to Q$ and the inclusion map $j : Q \to P$, so g_N factors as a composition :

$$U_P \to U_Q \xrightarrow{h} U_P.$$

The maps on geometric points $U_P(\kappa) \to U_Q(\kappa) \to U_P(\kappa)$ induced by g_N and h correspond to $j^{\mathrm{gp}*}$ and respectively $\nu^{\mathrm{gp}*}$ in the resulting commutative diagram

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
0 \arrow[r] & H \arrow[r] \arrow[d, equal] & \mathrm{Hom}_{\mathbb{Z}}(P^{\mathrm{gp}}, \boldsymbol{\mu}_N(\kappa)) \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H \arrow[r] & \mathrm{Hom}_{\mathbb{Z}}(P^{\mathrm{gp}}, \kappa^\times) \arrow[r, "j^{\mathrm{gp}*}"] & \mathrm{Hom}_{\mathbb{Z}}(Q^{\mathrm{gp}}, \kappa^\times) \arrow[r] \arrow[d, "\nu^{\mathrm{gp}*}"] & 0 \\
 & & & \mathrm{Hom}_{\mathbb{Z}}(P^{\mathrm{gp}}, \kappa^\times) \arrow[d] & \\
 & & & 0 &
\end{tikzcd}$$

whose rows and column are exact. Hence, let τ and $\chi_\tau : P^{\mathrm{gp}} \to \kappa^\times$ be as in (7.3.31); the fibre $h^{-1}(\tau)$ corresponds to the set of all characters $\chi_{\tau'} : Q^{\mathrm{gp}} \to \kappa^\times$ whose restriction $\chi_{\tau'} \circ \nu^{\mathrm{gp}}$ to P^{gp} agrees with χ_τ. Then, with the notation of (7.3.37), we conclude that the restriction $(f_\nu)_{\mathrm{tr}} : (X_\nu, \underline{M}_\nu)_{\mathrm{tr}} \to U$ is the sought G_U-torsor. Notice as well that f_ν is an étale covering of $(X, \underline{M})$.

• Here is another handier description of the same submonoid Q of P. Notice that ρ is the same as a group homomorphism

$$\rho^\dagger : P^{\mathrm{gp}\vee} \to \mathrm{Hom}_{\mathbb{Z}}(\textstyle\prod_{\ell \neq p} \mathbb{Z}_\ell(1), G)$$

and let $L := \mathrm{Ker}\, \rho^\dagger$. Fix a generator ζ_N of $\boldsymbol{\mu}_N(\kappa)$; by definition

$$\begin{aligned} Q^{\mathrm{gp}} &= \{x \in P^{\mathrm{gp}} \mid t(x) \otimes \xi = 0 \text{ for all } t \in P^{\mathrm{gp}\vee} \text{ and } \xi \in \boldsymbol{\mu}_N \text{ such that } \overline{\rho}(t \otimes \xi) = 0\} \\ &= \{x \in P^{\mathrm{gp}} \mid t(x) \otimes \zeta_N = 0 \text{ for all } t \in P^{\mathrm{gp}\vee} \text{ such that } \overline{\rho}(t \otimes \zeta_N) = 0\} \\ &= \{x \in P^{\mathrm{gp}} \mid t(x) \in N\mathbb{Z} \text{ for all } t \in L\}. \end{aligned}$$

On the other hand, notice that the N-Frobenius of $P^{\mathrm{gp}\vee}$ factors through a map $\beta : P^{\mathrm{gp}\vee} \to L$ and the inclusion map $i : L \to P^{\mathrm{gp}\vee}$, and $\beta \circ i$ is the N-Frobenius map of L. Let $\omega : P^{\mathrm{gp}} \xrightarrow{\sim} (P^{\mathrm{gp}\vee})^\vee$ be the natural isomorphism. We may then write

$$\begin{aligned} Q^{\mathrm{gp}} &= \{x \in P^{\mathrm{gp}} \mid \omega(x)(t) \in N\mathbb{Z} \text{ for all } t \in L\} \\ &= \{x \in P^{\mathrm{gp}} \mid \omega(x) \circ i \in N \cdot L^\vee = i^\vee \circ \beta^\vee(L^\vee)\} \\ &= \omega^{-1}(\mathrm{Im}\, \beta^\vee). \end{aligned}$$

In other words, ω induces a natural isomorphism $Q^{\mathrm{gp}} \xrightarrow{\sim} L^\vee$, that identifies $\beta^\vee$ with the map $j^{\mathrm{gp}} : Q^{\mathrm{gp}} \to P^{\mathrm{gp}}$, and then necessarily also the map $i^\vee$ with ν. Now, the image of Q inside

$L^\vee$ can be recovered just as $L^\vee \cap P^{\vee\vee}_{\mathbb{Q}}$ (the intersection here takes place in $L^\vee_{\mathbb{Q}}$, which contains $(P^{\mathrm{gp}\vee})^\vee$, via the injective map $i^\vee$: details left to the reader).

7.3.40. Our chief supply of tamely ramified coverings comes from the following source. Let $f : (Y, \underline{N}) \to (X, \underline{M})$ be an étale morphism of log schemes, whose underlying morphism of schemes is finite, and suppose that $(X, \underline{M})$ is regular. Then $(Y, \underline{N})$ is regular, and both X and Y are normal schemes (theorem 6.5.44 and corollary 6.5.29). Moreover, lemma 3.4.41(ii) and proposition 7.3.12 imply that $(Y, \underline{N})_{\mathrm{tr}} = f^{-1}(X, \underline{M})_{\mathrm{tr}}$, hence the restriction of f

$$f_{\mathrm{tr}} : (Y, \underline{N})_{\mathrm{tr}} \to (X, \underline{M})_{\mathrm{tr}}$$

is a finite étale morphism of schemes (corollary 6.3.27(i)). Furthermore, it follows easily from corollary 6.5.36(i) that the closed subset $X \backslash (X, \underline{M})_{\mathrm{tr}}$ is a union of irreducible closed subsets of codimension 1 in X (and the union is locally finite on the Zariski topology of X); the same holds also for $Y \setminus (Y, \underline{N})_{\mathrm{tr}}$, especially Y is the normalization of $(Y, \underline{N})_{\mathrm{tr}}$ over X. Finally, example 7.3.16 implies that f_{tr} is tamely ramified along $X \backslash (X, \underline{M})_{\mathrm{tr}}$. It is then clear that the rule $f \mapsto f_{\mathrm{tr}}$ defines a functor :

$$F_{(X,\underline{M})} : \mathbf{Cov}(X, \underline{M}) \to \mathbf{Tame}(X, (X, \underline{M})_{\mathrm{tr}}).$$

It follows easily from remark 6.5.54 that $F_{(X,\underline{M})}$ is fully faithful; therefore, any choice of a geometric point ξ of $(X, \underline{M})_{\mathrm{tr}}$ determines a surjective group homomorphism :

$$\pi_1((X, \underline{M})_{\mathrm{tr,\acute{e}t}}, \xi) \to \pi_1((X, \underline{M})_{\mathrm{\acute{e}t}}, \xi) \tag{7.3.41}$$

([42, Exp.V, Prop.6.9]).

Proposition 7.3.42. *In the situation of* (7.3.26), *suppose that* $(X, \underline{M})$ *is a regular log scheme. Then the map* (7.3.29) *factors through* (7.3.41), *and induces an isomorphism :*

$$\pi_1((X, \underline{M})_{\mathrm{\acute{e}t}}, \xi) \xrightarrow{\sim} \underline{M}^{\mathrm{gp}\vee}_{\overline{x}} \otimes_{\mathbb{Z}} \prod_{\ell \neq p} \mathbb{Z}_\ell(1). \tag{7.3.43}$$

Proof. The discussion of (7.3.39) shows that (7.3.29) factors through (7.3.41), and proposition 7.3.35 implies that (7.3.43) is surjective. Next, let $f : (Y, \underline{N}) \to (X, \underline{M})$ be an étale covering; according to proposition 7.3.12, f admits a Kummer chart $(\omega_P, \omega_Q, \vartheta)$ with Q fine, sharp and saturated, such that the order k of $\mathrm{Coker}\, \vartheta^{\mathrm{gp}}$ is invertible in $\mathscr{O}_Y$. Moreover, the induced morphism of X-schemes $Y \to X \times_{\mathrm{Spec}\, \mathbb{Z}[P]} \mathrm{Spec}\, \mathbb{Z}[Q]$ is an isomorphism. With this notation, denote by G the cokernel of the induced group homomorphism $\mathrm{Hom}_{\mathbb{Z}}(\vartheta^{\mathrm{gp}}, \boldsymbol{k}_{P^{\mathrm{gp}}})$; there follows a map $\rho : P^{\mathrm{gp}} \otimes_{\mathbb{Z}} \prod_{\ell \neq p} \mathbb{Z}_\ell(1) \to G$, and by inspecting the construction, one may check that the corresponding G_U-torsor, constructed as in (7.3.39), is isomorphic to $f \times_X \mathbf{1}_U$ (details left to the reader). This shows the injectivity of (7.3.43), and completes the proof of the proposition. $\square$

The following result is the logarithmic version of the classical Abhyankar's lemma.

Theorem 7.3.44. *The functor* $F_{(X,\underline{M})}$ *is an equivalence.*

Proof. First we show how to reduce to the case where $\tau = \mathrm{\acute{e}t}$.

Claim 7.3.45. Let $(X_{\mathrm{Zar}}, \underline{M})$ be a regular log structure on the Zariski site of X, and suppose that the theorem holds for $\widetilde{u}^*(X_{\mathrm{Zar}}, \underline{M})$. Then the theorem holds for $(X_{\mathrm{Zar}}, \underline{M})$ as well.

Proof of the claim. Let $g : V \to (X_{\mathrm{Zar}}, \underline{M})_{\mathrm{tr}}$ be an étale covering whose normalization over X is tamely ramified along $X \setminus (X_{\mathrm{Zar}}, \underline{M})_{\mathrm{tr}}$. By assumption, there exists an étale covering $f : (Y_{\mathrm{\acute{e}t}}, \underline{N}) \to \widetilde{u}^*(X_{\mathrm{Zar}}, \underline{M})$ such that $f_{\mathrm{tr}} = g$; by lemma 7.3.11, we may then find a morphism of log schemes of Kummer type $f_{\mathrm{Zar}} : \widetilde{u}_*(Y_{\mathrm{\acute{e}t}}, \underline{N}) \to (X_{\mathrm{Zar}}, \underline{M})$ such that $\widetilde{u}^* f_{\mathrm{Zar}} = f$, therefore f_{Zar} is an étale covering of $(X_{\mathrm{Zar}}, \underline{M})$ (corollary 6.3.27(iii)), and clearly $(f_{\mathrm{Zar}})_{\mathrm{tr}} = g$, so the functor $F_{(X_{\mathrm{Zar}}, \underline{M})}$ is essentially surjective. Full faithfulness for the same functor is derived formally from the full faithfulness of the functor $F_{\widetilde{u}^*(X_{\mathrm{Zar}},\underline{M})}$, and that of the functor (7.3.10) : details left to the reader. $\Diamond$

Henceforth we assume that $\tau = \text{ét}$.

Claim 7.3.46. Let $g : V \to (X, \underline{M})_{\mathrm{tr}}$ be an object of $\mathbf{Tame}(X, (X, \underline{M})_{\mathrm{tr}})$. Then g lies in the essential image of $F_{(X,\underline{M})}$ if (and only if) there exists an étale covering $(U_\lambda \to X \mid \lambda \in \Lambda)$ of X, such that, for every $\lambda \in \Lambda$, the étale covering $g \times_X U_\lambda$ lies in the essential image of $F_{(U_\lambda,\underline{M}_{|U_\lambda})}$.

Proof of the claim. We have to exhibit a finite étale covering $f : (Y, \underline{N}) \to (X, \underline{M})$ such that $f_{\mathrm{tr}} = g$. However, given such f, theorem 6.5.44 and corollary 6.5.29 imply that Y is the normalization of V over X, and proposition 6.5.52 says that $\underline{N} = j_*\mathscr{O}_V^\times \cap \mathscr{O}_Y$, where $j : V \to Y$ is the open immersion; then $\log f : f^*\underline{M} \to \underline{N}$ is completely determined by f, hence by g. Thus, we come down to showing that :

(i) $\underline{N} := j_*\mathscr{O}_V^\times \cap \mathscr{O}_Y$ is a regular log structure on the normalization Y of V over X.

(ii) The unique morphism $f : (Y, \underline{N}) \to (X, \underline{M})$ is an étale covering.

However, [61, §33, Lemma 1] implies that Y is finite over X. To show (ii), it then suffices to prove that each restriction $f_\lambda := f \times_X \mathbf{1}_{U_\lambda}$ is étale (proposition 6.3.24(iii)). Likewise, (i) holds, provided the restriction $\underline{N}_{|Y_\lambda}$ is a regular log structure on $Y_\lambda := Y \times_X U_\lambda$, for every $\lambda \in \Lambda$. Let $j_\lambda : V_\lambda := V \times_X U_\lambda \to Y_\lambda$ be the induced open immersion. It is clear that $\underline{N}_{|Y_\lambda} = j_{\lambda*}\mathscr{O}_{V_\lambda}^\times \cap \mathscr{O}_{Y_\lambda}$, and Y_λ is the normalization of V_λ over U_λ ([33, Ch.IV, Prop.17.5.7]); moreover, $(U_\lambda, \underline{M}_{|U_\lambda})$ is again regular, so our assumption implies that $(Y_\lambda, \underline{N}_{|Y_\lambda})$ is the unique regular log structure on Y_λ whose trivial locus is V_λ, and that f_λ is indeed étale. $\Diamond$

Claim 7.3.47. If $F_{(X(\xi),\underline{M}(\xi))}$ is essentially surjective for every geometric point ξ of X, the same holds for $F_{(X,\underline{M})}$.

Proof of the claim. Indeed, let $g : V \to (X, \underline{M})_{\mathrm{tr}}$ be an object of $\mathbf{Tame}(X, (X, \underline{M})_{\mathrm{tr}})$, and ξ any geometric point of X. By claim 7.3.46 it suffices to find an étale neighborhood $U \to X$ of ξ, such that $g \times_X \mathbf{1}_U$ lies in the essential image of $F_{(U,\underline{M}_{|U})}$. Denote by Y the normalization of X in V, which is a finite X-scheme ([61, §33, Lemma 1]). The scheme $Y(\xi) := Y \times_X X(\xi)$ decomposes as a finite disjoint union of strictly local open and closed subschemes $Y_1(\xi), \dots, Y_n(\xi)$, and we may find an étale neighborhood $U \to X$ of ξ, and a decomposition of $Y \times_X U$ by open and closed subschemes $Y_1, \dots, Y_n$, with isomorphisms of $X(\xi)$-schemes $Y_i(\xi) \xrightarrow{\sim} Y_i \times_X X(\xi)$ for every $i = 1, \dots, n$ ([32, Ch.IV, Cor.8.3.12]). We are then reduced to showing that all the restrictions $Y_i \to U$ lie in the essential image of $F_{(U,\underline{M}_{|U})}$. Thus, we may replace X by U, and Y by any Y_i, after which we may assume that $Y(\xi)$ is strictly local. Proposition 6.5.32 and theorem 6.5.46 imply that $(X(\xi), \underline{M}(\xi))$ is a regular log scheme. Clearly $g_\xi := g \times_X \mathbf{1}_{X(\xi)}$ is an object of $\mathbf{Tame}(X(\xi), (X(\xi), \underline{M}(\xi))_{\mathrm{tr}})$, so by assumption there exists a finite étale covering $h : (Z, \underline{N}) \to (X(\xi), \underline{M}(\xi))$ such that $h_{\mathrm{tr}} = g_\xi$. By theorem 6.5.44 and corollary 6.5.29 we know that Z is a normal scheme, and we deduce that $Z = Y(\xi)$ ([33, Ch.IV, Prop.17.5.7]).

According to proposition 7.3.12, the morphism h admits a fine and saturated Kummer chart $(\omega_P, \omega_Q, \vartheta : P \to Q)$, such that the order d of $\mathrm{Coker}\,\vartheta^{\mathrm{gp}}$ is invertible in $\mathscr{O}_{X,\xi}$, and such that the induced map $Y(\xi) \to X(\xi) \times_{\mathrm{Spec}\,P} \mathrm{Spec}\,Q$ is an isomorphism. By proposition 6.2.28, there exists an étale neighborhood $U \to X$ of ξ, and a coherent log structure $\underline{N}'$ on $Y' := Y \times_X U$ with an isomorphism $Y(\xi) \times_{Y'} (Y', \underline{N}') \xrightarrow{\sim} (Y(\xi), \underline{N})$. Moreover, let $h' : Y' \to U$ be the projection; after shrinking U, the map $\log h$ descends to a morphism of log structures $h'^*\underline{M}_{|U} \to \underline{N}'$, whence a morphism $(h', \log h') : (Y', \underline{N}') \to (U, \underline{M}_{|U})$ of log schemes, such that $h' \times_U \mathbf{1}_{X(\xi)} = h$. After further shrinking U, we may also assume that h' admits a Kummer chart $(\omega'_P, \omega'_Q, \vartheta)$ (corollary 6.2.34), that d is invertible in $\mathscr{O}_U$, and that the induced morphism $Y' \to U \times_{\mathrm{Spec}\,P} \mathrm{Spec}\,Q$ is an isomorphism ([32, Ch.IV, Cor.8.8.24]). Then h' is an étale covering, by proposition 7.3.12, and by construction, $F_{(U,\underline{M}_{|U})}(h') = g \times_X \mathbf{1}_U$, as desired. $\Diamond$

Claim 7.3.48. Assume that X is strictly local, denote by x the closed point of X, set $U := (X, \underline{M})_{\mathrm{tr}}$, and $P := \underline{M}^{\sharp}_{x}$. Let $h : V \to U$ be any connected étale covering, tamely ramified along $X \setminus U$; then h lies in the essential image of $F_{(X,\underline{M})}$, provided there exists a fine, sharp and saturated monoid Q, and a morphism $\nu : P \to Q$ of Kummer type, such that

$$h_\nu := h \times_X X_\nu : V \times_X X_\nu \to U \times_X X_\nu$$

admits a section (notation of (7.3.37)).

Proof of the claim. Suppose first that the order of $\mathrm{Coker}\,\nu^{\mathrm{gp}}$ is invertible in $\mathscr{O}_X$; in this case, f_ν is an étale covering of log schemes (proposition 7.3.12), hence $(f_\nu)_{\mathrm{tr}}$ is an étale covering of the scheme U. Then, by composing a section of h_ν with the projection $V \times_X X_\nu \to V$ we deduce a morphism $U \times_X X_\nu \to V$ of étale coverings of U. Such morphism shall be open and closed, hence surjective, since V is connected; hence the $\pi_1(U, \xi)$-set $h^{-1}(\xi)$ will be a quotient of $f_\nu^{-1}(\xi)$, on which $\pi_1(U, \xi)$ acts through (7.3.29), as required. Next, for a general morphism ν of Kummer type, let $L \subset Q^{\mathrm{gp}}$ be the largest subgroup such that $\nu^{\mathrm{gp}}(P^{\mathrm{gp}}) \subset L$, and $(L : P^{\mathrm{gp}})$ is invertible in $\mathscr{O}_X$. Set $Q' := L \cap Q$, and notice that $Q'^{\mathrm{gp}} = L$. Indeed, every element of L can be written in the form $x = b^{-1}a$, for some $a, b \in Q$; then choose $n > 0$ such that $b^n \in \nu P$, write $x = b^{-n} \cdot (b^{n-1}a)$ and remark that $b^{-n}, b^{n-1}a \in Q'$. The morphism ν factors as the composition of $\nu' : P \to Q'$ and $\psi : Q' \to Q$, and therefore f_ν factors through a morphism $f_\psi : X_\nu \to X_{\nu'}$. In view of the previous case, we are reduced to showing that the morphism $h_{\nu'}$ already admits a section, hence we may replace $(X, \underline{M})$ by $(X_{\nu'}, \underline{M}_{\nu'})$ (which is still regular and strictly local, by claim 7.3.36), h by $h_{\nu'}$, and ν by ψ, after which we may assume that the order of $\mathrm{Coker}\,\nu^{\mathrm{gp}}$ is p^m for some integer $m > 0$, where p is the characteristic of the residue field $\kappa(x)$, and then we need to show that h already admits a section.

Next, using again claim 7.3.36 and an easy induction, we may likewise reduce to the case where $m = 1$. Say that $X = \mathrm{Spec}\,A$, and suppose first that A is a $\mathbb{F}_p$-algebra (where $\mathbb{F}_p$ is the finite field with p elements), so that $X_\nu = \mathrm{Spec}\,A \otimes_{\mathbb{F}_p[P]} \mathbb{F}_p[Q]$; it is easily seen that the ring homomorphism $\mathbb{F}_p[\nu] : \mathbb{F}_p[P] \to \mathbb{F}_p[Q]$ is invertible up to Φ, in the sense of [36, Def.3.5.8(i)]. Especially, $\mathrm{Spec}\,\mathbb{F}_p[\nu]$ is integral, surjective and radicial, hence the same holds for f_ν, and therefore the morphism of sites :

$$f_\nu^* : X_{\mathrm{\acute{e}t}} \to X_{\nu,\mathrm{\acute{e}t}}$$

is an equivalence of categories (lemma 7.1.7(i)). It follows that in this case, h admits a section if and only if the same holds for h_ν.

Next, suppose that the field of fractions K of A has characteristic zero; we may write $X_\nu \times_X \mathrm{Spec}\,K = \mathrm{Spec}\,K_\nu$ and $V \times_X \mathrm{Spec}\,K = \mathrm{Spec}\,E$ for two field extensions K_ν and E of K, such that $[K_\nu : K] = p$, and the section of h_ν yields a map $E \to K_\nu$ of K-algebras. Therefore we have either $E = K$ (in which case $V = U$, and then we are done), or else $E = K_\nu$, in which case $V = (X_\nu, \underline{M}_\nu)_{\mathrm{tr}}$, since both these schemes are normal and finite over U.

Hence, let us assume that $V = (X_\nu, \underline{M}_\nu)$, and pick any point $\eta \in X$ of codimension one, whose residue field $\kappa(\eta)$ has characteristic p. Then $X_\nu \times_X X(\eta) = \mathrm{Spec}\,B$, where $B := \mathscr{O}_{X,\eta} \otimes_{\mathbb{Z}[P]} \mathbb{Z}[Q]$, and $\overline{B} := B \otimes_A \kappa(\eta) = \kappa(\eta) \otimes_{\mathbb{F}_p[P]} \mathbb{F}_p[Q]$. Since the map $\mathbb{F}_p[P] \to \mathbb{F}_p[Q]$ is invertible up to Φ, we easily deduce that $\overline{B}$ is local, and its residue field is a purely inseparable extension of $\kappa(\eta)$, say of degree d. Therefore $f_\nu^{-1}\eta$ consists of a single point η', and $\mathscr{O}_{X_\nu,\eta'} \simeq B$ is a normal and finite $\mathscr{O}_{X,\eta}$-algebra, hence it is a discrete valuation ring. Let e denote the ramification index of the extension $\mathscr{O}_{X,\eta} \to \mathscr{O}_{X_\nu,\eta'}$; then $ed = p$ ([14, Ch.VI, §8, n.5, Cor.1]). Suppose that $e = p$; in view of [36, Lemma 6.2.5], the ramification index of the induced extension of strict henselizations $\mathscr{O}^{\mathrm{sh}}_{X,\eta} \to \mathscr{O}^{\mathrm{sh}}_{X_\nu,\eta'}$ is still equal to p, which is impossible, since h is tamely ramified along $X \setminus U$. In case $e = 1$, we must have $d = p$, and since the residue field $\kappa(\eta)^{\mathrm{s}}$ of $\mathscr{O}^{\mathrm{sh}}_{X,\eta}$ is a separable closure of $\kappa(\eta)$, it follows easily that the residue field of $\mathscr{O}^{\mathrm{sh}}_{X_\nu,\eta'}$ must be a purely inseparable extension of $\kappa(\eta)^{\mathrm{s}}$ of degree p, which again contradicts the tameness of h. $\diamondsuit$

Claim 7.3.49. If $(X,\underline{M}) = (X,\underline{M})_2$, then $F_{(X,\underline{M})}$ is essentially surjective (notation of definition 6.2.7(i)).

Proof of the claim. By claim 7.3.47, we may assume that X is strictly henselian. In this case, let U, P and $h : V \to U$ be as in claim 7.3.48. By assumption $d := \dim P \leq 2$, and we have to show that $h \times_X X_\nu$ admits a section, for a suitable Kummer morphism $\nu : P \to Q$ (notation of (7.3.37)). If $d = 0$, then $P = \{1\}$, in which case $U = X$ is strictly local, so its fundamental group is trivial, and there is nothing to prove. In case $d = 1$, then $P \simeq \mathbb{N}$ (theorem 3.4.16(ii)), in which case X is a regular scheme (corollary 6.5.35), and $X\setminus U$ is a regular divisor (remark 6.5.31) and then the assertion follows from the classical Abhyankar's lemma ([42, Exp.XIII, Prop.5.2]). For $d = 2$, we may find $e_1, e_2 \in P$, and an integer $N > 0$ such that

$$\mathbb{N}e_1 \oplus \mathbb{N}e_2 \subset P \subset P' := \mathbb{N}\frac{e_1}{N} \oplus \mathbb{N}\frac{e_2}{N}$$

(see example 3.4.17(i)). Especially, the inclusion $\nu : P \to P'$ is a morphism of Kummer type. Since a composition of morphisms of Kummer type is obviously of Kummer type, claims 7.3.48 and 7.3.36 imply that we may replace $(X,\underline{M})$ by $(X_\nu,\underline{M}_\nu)$ and h by h_ν (notation of (7.3.37)), after which we may assume that P is isomorphic to $\mathbb{N}^{\oplus 2}$. In this case, X is again regular and $X\setminus U$ is a strictly normal crossing divisor, so the assertion follows again from [42, Exp.XIII, Prop.5.2]. $\diamondsuit$

Claim 7.3.50. In the situation of (7.3.1), suppose that $(Y,\underline{N})$ is regular. Then the functor f^*_{tr} : $\mathbf{Cov}(Y,\underline{N})_{\mathrm{tr}} \to \mathbf{Cov}(Y',\underline{N}')_{\mathrm{tr}}$ restricts to an equivalence :

$$\mathbf{Tame}(Y,(Y,\underline{N})_{\mathrm{tr}}) \stackrel{\sim}{\to} \mathbf{Tame}(Y',(Y',\underline{N}')_{\mathrm{tr}}). \tag{7.3.51}$$

Proof of the claim. Arguing as in the proof of proposition 7.3.2, we may reduce to the case where $\varphi : T' \to T$ is the saturated blow up of an ideal generated by two elements of $\Gamma(T,\mathscr{O}_T)$.

Notice that $(f,\log f)$ is an étale morphism (proposition 6.6.14(v)), and it restricts to an isomorphism $f_{\mathrm{tr}} : (Y',\underline{N}')_{\mathrm{tr}} \stackrel{\sim}{\to} (Y,\underline{N})_{\mathrm{tr}}$ (remark 6.6.17). Especially, $(Y',\underline{N}')$ is regular, and f^*_{tr} is trivially an equivalence from the étale coverings of $(Y,\underline{N})_{\mathrm{tr}}$ to those of $(Y',\underline{N}')_{\mathrm{tr}}$.

Let $g : V \to (Y,\underline{N})_{\mathrm{tr}}$ be an object of $\mathbf{Tame}(Y,(Y,\underline{N})_{\mathrm{tr}})$. By claim 7.3.49 and lemma 6.5.37, the functor $F_{\widetilde{u}^*(Y,\underline{N})_2}$ is essentially surjective, and then the same holds for $F_{(Y,\underline{N})_2}$, in view of claim 7.3.45; hence we may find an étale covering $(W,\underline{Q}) \to (Y,\underline{N})_2$, and an isomorphism $V \stackrel{\sim}{\to} (Y,\underline{N})_{\mathrm{tr}} \times_Y W$ of $(Y,\underline{N})_{\mathrm{tr}}$-schemes. On the other hand, example 3.5.56(ii) shows that φ restricts to a morphism $T'_1 \to T_2$ (notation of (3.5.16)), consequently f restricts to a morphism $(Y',\underline{N}')_1 \to (Y,\underline{N})_2$. There follows a well defined étale covering :

$$(Y',\underline{N}')_1 \times_{(Y,\underline{N})_2} (W,\underline{Q}) \to (Y',\underline{N}')_1 \times_{Y'} (Y',\underline{N}').$$

whose image under $F_{(Y',\underline{N}')}$ is $f^*_{\mathrm{tr}}(g)$. However, $(Y',\underline{N}')_1$ contains all the points of height one of $Y'\setminus(Y',\underline{N}')_{\mathrm{tr}}$ (corollary 6.5.36(i)), hence $f^*_{\mathrm{tr}}(g)$ is tamely ramified along $Y'\setminus(Y',\underline{N}')_{\mathrm{tr}}$ (see (7.3.40)). This shows that (7.3.51) is well defined, and clearly this functor is fully faithful, since the same holds for f^*_{tr}. $\diamondsuit$

Suppose again that $(X,\underline{M})$ is strictly henselian, define P, U and $h : V \to U$ as in claim 7.3.48, and set $T_P := (\mathrm{Spec}\, P)^\sharp$; it is easily seen that the counit of adjunction $\widetilde{u}^*\widetilde{u}_*(X,\underline{M}) \to (X,\underline{M})$ is an isomorphism (cp. (6.6.48)), hence $\widetilde{u}_*(X,\underline{M})$ is regular (lemma 6.5.37). Let

$$\underline{X} := (\widetilde{u}_*(X,\underline{M}), T_P, \pi_X)$$

be the object of $\mathscr{K}$ arising from $\widetilde{u}_*(X,\underline{M})$, as in (6.6.7); by theorem 6.6.32 (and its proof), we may find an integral proper simplicial subdivision $\varphi : F \to T_P$, such that the morphism of schemes underlying $(f,\varphi):\varphi^*\underline{X}\to\underline{X}$ is a resolution of singularities for X. Let k be the ramification index of the covering h; we define $(X_{(k)},\underline{M}_{(k)})$, $(X_\varphi,\underline{M}_\varphi)$, $(Y,\underline{N})$, g_X and $\boldsymbol{k}_X$ as in example 7.3.38 : this makes sense, since, in the current situation, $\underline{X}$ is isomorphic to $X\times_S\underline{S}$ (where $\underline{S}$ is defined as in example 6.6.5(i)). Moreover, by the same token, the morphism of schemes $Y\to X_{(k)}$ is a resolution of singularities.

Set as well $U_\varphi := (X_\varphi,\underline{M}_\varphi)_{\mathrm{tr}}$ and $U_{(k)} := (X_{(k)},\underline{M}_{(k)})_{\mathrm{tr}}$; by claim 7.3.50 and lemma 7.3.18(i), we have an essentially commutative diagram of functors :

$$\begin{tikzcd}
\mathbf{Tame}(X,U) \arrow[r, "f^*_{\mathrm{tr}}"] \arrow[d, "\boldsymbol{k}^*_X"'] & \mathbf{Tame}(X_\varphi,U_\varphi) \arrow[d, "g^*_X"] \\
\mathbf{Tame}(X_{(k)},U_{(k)}) \arrow[r] & \mathbf{Tame}(Y,(Y,\underline{N})_{\mathrm{tr}})
\end{tikzcd}$$

whose horizontal arrows are equivalences. In light of claim 7.3.48, we are reduced to showing that $\boldsymbol{k}^*_X(h)$ admits a section. Set $h_\varphi := f^*_{\mathrm{tr}}(h)$; then it suffices to show that $g^*_X(h_\varphi)$ admits a section, and notice that the ramification index of h_φ along $X_\varphi\backslash U_\varphi$ divides k (lemma 7.3.18(ii)).

Let $\overline{V}\to X_\varphi$ be the normalization of h_φ over X_φ, and w a geometric point of $\overline{V}\times_{X_\varphi}Y$ whose support has height one; denote by y (resp. v, resp. x) the image of w in Y (resp. in $\overline{V}$, resp. in X_φ), and suppose that the support of x lies in U_φ. Denote by $K^{\mathrm{sh}}(x)$ the field of fractions of the strict henselization $\mathscr{O}^{\mathrm{sh}}_{X,x}$ of $\mathscr{O}_{X,x}$, and define likewise $K^{\mathrm{sh}}(w)$, $K^{\mathrm{sh}}(y)$ and $K^{\mathrm{sh}}(v)$. There follow a commutative diagram of inclusions of valued fields :

$$\begin{tikzcd}
K^{\mathrm{sh}}(x) \arrow[r] \arrow[d] & K^{\mathrm{sh}}(y) \arrow[d] \\
K^{\mathrm{sh}}(v) \arrow[r] & K^{\mathrm{sh}}(w)
\end{tikzcd}$$

which is cocartesian in the category of fields (*i.e.* $K^{\mathrm{sh}}(w)$ is the compositum of $K^{\mathrm{sh}}(y)$ and $K^{\mathrm{sh}}(v)$: cp. the proof of lemma 7.3.18(i)). By the foregoing, the ramification index of $\mathscr{O}^{\mathrm{sh}}_{\overline{V},v}$ over $\mathscr{O}^{\mathrm{sh}}_{X_\varphi,x}$ divides k; on the other hand, example 7.3.38(ii) shows that the ramification index of $\mathscr{O}^{\mathrm{sh}}_{Y,y}$ over $\mathscr{O}^{\mathrm{sh}}_{X_\varphi,x}$ equals k. It then follows (*e.g.* from [36, Claim 6.2.15]) that $K^{\mathrm{sh}}(v)\subset K^{\mathrm{sh}}(y)$, and therefore $K^{\mathrm{sh}}(w)=K^{\mathrm{sh}}(y)$; this shows that the ramification index of $g^*_X(h_\varphi)$ along $Y\setminus(Y,\underline{N})_{\mathrm{tr}}$ equals 1, therefore $g^*_X(h_\varphi)$ is the restriction of an étale covering $\overline{h}$ of Y (lemma 7.3.18(iii)). By proposition 7.3.2(ii), $\overline{h}$ is the pull-back of an étale covering of $X_{(k)}$. Since $X_{(k)}$ is strictly local (claim 7.3.36), it follows that $\overline{h}$ admits a section, hence the same holds for $g^*_X(h_\varphi)$, as required. □

7.3.52. As an application of theorem 7.3.44, we shall determine the fundamental group of the “punctured” scheme obtained by removing the closed point from a strictly local regular log scheme $(X,\underline{M})$ of dimension ≥ 2. Indeed, let $x\in X$ be the closed point, and set $r:=\dim\underline{M}_x$. Also, let $\overline{y}$ be any geometric point of X, localized at a point $y\in U:=X\setminus\{x\}$, and ξ a geometric point of X localized at the maximal point. Let p (resp. p') denote the characteristic exponent of $\kappa(x)$ (resp. of $\kappa(y)$). Pick any lifting of ξ to a geometric point ξ_y of $X(\overline{y})$; since $(X(\overline{y}),\underline{M}(\overline{y}))$ is regular (theorem 6.5.46), according to proposition 7.3.42, the vertical arrows of the commutative diagram in (7.3.30) factor through the surjections (7.3.41), and we get a commutative diagram of group homomorphisms :

$$\begin{array}{ccc} \pi_1((X(\overline{y}),\underline{M}(\overline{y}))_{\text{ét}},\xi_y) & \xrightarrow{\varphi_y} & \pi_1((X,\underline{M})_{\text{ét}},\xi) \\ \downarrow & & \downarrow \\ \underline{M}^{\text{gp}\vee}_{\overline{y}}\otimes_{\mathbb{Z}}\prod_{\ell\neq p'}\mathbb{Z}_\ell(1) & \longrightarrow & \underline{M}^{\text{gp}\vee}_{x}\otimes_{\mathbb{Z}}\prod_{\ell\neq p}\mathbb{Z}_\ell(1) \end{array}$$

whose vertical arrows are the isomorphisms (7.3.43), and whose top (resp. bottom) horizontal arrow is induced by the natural morphism $X(\overline{y})\to X$ (resp. by the specialization map $\underline{M}_x\to\underline{M}_{\overline{y}}$). Now, by corollary 6.5.29, the scheme U is connected and normal, hence the restriction functor $\mathbf{Cov}(U)\to\mathbf{Tame}(X,(X,\underline{M})_{\text{tr}})$ is fully faithful (lemma 7.1.7(ii.b)); by [42, Exp.V, Prop.6.9] and theorem 7.3.44, it follows that the induced group homomorphism

$$\pi_1((X,\underline{M})_{\text{ét}},\xi)\to\pi_1(U_{\text{ét}},\xi) \tag{7.3.53}$$

is surjective (see (7.3.40)). Clearly, the image of φ_y lies in the kernel of (7.3.53). Especially :

$$(U,\underline{M}_{|U})_r\neq\varnothing\Rightarrow\pi_1(U_{\text{ét}},\xi)=\{1\}$$

since, for any geometric point $\overline{y}$ of $(X,\underline{M})_r$, the specialization map $\underline{M}_x\to\underline{M}_{\overline{y}}$ is an isomorphism (notation of definition 6.2.7(i)). Thus, suppose that $(X,\underline{M})_r=\{x\}$, set $P:=\underline{M}^\sharp_x$, and denote by $\psi:X\to\operatorname{Spec}P$ the natural continuous map; we have $\underline{M}^\sharp_{\overline{y}}=P_{\psi(y)}$, and if we let $F_y:=P\setminus\psi(y)$, we get a short exact sequence of free abelian groups of finite rank :

$$0\to F_y^{\text{gp}}\to\underline{M}^{\sharp\text{gp}}_x\to\underline{M}^{\sharp\text{gp}}_{\overline{y}}\to 0. \tag{7.3.54}$$

It is easily seen that ψ is surjective; hence, for any $\mathfrak{p}\in\operatorname{Spec}P$ of height $r-1$, pick $y\in\psi^{-1}(\mathfrak{p})$. With this choice, we have $F_y^{\text{gp}}=\mathbb{Z}$; considering $(7.3.54)^\vee$, we deduce that $\pi_1(U_{\text{ét}},\xi)$ is a quotient of $\widehat{\mathbb{Z}}'(1):=\prod_{\ell\neq p}\mathbb{Z}_\ell(1)$. More precisely, let $(\operatorname{Spec}P)_{r-1}$ be the set of prime ideals of P of height $r-1$; then we have :

Theorem 7.3.55. *If* $(X,\underline{M})_r=\{x\}$, *we have a natural identification :*

$$\frac{P^\vee}{\sum_{\mathfrak{p}\in(\operatorname{Spec}P)_{r-1}}P^\vee_{\mathfrak{p}}}\otimes_{\mathbb{Z}}\widehat{\mathbb{Z}}'(1)\xrightarrow{\sim}\pi_1(U_{\text{ét}},\xi) \tag{7.3.56}$$

and these two abelian groups are cyclic of finite order prime to p.

Proof. The foregoing discussion already yields a natural surjective map as stated; it remains only to check the injectivity, and to verify that the source is a cyclic finite group.

However, since $d\geq 2$, and $(X,\underline{M})_r=\{x\}$, corollary 6.5.36(i) implies that $r\geq 2$, in which case $(\operatorname{Spec}P)_{r-1}$ must contain at least two distinct elements, say $\mathfrak{p}$ and $\mathfrak{q}$. Let $F:=P\setminus\mathfrak{p}$; the image H of $(P_\mathfrak{q})^{\sharp\text{gp}\vee}$ in $F^{\text{gp}\vee}$ is clearly a non-trivial subgroup, hence $F^{\text{gp}\vee}/H$ is a cyclic finite group, and then the same holds for the source of (7.3.56).

Next, it follows easily from lemma 7.1.7(ii.b) that (7.3.53) identifies $\pi_1(U_{\text{ét}},\xi)$ with the quotient of $\pi_1((X,\underline{M})_{\text{ét}},\xi)$ by the sum of the images of the maps φ_y, for y ranging over all the points of U. However, it is clear that the same sum is already spanned by the sum of the images of the φ_y such that $y\in(X,\underline{M})_{r-1}$; whence the theorem. □

Example 7.3.57. Let $P\subset\mathbb{N}^{\oplus 2}$ be the submonoid of all pairs (a,b) such that $a+b\in 2\mathbb{N}$. Clearly P is fine and saturated of dimension 2. Let K be any field; the K-scheme $X:=\operatorname{Spec}K[P]$ is the singular quadric in $\mathbb{A}^3_K$ cut by the equation $XY-Z^2=0$. It is easily seen that $\operatorname{Spec}(K,P)_2$ consists of a single point x (the vertex of the cone); let $\overline{x}$ be a geometric point of X localized at x, and $U\subset X(\overline{x})$ the complement of the closed point. Then U is a normal K-scheme of dimension 1, and we have a natural isomorphism

$$\pi_1(U,\xi)\xrightarrow{\sim}\mathbb{Z}/2\mathbb{Z}.$$

Indeed, a simple inspection shows that P admits exactly two prime ideals of height one, namely $\mathfrak{p} := P \setminus (2\mathbb{N} \oplus \{0\})$ and $\mathfrak{q} := P \setminus (\{0\} \oplus 2\mathbb{N})$. Then, $P_\mathfrak{p} = \{(a,b) \in \mathbb{Z} \oplus \mathbb{N} \mid a+b \in 2\mathbb{N}\}$, and similarly $P_\mathfrak{q}$ is a submonoid of $\mathbb{N} \oplus \mathbb{Z}$. The quotients $P^\sharp_\mathfrak{p}$ and $P^\sharp_\mathfrak{q}$ are both isomorphic to $\mathbb{N}$, and are both generated by the class of $(1,1)$. Let $\varphi : P^\sharp_\mathfrak{p} \to \mathbb{Z}$ be a map of monoids; then the image of φ in $P^{\mathrm{gp}\vee}$ is the unique map of monoids $P \to \mathbb{Z}$ given by the rule : $(2,0) \mapsto 0$, $(1,1) \mapsto \varphi(1,1)$, $(0,2) \mapsto 2\varphi(1,1)$. Likewise, a map $\psi : P^\sharp_\mathfrak{q} \to \mathbb{Z}$ gets sent to the morphism $P \to \mathbb{Z}$ such that $(2,0) \mapsto 2\psi(1,1)$, $(1,1) \mapsto \psi(1,1)$, and $(0,2) \mapsto 0$. We see therefore that $(P_\mathfrak{p})^{\sharp\mathrm{gp}\vee} + (P_\mathfrak{q})^{\sharp\mathrm{gp}\vee}$ is a subgroup of index two in $P^{\mathrm{gp}\vee}$, and the contention follows from theorem 7.3.55.

7.4. **Local acyclicity of smooth morphisms of log schemes.** In this section we consider a smooth and saturated morphism $f : (X, \underline{M}) \to (Y, \underline{N})$ of fine log schemes. We fix a geometric point $\overline{x}$ of X, localized at a point x, and let $\overline{y} := f(\overline{x})$. We shall suppose as well that Y is strictly local and normal, that $(Y, \underline{N})_{\mathrm{tr}}$ is a dense open subset of Y, and that $\overline{y}$ is localized at the closed point y of Y. Let $\overline{\eta}$ be a strict geometric point of Y, localized at the generic point η; to ease notation, set :

$$U := f^{-1}_{\overline{x}}(\eta) \qquad U_{\mathrm{tr}} := (X(\overline{x}), \underline{M}(\overline{x}))_{\mathrm{tr}} \cap U \qquad \overline{U} := U \times_{|\eta|} |\overline{\eta}| \qquad \overline{U}_{\mathrm{tr}} := U_{\mathrm{tr}} \times_{|\eta|} |\overline{\eta}|$$

and notice that U_{tr} is a dense open subset of U, by virtue of proposition 6.7.17(ii,iv). Choose a geometric point ξ of $\overline{U}_{\mathrm{tr}}$, and let ξ' be the image of ξ in U_{tr}. There follows a short exact sequence of topological groups :

$$1 \to \pi_1(\overline{U}_{\mathrm{tr,ét}}, \xi) \to \pi_1(U_{\mathrm{tr,ét}}, \xi') \xrightarrow{\alpha} \pi_1([\eta|_{\mathrm{ét}}, \overline{\eta}) \to 1.$$

On the other hand, let p be the characteristic exponent of $\kappa := \kappa(\overline{y})$; for every integer e such that $(e,p) = 1$, the discussion of (7.3.26) yields a commutative diagram :

(7.4.1)
$$\begin{tikzcd}
\underline{N}_{\overline{y}} \arrow[r] \arrow[d, "\log f_{\overline{x}}"'] & \kappa(\eta)^\times \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathrm{cont}}(\pi_1(|\eta|_{\mathrm{ét}}, \overline{\eta}), \boldsymbol{\mu}_e(\kappa)) \arrow[d, "\mathrm{Hom}_{\mathrm{cont}}(\alpha, \boldsymbol{\mu}_e(\kappa))"] \\
\underline{M}_{\overline{x}} \arrow[r] & \Gamma(U_{\mathrm{tr}}, \mathscr{O}^\times_U) \arrow[r] & \mathrm{Hom}_{\mathrm{cont}}(\pi_1(U_{\mathrm{tr,ét}}, \xi'), \boldsymbol{\mu}_e(\kappa))
\end{tikzcd}$$

such that the composition of the two top (rep. bottom) horizontal arrows factors through $\underline{N}^\sharp_{\overline{y}}$ (resp. $\underline{M}^\sharp_{\overline{x}}$). There follows a system of natural group homomorphisms :

(7.4.2)
$$\mathrm{Coker}\,(\log f)^{\mathrm{gp}}_{\overline{x}} \to \mathrm{Hom}_{\mathrm{cont}}(\pi_1(\overline{U}_{\mathrm{tr,ét}}, \xi), \boldsymbol{\mu}_e(\kappa)) \qquad \text{where } (e,p) = 1$$

which assemble into a group homomorphism :

(7.4.3)
$$\pi_1(\overline{U}_{\mathrm{tr,ét}}, \xi) \to \mathrm{Coker}\,(\log f^{\mathrm{gp}}_{\overline{x}})^\vee \otimes_\mathbb{Z} \prod_{\ell \neq p} \mathbb{Z}_\ell(1).$$

7.4.4. We wish to give a second description of the map (7.4.3), analogous to the discussion in (7.3.31). To this aim, let R be a ring; for any monoid P, and any integer $e > 0$, denote by $\boldsymbol{e}_P : P \to P$ the e-Frobenius map of P. Let $\lambda : P \to Q$ be a local morphism of finitely generated monoids. For any integer $e > 0$, we get a commutative diagram of monoids :

(7.4.5)
$$\begin{tikzcd}
P \arrow[r, "\boldsymbol{e}_P"] \arrow[d, "\lambda"'] & P \arrow[d, "\lambda_e"] \arrow[drr, "\lambda"] & & \\
Q \arrow[r, "\mu'"] & Q' \arrow[rr, "\boldsymbol{e}_{Q|P}"] & & Q
\end{tikzcd}$$

whose square subdiagram is cocartesian, and such that $\boldsymbol{e}_{Q|P} \circ \mu' = \boldsymbol{e}_Q$. The latter induces a commutative diagram of log schemes :

$$\begin{tikzcd}
\mathrm{Spec}(R,Q) \arrow[r, "g_{Q|P}"] \arrow[dr, "g"'] & \mathrm{Spec}(R,Q') \arrow[r, "g'"] \arrow[d, "g_e"'] & \mathrm{Spec}(R,Q) \arrow[d, "g"] \\
 & \mathrm{Spec}(R,P) \arrow[r, "g_P"] & \mathrm{Spec}(R,P)
\end{tikzcd}$$

whose square subdiagram is cartesian, and such that $g_Q := g' \circ g_{Q|P}$ is the morphism induced by $\boldsymbol{e}_Q$. Also, by example 6.6.5, we have a commutative diagram of monoidal spaces :

$$\begin{tikzcd}
\mathrm{Spec}(R,Q) \arrow[r, "g_{Q|P}"] \arrow[d, "\psi_Q"'] & \mathrm{Spec}(R,Q') \arrow[r, "g_e"] \arrow[d, "\psi_{Q'}"] & \mathrm{Spec}(R,P) \arrow[d, "\psi_P"] \\
T_Q \arrow[r, "\varphi"] & T_{Q'} \arrow[r] & T_P
\end{tikzcd}$$

(where $\varphi := (\mathrm{Spec}\, \boldsymbol{e}_{Q|P})^\sharp$) which determines morphisms in the category $\mathscr{K}$:

$$(\mathrm{Spec}(R,Q), T_Q, \psi_Q) \to (\mathrm{Spec}(R,Q'), T_{Q'}, \psi_{Q'}) \to (\mathrm{Spec}(R,P), T_P, \psi_P).$$

7.4.6. Now, suppose that the closed point $\mathfrak{m}_Q$ of T_Q lies in the strict locus of $(\mathrm{Spec}\,\lambda)^\sharp$ (which just means that $\lambda^\sharp$ is an isomorphism). Notice that the functor $M \mapsto M^\sharp$ commutes with colimits (since it is a left adjoint); taking into account lemma 3.1.12, we deduce that the square subdiagram of (7.4.5)$^\sharp$ is still cocartesian, and therefore $\boldsymbol{e}^\sharp_{Q|P}$ is an isomorphism, so the same holds for φ.

More generally, let $\mathfrak{q} \in \mathrm{Spec}\, Q$ be any prime ideal in the strict locus of $(\mathrm{Spec}\,\lambda)^\sharp$; set $\mathfrak{p} := \lambda^{-1}\mathfrak{q}$, $\mathfrak{r} := \varphi(\mathfrak{q})$, and recall that $T_{Q_\mathfrak{q}} := \mathrm{Spec}\, Q_\mathfrak{q}$ is naturally an open subset of T_Q (see (3.5.16)). A simple inspection shows that the restriction $T_{Q_\mathfrak{q}} \to T_{Q'_\mathfrak{r}}$ of φ is naturally identified with the morphism of affine fans $(\mathrm{Spec}\, \boldsymbol{e}_{Q_\mathfrak{q}|P_\mathfrak{p}})^\sharp$. Since $\mathfrak{q}$ is the closed point of $T_{Q_\mathfrak{q}}$, the foregoing shows that $T_{Q_\mathfrak{q}}$ lies in the strict locus of φ; in other words, $\mathrm{Str}(\varphi)$ is an open subset of T_Q, and we have

$$\mathrm{Str}((\mathrm{Spec}\,\lambda)^\sharp) \subset \mathrm{Str}(\varphi). \tag{7.4.7}$$

Moreover, recall that $\mathrm{Spec}\, \boldsymbol{e}_Q : T_Q \to T_Q$ is the identity on the underlying topological space (see example 3.5.10(i)), and by construction it factors through φ, so the latter is injective on the underlying topological spaces. Especially, $\mathrm{Str}(\varphi) = \varphi^{-1}\varphi(\mathrm{Str}(\varphi))$, and therefore

$$\mathrm{Str}(g_{Q|P}) = \psi_Q^{-1}(\mathrm{Str}(\varphi)) = g_{Q|P}^{-1}(\psi_{Q'}^{-1}\varphi(\mathrm{Str}(\varphi))). \tag{7.4.8}$$

Hence, set $g := \mathrm{Spec}(R,\lambda)$; from (7.4.7) and (6.6.3) we obtain :

$$\mathrm{Str}(g) \subset \mathrm{Str}(g_{Q|P})$$

and together with (7.4.8) we deduce that :

- $\mathrm{Str}(g_{Q|P})$ is an open subset of $\mathrm{Spec}\, R[Q]$.
- $\psi_{Q'}^{-1}\varphi(\mathrm{Str}(\varphi))$ is a locally closed subscheme of $\mathrm{Spec}\, R[Q']$.
- The restriction $\mathrm{Str}(g) \to \psi_{Q'}^{-1}\varphi(\mathrm{Str}(\varphi))$ of $g_{Q|P}$ is a finite morphism.

Lastly, suppose that e is invertible in R; in this case, the morphisms g_P and g_Q are étale (proposition 6.3.34), so the same holds for $g_{Q|P}$ (corollary 6.3.25(iii)). From corollary 6.3.27(i) we deduce that the restriction $\mathrm{Str}(g) \to \psi_{Q'}^{-1}\varphi(\mathrm{Str}(\varphi))$ of $g_{Q|P}$ is a (finite) étale covering.

7.4.9. In the situation of (7.4.4), suppose additionally, that R is a $\mathbb{Z}[1/e, \boldsymbol{\mu}_e]$-algebra (where $\boldsymbol{\mu}_e$ is the e-torsion subgroup of $\mathbb{C}^\times$), P and Q are fine monoids, and λ is integral, so that Q' is also fine. Set

$$G_P := \mathrm{Hom}_{\mathbb{Z}}(P^{\mathrm{gp}}, \boldsymbol{\mu}_e) \qquad G_Q := \mathrm{Hom}_{\mathbb{Z}}(Q^{\mathrm{gp}}, \boldsymbol{\mu}_e) \qquad G_{Q|P} := \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Coker}\,\lambda^{\mathrm{gp}}, \boldsymbol{\mu}_e).$$

Notice that the trivial locus $\mathrm{Spec}(R,P)_{\mathrm{tr}}$ is the open subset $\mathrm{Spec}\,R[P^{\mathrm{gp}}]$, and likewise for $\mathrm{Spec}(R,Q)_{\mathrm{tr}}$ and $\mathrm{Spec}(R,Q')_{\mathrm{tr}}$; therefore, (7.4.5)$^{\mathrm{gp}}$ induces a cartesian diagram of schemes

$$\begin{tikzcd}
\mathrm{Spec}(R,Q')_{\mathrm{tr}} \arrow[r, "g'_{\mathrm{tr}}"] \arrow[d, "g_{e,\mathrm{tr}}"'] & \mathrm{Spec}(R,Q)_{\mathrm{tr}} \arrow[d, "g_{\mathrm{tr}}"] \\
\mathrm{Spec}(R,P)_{\mathrm{tr}} \arrow[r, "g_{P,\mathrm{tr}}"] & \mathrm{Spec}(R,P)_{\mathrm{tr}}.
\end{tikzcd}$$

Fix a geometric point τ'_Q of $\mathrm{Spec}(R,Q')_{\mathrm{tr}}$, and let $\tau_Q := g'(\tau'_Q)$, $\tau_P := g(\tau_Q)$, $\tau'_P := g_e(\tau'_Q)$. It was shown in (7.3.31) that $g_P^{-1}(\tau_P)$ is a G_P-torsor, so $g_{P,\mathrm{tr}}$ is a Galois étale covering, corresponding to a continuous representation (7.3.32) into the same group. Hence $g'^{-1}(\tau_Q)$ is a G_P-torsor, and g'_{tr} is a Galois étale covering, whose corresponding representation of $\pi_1(\mathrm{Spec}(R,Q)_{\mathrm{tr,ét}}, \tau_Q)$ is obtained by composing (7.3.32) with the natural continuous group homomorphism

$$\pi_1(g_{\mathrm{tr}}, \tau_Q) : \pi_1(\mathrm{Spec}(R,Q)_{\mathrm{tr,ét}}, \tau_Q) \to \pi_1(\mathrm{Spec}(R,P)_{\mathrm{tr,ét}}, \tau_P).$$

By the same token, $g_Q^{-1}(\tau_Q)$ is a G_Q-torsor, so also $g_{Q,\mathrm{tr}}$ is a Galois étale covering, and a simple inspection shows that the surjection

$$g_Q^{-1}(\tau_Q) \to g'^{-1}(\tau_Q)$$

induced by $g_{Q|P}$ is G_Q-equivariant, for the G_Q-action on the target obtained from the map

$$\mathrm{Hom}_{\mathbb{Z}}(\lambda^{\mathrm{gp}}, \boldsymbol{\mu}_e) : G_Q \to G_P.$$

The situation is summarized by the commutative diagram of continuous group homomorphisms

$$\begin{tikzcd}
\pi_1(\mathrm{Spec}(R,Q')_{\mathrm{tr,ét}}, \tau'_Q) \arrow[r] \arrow[d, "\pi_1(g_{e,\mathrm{tr}}, \tau'_Q)"'] & \pi_1(\mathrm{Spec}(R,Q)_{\mathrm{tr,ét}}, \tau_Q) \arrow[r] \arrow[d, "\pi_1(g_{\mathrm{tr}}, \tau_Q)"'] & G_Q \arrow[d, "\mathrm{Hom}_{\mathbb{Z}}(\lambda^{\mathrm{gp}}, \boldsymbol{\mu}_e)"] \\
\pi_1(\mathrm{Spec}(R,P)_{\mathrm{tr,ét}}, \tau'_P) \arrow[r] & \pi_1(\mathrm{Spec}(R,P)_{\mathrm{tr,ét}}, \tau_P) \arrow[r] & G_P
\end{tikzcd}$$

whose horizontal right-most arrows are the maps (7.3.32). Consequently, $g_{Q|P}^{-1}(\tau'_Q)$ is a $G_{Q|P}$-torsor, and $g_{Q|P,\mathrm{tr}}$ is a Galois étale covering, classified by a continuous group homomorphism

$$\mathrm{Ker}\,\pi_1(g_{e,\mathrm{tr}}, \tau'_Q) \to \mathrm{Ker}\,\pi_1(g_{\mathrm{tr}}, \tau_Q) \to G_{Q|P}. \tag{7.4.10}$$

7.4.11. Let us return to the situation of (7.4), and assume additionally, that both $(X, \underline{M})$ and $(Y, \underline{N})$ are fs log schemes. Take $R := \mathscr{O}_{Y,\overline{y}}$ in (7.4.4); by corollary 6.3.42 and theorem 6.1.35(iii), we may assume that there exist

- a local and saturated morphism $\lambda : P \to Q$ of fine and saturated monoids, such that P is sharp, $Q^\times$ is a free abelian group of finite type, say of rank r, and $\mathrm{Ass}_{\mathbb{Z}}\,\mathrm{Coker}\,\lambda^{\mathrm{gp}}$ does not contain the characteristic exponent of $\kappa(\overline{y})$;
- a morphism of schemes $\pi : Y \to \mathrm{Spec}\,R[P]$, which is a section of the projection $\mathrm{Spec}\,R[P] \to Y$, such that

$$(Y, \underline{N}) = Y \times_{\mathrm{Spec}\,R[P]} \mathrm{Spec}(R,P) \qquad (X, \underline{M}) = Y \times_{\mathrm{Spec}\,R[P]} \mathrm{Spec}(R,Q)$$

  and f is obtained by base change from the morphism $g := \mathrm{Spec}(R, \lambda)$. Moreover, the induced chart $Q_X \to \underline{M}$ shall be local at the geometric point $\overline{x}$.

By claim 6.1.37, we may further assume that the projection $Q \to Q^\sharp$ admits a section $\sigma : Q^\sharp \to Q$, such that $\lambda(P)$ lies in the image of σ. In this case, g factors through the morphism $\mathrm{Spec}(R, P) \to \mathrm{Spec}(R, Q^\sharp)$ induced by $\lambda^\sharp$, and σ induces an isomorphism of log schemes :

$$\mathrm{Spec}(R, Q) = \mathbb{G}_{m,Y}^{\oplus r} \times_Y \mathrm{Spec}(R, Q^\sharp)$$

(where $\mathbb{G}_{m,Y}^{\oplus r}$ denotes the standard torus of rank r over Y). In this situation, X is smooth over $Y \times_{\mathrm{Spec}\, R[P]} \mathrm{Spec}\, R[Q^\sharp]$, and more precisely

$$(X, \underline{M}) = \mathbb{G}_{m,Y}^{\oplus r} \times_{\mathrm{Spec}\, R[P]} \mathrm{Spec}(R, Q^\sharp). \tag{7.4.12}$$

Summing up, after replacing Q by $Q^\sharp$, we may assume that Q is also sharp, and (7.4.12) holds with $\mathrm{Spec}(R, Q)$ instead of $\mathrm{Spec}(R, Q^\sharp)$. Moreover, the image of x in T_Q is the closed point $\mathfrak{m}_Q$.

Let $e > 0$ be an integer which is invertible in R; by inspecting the definition, it is easily seen that there exists a finite separable extension K_e of $\kappa(\eta) = \mathrm{Frac}(R)$, such that the normalization Y_e of Y in $\mathrm{Spec}\, K_e$ fits into a commutative diagram of schemes :

$$\begin{tikzcd} Y_e \arrow[r] \arrow[d, "\pi_e"'] & Y \arrow[d, "\pi"] \\ \mathrm{Spec}\, R[P] \arrow[r, "g_P"] & \mathrm{Spec}\, R[P] \end{tikzcd}$$

(whose top horizontal arrow is the obvious morphism); namely, π is defined by some morphism of monoids $\beta : P \to R$, and one takes for K_e any subfield of $\kappa(\overline{\eta})$ containing $\kappa(\eta)$ and the e-th roots of the elements of $\beta(P)$. Set $(Y_e, \underline{N}_e) := Y_e \times_{\mathrm{Spec}\, R[P]} \mathrm{Spec}(R, P)$, and define log schemes $(X_e, \underline{M}_e)$, $(X'_e, \underline{M}'_e)$ so that the two square subdiagrams of the diagram of log schemes

$$\begin{tikzcd} (X'_e, \underline{M}'_e) \arrow[r, "h_e"] \arrow[d] & (X_e, \underline{M}_e) \arrow[r] \arrow[d] & \mathbb{G}_{m,Y}^{\oplus r} \times_Y (Y_e, \underline{N}_e) \arrow[d, "\pi'_e"] \\ \mathrm{Spec}(R, Q) \arrow[r, "g_{Q|P}"] & \mathrm{Spec}(R, Q') \arrow[r, "g_e"] & \mathrm{Spec}(R, P) \end{tikzcd}$$

are cartesian (here π'_e is the composition of π_e and the projection $\mathbb{G}_{m,Y}^{\oplus r} \times_Y Y_e \to Y_e$), whence a commutative diagram :

$$\begin{tikzcd} (X'_e, \underline{M}'_e) \arrow[r, "h_e"] \arrow[dr, "f'_e"'] & (X_e, \underline{M}_e) \arrow[d, "f_e"] \\ & (Y_e, \underline{N}_e). \end{tikzcd} \tag{7.4.13}$$

Notice that both f_e and f'_e are smooth and saturated morphisms of fine log schemes. Also, by construction Y_e is strictly local, and $(Y_e, \underline{N}_e)_{\mathrm{tr}}$ is a dense subset of Y_e. In other words, f_e and f'_e are still of the type considered in (7.4). Moreover, h_e is étale, since the same holds for $g_{Q|P}$, and the discussion in (7.4.6) shows that the restriction

$$\mathrm{Str}(f'_e) \to X_e$$

is an étale morphism of schemes. Furthermore, the discussion in (7.4.9) shows that

$$h_{e,\mathrm{tr}} : (X'_e, \underline{M}'_e)_{\mathrm{tr}} \to (X_e, \underline{M}_e)_{\mathrm{tr}}$$

is a Galois étale covering.

7.4.14. More precisely, notice that $\lambda^\sharp = \log f^\sharp_{\overline{x}}$; combining with (7.4.10), we deduce a continuous group homomorphism :

$$\text{(7.4.15)} \qquad \mathrm{Ker}\,\pi_1(f_{e,\mathrm{tr}}, \xi'_e) \to \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Coker}\,(\log f)^{\mathrm{gp}}_{\overline{x}}, \boldsymbol{\mu}_e(\kappa))$$

where ξ'_e is the image in X_e of the geometric point ξ. The geometric point $\overline{y}$ lifts uniquely to a geometric point $\overline{y}_e$ of Y_e, localized at the closed point y_e, and the pair $(\overline{x}, \overline{y}_e)$ determines a unique geometric point $\overline{x}_e$ such that $f_e(\overline{x}_e) = \overline{y}_e$. Also, since the field extension $\kappa(y) \to \kappa(y_e)$ is purely inseparable, it is easily seen that the induced map $f_e^{-1}(y_e) \to f^{-1}(y)$ is a homeomorphism; there follows an isomorphism of $X(\overline{x})$-schemes ([33, Ch.IV, Prop.18.8.10]) :

$$X_e(\overline{x}_e) \xrightarrow{\sim} X(\overline{x}) \times_Y Y_e.$$

Let η_e be the generic point of Y_e; by construction, $\overline{\eta}$ lifts to a geometric point $\overline{\eta}_e$ of Y_e, localized at η_e, and we have continuous group homomorphisms :

$$\text{(7.4.16)} \qquad \pi_1(\overline{U}_{\mathrm{tr}}, \xi) \xrightarrow{\sim} \mathrm{Ker}\,(\pi_1(U_{e,\mathrm{tr}}, \xi'_e) \to \pi_1(\eta_e, |\overline{\eta}_e|)) \to \mathrm{Ker}\,\pi_1(f_{e,\mathrm{tr}}, \xi'_e).$$

The composition of (7.4.15) and (7.4.16) is a continuous group homomorphism

$$\text{(7.4.17)} \qquad \pi_1(\overline{U}_{\mathrm{tr}}, \xi) \to \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Coker}\,(\log f)^{\mathrm{gp}}_{\overline{x}}, \boldsymbol{\mu}_e(\kappa))$$

whence, finally, a pairing

$$\mathrm{Coker}\,(\log f)^{\mathrm{gp}}_{\overline{x}} \times \pi_1(\overline{U}_{\mathrm{tr,\acute{e}t}}, \xi) \to \boldsymbol{\mu}_e(\kappa).$$

We claim that this pairing agrees with the one deduced from (7.4.2). Indeed, by tracing back through the constructions, we see that (7.4.17) is the homomorphism arising from the Galois covering of $\overline{U}_{\mathrm{tr}}$, which is obtained from $g_{Q|P}$, after base change along the composition

$$\overline{U}_{\mathrm{tr}} \to X_e \to \mathrm{Spec}\,R[Q'].$$

On the other hand, the discussion of (7.3.31) shows that the homomorphism $\pi_1(U_{\mathrm{tr,\acute{e}t}}, \xi') \to G_Q$ arising from the bottom row of (7.4.1), classifies the Galois covering $C \to \overline{U}_{\mathrm{tr}}$ obtained from g_Q, by base change along the same map. By the same token, the top row of (7.4.1) corresponds to the G_P-Galois covering $C' \to |\eta|$ obtained by base change of g_P along the composition $|\eta| \to Y \to S_P$. The map $\log f_{\overline{x}}$ induces a morphism of schemes $C \to C' \times_{|\eta|} U_{\mathrm{tr}}$, and (7.4.2) corresponds to the $G_{Q|P}$-torsor obtained from a fibre of this morphism. Evidently, this torsor is isomorphic to $g^{-1}_{Q|P}(\tau'_Q)$, whence the contention.

7.4.18. In the situation of (7.4), recall that there is a natural bijection between the set of maximal points of $f^{-1}_{\overline{x}}(\overline{y})$, and the set Σ of maximal points of the closed fibre of the induced map

$$\text{(7.4.19)} \qquad \mathrm{Spec}\,\underline{M}_{\overline{x}} \to \mathrm{Spec}\,\underline{N}_{\overline{y}}$$

(proposition 6.7.14). For every $\mathfrak{q} \in \Sigma$, denote by $\eta_{\mathfrak{q}}$ the corresponding maximal point of $f^{-1}_{\overline{x}}(\overline{y})$, choose a geometric point $\overline{\eta}_{\mathfrak{q}}$ localized at $\eta_{\mathfrak{q}}$, and let $X(\overline{\eta}_{\mathfrak{q}})$ be the strict henselization of $X(\overline{x})$ at $\overline{\eta}_{\mathfrak{q}}$. Also, set

$$U_{\mathfrak{q}} := U \times_{X(\overline{x})} X(\overline{\eta}_{\mathfrak{q}}) \qquad \overline{U}_{\mathfrak{q}} := U_{\mathfrak{q}} \times_{|\eta|} |\overline{\eta}|$$

and notice that $\overline{U}_{\mathfrak{q}}$ is an irreducible normal scheme. Notice as well that f induces a strict morphism $(X(\overline{\eta}_{\mathfrak{q}}), \underline{M}(\overline{\eta}_{\mathfrak{q}})) \to (Y, \underline{N})$ (theorem 6.7.8(iii.a)), and therefore the log structure of $U_{\mathfrak{q}} \times_{X(\overline{\eta}_{\mathfrak{q}})} (X(\overline{\eta}_{\mathfrak{q}}), \underline{M}(\overline{\eta}_{\mathfrak{q}}))$ is trivial.

Recall that $Z := \overline{U} \setminus \overline{U}_{\mathrm{tr}}$ is a finite union of irreducible closed subsets of codimension one in $\overline{U}$, and $\mathscr{O}_{\overline{U},z}$ is a discrete valuation ring, for each maximal point $z \in Z$ (proposition 6.7.17(iv,v)); especially, the category $\mathbf{Tame}(\overline{U}, \overline{U}_{\mathrm{tr}})$ is well defined (definition 7.3.17(iii)). We denote

$$\mathbf{Tame}(f, \overline{x})$$

the full subcategory of $\mathbf{Tame}(\overline{U}, \overline{U}_{\mathrm{tr}})$ consisting of all the coverings $C \to \overline{U}_{\mathrm{tr}}$ such that, for every $\mathfrak{q} \in \Sigma$, the induced covering

$$C \times_{\overline{U}_{\mathrm{tr}}} \overline{U}_{\mathfrak{q}} \to \overline{U}_{\mathfrak{q}}$$

is trivial (*i.e.* $C \times_{\overline{U}_{\mathrm{tr}}} \overline{U}_{\mathfrak{q}}$ is a disjoint union of copies of $\overline{U}_{\mathfrak{q}}$). It is easily seen that $\mathbf{Tame}(f, \overline{x})$ is a Galois category (see [42, Exp.V, Déf.5.1]), and we obtain a fibre functor for this category, by restriction of the usual fibre functor $\varphi \mapsto \varphi^{-1}(\xi)$ defined on all étale coverings φ of $\overline{U}_{\mathrm{tr}}$; we denote by $\pi_1(\overline{U}_{\mathrm{tr}}/Y_{\mathrm{\acute{e}t}}, \xi)$ the corresponding fundamental group. According to [42, Exp.V, Prop.6.9], the fully faithful inclusion $\mathbf{Tame}(f, \overline{x}) \to \mathbf{Cov}(\overline{U}_{\mathrm{tr}})$ induces a continuous surjective group homomorphism

$$\pi_1(\overline{U}_{\mathrm{tr,\acute{e}t}}, \xi) \to \pi_1(\overline{U}_{\mathrm{tr}}/Y_{\mathrm{\acute{e}t}}, \xi). \tag{7.4.20}$$

Proposition 7.4.21. *The map* (7.4.3) *factors through* (7.4.20)*, and the induced group homomorphism :*

$$\pi_1(\overline{U}_{\mathrm{tr}}/Y_{\mathrm{\acute{e}t}}, \xi) \to \mathrm{Coker}\,(\log f^{\mathrm{gp}}_{\overline{x}})^\vee \otimes_{\mathbb{Z}} \prod_{\ell \neq p} \mathbb{Z}_\ell(1). \tag{7.4.22}$$

is surjective.

Proof. Let $\sigma_Y : Y \to Y^{\mathrm{qfs}}$ be the natural morphism of schemes exhibited in remark 6.2.36(iv), and set

$$(Y, \underline{N}') := Y \times_{Y^{\mathrm{qfs}}} (Y, \underline{N})^{\mathrm{qfs}} \qquad (X, \underline{M}') := (Y, \underline{N}') \times_{(Y,\underline{N})} (X, \underline{M}).$$

Since f is saturated, both $(X, \underline{M}')$ and $(Y, \underline{N}')$ are fs log schemes (see remark 6.2.36(i)); also, the morphism of schemes underlying the induced morphism of log schemes $f' : (X, \underline{M}') \to (Y, \underline{N}')$, agrees with that underlying f. Moreover, by construction we have $\underline{N}'^{\sharp}_{\overline{z}} = (\underline{N}^{\sharp}_{\overline{z}})^{\mathrm{sat}}$ for every geometric point $\overline{z}$ of Y, especially $(Y, \underline{N}')_{\mathrm{tr}} = (Y, \underline{N})_{\mathrm{tr}}$. Likewise, $\underline{M}'^{\sharp}_{\overline{z}} = (\underline{M}^{\sharp}_{\overline{z}})^{\mathrm{sat}}$ (lemma 3.2.12(iii,iv)), therefore $\mathrm{Str}(f') = \mathrm{Str}(f)$, and especially, $(X, \underline{M}')_{\mathrm{tr}} = (X, \underline{M})_{\mathrm{tr}}$. Furthermore, notice that the the natural map

$$\mathrm{Coker}\,(\log f)^{\mathrm{gp}}_{\overline{x}} \to \mathrm{Coker}\,(\log f')^{\mathrm{gp}}_{\overline{x}} \tag{7.4.23}$$

is surjective, and its kernel is a quotient of $(\underline{M}^{\mathrm{sat}}_{\overline{x}})^\times / \underline{M}^\times_{\overline{x}}$, especially it is a torsion subgroup. However, the cokernel of $(\log f)^{\mathrm{gp}}_{\overline{x}}$ equals the cokernel of $(\log f^{\sharp}_{\overline{x}})^{\mathrm{gp}}$, hence it is torsion-free (corollary 3.2.32(ii)), so (7.4.23) is an isomorphism. Thus, we may replace $\underline{N}$ (resp. $\underline{M}$) by $\underline{N}'$ (resp. $\underline{M}'$), and assume from start that f is a smooth, saturated morphism of fs log schemes.

In this case, in light of the discussion of (7.4.14), it suffices to prove that (7.4.17) is a surjection, and that it factors through $\pi_1(\overline{U}_{\mathrm{tr}}/Y_{\mathrm{\acute{e}t}}, \xi)$. To prove the surjectivity comes down to showing that $\overline{U}_{\mathrm{tr}} \times_{X_e} X'_e$ is a connected scheme. However, let x_e be the support of $\overline{x}_e$, and notice that $\psi_P \circ \pi_e(y_e) = \mathfrak{m}_P$, the closed point of T_P. Since x maps to the closed point of T_Q, we deduce easily that the image of x_e in $T_{Q'}$ is the closed point $\mathfrak{m}_{Q'}$, *i.e.* x_e lies in the closed subscheme $X_e \times_{S'_Q} \mathrm{Spec}\,\kappa\langle Q'/\mathfrak{m}_{Q'}\rangle$. Notice as well that $\boldsymbol{e}_{Q|P}$ is of Kummer type (see definition 3.4.40); by proposition 7.3.8(ii), it follows that there exists a unique geometric point $\overline{x}'_e$ of X'_e lying over $\overline{x}_e$, whence an isomorphism of $X_e(\overline{x}_e)$-schemes ([33, Ch.IV, Prop.18.8.10])

$$X'_e(\overline{x}'_e) \xrightarrow{\sim} X'_e \times_{X_e} X_e(\overline{x}_e).$$

Hence $\overline{U}_{\mathrm{tr}} \times_{X_e} X'_e$ is an open subset of $|\overline{\eta}_e| \times_{Y_e} X'_e(\overline{x}_e)$, and the latter is an irreducible scheme, by proposition 6.7.17(ii). We also deduce that the induced morphism

$$h_{e,\overline{x}} : (X'_e(\overline{x}'_e), \underline{M}'_e(\overline{x}'_e)) \to (X_e(\overline{x}_e), \underline{M}_e(\overline{x}_e))$$

is a finite étale covering of log schemes. From the discussion in (7.4.11), we see that the restriction of $h_{e,\overline{x}}$

$$\mathrm{Str}(h_{e,\overline{x}}) \to X_e(\overline{x}_e)$$

is an étale morphism, and $\mathrm{Str}(h_{e,\overline{x}})$ contains the strict locus of the induced morphism

$$f'_{e,\overline{x}} : (X'_e(\overline{x}'_e), \underline{M}'_e(\overline{x}'_e)) \to (Y_e, \underline{N}_e).$$

Since the field extension $\kappa(\overline{y}) \to \kappa(\overline{y}_e)$ is purely inseparable, $\overline{\eta}_{\mathfrak{q}}$ lifts uniquely to a geometric point $\eta_{e,\mathfrak{q}} \in X_e(\overline{x}_e)$, and as usual we deduce that the strict henselization $X_e(\overline{\eta}_{\mathfrak{q}})$ of $X_e(\overline{x}_e)$ at $\overline{\eta}_{e,\mathfrak{q}}$ is isomorphic, as an $X_e(\overline{x}_e)$-scheme, to $X_e(\overline{x}_e) \times_{X(\overline{x})} X(\overline{\eta}_{\mathfrak{q}})$. Moreover, if $\eta_{e,\mathfrak{q}}$ is the support of $\overline{\eta}_{e,\mathfrak{q}}$, a simple inspection shows that the fibre $h^{-1}_{e,\overline{x}}(\eta_{e,\mathfrak{q}})$ consists of maximal points of $f'^{-1}_{e,\overline{x}}(y_e)$. By theorem 6.7.8(iii.a), every point of $h^{-1}_{e,\overline{x}}(\eta_{e,\mathfrak{q}})$ lies in $\mathrm{Str}(f'_{e,\overline{x}})$, therefore the induced morphism $X'_e \times_{X_e} X_e(\overline{\eta}_{\mathfrak{q}}) \to X_e(\overline{\eta}_{\mathfrak{q}})$ is finite and étale. Taking into account (7.3.40), we conclude that the étale covering $X'_e \times_{X_e} \overline{U}_{\mathrm{tr}} \to \overline{U}_{\mathrm{tr}}$ is an object of $\mathbf{Tame}(f, \overline{x})$, whence the proposition. □

7.4.24. Say that $Y = \mathrm{Spec}\, R$; for every algebraic field extension K of $\kappa(\eta) = \mathrm{Frac}(R)$, let R_K be the normalization of R in K, set $|\eta_K| := \mathrm{Spec}\, K$ and

$$Y_K := \mathrm{Spec}\, R_K \quad (Y_K, \underline{N}_K) := Y_K \times_Y (Y, \underline{N}) \quad (X_K, \underline{M}_K) := Y_K \times_Y (X, \underline{M}).$$

Moreover, let $f_K : (X_K, \underline{M}_K) \to (Y_K, \underline{N}_K)$ be the induced morphism, and $\overline{y}_K$ any geometric point localized at the closed point y_K of the strictly local scheme Y_K; since the extension $\kappa(\overline{y}) \to \kappa(\overline{y}_K)$ is purely inseparable, there exists a unique geometric point $\overline{x}_K$ of X_K lifting $\overline{x}$, and we have an isomorphism of $(X(\overline{x}), \underline{M}(\overline{x}))$-schemes :

$$(X_K(\overline{x}_K), \underline{M}_K(\overline{x}_K)) \xrightarrow{\sim} X_K \times_X (X(\overline{x}), \underline{M}(\overline{x})) = Y_K \times_Y (X(\overline{x}), \underline{M}(\overline{x})).$$

Clearly the morphism f_K is again of the type considered in (7.4.18); especially, the maximal points of $f^{-1}_{K,\overline{x}_K}(\overline{y}_K)$ are in natural bijection with the elements of Σ, and it is natural to denote

$$U_K := U \times_Y |\eta_K| \qquad U_{K,\mathrm{tr}} := U_{\mathrm{tr}} \times_Y |\eta_K| \qquad U_{K,\mathfrak{q}} := U_{K,\mathrm{tr}} \times_{X(\overline{x})} X(\overline{\eta}_{\mathfrak{q}})$$

for every $\mathfrak{q} \in \Sigma$. Then, let $\overline{\eta}_{K,\mathfrak{q}}$ be the unique geometric point of $f^{-1}_{K,\overline{x}_K}(\overline{y}_K)$ lying over $\overline{\eta}_{\mathfrak{q}}$, and $\eta_{K,\mathfrak{q}}$ the support of $\overline{\eta}_{K,\mathfrak{q}}$; as usual, we have

$$X_K(\overline{\eta}_{K,\mathfrak{q}}) = X(\overline{\eta}_{\mathfrak{q}}) \times_Y Y_K \tag{7.4.25}$$

hence the above notation is consistent with the one introduced for the original morphism f. Furthermore, if z is any maximal point of $\overline{U}\backslash\overline{U}_{\mathrm{tr}}$, the image z_K of z in U_K is a maximal point of $U_K\backslash U_{K,\mathrm{tr}}$, and since the induced map $\mathscr{O}_{U_K,z_K} \to \mathscr{O}_{\overline{U},z}$ is faithfully flat, proposition 6.7.17(iv,v) easily implies that $\mathscr{O}_{U_K,z_K}$ is a discrete valuation ring. We may then denote

$$\mathbf{Tame}(f, \overline{x}, K)$$

the full subcategory of $\mathbf{Tame}(U_K, U_{K,\mathrm{tr}})$, consisting of those objects $C \to U_{K,\mathrm{tr}}$, such that the induced covering $C \times_{U_{K,\mathrm{tr}}} U_{K,\mathfrak{q}} \to U_{K,\mathfrak{q}}$ is trivial, for every $\mathfrak{q} \in \Sigma$. We have a natural functor

$$2\text{-}\operatorname*{colim}_K \mathbf{Tame}(f, \overline{x}, K) \to \mathbf{Tame}(f, \overline{x}) \tag{7.4.26}$$

where the 2-colimit ranges over the filtered family of all finite separable extensions K of $\kappa(\eta)$.

Lemma 7.4.27. *The functor* (7.4.26) *is an equivalence.*

Proof. Let $\overline{h} : \overline{C} \to \overline{U}_{\mathrm{tr}}$ be an object of the category $\mathbf{Tame}(f, \overline{x})$. According to [33, Ch.IV, Prop.17.7.8(ii)] and [32, Ch.IV, Prop.8.10.5], we may find a finite extension K of $\kappa(\eta)$, such that $\overline{h}$ descends to a finite étale morphism

$$h_K : C_K \to U_{K,\mathrm{tr}}.$$

Let C'_K (resp. $\overline{C}'$) denote the normalization of C_K (resp. of $\overline{C}$) over U_K (resp. over $\overline{U}$). Since the morphism $|\overline{\eta}| \to |\eta_K|$ is ind-étale, we have $\overline{C}' = C'_K \times_{|\eta_K|} |\overline{\eta}|$ ([33, Ch.IV, Prop.17.5.7]), and it follows easily that C_K is tamely ramified along the divisor $U_K \setminus U_{K,\mathrm{tr}}$. From (7.4.25) we get a natural isomorphism :

$$\overline{U}_{\mathfrak{q}} \xrightarrow{\sim} U_{K,\mathfrak{q}} \times_{|\eta_K|} |\overline{\eta}|.$$

Thus, after replacing K by a larger finite separable extension of $\kappa(\eta)$, we may assume that the induced morphism $C_K \times_{U_{K,\mathrm{tr}}} U_{K,\mathfrak{q}} \to U_{K,\mathfrak{q}}$ is a trivial étale covering, for every $\mathfrak{q} \in \Sigma$.

This shows that (7.4.26) is essentially surjective; likewise one shows the full faithfulness : the details shall be left to the reader. □

7.4.28. In the situation of (7.4.24), let K be an algebraic extension of $\kappa(\eta)$, and

$$h : C \to U_{K,\mathrm{tr}}$$

any object of $\mathbf{Tame}(U_K, U_{K,\mathrm{tr}})$, and denote by C' the normalization of $X_K(\overline{x}_K)$ in C. We claim that there exists a largest non-empty open subset

$$E(h) \subset X_K(\overline{x}_K)$$

such that the restriction $h^{-1}E(h) \to E(h)$ of h is étale. Indeed, in any case, h restricts to an étale morphism on a dense open subset containing $U_{K,\mathrm{tr}}$, and there exists a largest open subset $E' \subset C'$ such that $h_{|E'}$ is étale (claim 7.1.8 and lemma 7.1.7(ii.b)). Then it is easily seen that $E(h) := X_K(\overline{x}_K) \setminus h(C' \setminus E')$ will do.

Lemma 7.4.29. *With the notation of* (7.4.28)*, the category* $\mathbf{Tame}(f, \overline{x}, K)$ *is the full subcategory of* $\mathbf{Tame}(U_K, U_{K,\mathrm{tr}})$ *consisting of those objects* $h : C \to U_{K,\mathrm{tr}}$ *such that* $E(h)$ *contains the maximal points of* $X_K(\overline{x}_K) \times_{Y_K} |y_K|$.

Proof. In view of claim 7.1.9, this characterization is a rephrasing of the definition of the category $\mathbf{Tame}(f, \overline{x}, K)$. □

7.4.30. Keep the notation of (7.4.28), and suppose that h is an object of $\mathbf{Tame}(f, \overline{x}, K)$. Fix $\mathfrak{q} \in \Sigma$; then lemma 7.4.29 says that $\eta_{K,\mathfrak{q}} \in E(h)$. Thus, we obtain a functor

$$\mathbf{Tame}(f, \overline{x}, K) \to \mathbf{Cov}(|\eta_{K,\mathfrak{q}}|) \qquad : \qquad C \mapsto C' \times_{X_K(\overline{x}_K)} |\eta_{K,\mathfrak{q}}|.$$

However, the natural morphism $|\eta_{K,\mathfrak{q}}| \to |\eta_{\mathfrak{q}}|$ is radicial, hence it induces an equivalence

$$\mathbf{Cov}(|\eta_{\mathfrak{q}}|) \xrightarrow{\sim} \mathbf{Cov}(|\eta_{K,\mathfrak{q}}|)$$

(lemma 7.1.7(i)). Combining these two functors in the special special case where $K := \kappa(\overline{\eta})$, we get a functor

$$\mathbf{Tame}(f, \overline{x}) \to \mathbf{Cov}(|\eta_{\mathfrak{q}}|) \quad : \quad (C \to \overline{U}_{\mathrm{tr}}) \mapsto (C_{|\eta_{\mathfrak{q}}} \to |\eta_{\mathfrak{q}}|). \tag{7.4.31}$$

Now, the rule $\varphi \mapsto \varphi^{-1}(\overline{\eta}_{\mathfrak{q}})$ yields a fibre functor for the Galois category $\mathbf{Cov}(|\eta_{\mathfrak{q}}|)$; by composition with (7.4.31), we deduce a fibre functor for $\mathbf{Tame}(f, \overline{x})$, whose group of automorphisms we denote $\pi_1(\overline{U}_{\mathrm{tr}}/Y_{\text{ét}}, \overline{\eta}_{\mathfrak{q}})$. Also, set $F(\mathfrak{q}) := \underline{M}_{\overline{x}} \setminus \mathfrak{q}$, and notice that the structure map $\underline{M}_{\overline{x}} \to \mathscr{O}_{X(\overline{x}),\overline{x}}$ induces a group homomorphism

$$F(\mathfrak{q})^{\mathrm{gp}} \to \kappa(\eta_{\mathfrak{q}})^{\times}. \tag{7.4.32}$$

Lemma 7.4.33. *With the notation of* (7.4.30)*, we have :*

(i) *The natural map* Coker $(\log f_{\overline{x}}) \to F(\mathfrak{q})$ *induces a commutative diagram of groups*

$$\begin{tikzcd}
\pi_1(|\eta_{\mathfrak{q}}|_{\text{ét}}, \overline{\eta}_{\mathfrak{q}}) \arrow[r] \arrow[d, "\alpha"'] & \pi_1(\overline{U}_{\mathrm{tr}}/Y_{\text{ét}}, \overline{\eta}_{\mathfrak{q}}) \arrow[r] & \pi_1(\overline{U}_{\mathrm{tr}}/Y_{\text{ét}}, \xi) \arrow[d, "\beta"] \\
F(\mathfrak{q})^{\mathrm{gp}\vee} \otimes_{\mathbb{Z}} \prod_{\ell \neq p} \mathbb{Z}_\ell(1) \arrow[rr, "\gamma"] & & \mathrm{Coker}\,(\log f^{\mathrm{gp}}_{\overline{x}})^{\vee} \otimes_{\mathbb{Z}} \prod_{\ell \neq p} \mathbb{Z}_\ell(1)
\end{tikzcd}$$

where β *is* (7.4.22)*, and* α *is deduced from* (7.4.32)*, as in the discussion of* (7.3.26).

(ii) *α is surjective, and γ is an isomorphism.*

Proof. (i): The proof amounts to unwinding the definitions, and shall be left as an exercice for the reader. Notice that the second arrow on the top row is only well-defined up to inner automorphisms, but since the groups on the bottom row are abelian, the ambiguity does not affect the statement.

(ii): Notice that $\log f_{\overline{x}}$ restricts to a map of monoids $\mathscr{O}^\times_{Y,\overline{y}} \to F(\mathfrak{q})$, which induces an isomorphism $\mathrm{Coker}(\log f)^\sharp \xrightarrow{\sim} F(\mathfrak{q})^\sharp$; we deduce that γ is an isomorphism. Next, let Z be the topological closure of $\eta_\mathfrak{q}$ in $X(\overline{x})$, and endow Z with its reduced subscheme structure; set also $(Z, \underline{M}(Z)) := Z \times_{X(\overline{x})} (X, \underline{M})$. The map α factors as a composition

$$\pi_1(|\eta_\mathfrak{q}|_{\mathrm{ét}}, \overline{\eta}_\mathfrak{q}) \to \pi_1((Z, \underline{M}(Z))_{\mathrm{tr,ét}}, \overline{\eta}_\mathfrak{q}) \to F(\mathfrak{q})^{\mathrm{gp}\vee} \otimes_\mathbb{Z} \prod_{\ell \neq p} \mathbb{Z}_\ell(1)$$

where the first map is surjective, by lemma 7.1.7. Lastly, notice that $\underline{M}(Z)_{\mathrm{red},\overline{x}} = F(\mathfrak{q})_\circ$; by propositions 7.3.42 and 6.7.14(ii), it follows that the second map is surjective as well, so the proof of (ii) is complete. $\square$

7.4.34. In the situation of (6.3.44), suppose that Y_i is a strictly local normal scheme for every $i \in I$, and the transition morphisms $Y_j \to Y_i$ are local and dominant, for every morphism $i \to j$ in I. Let $\overline{x}$ be a geometric point of X, and denote by $\overline{x}_i$ the image of $\overline{x}$ in X_i, for every $i \in I$. Suppose that the image $\overline{y}$ of $\overline{x}$ in Y is localized at the closed point. Also, let $\overline{\eta}$ be a strict geometric point of Y, localized at the generic point η_i, and denote by $\overline{\eta}_i$ (resp. $\overline{y}_i$) the strict image of $\overline{\eta}$ (resp. $\overline{y}$) in Y_i (see definition 7.1.10(iii)).

Lemma 7.4.35. *In the situation of* (7.4.34), *suppose that* $(g, \log g) : (X, \underline{M}) \to (Y, \underline{N})$ *is a smooth and saturated morphism of fine log schemes. Then there exists* $i \in I$, *and a smooth and saturated morphism* $(g_i, \log g_i) : (X_i, \underline{M}_i) \to (Y_i, \underline{N}_i)$ *of fine log schemes, such that* $\log g = \pi_i^* \log g_i$.

Proof. By corollary 6.3.45, we can descend $(g, \log g)$ to a smooth morphism $(g_i, \log g_i)$ of fine log schemes, and after replacing I by I/i, we may assume that $i = 0$. Then the contention follows from corollary 6.2.34(ii). $\square$

7.4.36. Keep the situation of (7.4.34), and suppose that $(g_0, \log g_0) : (X_0, \underline{M}_0) \to (Y_0, \underline{N}_0)$ is a smooth and saturated morphism of fine log schemes; set

$$(X_i, \underline{M}_i) := X_i \times_{X_0} (X_0, \underline{M}_0) \qquad (Y_i, \underline{N}_i) := Y_i \times_{Y_0} (Y_0, \underline{Y}_0)$$

and denote $(g_i, \log g_i) : (X_i, \underline{M}_i) \to (Y_i, \underline{N}_i)$ the induced morphism of log schemes, for every $i \in I$. Also, let $(g, \log g) : (X, \underline{M}) \to (Y, \underline{N})$ be the limit of the system of morphisms $((g_i, \log g_i) \mid i \in I)$. These are morphisms of the type considered in (7.4), so we may define $U_i := g^{-1}_{i,\overline{x}_i}(\eta_i)$, and introduce likewise the schemes $U_{i,\mathrm{tr}}$, $\overline{U}_i$ and $\overline{U}_{i,\mathrm{tr}}$ as in (7.4). Moreover, set $Z_i := \overline{U}_i \setminus \overline{U}_{i,\mathrm{tr}}$ for every $i \in I$; clearly $Z_i = Z_j \times_{X_j(\overline{x}_j)} X_i(\overline{x}_i)$ for every morphism $i \to j$ in I. Also, each Z_i is a finite union of irreducible subsets of codimension one, and for every $i \to j$ in I, the transition morphisms $X_i \to X_j$ restrict to maps

$$\mathrm{Max}\, Z_i \to \mathrm{Max}\, Z_j \qquad \mathrm{Max}\, X_i(\overline{x}_i) \times_{Y_i} |\overline{y}_i| \to \mathrm{Max}\, X_j(\overline{x}_j) \times_{Y_j} |\overline{y}_j|$$

Combining proposition 7.3.22(ii) and lemma 7.4.29, we deduce a fully faithful functor

$$\text{(7.4.37)} \qquad 2\text{-}\operatorname*{colim}_{i \in I} \mathbf{Tame}(g_i, \overline{x}_i) \to \mathbf{Tame}(g, \overline{x}).$$

Lemma 7.4.38. *The functor* (7.4.37) *is an equivalence.*

Proof. It remains only to show the essential surjectivity. Hence, let h be a given object of $\mathbf{Tame}(g,\overline{x})$; by proposition 7.3.22(ii), we know that there exists $j \in I$ such that h descends to an étale covering $h_j : V_j \to \overline{U}_{j,\mathrm{tr}}$, tamely ramified along Z_j, and after replacing I by I/i, we may assume that j is the final object of I, and define $h_i := \overline{U}_{i,\mathrm{tr}} \times_{\overline{U}_{j,\mathrm{tr}}} h_j$ for every $i \to j$ in I. Now, let $E' \subset E(h)$ be a constructible open subset containing the maximal points of $X(\overline{x}) \times_Y |\overline{y}|$. For every $i \in I$, let $\overline{Y}_i$ be the normalization of Y_i in $\operatorname{Spec} \kappa(\overline{\eta}_i)$, and set $\overline{X}_i := X_i(\overline{x}_i) \times_{Y_i} \overline{Y}_i$; according to [32, Ch.IV, Th.8.3.11], there exists $i \in I$ such that E' descends to a constructible open subset $E'_i \subset \overline{X}_i$, and then necessarily E'_i contains all the maximal points of $X_i(\overline{x}_i) \times_{Y_i} |\overline{y}_i|$. As usual, we may assume that i is the final object, so E'_i is defined for every $i \in I$. Lastly, since h extends to an étale covering on E', we see that h_i extends to an étale covering of E'_k, for some $k \in I$ ([33, Ch.IV, Prop.17.7.8(i)]). In view of lemma 7.4.29, the contention follows. $\square$

Theorem 7.4.39. *The map* (7.4.22) *is an isomorphism.*

Proof. Arguing as in the proof of proposition 7.4.21, we may assume that both $(X,\underline{M})$ and $(Y,\underline{N})$ are fs log schemes, and in view of proposition 7.4.21, we need only show that (7.4.22) is injective. This comes down to the following assertion. For every object $\overline{h} : \overline{C} \to \overline{U}_{\mathrm{tr}}$ of the category $\mathbf{Tame}(f,\overline{x})$, the induced action of $\pi_1(\overline{U}_{\mathrm{tr}},\xi)$ on $\overline{h}^{-1}(\xi)$ factors through the quotient $\mathrm{Coker}\,(\log f^{\mathrm{gp}}_{\overline{x}})^\vee \otimes_{\mathbb{Z}} \prod_{\ell\neq p} \mathbb{Z}_\ell(1)$.

• By lemma 7.4.27, there exists a finite separable extension K of $\kappa(\eta)$, and an object $h : C_K \to U_{K,\mathrm{tr}}$ of $\mathbf{Tame}(f,\overline{x},K)$, with an isomorphism $C_K \times_{U_{K,\mathrm{tr}}} \overline{U}_{\mathrm{tr}} \stackrel{\sim}{\to} \overline{C}$ of $\overline{U}_{\mathrm{tr}}$-schemes. Since $\log f_{\overline{x}} = \log f_{K,\overline{x}_K}$, the theorem will hold for the morphism f and the point $\overline{x}$, if and only if it holds for f_K and the point $\overline{x}_K$.

Claim 7.4.40. The theorem holds, if Y is noetherian of dimension one.

Proof of the claim. In this case, Y is the spectrum of a strictly henselian discrete valuation ring R, and the same then holds for Y_K. Hence, we may replace throughout f by f_K, and assume from start that there exists an object $h : C \to U_{\mathrm{tr}}$ of $\mathbf{Tame}(f,\overline{x},\kappa(\eta))$, with an isomorphism $\overline{C} \stackrel{\sim}{\to} C \times_{U_{\mathrm{tr}}} \overline{U}_{\mathrm{tr}}$ of $\overline{U}_{\mathrm{tr}}$-schemes. Endow Y with the fine log structure $\underline{N}'$ such that $\Gamma(Y,\underline{N}') = R\setminus\{0\}$; since $(Y,\underline{N})_{\mathrm{tr}}$ is dense in Y, we have a well defined morphism of log schemes $\pi : (Y,\underline{N}') \to (Y,\underline{N})$, which is the identity on the underlying schemes. Set $(X,\underline{M}') := (Y,\underline{N}') \times_{(Y,\underline{N})} (X,\underline{M})$. Then $(Y,\underline{N}')$ is a regular log scheme, and consequently the same holds for $(X,\underline{M}')$, by theorem 6.5.44.

Furthermore, π trivially restricts to a strict morphism on the open subset $|\eta|$, hence the induced morphism $(X,\underline{M}') \times_Y |\eta| \to (X,\underline{M}) \times_Y |\eta|$ is an isomorphism, especially U_{tr} is the trivial locus of $(X(\overline{x}),\underline{M}'(\overline{x})) \times_Y |\eta|$. However, it is easily seen that $(X(\overline{x}),\underline{M}'(\overline{x}))_{\mathrm{tr}}$ does not intersect the closed fibre $f^{-1}_{\overline{x}}(\overline{y})$, so finally $U_{\mathrm{tr}} = (X(\overline{x}),\underline{M}'(\overline{x}))_{\mathrm{tr}}$.

From theorem 7.3.44, we deduce that h extends to an étale covering of $(X(\overline{x}),\underline{M}'(\overline{x}))$. Then, arguing as in (7.4) we get a commutative diagram of groups :

$$\begin{tikzcd}
\pi_1(\overline{U}_{\mathrm{tr,\acute{e}t}},\xi) \arrow[r] \arrow[d] & \mathrm{Coker}\,(\log f^{\mathrm{gp}}_{\overline{x}})^\vee \otimes_{\mathbb{Z}} \prod_{\ell\neq p} \mathbb{Z}_\ell(1) \arrow[d] \\
\pi_1(U_{\mathrm{tr,\acute{e}t}},\xi') \arrow[r, "\alpha"] & \underline{M}^{\mathrm{gp}\vee}_{\overline{x}} \otimes_{\mathbb{Z}} \prod_{\ell\neq p} \mathbb{Z}_\ell(1)
\end{tikzcd}$$

whose top horizontal arrow is (7.4.3), and whose right vertical arrow is deduced from the projection $\underline{M}^{\mathrm{gp}}_{\overline{x}} \to \mathrm{Coker}\,(\log f^{\mathrm{gp}}_{\overline{x}})$. Lastly, proposition 7.3.42 shows that the natural action of $\pi_1(U_{\mathrm{tr,\acute{e}t}},\xi')$ on $h^{-1}(\xi')$ factors through α, so the claim follows. $\Diamond$

• Next, suppose that Y is an arbitrary normal, strictly local scheme. Ths discussion in (7.4.14) implies that, in order to prove the theorem, it suffices to find an integer $e > 0$, a

$(X,\underline{M})$-scheme $(X'_e,\underline{M}'_e)$ as in (7.4.11), and a geometric point $\overline{x}'_e$ of X'_e lying over $\overline{x}$, such that $h\times_{X(\overline{x})}X'_e(\overline{x}'_e)$ is a trival covering. To this aim, we write Y as the limit of a cofiltered system $(Y_i \mid i\in I)$ of strictly local excellent and normal schemes (lemma 7.1.30), and we denote by η_i the generic point of Y_i, for every $i\in I$. By lemma 7.4.35, we may then descend $(f,\log f)$ to a smooth and saturated morphism $(f_i,\log f_i):(X_i,\underline{M}_i)\to(Y_i,\underline{N}_i)$, for some $i\in I$, and as usual, we may assume that i is the final object of I. Let $\overline{x}_i$ be the image of $\overline{x}$ in X_i; by lemmata 7.4.38 and 7.4.27, the object h of $\mathbf{Tame}(f,\overline{x},\kappa(\eta))$ descends to an object h_i of $\mathbf{Tame}(f_i,\overline{x}_i,K)$, for some $i\in I$, and some finite separable extension K of $\kappa(\eta_i)$. It suffices therefore to find $e>0$, a $(X_i,\underline{M}_i)$-scheme $(X'_{i,e},\underline{M}'_{i,e})$, and a geometric point $\overline{x}'_{i,e}$ of $X'_{i,e}$ lying over $\overline{x}_i$, such that $h_i\times_{X_i(\overline{x}_i)}X'_{i,e}(\overline{x}'_{i,e})$ is a trivial covering. In other words, we may replace throughout Y by $Y_{i,K}$, and assume from start that Y is excellent, and $\overline{h}$ descends to an object $h:C\to U_{\mathrm{tr}}$ of $\mathbf{Tame}(f,\overline{x},\kappa(\eta))$.

• By [27, Ch.II, Prop.7.1.7] we may find a discrete valuation ring V and a local injective morphism $R\to\mathsf{V}$ inducing an isomorphism on the respective fields of fractions. Let V^{sh} be the strict henselization of V (at a geometric point whose support is the closed point), and set

$$\mathsf{Y}:=\operatorname{Spec}\mathsf{V}^{\mathrm{sh}}\quad(\mathsf{Y},\underline{\mathsf{N}}):=\mathsf{Y}\times_Y(Y,\underline{N})\quad(\mathsf{X},\underline{\mathsf{M}}):=\mathsf{Y}\times_Y(X,\underline{M}).$$

Also, let $\mathsf{f}:(\mathsf{X},\underline{\mathsf{M}})\to(\mathsf{Y},\underline{\mathsf{N}})$ be the induced morphism. Denote by $\overline{\mathsf{y}}$ a geometric point localized at the closed point y of Y; also, pick any geometric point $\overline{\mathsf{x}}$ of X, whose image in X is $\overline{x}$; the induced morphism

$$(\mathsf{X}(\overline{\mathsf{x}}),\underline{\mathsf{M}}(\overline{\mathsf{x}}))\to(X(\overline{x}),\underline{M}(\overline{x}))\tag{7.4.41}$$

restricts to a flat morphism $\mathsf{f}^{-1}_{\overline{\mathsf{x}}}(\overline{\mathsf{y}})\to f^{-1}_{\overline{x}}(\overline{y})$ and from proposition 6.7.14, we see that the latter induces a bijection between the sets of maximal points of the two fibres. On the other hand, let $\overline{\eta}_{\mathsf{V}}$ denote a geometric point localized at the generic point η_{V} of Y; then (7.4.41) restricts to an ind-étale morphism $\mathsf{f}^{-1}_{\overline{\mathsf{x}}}(\eta_{\mathsf{V}})\to f^{-1}_{\overline{x}}(\eta)$. Hence, set

$$\mathsf{U}_{\mathrm{tr}}:=(\mathsf{X}(\overline{\mathsf{x}}),\underline{\mathsf{M}}(\overline{\mathsf{x}}))_{\mathrm{tr}}\times_{\mathsf{Y}}|\eta_{\mathsf{V}}|.$$

From lemma 7.4.29, it follows easily that the covering $C\times_{U_{\mathrm{tr}}}\mathsf{U}_{\mathrm{tr}}\to\mathsf{U}_{\mathrm{tr}}$ is an object of $\mathbf{Tame}(\mathsf{f},\overline{\mathsf{x}},\kappa(\eta_{\mathsf{V}}))$.

For any integer $e>0$ invertible in R, pick a $(Y,\underline{N})$-scheme $(Y_e,\underline{N}_e)$ as in (7.4.11), so that we may define the étale morphism $(X'_e,\underline{M}'_e)\to(X_e,\underline{M}_e)$ of $(Y_e,\underline{N}_e)$-schemes as in (7.4.13). Notice that the morphism $(X'_e,\underline{M}'_e)\to(Y_e,\underline{N}_e)$ is again of the type considered in (7.4), and there exists, up to isomorphism, a unique geometric point $\overline{x}'_e$ of X'_e lifting $\overline{x}$; moreover, for any geometric point $\overline{y}_e$ supported at y_e, the induced map

$$\operatorname{Spec}\underline{M}'_{e,\overline{x}'_e}\to\operatorname{Spec}\underline{N}'_{e,\overline{y}_e}$$

is naturally identified with (7.4.19). Likewise, pick a $(\mathsf{Y},\underline{\mathsf{N}})$-scheme $(\mathsf{Y}_e,\underline{\mathsf{N}}_e)$ in the same fashion, and denote by η_e (resp. $\eta_{\mathsf{V},e}$) the generic point of Y_e (resp. of Y_e), and by $\mathsf{y}_e\in\mathsf{Y}_e$ (resp. $y_e\in Y_e$) the closed point. We may choose Y_e so that $\kappa(\eta_{\mathsf{V},e})$ contains $\kappa(\eta_e)$, in which case we have a strict morphism

$$(\mathsf{Y}_e,\underline{\mathsf{N}}_e)\to(Y_e,\underline{N}_e)$$

of log schemes, and we may set $(\mathsf{X}'_e,\underline{\mathsf{M}}'_e):=\mathsf{Y}_e\times_{Y_e}(X'_e,\underline{M}'_e)$. Again, there exists, up to isomorphism, a unique geometric point $\overline{\mathsf{x}}'_e$ of X'_e lifting $\overline{\mathsf{x}}$, and by claim 7.4.40, we may assume that both e and $\kappa(\eta_{\mathsf{V},e})$ have been chosen large enough, so that the base change

$$h\times_{X(\overline{x})}\mathsf{X}'_e(\overline{\mathsf{x}}'_e)$$

shall be a trivial étale covering. Hence, we may replace Y by Y_e, Y by Y_e, X by X'_e, and h by $h\times_{X(\overline{x})}X'_e(\overline{x}'_e)$, and assume from start that $C\times_{U_{\mathrm{tr}}}\mathsf{U}_{\mathrm{tr}}$ is a trivial covering of U_{tr}. The theorem will follow, once we show that – in this case – h is a trivial étale covering.

Let C' (resp. C') be the normalization of $X(\overline{x})$ in C (resp. of $\mathsf{X}(\overline{\mathsf{x}})$ in $C \times_{U_{\mathrm{tr}}} \mathsf{U}_{\mathrm{tr}}$), and define $E(h)$ as in (7.4.28). Then C' is a trivial étale covering of $\mathsf{X}(\overline{\mathsf{x}})$, and since $X(\overline{x})$ is excellent, the induced morphism $h' : C' \to X(\overline{x})$ is finite.

Claim 7.4.42. For every maximal point $\eta_\mathfrak{q}$ of $f_{\overline{x}}^{-1}(\overline{y})$, the induced covering $C_{|\eta_\mathfrak{q}} \to |\eta_\mathfrak{q}|$ is trivial (notation of (7.4.31)).

Proof of the claim. Let $Z_\mathfrak{q}$ denote the topological closure of $\{\eta_\mathfrak{q}\}$ in $X(\overline{x})$, and endow $Z_\mathfrak{q}$ with its reduced subscheme structure; then $E_\mathfrak{q} := E(h) \cap Z_\mathfrak{q}$ is non-empty (lemma 7.4.29), and geometrically normal (proposition 6.7.14(ii) and corollary 6.5.29). Also, (7.4.41) induces an isomorphism of $\kappa(\mathsf{y})$-schemes

$$E_\mathfrak{q} \times_{X(\overline{x})} \mathsf{X}(\overline{x}) \xrightarrow{\sim} E_\mathfrak{q} \times_{\kappa(y)} \kappa(\mathsf{y}).$$

Moreover, $Z_\mathfrak{q}$ is strictly local, and $Z_\mathfrak{q} \times_{X(\overline{x})} \mathsf{X}(\overline{\mathsf{x}})$ is the strict henselization of $Z_\mathfrak{q} \times_{\kappa(y)} \kappa(\mathsf{y})$ at the point x. Furthermore, the morphism $h'' := h' \times_{X(\overline{x})} E_\mathfrak{q}$ is an étale covering of $E_\mathfrak{q}$, and $h'' \times_{\kappa(y)} \kappa(\mathsf{y})$ is naturally identified with the restriction of C' to the subscheme $E_\mathfrak{q} \times_{X(\overline{x})} \mathsf{X}(\overline{\mathsf{x}})$ ([33, Ch.IV, Prop.17.5.7]), hence it is a trivial covering. By example 7.2.6, it follows that h'' is trivial as well. Since $C_{|\eta_\mathfrak{q}}$ is the restriction of h'' to $|\eta_\mathfrak{q}|$, the claim follows. $\Diamond$

Clearly (7.4.41) maps each stratum $\mathsf{U}_\mathfrak{q}$ of the logarithmic stratification of $\mathsf{X}(\overline{\mathsf{x}})$, to the corresponding stratum $U_\mathfrak{q}$ of the logarithmic stratification of $X(\overline{x})$ (see (6.5.49)). More precisely, since (7.4.41) $\times_Y |\eta|$ is ind-étale, proposition 6.7.17(iii) implies that the generic point of $\mathsf{U}_\mathfrak{q} \times_\mathsf{Y} |\eta_\mathsf{V}|$ gets mapped to the generic point of $U_\mathfrak{q} \times_Y |\eta|$. We conclude that $E(h)$ contains the generic point of every stratum $U_\mathfrak{q} \times_Y |\eta|$.

Claim 7.4.43. $X(\overline{x}) \times_Y |\eta| \subset E(h)$.

Proof of the claim. Notice first that $(X, \underline{M}) \times_Y |\eta|$ is a regular log scheme (corollary 6.5.45).

For any geometric point ξ of $X(\overline{x})$, denote by $(X(\xi), \underline{M}(\xi))$ the strict henselization of $(X(\overline{x}), \underline{M}(\overline{x}))$ at ξ, and set $C'(\xi) := C' \times_{X(\overline{x})} X(\xi)$. Now, suppose that the support of ξ lies in the stratum $U_\mathfrak{q} \times_Y |\eta|$, and let $\xi_\mathfrak{q}$ be a geometric point localized at the generic point of $U_\mathfrak{q}$. By assumption, $C'(\xi)$ is a finite $X(\xi)$-scheme, tamely ramified along the non-trivial locus of $(X(\xi), \underline{M}(\xi))$. Likewise, $C'(\xi_\mathfrak{q})$ is tamely ramified along the non-trivial locus of $(X(\xi_\mathfrak{q}), \underline{M}(\xi_\mathfrak{q}))$. Pick any strict specialization map $(X(\xi_\mathfrak{q}), \underline{M}(\xi_\mathfrak{q})) \to (X(\xi), \underline{M}(\xi))$ (see (6.7.11)); it induces a functor

$$\mathbf{Cov}(X(\xi), \underline{M}(\xi)) \to \mathbf{Cov}(X(\xi_\mathfrak{q}), \underline{M}(\xi_\mathfrak{q})) \qquad (7.4.44)$$

and theorem 7.3.44 implies that $C'(\xi)$ is an object of the source of (7.4.44), which is mapped, under this functor, to the object $C'(\xi_\mathfrak{q})$. By proposition 7.3.42, for any geometric point ξ of $X(\overline{x}) \times_Y |\eta|$, the category $\mathbf{Cov}(X(\xi), \underline{M}(\xi))$ is equivalent to the category of finite sets with a continuous action of $(\underline{M}_\xi)^{\mathrm{gp}\vee} \otimes_\mathbb{Z} \prod_{\ell \neq p} \mathbb{Z}_\ell(1)$. On the other hand, clearly $\underline{M}(\overline{x})^\sharp$ restricts to a constant sheaf of monoids on $(U_\mathfrak{q})_\tau$. In view of (7.3.52), we deduce that (7.4.44) is an equivalence; lastly, we have seen that the induced morphism $C'(\xi_\mathfrak{q}) \to X(\xi_\mathfrak{q})$ is étale, *i.e.* is a trivial covering, therefore the same holds for the morphism $C'(\xi) \to X(\xi)$, and consequently the support of ξ lies in $E(h)$ (claim 7.1.9). Since ξ is arbitrary, the assertion follows. $\Diamond$

Claim 7.4.45. There exists a non-empty open subset $U_Y \subset Y$ such that $X(\overline{x}) \times_Y U_Y \subset E(h)$.

Proof of the claim. Since $X(\overline{x})$ is a noetherian scheme, $E(h)$ is a constructible open subset, hence $Z := X(\overline{x}) \setminus E(h)$ is a constructible closed subset of $X(\overline{x})$. The subset $f_{\overline{x}}(Z)$ is pro-constructible ([30, Ch.IV, Prop.1.9.5(vii)]) and does not contain η (by claim 7.4.43), hence neither does its topological closure W ([30, Ch.IV, Th.1.10.1]). It is easily seen that $U_Y := Y \setminus W$ will do. $\Diamond$

Claim 7.4.46. Let U_Y be as in claim 7.4.45. We have :

(i) There exists an irreducible closed subset Z of Y of dimension one, such that $Z \cap (Y, \underline{N})_{\mathrm{tr}} \cap U_Y \neq \varnothing$.

(ii) For any Z as in (i), the induced functor $\mathbf{Cov}(E(h)) \to \mathbf{Cov}(Z \times_Y E(h))$ is fully faithful.

Proof of the claim. (i): More generally, let A be any local noetherian domain of Krull dimension $d \geq 1$, and $W \subset \operatorname{Spec} A$ a proper closed subset. Then we show that there exists an irreducible closed subset $Z \subset \operatorname{Spec} A$ of dimension one, not contained in W. We may assume that $d > 1$ and $W \neq \varnothing$, and arguing by induction on d, we are reduced to showing that there exists an irreducible closed subset $Z' \subset \operatorname{Spec} A$ with $\dim Z' < d$, which is not contained in W. This is an easy exercise that we leave to the reader.

(ii): It suffices to check that conditions (i)–(iii) of proposition 7.1.36 hold for Z and the open subset $E(h)$. However, condition (i) is immediate, since the generic point η_Z of Z lies in U_Y. Likewise, condition (ii) holds trivially for the fibre over the point η_Z, so it suffices to consider the fibre over the closed point y of Z, in which case the assertion is just lemma 7.4.29. Lastly, condition (iii) follows directly from theorem 6.7.8(iii.b) and [33, Ch.IV, Prop.18.8.10, 18.8.12(i)]. $\diamondsuit$

Let Z be as in claim 7.4.46(i), and endow Z with its reduced subscheme structure. Let also Z' be the normalization of Z; then both Z and Z' are strictly local, and the morphism $Z' \to Z$ is radicial and surjective, hence the induced functor

$$\mathbf{Cov}(Z \times_Y E(h)) \to \mathbf{Cov}(Z' \times_Y E(h))$$

is an equivalence (lemma 7.1.7(i)). Taking into account claim 7.4.46, we are thus reduced to showing that the morphism

$$(Z' \times_Y E(h)) \times_{X(\overline{x})} C' \to Z' \times_Y E(h)$$

is a trivial étale covering. However, let $\eta_{Z'}$ be the generic point of Z', and set

$$(Z', \underline{N}') := Z' \times_Y (Y, \underline{N}) \qquad (X', \underline{M}') := Z' \times_Y (X, \underline{M}).$$

The open subset $(Z', \underline{N}')_{\mathrm{tr}}$ is dense in Z', by virtue of claim 7.4.46(i), so the induced morphism $f' : (X', \underline{M}') \to (Z', \underline{N}')$ is still of the type considered in (7.4), the geometric point $\overline{x}$ lifts uniquely (up to isomorphism) to a geometric point $\overline{x}'$ of X', and $h \times_Y Z'$ is an object of $\mathbf{Tame}(f', \overline{x}', \kappa(\eta_{Z'}))$ (lemma 7.3.18(i)). Hence, we may replace from start $(X, \underline{M})$ by $(X', \underline{M}')$, $(Y, \underline{N})$ by $(Z', \underline{N}')$, h by $h \times_Y Z'$, after which, we may assume that Y is noetherian and of dimension one. Moreover, taking into account claim 7.4.42, we may assume that the induced covering $C_{|\eta_{\mathfrak{q}}} \to |\eta_{\mathfrak{q}}|$ is trivial, for every maximal point $\eta_{\mathfrak{q}}$ of $f_{\overline{x}}^{-1}(\overline{y})$, and it remains to show that h is trivial under these assumptions.

To this aim, we look at the corresponding commutative diagram of groups, provided by lemma 7.4.33(i) : with the notation of *loc.cit.*, we see that in the current situation, β is an isomorphism as well, by claim 7.4.40, therefore lemma 7.4.33(ii) says that the group homomorphism $\pi_1(|\eta_{\mathfrak{q}}|_{\mathrm{ét}}, \overline{\eta}_{\mathfrak{q}}) \to \pi_1(\overline{U}_{\mathrm{tr}}/Y_{\mathrm{ét}}, \xi)$ is surjective, for any maximal point $\eta_{\mathfrak{q}}$ of $f_{\overline{x}}^{-1}(\overline{y})$. From this, we deduce that $\overline{h}$ is a trivial covering, and therefore there exists an étale covering $C_Y \to |\eta|$ with an isomorphism $C \overset{\sim}{\to} C_Y \times_{|\eta|} U_{\mathrm{tr}}$ ([42, Exp.IX, Th.6.1]). Denote by C_Y^ν the normalization of Y in C_Y. Also, set $E' := E(h) \subset \operatorname{Str}(f_{\overline{x}})$. In light of theorem 6.7.8(iii.a) and lemma 7.4.29, it is easily seen that the restriction $E' \to Y$ of $f_{\overline{x}}$ is surjective; the latter is also a smooth morphism of schemes (corollary 6.3.27(i)). It follows that $C_Y^\nu \times_Y E'$ is the normalization of E' in C ([33, Ch.IV, Prop.17.5.7]), especially, it is an étale covering of E'. We then deduce that C_Y^ν is already a (trivial) étale covering of Y ([33, Ch.IV, Prop.17.7.1(ii)]), and then clearly h must be a trivial covering as well. $\square$

Remark 7.4.47. Theorem 7.4.39 is the local acyclicity result that gives the name to this section. However, the title is admittedly not self-explanatory, and its full justification would require the introduction of a more advanced theory of the *log-étale site*, that lies beyond the bounds of this treatise. In rough terms, we can try to describe the situation as follows. In lieu of the standard strict henselization, one should consider a suitable notion of strict *log henselization* for points of the *log-étale topoi* associated to log schemes. Then, for $f : X \to Y$ as in (7.4) with saturated log structures on both X and Y, and log-étale points $\widetilde{x}$ of X with image $\widetilde{y}$ in Y, one should look, not at our $f_{\overline{x}}$, but rather at the induced morphism $f_{\widetilde{x}}$ of strict log henselizations (of X at $\widetilde{x}$ and of Y at $\widetilde{y}$). The (suitably defined) *log geometric fibres* of $f_{\widetilde{x}}$ will be the *log Milnor fibres* of f at the log-étale point $\widetilde{x}$, and one can state for such fibres an acyclicity result : namely, the prime-to-p quotients of their (again, suitably defined) *log fundamental groups* vanish. The proof proceeds by reduction to our theorem 7.4.39, which, with hindsight, is seen to supply the essential geometric information encoded in the more sophisticated log-étale language.

8. THE ALMOST PURITY TOOLBOX

The sections of this rather eterogeneous chapter are each devoted to a different subject, and are linked to each other only very loosely, if they are at all. They have been lumped here together, because they each contribute a distinct self-contained little theory, that will find application in chapter 9, in one step or other of the proof of the almost purity theorem. The exception is section 8.7 : it studies a class of rings more general than the measurable algebras introduced in section 8.3; the results of section 8.7 will not be used elsewhere in this treatise, but they may be interesting for other purposes.

Section 8.2 develops the yoga of almost pure pairs (see definition 8.2.25(i)); the relevance to the almost purity theorem is clear, since the latter establishes the almost purity of certain pairs (X, Z) consisting of a scheme X and a closed subscheme $Z \subset X$. This section provides the means to perform various kinds of reductions in the proof of the almost purity theorem, allowing to replace the given pair (X, Z) by more tractable ones.

Section 8.3 introduces measurable (and more generally, ind-measurable) K^+-algebras, where K^+ is a fixed valuation ring of rank one : see definitions 8.3.3(ii) and 8.3.51. For modules over a measurable algebra, one can define a well-behaved real-valued normalized length. This length function is non-negative, and additive for short exact sequences of modules. Moreover, the length of an almost zero module vanishes (for the standard almost structure associated to K^+). Conversely, under some suitable assumptions, a module of normalized length zero will be almost zero. In the proof of the almost purity theorem we shall encounter certain cohomology modules whose normalized length vanishes, and the results of section 8.3 will enable us to prove that these modules are almost zero.

Section 8.4 discusses a category of topologically locally ringed spaces that contains the formal schemes of [26]. We call ω-admissible our generalized formal schemes, and they are obtained by gluing formal spectra of suitable ω-admissible topological rings (see definition 8.4.18). It turns out that the so-called Fontaine rings reviewed in section 4.6 carry a natural ω-admissible linear topology; hence the theory of section 8.4 attaches to such rings an ω-admissible formal scheme, endowed with a natural Frobenius automorphism. This construction – detailed in section 8.5 – lies at the heart of Faltings' proof of the almost purity theorem.

Lastly, section 8.6 studies some questions concerning the formation of quotients of affine almost schemes under a finite group action. These results will be used in the proof of theorem 9.4.34, the last conspicuous stepping stone on the path to almost purity.

8.1. Non-flat almost structures.

This section contains some material that complements the generalities of [36, §2.4, §2.5] : indeed, whereas many of the preliminaries in *loc.cit.* make no assumptions on the basic setup $(V, \mathfrak{m})$ that underlies the whole discussion, for the more advanced results it is usually required that $\widetilde{\mathfrak{m}} := \mathfrak{m} \otimes_V \mathfrak{m}$ is a flat V-module. We shall show how, with more work, one can remove this condition (or at least, weaken it significantly) and still recover most of the useful almost ring theory of [36]. This extension shall be applied in section 9.6, in order to state and prove the most general case of almost purity.

For future reference, we recall the following :

Lemma 8.1.1. *Let $\mathscr{C}$ be any Grothendieck abelian category, $X^\bullet, Y^\bullet \in \mathrm{Ob}(\mathsf{D}(\mathscr{C}))$ any two complexes.*

(i) *If $X^\bullet \in \mathrm{Ob}(\mathsf{D}^{\leq b}(\mathscr{C}))$, the natural map*
$$\mathrm{Hom}_{\mathsf{D}^{\leq b}(\mathscr{C})}(X^\bullet, \tau_{\leq b} Y^\bullet) \to \mathrm{Hom}_{\mathsf{D}(\mathscr{C})}(X^\bullet, Y^\bullet)$$
is an isomorphism.

(ii) *If $Y^\bullet \in \mathrm{Ob}(\mathsf{D}^{\geq a}(\mathscr{C}))$, the natural map*
$$\mathrm{Hom}_{\mathsf{D}(\mathscr{C})}(X^\bullet, Y^\bullet) \to \mathrm{Hom}_{\mathsf{D}^{\geq a}(\mathscr{C})}(\tau_{\geq a} X^\bullet, Y^\bullet)$$
is an isomorphism.

(iii) *If $X^\bullet \in \mathrm{Ob}(\mathsf{D}^{\leq b}(\mathscr{C}))$ and $Y^\bullet \in \mathrm{Ob}(\mathsf{D}^{\geq a}(\mathscr{C}))$, then the complex $R\mathrm{Hom}^\bullet_{\mathscr{C}}(X^\bullet, Y^\bullet)$ lies in $\mathsf{D}^{\geq a-b}(\mathscr{C})$.*

Proof. (Here $\tau_{\geq a}$ denotes the usual truncation functor $\mathsf{D}(R\text{-}\mathbf{Mod}) \to \mathsf{D}^{\geq a}(R\text{-}\mathbf{Mod})$, and recall that a Grothendieck abelian category has enough injectives, and admits a derived category $\mathsf{D}(\mathscr{C})$.) We show only assertion (i) : *mutatis mutandi*, the same argument applies to prove (ii). In light of the distinguished triangle
$$\tau_{\leq b} Y^\bullet \to Y^\bullet \to \tau_{>b} Y^\bullet \to (\tau_{\leq b} Y^\bullet)[1] = \tau_{<b}(Y^\bullet[1])$$
we reduce to checking that
$$\mathrm{Hom}_{\mathsf{D}(\mathscr{C})}(X^\bullet, \tau_{>b} Y^\bullet) = 0 = \mathrm{Hom}_{\mathsf{D}(\mathscr{C})}(X^\bullet, \tau_{>b+1}(Y^\bullet[-1])).$$
We show the first vanishing : the same argument applies also to the second one. We pick an injective resolution $\tau_{>b} Y^\bullet \xrightarrow{\sim} J^\bullet$ such that $J^q = 0$ for every $q \leq b$, and consider the spectral sequence
$$E_1^{pq} := \mathrm{Hom}_{\mathscr{C}}(H^p X^\bullet, J^q) \Rightarrow \mathrm{Hom}_{\mathsf{D}(\mathscr{C})}(X^\bullet, (\tau_{>b} Y^\bullet)[q-p]).$$
Under the current assumptions we have $E_1^{p,-p} = 0$ for every $p \in \mathbb{Z}$, whence the contention.

(iii): In view of the natural isomorphism
$$R^i\mathrm{Hom}_{\mathscr{C}}(X^\bullet, Y^\bullet) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{D}(\mathscr{C})}(X^\bullet, Y^\bullet[i]) \qquad \text{for every } i \in \mathbb{Z}$$
the assertion follows from both (i) and (ii). □

8.1.2. Let $(V, \mathfrak{m})$ be any basic setup (as defined in [36, §2.1.1]), and R any V-algebra. For every interval $I \subset \mathbb{N}$, we have a localization functor

(8.1.3) $$\mathsf{C}^I(R\text{-}\mathbf{Mod}) \to \mathsf{C}^I(R^a\text{-}\mathbf{Mod}) \qquad K^\bullet \mapsto K^{\bullet a}$$

from complexes of R-modules, to complexes of R^a-modules, which is obviously exact, hence it induces a derived localization functor :

(8.1.4) $$\mathsf{D}^I(R\text{-}\mathbf{Mod}) \to \mathsf{D}^I(R^a\text{-}\mathbf{Mod}).$$

The functor (8.1.3) admits a left (resp. right) adjoint

(8.1.5) $$\mathsf{C}^I(R^a\text{-}\mathbf{Mod}) \to \mathsf{C}^I(R\text{-}\mathbf{Mod}) \qquad K^\bullet \mapsto K^\bullet_! \qquad (\text{ resp. } K^\bullet \mapsto K^\bullet_* \text{ })$$

defined by applying termwise to $K^\bullet$ the functor $M \mapsto M_!$ (resp. $M \mapsto M_*$) for R^a-modules given by [36, §2.2.10, §2.2.21]. However, if $\widetilde{\mathfrak{m}}$ is not flat, the functor $M \mapsto M_!$ is obviously not exact, and the localization functor $R\text{-}\mathbf{Mod} \to R^a\text{-}\mathbf{Mod}$ does not send injectives to injectives (cp. [36, Cor.2.2.24]). This makes it trickier to deal with constructions in the derived category;

for instance, if $\widetilde{\mathfrak{m}}$ is flat, we get a left adjoint to (8.1.4), simply by deriving trivially the exact functor $M \mapsto M_!$. This fails in the general case, but we shall see later that a suitable derived version of the construction of $M_!$ is still available.

8.1.6. Let $I \subset \mathbb{N}$ be any interval. As a first step, denote by Σ_I the multiplicative set of morphisms φ in $\mathsf{D}^I(R\text{-}\mathbf{Mod})$ such that φ^a is an isomorphism in $\mathsf{D}^I(R^a\text{-}\mathbf{Mod})$; as already pointed out in [36, §2.4.9], Σ_I is locally small, hence the localized category $\Sigma_I^{-1}\mathsf{D}^I(R\text{-}\mathbf{Mod})$ exists, and obviously the derived localization functor factors through a natural functor

$$\Sigma_I^{-1}\mathsf{D}^I(R\text{-}\mathbf{Mod}) \to \mathsf{D}^I(R^a\text{-}\mathbf{Mod}). \tag{8.1.7}$$

Lemma 8.1.8. *For every interval I, the functor* (8.1.7) *is an equivalence.*

Proof. A proof is sketched in [36, §2.4.9], in case $\widetilde{\mathfrak{m}}$ is flat, but in fact this assumption is superfluous. Indeed, since the unit of adjunction $M \to M_!^a$ is an isomorphism ([36, Prop.2.2.23(ii)]), it is clear that the functor $K^\bullet \mapsto K_!^\bullet$ of (8.1.5) descends to a well-defined functor

$$\mathsf{D}^I(R^a\text{-}\mathbf{Mod}) \to \Sigma_I^{-1}\mathsf{D}^I(R\text{-}\mathbf{Mod}) \tag{8.1.9}$$

such that the composition (8.1.7) $\circ$ (8.1.9) is naturally isomorphic to the identity automorphism of $\mathsf{D}^I(R^a\text{-}\mathbf{Mod})$. A simple inspection shows that, likewise, (8.1.9) $\circ$ (8.1.7) is naturally isomorphic to the identity of $\Sigma_I^{-1}\mathsf{D}^I(R\text{-}\mathbf{Mod})$, whence the contention. $\square$

8.1.10. We sketch a few generalities on derived tensor products, since we shall use these functors to construct useful objects in various derived categories. Recall first that the usual tensor product $-\otimes_R -$ on R-modules, descends to a bifunctor $-\otimes_{R^a} -$ ([36, §2.2.6, §2.2.12]), and if M is a flat R-module, then M^a is a flat R^a-module ([36, Lemma 2.4.7]). It follows that the category $R^a\text{-}\mathbf{Mod}$ has enough flat objects, so every bounded above complex of R^a-modules admits a bounded above flat resolution. Now, given bounded above complexes $K^\bullet, L^\bullet$ of R^a-modules, set

$$K^\bullet \overset{\mathbf{L}}{\otimes}_{R^a} L^\bullet := (K_!^\bullet \overset{\mathbf{L}}{\otimes}_R L_!^\bullet)^a$$

which is a well defined object of $\mathsf{D}(R^a\text{-}\mathbf{Mod})$.

- We claim that this rule yields a well defined functor

$$-\overset{\mathbf{L}}{\otimes}_{R^a} - : \mathsf{D}^-(R^a\text{-}\mathbf{Mod}) \times \mathsf{D}^-(R^a\text{-}\mathbf{Mod}) \to \mathsf{D}^-(R^a\text{-}\mathbf{Mod}).$$

Indeed, suppose $\varphi^\bullet : K_1^\bullet \to K_2^\bullet$ is a morphism in $\mathsf{C}(R^a\text{-}\mathbf{Mod})$, inducing an isomorphism in $\mathsf{D}(R^a\text{-}\mathbf{Mod})$, and set $C^\bullet := \mathrm{Cone}\,(\varphi_!^\bullet)$; clearly, $C^{\bullet a} = 0$ in $\mathsf{D}(R^a\text{-}\mathbf{Mod})$. Now, pick any flat resolution $P^\bullet \to L_!^\bullet$, so that $K_{i!}^\bullet \otimes_R P^\bullet$ computes $K_{i!}^\bullet \overset{\mathbf{L}}{\otimes}_R L_!^\bullet$, for $i = 1, 2$. We get natural isomorphism :

$$\mathrm{Cone}(\varphi^\bullet \overset{\mathbf{L}}{\otimes}_{R^a} L^\bullet) \overset{\sim}{\to} (C^\bullet \otimes_R P^\bullet)^a \overset{\sim}{\to} C^{\bullet a} \otimes_{R^a} P^{\bullet a} = 0.$$

so the derived tensor product depends only on the image of $K^\bullet$ in $\mathsf{D}^-(R^a\text{-}\mathbf{Mod})$; likewise for the argument $L^\bullet$, whence the contention.

- Next, suppose that $P^\bullet \to K^\bullet$ is a bounded above resolution, with $P^\bullet$ a complex of flat R^a-modules; we claim that there is a natural isomorphism in $\mathsf{D}(R^a\text{-}\mathbf{Mod})$

$$K^\bullet \overset{\mathbf{L}}{\otimes}_{R^a} L^\bullet \overset{\sim}{\to} P^\bullet \otimes_{R^a} L^\bullet.$$

Indeed, pick any bounded above flat resolution $Q^\bullet \to L_!^\bullet$; we have natural isomorphisms

$$K^\bullet \overset{\mathbf{L}}{\otimes}_{R^a} L^\bullet \overset{\sim}{\to} (K_!^\bullet \otimes_R Q^\bullet)^a \overset{\sim}{\to} K^\bullet \otimes_{R^a} Q^{\bullet a} \overset{\sim}{\to} P^\bullet \otimes_{R^a} Q^{\bullet a}$$

in $\mathsf{D}(R^a\text{-}\mathbf{Mod})$, where the last holds, since $Q^{\bullet a}$ is a complex of flat R^a-modules. Finally, the induced map

$$P^\bullet \otimes_{R^a} Q^{\bullet a} \to P^\bullet \otimes_{R^a} L^\bullet$$

is also an isomorphism in $\mathsf{D}(R^a\text{-}\mathbf{Mod})$, since $P^\bullet$ is a complex of flat R^a-modules, so the claim follows.

• Notice that, for any $K^\bullet \in \mathrm{Ob}(\mathsf{D}^-(R\text{-}\mathbf{Mod}))$ and any flat resolution $P^\bullet \to K^\bullet$, the induced morphism $P^{\bullet a} \to K^{\bullet a}$ is a flat resolution; it follows that, for any $L^\bullet \in \mathrm{Ob}(\mathsf{D}(R\text{-}\mathbf{Mod}))$ we get a natural isomorphism

(8.1.11) $$(K^\bullet \overset{\mathbf{L}}{\otimes}_R L^\bullet)^a \xrightarrow{\sim} K^{\bullet a} \overset{\mathbf{L}}{\otimes}_{R^a} L^{\bullet a} \qquad \text{in } \mathsf{D}(R^a\text{-}\mathbf{Mod}).$$

Remark 8.1.12. Clearly, for the derived tensor products of R^a-modules, one has the same commutativity and associativity isomorphisms as the ones detailed in remark 4.1.13(i) for usual modules, as well as the vanishing properties of remark 4.1.13(ii).

We are now ready to return to the question of the existence of adjoints to derived localization. The key point is the following :

Lemma 8.1.13. *In the situation of* (8.1.2), *let $K^\bullet$ be any complex of R-modules, $i \in \mathbb{Z}$ any integer and suppose that :*

(a) $K^\bullet \in \mathrm{Ob}(\mathsf{D}^{\leq i}(R\text{-}\mathbf{Mod}))$
(b) $K^{\bullet a} \in \mathrm{Ob}(\mathsf{D}^{\leq i-1}(R^a\text{-}\mathbf{Mod}))$.

Then $\mathfrak{m} \overset{\mathbf{L}}{\otimes}_V K^\bullet \in \mathrm{Ob}(\mathsf{D}^{\leq i-1}(R\text{-}\mathbf{Mod}))$.

Proof. We apply the standard spectral sequence

$$E^2_{pq} := \mathrm{Tor}^V_p(\mathfrak{m}, H^q K^\bullet) \Rightarrow H^{q-p}(\mathfrak{m} \overset{\mathbf{L}}{\otimes}_V K^\bullet).$$

Indeed, (a) says that $H^q K^\bullet = 0$ for every $q > i$, and (b) says that $(H^i K^\bullet)^a = 0$, and therefore $\mathfrak{m}_V \otimes_V H^i K^\bullet = 0$ ([36, Rem.2.1.4(i)]). In either case, we conclude that $E^2_{pq} = 0$ whenever $q - p \geq i$, whence the lemma. □

8.1.14. Now, let us define inductively :

$$\mathfrak{M}^\bullet_0 := V[0] \quad \text{and} \quad \mathfrak{M}^\bullet_{i+1} := \mathfrak{m} \overset{\mathbf{L}}{\otimes}_V \mathfrak{M}^\bullet_i \quad \text{for every } i \in \mathbb{N}.$$

A simple induction shows that $\mathfrak{M}^\bullet_i \in \mathsf{D}^{\leq 0}(V\text{-}\mathbf{Mod})$ for every $i \in \mathbb{N}$, so all these derived tensor product are well defined in $\mathsf{D}^{\leq 0}(V\text{-}\mathbf{Mod})$. Moreover, from the short exact sequence of V-modules

$$\Sigma \quad : \quad 0 \to \mathfrak{m} \to V \to V/\mathfrak{m} \to 0$$

we obtain a distiguished triangle

$$\mathfrak{M}^\bullet_i \overset{\mathbf{L}}{\otimes}_V \Sigma \quad : \quad \mathfrak{M}^\bullet_{i+1} \to \mathfrak{M}^\bullet_i \to \mathfrak{M}^\bullet_i \overset{\mathbf{L}}{\otimes}_V (V/\mathfrak{m}) \to \mathfrak{M}^\bullet_{i+1}[1] \qquad \text{for every } i \in \mathbb{N}.$$

Especially, we get an inverse system of morphisms in $\mathsf{D}^{\leq 0}(V\text{-}\mathbf{Mod})$:

$$\cdots \to \mathfrak{M}^\bullet_{i+1} \xrightarrow{\pi^\bullet_i} \mathfrak{M}^\bullet_i \xrightarrow{\pi^\bullet_{i-1}} \mathfrak{M}^\bullet_{i-1} \to \cdots \to \mathfrak{M}^\bullet_0 := V[0].$$

Also, from (8.1.11) we deduce natural isomorphisms in $\mathsf{D}^{\leq 0}(V^a\text{-}\mathbf{Mod})$:

(8.1.15) $$\mathfrak{M}^{\bullet a}_i \xrightarrow{\sim} V^a[0] \qquad \text{for every } i \in \mathbb{N}$$

and under these identifications, the morphism $\pi^{\bullet a}_i$ corresponds to the identity automorphism of $V^a[0]$ (details left to the reader). Furthermore, we deduce the following derived version of [36, Rem.2.1.4(i)] :

Proposition 8.1.16. *Let $a, b \in \mathbb{Z}$ be any integers with $a \leq b$. We have :*

(i) *For every* $K^\bullet \in \mathrm{Ob}(\mathsf{D}^{[a,b]}(R\text{-}\mathbf{Mod}))$, *the following conditions are equivalent :*
   (a) $K^{\bullet a} \simeq 0$ *in* $\mathsf{D}^{[a,b]}(R^a\text{-}\mathbf{Mod})$.
   (b) $\tau_{\geq a}(\mathfrak{M}^\bullet_{b-a+1} \stackrel{\mathbf{L}}{\otimes}_V K^\bullet) \simeq 0$ *in* $\mathsf{D}^{[a,b]}(R\text{-}\mathbf{Mod})$.

(ii) *For every morphism* $\varphi^\bullet : K^\bullet \to L^\bullet$ *in* $\mathsf{D}^{\leq b}(R\text{-}\mathbf{Mod})$, *the following conditions are equivalent :*
   (a) $\varphi^{\bullet a}$ *is an isomorphism in* $\mathsf{D}^{[a,b]}(R^a\text{-}\mathbf{Mod})$.
   (b) $\tau_{\geq a}(\mathfrak{M}^\bullet_{b-a+2} \stackrel{\mathbf{L}}{\otimes}_V \varphi^\bullet)$ *is an isomorphism in* $\mathsf{D}^{[a,b]}(R\text{-}\mathbf{Mod})$.

Proof. (i): From (8.1.15), it is easily seen that (b)⇒(a). The other direction follows straightforwardly from lemma 8.1.13, via an easy descending induction on b.

(ii): Again, the direction (b)⇒(a) is immediate from (8.1.15). For the other direction, denote by $C^\bullet$ the cone of $\varphi^\bullet$; then $C^\bullet \in \mathrm{Ob}(\mathsf{D}^{[a-1,b]}(R\text{-}\mathbf{Mod}))$, and $C^{\bullet a} \simeq 0$ in $\mathsf{D}^{[a-1,b]}(R^a\text{-}\mathbf{Mod})$. From (i) we deduce that $\tau_{\geq a-1}(\mathfrak{M}^\bullet_{b-a+2} \stackrel{\mathbf{L}}{\otimes}_V C^\bullet) \simeq 0$ in $\mathsf{D}^{[a-1,b]}(R\text{-}\mathbf{Mod})$. Since the derived tensor product is a triangulated functor, the assertion follows easily : details left to the reader. □

Corollary 8.1.17. *With the notation of* (8.1.14), *the morphism*

$$\tau_{\geq 2-i}\pi^\bullet_i : \tau_{\geq 2-i}\mathfrak{M}^\bullet_{i+1} \to \tau_{\geq 2-i}\mathfrak{M}^\bullet_i$$

is an isomorphism in $\mathsf{D}^{[2-i,0]}(V\text{-}\mathbf{Mod})$ *for every integer* $i \geq 2$.

Proof. By construction, we have a natural isomorphism :

$$\mathrm{Cone}(\tau_{\geq 2-i}\pi^\bullet_i) \xrightarrow{\sim} \tau_{\geq 1-i}(\mathfrak{M}^\bullet_i \stackrel{\mathbf{L}}{\otimes}_V (V/\mathfrak{m})) \qquad \text{in } \mathsf{D}^{[1-i,0]}(V\text{-}\mathbf{Mod})$$

in light of which, the assertion is an immediate consequence of proposition 8.1.16(i). □

Proposition 8.1.18. *In the situation of* (8.1.2), *we have :*

(i) *The functor*

$$\mathsf{D}^+(R^a\text{-}\mathbf{Mod}) \to \mathsf{D}^+(R\text{-}\mathbf{Mod}) \quad : \quad K^\bullet \mapsto K^\bullet_{[*]} := R\mathrm{Hom}^\bullet_{R^a}(R^a[0], K^\bullet) \tag{8.1.19}$$

is right adjoint to the localization functor.

(ii) *Let* $a, b, i \in \mathbb{Z}$ *be any three integers with* $b \geq a$ *and* $i \geq b-a+2$. *Then*
   (a) *The functor*

$$\mathsf{D}^{[a,b]}(R^a\text{-}\mathbf{Mod}) \to \mathsf{D}^{[a,b]}(R\text{-}\mathbf{Mod}) \quad : \quad K^\bullet \mapsto K^\bullet_{[!]} := \tau_{\geq a}(\mathfrak{M}^\bullet_i \stackrel{\mathbf{L}}{\otimes}_V K^\bullet_{[*]})$$

   is left adjoint to the localization functor.
   (b) *The functor*

$$\mathsf{D}^{[a,b]}(R^a\text{-}\mathbf{Mod}) \to \mathsf{D}^{[a,b]}(R\text{-}\mathbf{Mod}) \quad : \quad K^\bullet \mapsto \tau_{\leq b}K^\bullet_{[*]}$$

   is right adjoint to the localization functor.

(iii) *For every* $K^\bullet \in \mathrm{Ob}(\mathsf{D}^+(R^a\text{-}\mathbf{Mod}))$ *(resp.* $L^\bullet \in \mathrm{Ob}(\mathsf{D}^{[a,b]}(R^a\text{-}\mathbf{Mod}))$*) the counit of adjunction is an isomorphism*

$$(K^\bullet_{[*]})^a \xrightarrow{\sim} K^\bullet \qquad (\textit{ resp. } (\tau_{\leq b}L^\bullet_{[*]})^a \xrightarrow{\sim} L^\bullet \ ).$$

(iv) *For every* $L^\bullet \in \mathrm{Ob}(\mathsf{D}^{[a,b]}(R^a\text{-}\mathbf{Mod}))$, *the unit of adjunction is an isomorphism*

$$L^\bullet \to (L^\bullet_{[!]})^a.$$

Proof. (i): This is analogous to lemma 5.1.25(iii). Recall the construction : we know that the category R^a-$\mathbf{Mod}$ admits enough injectives ([36, 2.2.18]), hence (8.1.19) can be represented by $I^\bullet_*$, where $K^\bullet \xrightarrow{\sim} I^\bullet$ is any injective resolution of $K^\bullet$, and $I^\bullet_*$ is obtained by applying term-wise to $I^\bullet$ the functor $M \mapsto M_*$ of [36, §2.2.10]. Indeed, taking into account [36, Cor.2.2.19] we get natural isomorphisms :

$$\begin{aligned}\mathrm{Hom}_{\mathsf{D}^+(R\text{-}\mathbf{Mod})}(L^\bullet, I^\bullet_*) &\xrightarrow{\sim} H^0\mathrm{Hom}^\bullet_R(L^\bullet, I^\bullet_*) \\ &\xrightarrow{\sim} H^0\mathrm{Hom}^\bullet_{R^a}(L^{\bullet a}, I^\bullet) \\ &\xrightarrow{\sim} \mathrm{Hom}_{\mathsf{D}(R^a\text{-}\mathbf{Mod})}(L^{\bullet a}, I^\bullet) \\ &\xrightarrow{\sim} \mathrm{Hom}_{\mathsf{D}(R^a\text{-}\mathbf{Mod})}(L^{\bullet a}, K^\bullet)\end{aligned}$$

for every bounded below complex $L^\bullet$ of R-modules.

(ii.b): Let $K^\bullet \in \mathsf{D}^{[a,b]}(R^a\text{-}\mathbf{Mod})$; we may find an injective resolution $K^\bullet \xrightarrow{\sim} I^\bullet$ such that $I^j = 0$ for every $j < a$, in which case $I^\bullet_* \in \mathsf{D}^{\geq a}(R\text{-}\mathbf{Mod})$, and $\tau_{\leq b} I^\bullet_*$ represents $\tau_{\leq b} K^\bullet_{[*]}$ in $\mathsf{D}^{[a,b]}(R\text{-}\mathbf{Mod})$. Then, in view of (i), the assertion is reduced to lemma 8.1.1(i).

(iii) follows by direct inspection of the definitions, taking into account that, for every R^a-module M, the counit of adjunction $(M_*)^a \to M$ is an isomorphism ([36, Prop.2.2.14(iii)]) : details left to the reader.

(ii.a): To ease notation, set

$$\omega L^\bullet := \tau_{\geq a}(\mathfrak{M}^\bullet_i \overset{\mathbf{L}}{\otimes}_V L^\bullet) \qquad \text{for every } L^\bullet \in \mathrm{Ob}(\mathsf{D}^{[a,b]}(R\text{-}\mathbf{Mod})).$$

Then $\omega L^\bullet$ is naturally an object of $\mathsf{D}^{[a,b]}(R\text{-}\mathbf{Mod})$, as explained in remark 4.1.13(iii), and likewise for $K^\bullet_{[!]}$, if $K^\bullet$ is any object of $\mathsf{D}^{[a,b]}(R^a\text{-}\mathbf{Mod})$. We begin with the following :

Claim 8.1.20. Let $K^\bullet, L^\bullet \in \mathrm{Ob}(\mathsf{D}^{[a,b]}(R\text{-}\mathbf{Mod}))$ be any two objects. Then the natural map

$$\mathrm{Hom}_{\mathsf{D}^{[a,b]}(R\text{-}\mathbf{Mod})}(\omega K^\bullet, L^\bullet) \to \mathrm{Hom}_{\mathsf{D}^{[a,b]}(R^a\text{-}\mathbf{Mod})}(K^{\bullet a}, L^{\bullet a})$$

is an isomorphism.

Proof of the claim. Arguing as in [36, §2.2.2], and taking into account lemma 8.1.8, we reduce to showing that, for any $K^\bullet \in \mathsf{D}^{[a,b]}(R\text{-}\mathbf{Mod})$, the natural morphism

$$\varphi_{K^\bullet} : \omega K^\bullet \to K^\bullet$$

is initial in the full subcategory of $\mathsf{D}^{[a,b]}(R\text{-}\mathbf{Mod})/K^\bullet$ whose objects are the morphisms $\psi : L^\bullet \to K^\bullet$ that lie in $\Sigma_{[a,b]}$. However, for any such ψ, we have a commutative diagram in $\mathsf{D}^{[a,b]}(R\text{-}\mathbf{Mod})$:

$$\begin{array}{ccc} \omega L^\bullet & \xrightarrow{\varphi_{L^\bullet}} & L^\bullet \\ \downarrow & & \downarrow{\scriptstyle\psi} \\ \omega K^\bullet & \xrightarrow{\varphi_{K^\bullet}} & K^\bullet \end{array}$$

whose left vertical arrow is an isomorphism, by proposition 8.1.16(ii). There follows a morphism $\varphi_{K^\bullet} \to \psi$, and we have to check that this is the unique morphism from $\varphi_{K^\bullet}$ to ψ. However, say that $\alpha, \beta : \varphi_{K^\bullet} \to \psi$ are two such morphisms; then their difference is a morphism $\gamma := \alpha - \beta : \omega K^\bullet \to L^\bullet$ such that $\psi \circ \gamma = 0$, so γ factors through a morphism $\overline{\gamma} : \omega K^\bullet \to \mathrm{Cone}\,\psi[-1]$. Set $C^\bullet := \tau_{\leq b}\mathrm{Cone}\,\psi[-1]$; then $C^\bullet \in \mathsf{D}^{[a,b]}(R\text{-}\mathbf{Mod})$, and according to lemma 8.1.1(i), $\overline{\gamma}$ lifts uniquely to a morphism $\omega K^\bullet \to C^\bullet$ that we denote again $\overline{\gamma}$. We deduce a commutative diagram

$$\begin{tikzcd} \omega \circ \omega K^\bullet \arrow[r] \arrow[d, "\varphi_{\omega_{K^\bullet}}"'] & \omega C^\bullet \arrow[d, "\varphi_{C^\bullet}"] \\ \omega K^\bullet \arrow[r, "\overline{\gamma}"] & C^\bullet. \end{tikzcd}$$

Now, by construction $C^{\bullet a} = 0$, therefore $\omega C^\bullet = 0$ (proposition 8.1.16(i)); on the other hand, $\varphi_{\omega_{K^\bullet}}$ is an isomorphism, by proposition 8.1.16(ii). We conclude that $\overline{\gamma} = 0$, whence $\alpha = \beta$, as sought. ◊

Assertion (ii.a) is an immediate consequence of (iii) and claim 8.1.20; from this, also (iv) is immediate : details left to the reader. □

Remark 8.1.21. Let $a, b, i \in \mathbb{Z}$ be any three integers such that $a \leq b$ and $i \geq b - a + 2$. From propositions 8.1.18(ii.a,iii) and 8.1.16(ii) we deduce a natural isomorphism

$$\tau_{\geq a}(\mathfrak{M}_i^\bullet \overset{\mathbf{L}}{\otimes}_V K^\bullet) \xrightarrow{\sim} (K^{\bullet a})_{[!]} \qquad \text{for every } K^\bullet \in \mathrm{Ob}(\mathsf{D}^{[a,b]}(R\text{-}\mathbf{Mod}))$$

(details left to the reader), which allows to compute $(K^{\bullet a})_{[!]}$ purely in terms of $K^\bullet$ and operations within $\mathsf{D}(R\text{-}\mathbf{Mod})$. It turns out that an analogous isomorphism is available also for $K^\bullet_{[*]}$: this is contained in the following

Lemma 8.1.22. *Let $a, b, i \in \mathbb{N}$ be any integers such that $a \leq b$ and $i \geq b - a + 2$. For every $K^\bullet \in \mathrm{Ob}(\mathsf{D}^{[a,b]}(R\text{-}\mathbf{Mod}))$, we have a natural isomorphism :*

$$\tau_{\leq b} R\mathrm{Hom}^\bullet_V(\mathfrak{M}_i^\bullet, K^\bullet). \xrightarrow{\sim} \tau_{\leq b} K^{\bullet a}_{[*]}.$$

Proof. Theorem 5.1.27(ii) yields a natural isomorphism

$$R\mathrm{Hom}^\bullet_V(\mathfrak{M}_i^\bullet, K^\bullet) \xrightarrow{\sim} R\mathrm{Hom}^\bullet_R(\mathfrak{M}_i^\bullet \overset{\mathbf{L}}{\otimes}_V R[0], K^\bullet).$$

To compute the right-hand side, we may fix an injective resolution $K^\bullet \xrightarrow{\sim} I^\bullet$; the complex $I^{\bullet a}$ is not necessarily injective, but we can find an injective resolution $\varphi : I^{\bullet a} \xrightarrow{\sim} J^\bullet$ (in the category of bounded below complexes of R^a-modules). In view of (8.1.11) and (8.1.15), the morphism φ induces a natural transformation

$$\psi : R\mathrm{Hom}^\bullet_R(\mathfrak{M}_i^\bullet \overset{\mathbf{L}}{\otimes}_V R[0], K^\bullet) \to R\mathrm{Hom}^\bullet_{R^a}(R^a[0], K^{\bullet a}) = (K^{\bullet a})_{[*]}$$

and it suffices to show that, for every $j \leq b$, the map

$$H^j\psi : \mathrm{Hom}_{\mathsf{D}(R\text{-}\mathbf{Mod})}(\mathfrak{M}_i^\bullet \overset{\mathbf{L}}{\otimes}_V R[0], K^\bullet[j]) \to \mathrm{Hom}^\bullet_{\mathsf{D}(R^a\text{-}\mathbf{Mod})}(R^a[0], K^{\bullet a}[j])$$

is an isomorphism. However, by lemma 8.1.1(i), the latter is the same as a map

$$\mathrm{Hom}_{\mathsf{D}(R\text{-}\mathbf{Mod})}(\mathfrak{M}_i^\bullet \overset{\mathbf{L}}{\otimes}_V R[0], \tau_{\leq 0} K^\bullet[j]) \to \mathrm{Hom}^\bullet_{\mathsf{D}(R^a\text{-}\mathbf{Mod})}(R^a[0], \tau_{\leq 0} K^{\bullet a}[j]) \tag{8.1.23}$$

and a direct inspection shows that (8.1.23) agrees with the map arising in claim 8.1.20, for every $j \leq b$. Especially, for every such j, the map (8.1.23) is an isomorphism, as sought. □

Proposition 8.1.24. *Let $a, b, c \in \mathbb{Z}$ be any three integers such that $a \leq b$, and $K^\bullet, L^\bullet$ any two objects of $\mathsf{D}^{[a,b]}(R\text{-}\mathbf{Mod}))$. Suppose that*

(a) $\mathrm{Hom}_{\mathsf{D}(R\text{-}\mathbf{Mod})}(K^\bullet, X[-j]) = 0$ *for all $j \in [c, b]$ and all R-modules X with $X^a = 0$.*
(b) $\mathrm{Hom}_{\mathsf{D}(R\text{-}\mathbf{Mod})}(Y[-j], L^\bullet) = 0$ *for all $j \in [a, c]$ and all R-modules Y with $Y^a = 0$.*

Then the natural map

$$\mathrm{Hom}_{\mathsf{D}(R\text{-}\mathbf{Mod})}(K^\bullet, L^\bullet) \to \mathrm{Hom}_{\mathsf{D}(R^a\text{-}\mathbf{Mod})}(K^{\bullet a}, L^{\bullet a}) \tag{8.1.25}$$

is an isomorphism.

Proof. We start out with the following observation :

Claim 8.1.26. Consider the following conditions :

(a') $\mathrm{Hom}_{\mathsf{D}(R\text{-}\mathbf{Mod})}(K^\bullet, X^\bullet) = 0$ for every $X^\bullet \in \mathrm{Ob}(\mathsf{D}^{[c,b]}(R\text{-}\mathbf{Mod}))$ such that $X^{\bullet a} = 0$.

(b') $\mathrm{Hom}_{\mathsf{D}(R\text{-}\mathbf{Mod})}(Y^\bullet, L^\bullet) = 0$ for every $Y^\bullet \in \mathrm{Ob}(\mathsf{D}^{[a,c]}(R\text{-}\mathbf{Mod}))$ such that $Y^{\bullet a} = 0$.

Then (a)⇔(a') and (b)⇔(b').

Proof of the claim. Obviously (a')⇒(a). For the converse, one argues by decreasing induction on $c \leq b$. Indeed, the case $c = b$ is immediate. Then, suppose that the sought equivalence has already been established for some $d \leq b$; if $X^\bullet \in \mathsf{D}^{[d-1,b]}$ and $X^{\bullet a} = 0$, and if we know that (a) holds with $c := d - 1$, we set $H^\bullet := H^c X^\bullet[-c]$, and consider the distinguished triangle

$$H^\bullet \to X^\bullet \to \tau_{\geq d} X^\bullet \to H^\bullet[1].$$

By inductive assumption, we have $\mathrm{Hom}_{\mathsf{D}(R\text{-}\mathbf{Mod})}(K^\bullet, \tau_{\geq d} X^\bullet) = 0$, and (a) says that

$$\mathrm{Hom}_{\mathsf{D}(R\text{-}\mathbf{Mod})}(K^\bullet, H^\bullet) = 0.$$

It then follows that $\mathrm{Hom}_{\mathsf{D}(R\text{-}\mathbf{Mod})}(K^\bullet, X^\bullet) = 0$, which shows that the equivalence holds for c.

The proof of the equivalence (b)⇔(b') is wholly analogous. ◊

Fix an integer $i \geq b - a + 2$, and set

$$C^\bullet := \mathrm{Cone}(\pi_0^\bullet \circ \cdots \circ \pi_i^\bullet : \mathfrak{M}_i^\bullet \to V[0])$$

(notation of (8.1.14)); notice that $C^\bullet \in \mathsf{D}^{\leq 1}(R\text{-}\mathbf{Mod})$, and $C^{\bullet a} = 0$. By virtue of condition (b), claim 8.1.26 and lemma 8.1.1(ii), it follows that :

$$R^j\mathrm{Hom}^\bullet_R(C^\bullet, L^\bullet) = \mathrm{Hom}_{\mathsf{D}(R\text{-}\mathbf{Mod})}(\tau_{\geq a}(C^\bullet[-j]), L^\bullet) = 0 \qquad \text{for every } j < c$$

In other words, $D^\bullet := R\mathrm{Hom}^\bullet_R(C^\bullet, L^\bullet) \in \mathsf{D}^{\geq c}(R\text{-}\mathbf{Mod})$, and clearly $D^{\bullet a} = 0$; also notice the induced distinguished triangle

$$\Sigma \quad : \quad D^\bullet \to L^\bullet \to R\mathrm{Hom}^\bullet_R(\mathfrak{M}_i^\bullet, L^\bullet) \to D^\bullet[-1].$$

Now, condition (a), claim 8.1.26 and lemma 8.1.1(i) imply that

$$\mathrm{Hom}_{\mathsf{D}(R\text{-}\mathbf{Mod})}(K^\bullet, D^\bullet[j]) = \mathrm{Hom}_{\mathsf{D}(R\text{-}\mathbf{Mod})}(K^\bullet, \tau_{\leq b} D^\bullet[j]) = 0 \qquad \text{for every } j \leq 0$$

whence, by considering the distinguished triangle $R\mathrm{Hom}^\bullet_R(K^\bullet, \Sigma)$, natural isomorphisms

$$\begin{aligned}
\mathrm{Hom}_{\mathsf{D}(R\text{-}\mathbf{Mod})}(K^\bullet, L^\bullet) &\xrightarrow{\sim} \mathrm{Hom}_{\mathsf{D}(R\text{-}\mathbf{Mod})}(K^\bullet, R\mathrm{Hom}^\bullet_R(\mathfrak{M}_i^\bullet, L)) \\
&\xrightarrow{\sim} \mathrm{Hom}_{\mathsf{D}(R\text{-}\mathbf{Mod})}(\mathfrak{M}_i^\bullet \overset{\mathbf{L}}{\otimes}_V K^\bullet, L^\bullet) && \text{(by [75, Th.10.8.7])} \\
&\xrightarrow{\sim} \mathrm{Hom}_{\mathsf{D}(R\text{-}\mathbf{Mod})}(\tau_{\geq a}\mathfrak{M}_i^\bullet \overset{\mathbf{L}}{\otimes}_V K^\bullet, L^\bullet) && \text{(by lemma 8.1.1(ii))} \\
&\xrightarrow{\sim} \mathrm{Hom}_{\mathsf{D}(R^a\text{-}\mathbf{Mod})}(K^{\bullet a}, L^{\bullet a}) && \text{(by claim 8.1.20)}
\end{aligned}$$

whose composition, after a simple inspection, is seen to agree with the map (8.1.25). □

Remark 8.1.27. (i) For every interval $I \subset \mathbb{N}$, denote by

$$\Phi_I : \mathsf{D}^I(R/\mathfrak{m}R\text{-}\mathbf{Mod}) \to \mathsf{D}^I(R\text{-}\mathbf{Mod})$$

the forgetful functor. It follows easily from lemmata 5.1.25(iii) and 8.1.1(i), that, for every interval $[a, b]$, the functor $\Phi_{[a,b]}$ admits the right adjoint

$$\mathsf{D}^{[a,b]}(R\text{-}\mathbf{Mod}) \to \mathsf{D}^{[a,b]}(R/\mathfrak{m}R\text{-}\mathbf{Mod}) \qquad K^\bullet \mapsto \Psi^r_{[a,b]} K^\bullet := \tau_{\leq b} R\mathrm{Hom}^\bullet_R(R/\mathfrak{m}R[0], K^\bullet).$$

(ii) Likewise, theorem 5.1.27 and lemma 8.1.1(ii) imply that $\Phi_{[a,b]}$ admits the left adjoint

$$\mathsf{D}^{[a,b]}(R\text{-}\mathbf{Mod}) \to \mathsf{D}^{[a,b]}(R/\mathfrak{m}R\text{-}\mathbf{Mod}) \qquad K^\bullet \mapsto \Psi^l_{[a,b]} K^\bullet := \tau_{\geq a}(K^\bullet \otimes_R R/\mathfrak{m}R[0]).$$

(iii) Moreover, arguing as in the proof of claim 8.1.26 it is easily seen that condition (a) of proposition 8.1.24 is equivalent to (a”) $\mathrm{Hom}_{\mathsf{D}(R\text{-}\mathbf{Mod})}(K^\bullet, \Phi_{[c,b]}X^\bullet) = 0$ for every $X^\bullet \in \mathrm{Ob}(\mathsf{D}^{[c,b]}(R/\mathfrak{m}R\text{-}\mathbf{Mod}))$.

(iv) Likewise, condition (b) is equivalent to

(b”) $\mathrm{Hom}_{\mathsf{D}(R\text{-}\mathbf{Mod})}(\Phi_{[a,c]}Y^\bullet, L^\bullet) = 0$ for every $Y^\bullet \in \mathrm{Ob}(\mathsf{D}^{[a,c]}(R/\mathfrak{m}R\text{-}\mathbf{Mod}))$.

Proposition 8.1.28. *Let $a, b \in \mathbb{Z}$ be any two integers such that $a \leq b$. For every object $K^\bullet$ of $\mathsf{D}^{[a,b]}(R\text{-}\mathbf{Mod})$, the following conditions are equivalent :*

(a) *$K^\bullet$ lies in the essential image of the left adjoint functor $X^\bullet \mapsto X^\bullet_{[!]}$.*

(b) $K^\bullet \overset{\mathbf{L}}{\otimes}_R R/\mathfrak{m}R[0] \in \mathrm{Ob}(\mathsf{D}^{<a-1}(R/\mathfrak{m}R\text{-}\mathbf{Mod}))$.

Proof. We start out with the following :

Claim 8.1.29. We may assume that $V = R$.

Proof of the claim. Clearly condition (b) does not depend on the underlying ring V. It suffices then to remark that condition (a) depends only on the basic setup $(R, \mathfrak{m}R)$ (as opposed to the original basic setup $(V, \mathfrak{m})$). Indeed, notice that there is a natural equivalence

$$\Omega : R^a\text{-}\mathbf{Mod} \xrightarrow{\sim} (R, \mathfrak{m}R)^a\text{-}\mathbf{Mod}$$

(where R^a denotes, as in the foregoing, the image of R in the category of $(V, \mathfrak{m})$-algebras), and the induced equivalence of the respective derived categories fits into an essentially commutative diagram

$$\begin{tikzcd} & \mathsf{D}(R\text{-}\mathbf{Mod}) \arrow[dl] \arrow[dr] & \\ \mathsf{D}(R^a\text{-}\mathbf{Mod}) \arrow[rr, "\mathsf{D}(\Omega)"] & & \mathsf{D}((R, \mathfrak{m}R)^a\text{-}\mathbf{Mod}) \end{tikzcd}$$

whose downward arrows are the forgetful functors. Especially, the left (resp. right) adjoints of these two forgetful functors share the same essential images. $\Diamond$

Henceforth, we assume that $V = R$ (and therefore, $\mathfrak{m} = \mathfrak{m}R$). Fix an integer $i \geq b - a + 2$, set $L^\bullet := \tau_{\geq a-1}(\mathfrak{M}_i^\bullet \overset{\mathbf{L}}{\otimes}_V K^\bullet)$, and notice first that, taking into account proposition 8.1.18(iv) and remark 8.1.21, condition (a) is equivalent to :

(c) The morphism $\pi_0^\bullet \circ \cdots \circ \pi_{i-1}^\bullet : \mathfrak{M}_i \to V[0]$ induces an isomorphism $\tau_{\geq a}L^\bullet \to K^\bullet$.

(c)⇒(b): Indeed, set $H := H^{a-1}L^\bullet$; if (c) holds, we have a distinguished triangle

$$H[a-1] \to L^\bullet \to K^\bullet \to H[a]$$

whence a distinguished triangle in $\mathsf{D}(V/\mathfrak{m}\text{-}\mathbf{Mod})$

$$H[a-1] \overset{\mathbf{L}}{\otimes}_V V/\mathfrak{m}[0] \to L^\bullet \overset{\mathbf{L}}{\otimes}_V V/\mathfrak{m}[0] \to K^\bullet \overset{\mathbf{L}}{\otimes}_V V/\mathfrak{m}[0] \to H[a] \overset{\mathbf{L}}{\otimes}_V V/\mathfrak{m}[0]. \tag{8.1.30}$$

However, we have natural isomorphisms

$$\begin{aligned} \tau_{\geq a-1}(L^\bullet \overset{\mathbf{L}}{\otimes}_V V/\mathfrak{m}[0]) &\xrightarrow{\sim} \tau_{\geq a-1}((\mathfrak{M}_i^\bullet \overset{\mathbf{L}}{\otimes}_V K^\bullet) \overset{\mathbf{L}}{\otimes}_V V/\mathfrak{m}[0]) && \text{(by remark 4.1.13(ii))} \\ &\xrightarrow{\sim} \tau_{\geq a-1}(\mathfrak{M}_i^\bullet \overset{\mathbf{L}}{\otimes}_V (K^\bullet \overset{\mathbf{L}}{\otimes}_V V/\mathfrak{m}[0])) && \text{(by remark 4.1.13(i))} \\ &\xrightarrow{\sim} \tau_{\geq a-1}(\mathfrak{M}_i^\bullet \overset{\mathbf{L}}{\otimes}_V \tau_{\geq a-1}(K^\bullet \overset{\mathbf{L}}{\otimes}_V V/\mathfrak{m}[0])) && \text{(by remark 4.1.13(ii))} \\ &\xrightarrow{\sim} 0 && \text{(by proposition 8.1.16(i))} \end{aligned}$$

whence (b), after considering the distinguished triangle $\tau_{\geq a-1}$(8.1.30).

(b)⇒(a): We remark

Claim 8.1.31. Condition (b) is equivalent to condition (a”) of remark 8.1.27(iii), with $c := a-1$.

Proof of the claim. Indeed, (b) holds if and only if $\Psi^l_{[a-1,b]}K^\bullet = 0$ in $\mathsf{D}^{[a-1,b]}(R/\mathfrak{m}R\text{-}\mathbf{Mod})$ (notation of remark 8.1.27(ii)). If the latter condition holds, then clearly (a”) holds with $c := a-1$. Conversely, if (a”) holds for this value of c, then $\mathrm{Hom}_{\mathsf{D}(R/\mathfrak{m}R\text{-}\mathbf{Mod})}(\Psi^l_{[a-1,b]}K^\bullet, X^\bullet) = 0$ for every $X^\bullet \in \mathsf{D}^{[a-1,b]}(R/\mathfrak{m}R\text{-}\mathbf{Mod})$; especially, the identity automorphism of $\Psi^l_{[a-1,b]}K^\bullet$ factors through 0, so $\Psi^l_{[a-1,b]}K^\bullet$ vanishes. ◊

From claim 8.1.31 and remark 8.1.27(iii) we deduce that, if (b) holds, condition (a) of proposition 8.1.24 holds for $c := a-1$, and condition (b) of the same proposition holds trivially for this value of c, for every $L^\bullet \in \mathsf{D}^{[a;b]}(R\text{-}\mathbf{Mod})$. We conclude that the natural map

$$\mathrm{Hom}_{\mathsf{D}(R\text{-}\mathbf{Mod})}(K^\bullet, L^\bullet) \to \mathrm{Hom}_{\mathsf{D}(R^a\text{-}\mathbf{Mod})}(K^{\bullet a}, L^{\bullet a}) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{D}(R\text{-}\mathbf{Mod})}(K^{\bullet a}_{[!]}, L^\bullet)$$

is an isomorphism, for every $L^\bullet \in \mathsf{D}^{[a;b]}(R\text{-}\mathbf{Mod})$, whence (a). □

Proposition 8.1.32. *Let $a, b \in \mathbb{Z}$ be any two integers such that $a \leq b$. For every $L^\bullet \in \mathrm{Ob}(\mathsf{D}^{[a,b]}(R\text{-}\mathbf{Mod}))$, the following conditions are equivalent :*

(a) *$L^\bullet$ lies in the essential image of the right adjoint functor $X^\bullet \mapsto \tau_{\leq b}X^\bullet_{[*]}$.*

(b) *$R\mathrm{Hom}^\bullet_R(R/\mathfrak{m}R[0], L^\bullet) \in \mathrm{Ob}(\mathsf{D}^{>b+1}(R/\mathfrak{m}R\text{-}\mathbf{Mod}))$.*

Proof. By the same argument as in the proof of claim 8.1.29, we reduce to the case where $V = R$. Next, fix $i \in \mathbb{N}$ such that $i \geq b-a+2$, define

$$K^\bullet := R\mathrm{Hom}^\bullet_V(\mathfrak{M}^\bullet_i, L^\bullet) \qquad \mathfrak{P}_i := \mathfrak{M}_i \overset{\mathbf{L}}{\otimes}_V V/\mathfrak{m}[0]$$

and notice that

(8.1.33) $$\tau_{\geq a-b-1}\mathfrak{P}_i = 0 \qquad \text{in } \mathsf{D}^{[a-b-1,0]}(V/\mathfrak{m}\text{-}\mathbf{Mod})$$

due to proposition 8.1.16(i). Morever, in view of proposition 8.1.18(iii) and lemma 8.1.22, condition (a) is equivalent to :

(c) The morphism $\pi^\bullet_0 \circ \cdots \circ \pi^\bullet_{i-1} : \mathfrak{M}_i \to V[0]$ induces an isomorphism $L^\bullet \xrightarrow{\sim} \tau_{\leq b}K^\bullet$.

(c)⇒(b): We argue as in the proof of proposition 8.1.28; namely, set $H := H^{b+1}K^\bullet$; if (c) holds, we obtain a distinguished triangle

(8.1.34) $$H[-b-2] \to L^\bullet \to \tau_{\leq b+1}K^\bullet \to H[-b-1]$$

and by considering the induced distinguished triangle $\tau_{\leq b+1}R\mathrm{Hom}^\bullet_V(V/\mathfrak{m}[0], (8.1.34))$, we reduce to observing that

$$\begin{aligned}
\tau_{\leq b+1}R\mathrm{Hom}^\bullet_V(V/\mathfrak{m}[0], \tau_{\leq b+1}K^\bullet) &\xrightarrow{\sim} \tau_{\leq b+1}R\mathrm{Hom}^\bullet_V(V/\mathfrak{m}[0], K^\bullet) && \text{(by lemma 8.1.1(iii))} \\
&\xrightarrow{\sim} \tau_{\leq b+1}R\mathrm{Hom}^\bullet_V(\mathfrak{P}^\bullet_i, L^\bullet) && \text{(by [75, Th.10.8.7])} \\
&\xrightarrow{\sim} \tau_{\leq b+1}R\mathrm{Hom}^\bullet_V(\tau_{\geq a-b-1}\mathfrak{P}^\bullet_i, L^\bullet) && \text{(by lemma 8.1.1(iii))} \\
&\xrightarrow{\sim} 0 && \text{(by (8.1.33)).}
\end{aligned}$$

(b)⇒(a): Again, we proceed as in the proof of the corresponding assertion in proposition 8.1.28; namely, arguing as in the proof of claim 8.1.31, we see that condition (b) is equivalent to condition (b”) of remark 8.1.27(iv), with $c := b+1$. Hence, if (b) holds, condition (b) of proposition 8.1.24 holds for $c := b+1$, and notice that condition (a) of *loc.cit.* holds trivially for this value of c, for every $K^\bullet \in \mathsf{D}^{[a;b]}(R\text{-}\mathbf{Mod})$. We conclude that the natural map

$$\mathrm{Hom}_{\mathsf{D}(R\text{-}\mathbf{Mod})}(K^\bullet, L^\bullet) \to \mathrm{Hom}_{\mathsf{D}(R^a\text{-}\mathbf{Mod})}(K^{\bullet a}, L^{\bullet a}) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{D}(R\text{-}\mathbf{Mod})}(K^\bullet, \tau_{\leq b}L^{\bullet a}_{[*]})$$

is an isomorphism, for every $K^\bullet \in \mathsf{D}^{[a;b]}(R\text{-}\mathbf{Mod})$, whence (a). □

8.1.35. Let A be any V^a-algebra; recall that the localization functor $V\text{-}\mathbf{Alg} \to V^a\text{-}\mathbf{Alg}$ admits a left adjoint $R \mapsto R_{!!}$, whose restriction to the subcategory of $A_{!!}$-algebras yields a left adjoint for the localization functor $A_{!!}\text{-}\mathbf{Alg} \to A\text{-}\mathbf{Alg}$ ([36, Prop.2.2.29]). In [36], we have studied the deformation theory of A-algebras by means of this left adjoint, under the assumption that $\widetilde{\mathfrak{m}}$ is V-flat; here we wish to show that the same can be repeated in the current setting, if one makes appeal instead to the results of the foregoing paragraphs. To begin with, we remark :

Proposition 8.1.36. *Let $A \to B$ be any morphism of V^a-algebras, N any $B_{!!}$-module. We have:*

(i) *If the unit of adjunction $N \to N^a_*$ is injective, the natural map*

$$\mathrm{Exal}_{A_{!!}}(B_{!!}, N) \to \mathrm{Exal}_A(B, N^a) \tag{8.1.37}$$

is a bijection (notation of [36, §2.5.7]*).*

(ii) *If $N^a = 0$, then $\mathrm{Exal}_{A_{!!}}(B_{!!}, N) = 0$.*

Proof. (i): Let

$$\Sigma \quad : \quad 0 \to N^a \to E \xrightarrow{\varphi} B \to 0$$

be any square-zero extension of A-algebras; there follows a square-zero extension

$$\Sigma_{!!} \quad : \quad 0 \to N^a_!/\mathrm{Ker}\,\varphi_{!!} \to E_{!!} \to B_{!!} \to 0$$

of $A_{!!}$-algebras. Under the stated assumption, the counit of adjunction $N^a_! \to N$ factors uniquely through a B-linear map $g_\varphi : N^a_!/\mathrm{Ker}\,\varphi_{!!} \to N$; then $g_\varphi * \Sigma_{!!}$ (defined as in [36, §2.5.5]) yields an element of $\mathrm{Exal}_{A_{!!}}(B_{!!}, N)$ whose image under (8.1.37) equals the class of Σ. Conversely, if

$$\Omega \quad : \quad 0 \to N \to F \xrightarrow{\psi} B_{!!} \to 0$$

is a square-zero extension of $A_{!!}$-algebras, then by adjunction we get a natural map

$$\Omega^a_{!!} \to \Omega$$

which in turns, by simple inspection, induces an isomorphism $g_{\psi^a} * \Omega^a_{!!} \xrightarrow{\sim} \Omega$ in the category of square-zero $A_{!!}$-algebra extensions of $B_{!!}$ (details left to the reader). The assertion follows.

(ii): Suppose $N^a = 0$, and let Ω be as in the foregoing; it follows that $\psi^a : F^a \to B$ is an isomorphism of A-algebras. By adjunction, the morphism $(\psi^a)^{-1}$ corresponds to a map of $A_{!!}$-algebras $\varphi : B_{!!} \to E$, and it is easily seen that $\psi \circ \varphi$ is the identity automorphism of $B_{!!}$, whence the assertion. ☐

Definition 8.1.38. Let $f : A \to B$ be a morphism of V^a-algebras. We set

$$\mathbb{L}^a_{B/A} := (\mathbb{L}_{B_{!!}/A_{!!}})^a$$

which is a simplicial complex of B-modules that we call the *almost cotangent complex* of f.

Remark 8.1.39. (i) In case $\widetilde{\mathfrak{m}}$ is a flat V-module, we have introduced in [36, Def.2.5.20] a simplicial $B_{!!}$-module $\mathbb{L}_{B/A}$; now, notice that the notation of *loc.cit.* agrees with the current one: indeed, [36, Prop.8.1.7(ii)] shows that complex $(\mathbb{L}_{B/A})^a$ obtained by applying the derived localization functor to $\mathbb{L}_{B/A}$ is naturally isomorphic (in $\mathsf{D}(s.B\text{-}\mathbf{Mod})$) to the complex of definition 8.1.38.

(ii) Depending on the context, we will want to regard $\mathbb{L}_{B/A}$ either as a simplicial object, or as a cochain complex, via the Dold-Kan isomorphism ([75, Th.8.4.1]). The resulting slight notational ambiguity should not be a source of confusion.

Theorem 8.1.40. *In the situation of definition* 8.1.38*, let N be any B-module. Then there are natural isomorphisms*

$$\mathrm{Der}_A(B, N) \xrightarrow{\sim} \mathrm{Ext}^0_B(\mathbb{L}_{B/A}, N)$$
$$\mathrm{Exal}_A(B, N) \xrightarrow{\sim} \mathrm{Ext}^1_B(\mathbb{L}_{B/A}, N).$$

(Notation of [36, Def.2.5.22(i)]*; so, here we view $\mathbb{L}_{B/A}$ as an object of $\mathsf{D}^{\leq 0}(B\text{-}\mathbf{Mod})$.)*

Proof. The first isomorphism follows easily from [56, II.1.2.4.2] and the natural isomorphism

$$\Omega_{B_{!!}/A_{!!}} \xrightarrow{\sim} (\Omega_{B/A})_! \qquad (8.1.41)$$

proved in [36, Lemma 2.5.29] (the proof in *loc.cit.* does not use the assumption that $\widetilde{\mathfrak{m}}$ is V-flat). Clearly we have

$$\mathrm{Hom}_{\mathsf{D}(B_{!!}\text{-}\mathbf{Mod})}(Y[0], N_*[0]) = 0 \qquad \text{for every } B_{!!}\text{-module } Y \text{ such that } Y^a = 0. \qquad (8.1.42)$$

On the other hand, we have :

Claim 8.1.43. $\mathrm{Hom}_{\mathsf{D}(B_{!!}\text{-}\mathbf{Mod})}(\mathbb{L}_{B_{!!}/A_{!!}}, X^\bullet) = 0$ for every $X^\bullet \in \mathrm{Ob}(\mathsf{D}^{[0,1]}(B_{!!}\text{-}\mathbf{Mod}))$ such that $X^{\bullet a} = 0$.

Proof of the claim. For every $X^\bullet \in \mathrm{Ob}(\mathsf{D}^{[0,1]}(B_{!!}\text{-}\mathbf{Mod}))$ we have a distinguished triangle

$$H^0X^\bullet[0] \to X^\bullet \to H^1X^\bullet[-1] \to (H^0X^\bullet)[1]$$

which reduces to considering the cases where $X^\bullet = M[j]$ for some almost zero $B_{!!}$-module M, and $j = 0, -1$. The case where $j = -1$ follows from [56, III.1.2.3] and proposition 8.1.36(ii). The case where $j = 0$ follows easily from [56, II.1.2.4.2] and (8.1.41) : details left to the reader. ◊

Now, (8.1.42) says that $L^\bullet := N_*[0]$ fulfills condition (b) of proposition 8.1.24, and claim 8.1.43 says that $K^\bullet := \mathbb{L}_{B_{!!}/A_{!!}}$ fulfills condition (a), so the natural map

$$\mathrm{Ext}^1_{B_{!!}}(\mathbb{L}_{B_{!!}/A_{!!}}, N_*) \to \mathrm{Ext}^1_B(\mathbb{L}^a_{B/A}, N)$$

is an isomorphism. Taking into account proposition 8.1.36(i) and [56, III.1.2.3], the theorem follows. □

8.1.44. For the further study the almost cotangent complex, we shall need some preliminaries concerning the derived functors of certain non-additive functors. This material generalizes the results of [36, §8.1], that were obtained under the assumption that $\widetilde{\mathfrak{m}}$ is V-flat.

Lemma 8.1.45. *Let $(V, \mathfrak{m})$ be any basic setup, R a simplicial V-algebra, $n \in \mathbb{N}$ an integer, M and N two R-modules such that $H_iM = H_iN = 0$ for every $i \geq n$. The following holds :*

(i) *If $M^a = 0$ in $\mathsf{D}(R^a\text{-}\mathbf{Mod})$, then $a \cdot \mathbf{1}_M = 0$ in $\mathsf{D}(R\text{-}\mathbf{Mod})$, for every $a \in \mathfrak{m}$.*

(ii) *If $\varphi : M \to N$ is a morphism of R-modules such that φ^a in an isomorphism in $\mathsf{D}(R^a\text{-}\mathbf{Mod})$, then for every $a \in \mathfrak{m}$ we may find a morphism $\psi : N \to M$ in $\mathsf{D}(R\text{-}\mathbf{Mod})$, such that $\psi \circ \varphi = a \cdot \mathbf{1}_M$ and $\varphi \circ \psi = a \cdot \mathbf{1}_N$ in $\mathsf{D}(R\text{-}\mathbf{Mod})$.*

Proof. (i): For every R-module X, set

$$\tau_{\leq -1}X := \sigma \circ \omega X$$

(notation of remark 4.5.21(iii)). According to [56, I.3.2.1.9(ii)], there exists a natural sequence of morphisms

$$\tau_{\leq -1}X \to X \to s.H_0(X) \to \sigma(\tau_{\leq -1}X) \qquad \text{in } \mathsf{D}(R\text{-}\mathbf{Mod}) \qquad (8.1.46)$$

whose induced sequence of normalized complexes is a distinguished triangle in $\mathsf{D}(V\text{-}\mathbf{Mod})$ (*i.e.* a distinguished triangle of $\mathsf{D}(R\text{-}\mathbf{Mod})$, in the terminology of [56, I.3.2.2.4], and in view of [56, I.3.2.2.5]). Let now n and M be as in the lemma; we argue by induction on n. The case where $n = 0$ is trivial, so suppose that $n > 0$, and that the assertion has already been proven for all almost zero R-modules N such that $H_iN = 0$ for every $i \geq n - 1$. Especially, for $N := \tau_{\leq 1}X$ and $P := s.H_0(M)$ we have $a \cdot \mathbf{1}_{\omega N} = 0$ and $a \cdot \mathbf{1}_P = 0$ in $\mathsf{D}(R\text{-}\mathbf{Mod})$, for every $a \in \mathfrak{m}$. Since the adjunction $\sigma \circ \omega N \to N$ is an isomorphism in $\mathsf{D}(R\text{-}\mathbf{Mod})$ ([56, I.3.2.1.10]), we deduce that

$$\mathrm{Hom}_{\mathsf{D}(R\text{-}\mathbf{Mod})}(M, N)^a = 0 = \mathrm{Hom}_{\mathsf{D}(R\text{-}\mathbf{Mod})}(M, P)^a$$

whence $\mathrm{End}_{\mathsf{D}(R\text{-}\mathbf{Mod})}(M)^a = 0$, by virtue of (8.1.46) (with $X := M$) and [56, I.3.2.2.10]. The assertion follows.

(ii): Set $C := \mathrm{Cone}\,\varphi$; according to [56, I.3.2.2], we have a distinguished triangle

$$M \xrightarrow{\varphi} N \to C \to \sigma M \qquad \text{in } \mathsf{D}(R\text{-}\mathbf{Mod}) \tag{8.1.47}$$

whence – by [56, I.3.2.2.10] – an exact sequence of V-modules

$$\mathrm{Hom}_{\mathsf{D}(R\text{-}\mathbf{Mod})}(N, M) \xrightarrow{\alpha} \mathrm{End}_{\mathsf{D}(R\text{-}\mathbf{Mod})}(N) \xrightarrow{\beta} \mathrm{Hom}_{\mathsf{D}(R\text{-}\mathbf{Mod})}(N, C).$$

Now, let us write $a = \sum_{i=1}^n a_i b_i$ for some $a_1, b_1, \dots, a_n, b_n \in \mathfrak{m}$; the assumption on φ implies that $C^a = 0$ in $\mathsf{D}(R^a\text{-}\mathbf{Mod})$, therefore (i) yields $\beta(a_i \cdot \mathbf{1}_N) = a_i \cdot \beta(\mathbf{1}_N) = 0$, so there exists a morphism $\psi_i : N \to M$ in $\mathsf{D}(R\text{-}\mathbf{Mod})$ such that $\alpha(\psi_i) = a_i \cdot \mathbf{1}_N$, *i.e.* $\varphi \circ \psi_i = a_i \cdot \mathbf{1}_N$ for every $i = 1, \dots, n$. Likewise, by considering the long exact sequence $\mathbb{E}\mathsf{xt}^\bullet_R((8.1.47), M)$ provided by [56, I.3.2.2.10] we find, for every $i = 1, \dots, n$, a morphism $\psi'_i : N \to M$ such that $\psi'_i \circ \varphi = b_i \cdot \mathbf{1}_M$. Thus,

$$a_i \cdot \psi'_i = \psi'_i \circ \varphi \circ \psi_i = b_i \cdot \psi_i \qquad \text{for every } i = 1, \dots, n$$

and a simple computation shows that $\psi := \sum_{i=1}^n b_i \cdot \psi_i$ will do. □

Remark 8.1.48. Before considering non-additive functors, let us see how to define derived tensor products in $\mathsf{D}(R^a\text{-}\mathbf{Mod})$, for any simplicial V-algebra R. We proceed as in (8.1.10) : for given R^a-modules M, N, set

$$M \overset{\ell}{\otimes}_{R^a} N := (M_! \overset{\ell}{\otimes}_R N_!)^a.$$

(i) We claim that this rule yields a well defined functor

$$- \overset{\ell}{\otimes}_{R^a} - : \mathsf{D}(R^a\text{-}\mathbf{Mod}) \times \mathsf{D}(R^a\text{-}\mathbf{Mod}) \to \mathsf{D}(R^a\text{-}\mathbf{Mod}).$$

Indeed, say that $\varphi : M \to M'$ is a quasi-isomorphism of R^a-modules, and set $C := \mathrm{Cone}(\varphi_!)$. We need to check that $(\varphi_! \overset{\ell}{\otimes}_R N)^a$ is a quasi-isomorphism, for any R-module N; in light of remark 4.5.21(iv), it then suffices to show that $(C \overset{\ell}{\otimes}_R N)^a = 0$; but the latter R^a-module may be computed as

$$(C \otimes_R \perp^R_\bullet N)^a = C^a \otimes_{R^a} (\perp^R_\bullet N)^a$$

whence the claim, since $C^a = 0$ in $R^a\text{-}\mathbf{Mod}$.

(ii) Next, suppose that M is a flat R^a-module; then we claim that the natural morphism of R^a-modules

$$M \overset{\ell}{\otimes}_{R^a} N \to M \otimes_{R^a} N$$

is a quasi-isomorphism, for every R^a-module N. Indeed, by Eilenberg-Zilber's theorem 4.2.48, the assertion comes down to checking that the augmented simplicial R^a-module

$$(M_! \otimes_R \perp^R_\bullet N_!)^a \to M \otimes_{R^a} N$$

is aspherical. But for every $k \in \mathbb{N}$, the k-th column of the latter is isomorphic to the augmented R^a-module

$$M[k] \otimes_{R^a[k]} (\perp^{R[k]}_\bullet N_![k])^a \to M[k] \otimes_{R^a[k]} N[k]$$

which is aspherical, since $M[k]$ is a flat $R^a[k]$-module, whence the assertion.

(iii) Just as in (8.1.10), the foregoing immediately implies that we have a natural isomorphism

$$(M \overset{\ell}{\otimes}_R N)^a \xrightarrow{\sim} M^a \overset{\ell}{\otimes}_{R^a} N^a \qquad \text{in } \mathsf{D}(R^a\text{-}\mathbf{Mod})$$

for any two R-modules M and N (details left to the reader).

(iv) Furthermore, we get suspension and loop functors σ and ω for R^a-modules, by the rule :

$$\sigma M := (\sigma M_!)^a \quad \text{and} \quad \omega M := (\omega M_*)^a \qquad \text{for every } R^a\text{-module } M$$

from which it follows that σ is left adjoint to ω. Then, it is clear that the assertions of remark 4.5.21(iii) hold as well for these functors.

(v) Likewise, we define the cone of a morphism $\varphi : M \to N$ of R^a-modules, by the rule

$$\mathrm{Cone}\,\varphi := (\mathrm{Cone}\,\varphi_!)^a$$

and then the assertion of remark 4.5.21(iv) holds also for morphisms of R^a-modules.

(vi) In view of (iv) and (v), it is then easy to check that also lemma 4.5.22 holds *verbatim* for $A := V$, and any two R^a-modules X, Y.

Remark 8.1.49. (i) In the same vein, we may define derived tensor products of R^a-algebras, for any simplicial V-algebra R. Namely, if S and S'' are any two R^a-algebras, we set

$$S \overset{\ell}{\otimes}_{R^a} S' := (S_{!!} \overset{\ell}{\otimes}_R S'_{!!})^a$$

(see (8.1.35)), where $\overset{\ell}{\otimes}_R$ denotes the derived tensor product for R-algebras, defined in example 4.5.15. In view of remarks 4.5.21(ii) and 8.1.48(i), it is easily seen that this rule defines a functor

$$- \overset{\ell}{\otimes}_{R^a} - : \mathsf{D}(R^a\text{-}\mathbf{Alg}) \times \mathsf{D}(R^a\text{-}\mathbf{Alg}) \to \mathsf{D}(R^a\text{-}\mathbf{Alg})$$

and moreover, the formation of these tensor products commutes with the forgetful functor $\mathsf{D}(R^a\text{-}\mathbf{Alg}) \to \mathsf{D}(R^a\text{-}\mathbf{Mod})$.

(ii) Moreover, they are computed by arbitrary flat resloutions : if S (or S') is a flat R^a-algebras, then the natural morphism

$$S \overset{\ell}{\otimes}_{R^a} S' \to S \otimes_{R^a} S'$$

is an isomorphism in $\mathsf{D}(R^a\text{-}\mathbf{Alg})$.

(iii) Furthermore, if $R^a \to S$ is a given morphism of simplicial V^a-algebras, we obtain a well defined functor

$$\mathsf{D}(R^a\text{-}\mathbf{Alg}) \to \mathsf{D}(S\text{-}\mathbf{Alg}) \qquad S' \mapsto S \overset{\ell}{\otimes}_{R^a} S'.$$

Namely, given an R^a-algebra S', we pick a resolution $P \to S'$ with P a flat R^a-algebra, and endow $S \otimes_{R^a} P$ with its natural S-algebra structure, which is independent, up to natural isomorphism, of the choice of P. All the verifications are exercises for the reader.

8.1.50. Resume the situation of (4.5.25), let R be any simplicial V-algebra, $d \in \mathbb{N}$ any integer, and suppose additionally that :

- T is *homogeneous of degree* d, *i.e.* we have $T(a \cdot \mathbf{1}_M) = a^d \cdot \mathbf{1}_{TM}$, for every object (A, M) of $V\text{-}\mathbf{Alg}.\mathbf{Mod}$, and every $a \in V$.
- The ideal $\mathfrak{m} \cdot H_0(R)$ of $H_0(R)$ satisfies condition $(\mathbf{B})$ of [36, §2.1.6].

Remark 8.1.51. (i) Let T and T' be two functors as in (8.1.50), with T (resp. T') homogeneous of degree d (resp. d'), and consider the functor

$$T \otimes T' : V\text{-}\mathbf{Alg}.\mathbf{Mod} \to V\text{-}\mathbf{Alg}.\mathbf{Mod} \qquad (A, M) \mapsto T(A, M) \otimes_A T'(A, M).$$

Then it easily seen that $T \otimes T'$ fulfills the conditions of (8.1.50), and it is homogeneous of degree $d + d'$.

(ii) Likewise, suppose that $f : T \to T'$ is a natural transformation of functors fulfilling the conditions of (8.1.50), for the same degree d; then the same holds for the functors $\mathrm{Ker}\, f$ and $\mathrm{Coker}\, f$.

(iii) In the situation of (8.1.50), let $f : H_0(R) \to B$ be any morphism of V-algebras, and $\varphi : M \to N$ any B-linear map such that φ^a is an isomorphism; arguing as in the proof of lemma 8.1.45(ii), it is easily seen that, for every $a \in \mathfrak{m}$ there exists a B-linear map $\psi : N \to M$ such that $\varphi \circ \psi = a \cdot \mathbf{1}_N$ and $\psi \circ \varphi = a \cdot \mathbf{1}_M$. Since $T_B(M)$ and $T_B(M')$ are two B-modules and $(\mathbf{B})$ holds for $\mathfrak{m} \cdot H_0(R)$ (hence also for $\mathfrak{m}B$), the homogeneity property of T implies that $T_B(\varphi)^a$ is an isomorphism as well (details left to the reader). Especially, the above holds with $B := R[p]$, for any $p \in \mathbb{N}$, and f the natural map given by the degeneracies of the simplicial V-algebra R. It is then clear that T_R induces a well defined functor

$$T_{R^a} : R^a\text{-}\mathbf{Mod} \to R^a\text{-}\mathbf{Mod}.$$

The following result shows that, in this situation, the construction of left derived functors descends likewise to R^a-modules.

Theorem 8.1.52. *In the situation of* (8.1.50)*, the following holds :*

(i) *Let $\varphi : M \to N$ be a morphism of R-modules, and $n \in \mathbb{N}$ an integer such that $H_i(\varphi)^a : (H_iM)^a \to (H_iN)^a$ is an isomorphism of V^a-modules, for every $i \leq n$. Then*

$$H_i(LT\varphi)^a : H_i(LTM)^a \to H_i(LTN)^a$$

is an isomorphism of V^a-modules, for every $i \leq n$.

(ii) *Especially, the functor T_R induces a well defined left derived functor*

$$LT_{R^a} : \mathsf{D}(R^a\text{-}\mathbf{Mod}) \to \mathsf{D}(R^a\text{-}\mathbf{Mod}).$$

Proof. Clearly (ii) follows from (i).

(i): In light of corollary 4.5.24(i), we may replace M and N by respectively $\mathrm{cosk}_n M$ and $\mathrm{cosk}_n N$, after which, we may assume that $H_iM = H_iN = 0$ for every $i > n$, so φ^a is an isomorphism in $\mathsf{D}(R^a\text{-}\mathbf{Mod})$. Next, by proposition 4.5.18, we may replace M and N by their respective standard free resolutions, in which case we are reduced to checking that the induced map $(T_R\varphi)^a$ is an isomorphism in $\mathsf{D}(R^a\text{-}\mathbf{Mod})$. Since T is homogeneous and $(\mathbf{B})$ holds for $\mathfrak{m} \cdot H_0(R)$, the latter assertion follows straightforwardly from lemma 8.1.45(ii). ☐

8.1.53. Next, we wish to decide how much of the foregoing theory can be salvaged, when we drop condition $(\mathbf{B})$. We will concentrate on the functors that are relevant to the later study of the cotangent complex. Thus, henceforth, for every integer $d \in \mathbb{N}$, we shall denote by T^d one of the three standard functors

$$\mathrm{Sym}^d, \Lambda^d, \Gamma^d : V\text{-}\mathbf{Alg}.\mathbf{Mod} \to V\text{-}\mathbf{Alg}.\mathbf{Mod}$$

(namely, the symmetric and antisymmetric d-th power functors, and the d-th divided power functor). Notice that the functor T^d fulfills the conditions of (8.1.50), for every $d \in \mathbb{N}$. Moreover, in all three cases we have natural identifications :

$$T^1 \xrightarrow{\sim} \mathbf{1}_{V\text{-}\mathbf{Alg}.\mathbf{Mod}}. \tag{8.1.54}$$

In general, we can no longer expect that T^d descends to almost modules; indeed, consider the following :

Example 8.1.55. Let $(V, \mathfrak{m})$ be as in (8.1.53), and $p \in \mathbb{N}$ any prime integer. The Frobenius map $\Phi_R : R/pR \to R/pR$ for V-algebras R, can be seen as a homogeneous polynomial law of degree p on the V-module V/pV. Denote by $\mathfrak{m}^{(p)} \subset V$ the ideal generated by $(x^p \mid x \in \mathfrak{m})$. Clearly Φ descends to a polynomial law

$$\overline{\Phi} : V/(pV + \mathfrak{m}) \to W_p := V/(pV + \mathfrak{m}^{(p)})$$

which is still homogeneous of degree p, so it factors through a unique V-linear map

$$\Gamma^p_A(V/(pV + \mathfrak{m})) \to W_p.$$

The latter is surjective, since its image contains the class of the unit element of V. Now, the proof of [36, Prop.2.1.7(ii)] shows that – if $(\mathbf{B})$ does not hold for $\mathfrak{m}$ – there exists some prime p such that $pV + \mathfrak{m}^{(p)}$ does not contain $\mathfrak{m}$, therefore $W_p^a \neq 0$. This shows that the functor

$$V\text{-}\mathbf{Mod} \to V^a\text{-}\mathbf{Mod} \qquad M \mapsto (\Gamma_A^p M)^a$$

does not factor through V^a-$\mathbf{Mod}$.

However, we will see that example 8.1.55 is, in a sense, the worst that can happen.

Lemma 8.1.56. *In the situation of* (8.1.53)*, we have :*

(i) *There are natural transformations*

$$T^{i+j} \to T^i \otimes T^j \to T^{i+j} \qquad \textit{for every } i,j \in \mathbb{N}$$

whose composition equals $\binom{i+j}{i} \cdot \mathbf{1}_{T^{i+j}}$. *(Notation of remark* 8.1.51(i)*.)*

(ii) *For every simplicial V-algebra R, the maps of* (i) *induce natural transformations*

$$LT_R^{i+j} \to LT_R^i \overset{\ell}{\otimes}_R LT_R^j \to LT_R^{i+j} \qquad \textit{for every } i,j \in \mathbb{N}$$

whose composition equals $\binom{i+j}{i} \cdot \mathbf{1}_{LT_R^{i+j}}$.

Proof. (i): For $T = \Lambda$, the sought morphism $\Lambda^{i+j} \to \Lambda^i \otimes \Lambda^j$ is the one denoted $\Delta_{i,j}$ in [36, §4.3.20], and the morphism $\Lambda^i \otimes \Lambda^j \to \Lambda^{i+j}$ is given by the rule : $x \otimes y \mapsto x \wedge y$, for every $(A, M) \in \mathrm{Ob}(V\text{-}\mathbf{Alg}.\mathbf{Mod})$, every $x \in \Lambda_A^i M$ and every $y \in \Lambda_A^j M$. The sought identity follows easily from the explicit formula given in [36, (4.3.21)] : details left to the reader.

For $T = \mathrm{Sym}$, the natural transformation $\mathrm{Sym}^{i+j} \to \mathrm{Sym}^i \otimes \mathrm{Sym}^j$ is given by a similar formula; namely, for every object (A, M) of V-$\mathbf{Alg}.\mathbf{Mod}$, every sequence of elements $x_1, \dots, x_{i+j} \in M$, and every subset $I \subset \{1, \dots, i+j\}$, set $x_I := \prod_{k \in I} x_k \in \mathrm{Sym}_A^k M$ (where the multiplication is formed in the graded ring $\mathrm{Sym}_A^\bullet M$); one checks easily that the rule

$$x_1 \cdots x_{i+j} \mapsto \sum_{I,J} x_I \otimes x_J$$

defines a well defined A-linear map on $\mathrm{Sym}_A^{i+j} M$ (where the sum ranges over all the partitions (I, J) of $\{1, \dots, i+j\}$ such that the cardinality of I equals i). Then one defines a map $\mathrm{Sym}_A^i M \otimes_A \mathrm{Sym}_A^j M \to \mathrm{Sym}_A^{i+j} M$ by the rule : $u \otimes v \mapsto u \cdot v$ for every $u \in \mathrm{Sym}_A^i M$ and $v \in \mathrm{Sym}_A^j M$. Again, the sought identity is verified by direct computation.

If $T = \Gamma$, then for every object (A, M) of V-$\mathbf{Alg}.\mathbf{Mod}$ we define a homogeneous polynomial law $M \rightsquigarrow \Gamma_A^i M \otimes_A \Gamma_A^j M$ of degree $i + j$, by the rule : $x \mapsto x^{[i]} \otimes x^{[j]}$ for every A-algebra B, and every $x \in B \otimes_A M$. This law yields a transformation $\Gamma^{i+j} \to \Gamma^i \otimes \Gamma^j$ as sought. Next, the multiplication of the graded ring functor $\Gamma^\bullet$ gives a transformation $\Gamma^i \otimes \Gamma^j \to \Gamma^{i+j}$. The composition of these two transformations is characterized as the unique transformation ψ of Γ^{i+j} such that $\psi_{(A,M)}(x^{[i+j]}) = x^{[i]} \cdot x^{[j]}$ for every object (A, M) of V-$\mathbf{Alg}.\mathbf{Mod}$, and every $x \in M$. Then, by corollary 4.6.81 we must have $\psi_{(A,M)} = \binom{i+j}{i} \mathbf{1}_M$, as required.

(ii): Recall that the three functors Λ, Γ and Sym transform free A-modules into free A-modules, for every V-algebra A. Therefore, the sought natural transformation is none else than the map

$$T_R^{i+j}(\perp_\bullet^R M) \to T_R^i(\perp_\bullet^R M) \otimes_R T_R^j(\perp_\bullet^R M) \to T_R^{i+j}(\perp_\bullet^R M)$$

given by (i), for any R-module M. □

Proposition 8.1.57. *In the situation of* (8.1.53)*, let $p \in \mathbb{N}$ be a prime integer, R a simplicial $V \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$-algebra, M an R-module such that $M^a = 0$ in $\mathsf{D}(R^a\text{-}\mathbf{Mod})$. Then*

$$(LT_R^d M)^a = 0 \qquad \textit{in } \mathsf{D}(R^a\text{-}\mathbf{Mod}), \textit{ for every } d \in \mathbb{N} \textit{ such that } (p,d) = 1.$$

Proof. In light of corollary 4.5.24(i), we may replace M by $\mathrm{cosk}_i M$, in which case lemma 8.1.45(i) implies that $a \cdot \mathbf{1}_M = 0$ for every $a \in \mathfrak{m}$. Now, we apply lemma 8.1.56(ii) with $i = 1$ and $j = d - 1$ (notice that $j \geq 0$, since $d > 0$); in view of (8.1.54), there result natural transformations

$$LT^d_R M \to M \otimes_R LT^{d-1}_R M \to LT^d_R M$$

whose composition is $d \cdot \mathbf{1}_{LT^d_R M}$. Since the image of d is invertible in R, it follows easily that $a \cdot \mathbf{1}_{LT^d_R M} = 0$ for every $a \in \mathfrak{m}$, whence the assertion. □

Theorem 8.1.58. *Let $d, n, p \in \mathbb{N}$ be any three integers, with $d > 0$ and p a prime. Let also R be a simplicial $V \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$-algebra, M an R-module, and suppose that*

(a) $H_i M = 0$ *for every* $i < n$.
(b) $M^a = 0$ *in* $\mathsf{D}(R^a\text{-}\mathbf{Mod})$.

Then we have :

(i) $H_i(L\Gamma^d_R M)^a = 0$ *for every* $i < n$.
(ii) $H_i(L\Lambda^d_R M)^a = 0$ *for every* $i < n + p - 1$.
(iii) $H_i(L\mathrm{Sym}^d_R M)^a = 0$ *for every* $i < n + 2(p - 1)$.

Proof. (i) is just a special case of corollary 4.5.24(ii), and for $d < p$, assertions (ii) and (iii) follow from the more general proposition 8.1.57, hence we may assume that $d \geq p$. For every $j = 0, \dots, d$, set

$$F^j := \Gamma^j \otimes \Lambda^{d-j} \qquad G^j := \Lambda^j \otimes \mathrm{Sym}^{d-j}.$$

By remark 8.1.51(i), these functors are homogeneous of degree d, and fulfill the conditions of (8.1.50). Moreover, since Γ, Λ and Sym transform free modules into free modules, we have natural isomorphisms of functors :

$$LF^j_R \xrightarrow{\sim} L\Gamma^j_R \overset{\ell}{\otimes}_R L\Lambda^{d-j}_R \quad LG^j_R \xrightarrow{\sim} L\Lambda^j_R \overset{\ell}{\otimes}_R L\mathrm{Sym}^{d-j}_R \qquad \text{for every } j = 0, \dots, d \tag{8.1.59}$$

(details left to the reader).

(ii): According to [56, I.4.3.1.7], there is a natural complex of functors :

$$0 \to F^d_R \xrightarrow{\partial_{d-1}} F^{d-1}_R \to \cdots \to F^1_R \xrightarrow{\partial_0} F^0_R \to 0 \tag{8.1.60}$$

which is exact on flat R-modules. Set $Z_p := \mathrm{Ker}\, \partial_{p-1}$; due to remark 8.1.51(ii) we may consider the derived functor LZ_p, and from corollary 4.5.24(ii) and assumption (a), we get

$$H_i(LZ_p M) = 0 \qquad \text{for every } i < n. \tag{8.1.61}$$

The evaluation of (8.1.60) on $(\perp^R_\bullet M)^\Delta$ gives an exact sequence of R-modules :

$$0 \to LZ_p M \to LF^{p-1}_R M \to \cdots \to LF^1_R M \to LF^0_R M \to 0.$$

Notice as well, that

$$(LF^j_R M)^a = 0 \qquad \text{in } \mathsf{D}(R^a\text{-}\mathbf{Mod}) \text{ for } j = 1, \dots, p-1 \tag{8.1.62}$$

in view of (8.1.59), remark 8.1.48(iii), and proposition 8.1.57. The assertion now follows from (8.1.61) and claim 4.5.39.

(iii): According to [56, I.4.3.1.7], there is a natural complex of functors :

$$\Sigma \quad : \quad 0 \to G^d_R \xrightarrow{\partial_{d-1}} G^{d-1}_R \to \cdots \to G^1_R \xrightarrow{\partial_0} G^0_R \to 0.$$

Set $Z'_p := \mathrm{Ker}\, \partial_{p-1}$; due to remark 8.1.51(ii) we may consider the derived functor LZ'_p, and since Σ is exact on flat R-modules, we obtain two exact sequences of R-modules :

$$\Sigma' \quad : \quad 0 \to LG^d_R M \to LG^{d-1}_R M \to \cdots \to LG^p_R M \to LZ'_p M \to 0$$
$$\Sigma'' \quad : \quad 0 \to LZ'_p M \to LG^{p-1}_R M \to \cdots \to LG^1_R M \to LG^0_R M \to 0.$$

On the one hand, in light of (ii), remark 8.1.48(iii,vi) and (8.1.59), we see that

$$\mathrm{cosk}_{n+p-1}(LG^j_R M)^a = 0 \qquad \text{in } \mathsf{D}(R^a\text{-}\mathbf{Mod}) \text{ for every } j = 1, \dots, d.$$

By applying claim 4.5.39 to the exact sequence $\mathrm{cosk}_{n+p-1}\Sigma'^a$, we deduce that $H_i(LZ'_p M)^a = 0$ for every $i < n+p-1$. On the other hand, (8.1.59) and proposition 8.1.57 imply that

$$(LG^j_R M)^a = 0 \qquad \text{in } \mathsf{D}(R^a\text{-}\mathbf{Mod}) \text{ for } j = 1, \dots, p-1$$

so the assertion follows, after applying claim 4.5.39 to the exact sequence Σ''. ☐

8.1.63. Let now $\varphi : R \to S$ be any morphism of simplicial V^a-algebras; pick any resolution $\rho : P \to S$ with P a flat R-algebra, and consider the composition

$$P \otimes_R S \xrightarrow{\rho \otimes_R S} S \otimes_R S \xrightarrow{\mu_S} S \tag{8.1.64}$$

where μ_S is the multiplication law of S. Then (8.1.64) represents a morphism

$$\Delta(\varphi) : S \overset{\ell}{\otimes}_R S \to S \qquad \text{in } \mathsf{D}(R\text{-}\mathbf{Alg})$$

which is independent (up to unique isomorphism) of the choice of P. Notice that, if φ is an isomorphism in $\mathsf{D}(s.V^a\text{-}\mathbf{Alg})$, then the same holds for $\varphi \overset{\ell}{\otimes}_R \varphi : R \to S \overset{\ell}{\otimes}_R S$; also, we have

$$\varphi = \Delta(\varphi) \circ (\varphi \overset{\ell}{\otimes}_R \varphi) \qquad \text{in } \mathsf{D}(s.V^a\text{-}\mathbf{Alg}).$$

More generally, set $C := \mathrm{Cone}\,\varphi$, and endow $S \overset{\ell}{\otimes}_R S$ with the S-module structure deduced from its left tensor factor; then, by considering the section $S \overset{\ell}{\otimes}_R \varphi$ of $\Delta(\varphi)$, we get a natural decomposition

$$S \overset{\ell}{\otimes}_R S \overset{\sim}{\to} S \oplus (S \overset{\ell}{\otimes}_R C) \qquad \text{in } \mathsf{D}(S\text{-}\mathbf{Mod})$$

which identifies $\Delta(\varphi)$ with the natural projection, whence a natural isomorphism

$$\mathrm{Cone}\,\Delta(\varphi) \overset{\sim}{\to} \sigma S \overset{\ell}{\otimes}_R C. \tag{8.1.65}$$

Thus, say that $C = \sigma^k C'$ in $\mathsf{D}(R\text{-}\mathbf{Mod})$, for some R-module C' with $H_0 C' \neq 0$; there follows a distinguished triangle in $\mathsf{D}(R\text{-}\mathbf{Mod})$

$$\sigma^k C' \to \sigma^k S \overset{\ell}{\otimes}_R C' \to \sigma^{2k} C' \overset{\ell}{\otimes}_R C' \to \sigma^{k+1} C'.$$

Especially, if $k \geq 2$, then $H_k C = H_k(S \overset{\ell}{\otimes}_R C)$, and we see that, in this case, $\mathrm{Cone}\,\Delta(\varphi) = \sigma^{k+1} C''$ in $\mathsf{D}(S\text{-}\mathbf{Mod})$, for some C'' such that $H_0 C'' \neq 0$. However, it is clear from (8.1.65) that $\mathrm{Cone}\,\Delta^2(\varphi) = \sigma^2 C''$ for some object C'' of $\mathsf{D}(S\text{-}\mathbf{Mod})$. We conclude that, if $\Delta^n(\varphi)$ is an isomorphism in $\mathsf{D}(S\text{-}\mathbf{Mod})$ for some $n \geq 2$, then the same holds already for $\Delta^2(\varphi)$.

Definition 8.1.66. In the situation of (8.1.63), we say that φ is a *weakly étale morphism*, if $\Delta^2(\varphi)$ is an isomorphism in $\mathsf{D}(S\text{-}\mathbf{Alg})$.

8.1.67. Let $\varphi : R' \to R$ be any morphism of simplicial V^a-algebras, and S, T two R-algebras. Since the standard resolution $F_R(S) := F_{R_{!!}}(S_{!!})^a$ of example 4.5.15 is clearly functorial in both R and S, we have a natural map

$$F_{R'}(S)^\Delta \to F_R(S)^\Delta$$

of simplicial R'-algebras, whence a natural morphism

$$S \overset{\ell}{\otimes}_{R'} T \to F_R(S)^\Delta \otimes_{R'} T \to S \overset{\ell}{\otimes}_R T. \tag{8.1.68}$$

Lemma 8.1.69. *In the situation of* (8.1.67), *suppose that* φ *is a quasi-isomorphism. Then the same holds for* (8.1.68).

Proof. Let $\psi : F_{R'}(S)^{\Delta} \otimes_{R'} R \to F_R(S)^{\Delta}$ be the natural morphism; by construction, (8.1.68) factors as the composition of $\psi \otimes_R \mathbf{1}_T$ and the natural isomorphism

$$F_{R'}(S)^{\Delta} \otimes_{R'} T \stackrel{\sim}{\to} (F_{R'}(S)^{\Delta} \otimes_{R'} R) \otimes_R T.$$

Since both $F_{R'}(S)^{\Delta} \otimes_{R'} R$ and $F_R(S)^{\Delta}$ are flat R-algebras, remark 8.1.49(ii) then reduces to checking that ψ is a quasi-isomorphism. Since the natural maps $\beta : F_R(S)^{\Delta} \to S$ and $\alpha : F_{R'}(S)^{\Delta} \to F_{R'}(\Delta) \otimes_{R'} R$ are quasi-isomorphisms, it suffices to show that the composition $\beta \circ \psi \circ \alpha$ is a quasi-isomorphism. But the latter is none else than the standard resolution $F_{R'}(S) \to S$, whence the lemma. □

Proposition 8.1.70. *Let $\varphi : R \to S$ and $\varphi' : R' \to S'$ be two morphisms of simplicial V-algebras, and suppose we have a commutative diagram in* $\mathsf{D}(R\text{-}\mathbf{Alg})$

(8.1.71)
$$\begin{array}{ccc} R & \xrightarrow{\varphi} & S \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\psi'} \\ R' & \xrightarrow{\varphi'} & S' \end{array}$$

where ψ and ψ' are isomorphisms (in $\mathsf{D}(R\text{-}\mathbf{Alg})$*). Then φ^a is weakly étale if and only if the same holds for φ'^a.*

Proof. Denote by $\mathsf{Hot}(V\text{-}\mathbf{Alg})$ the homotopy category of simplicial V-algebras, and recall that the multiplicative system of quasi-isomorphisms in $\mathsf{Hot}(V\text{-}\mathbf{Alg})$ admits a right calculus of fractions ([56, I.3.1.8(ii)]). We begin with the following special case :

Claim 8.1.72. Suppose that ψ and ψ' are also morphisms of simplicial V-algebras, and that (8.1.71) commutes in the category $\mathsf{Hot}(V\text{-}\mathbf{Alg})$. Then the proposition holds.

Proof of the claim. Indeed, in this situation it follows easily from lemma 8.1.69 that φ^a (resp. φ'^a) is weakly étale if and only if the same holds for $\psi'^a \circ \varphi^a$ (resp. for $\varphi'^a \circ \psi^a$), so we are reduced to the case where $R = R'$, $S = S'$, and $\varphi, \varphi' : R \to S$ are two homotopic morphisms of simplicial V-algebras. In this case, there exist morphisms of simplicial V-algebras $\varphi'' : R \to S''$ and $d_0, d_1 : S'' \to S$, such that d_0 and d_1 are quasi-isomorphisms, and $\varphi = d_0 \circ \varphi''$, $\varphi' = d_1 \circ \varphi''$ (see [56, I.2.3.2]). Then the claim follows by applying repeatedly lemma 8.1.69. ◊

Next, there exists a simplicial V-algebra S'' and morphisms $\beta : S'' \to S$ and $\gamma : S'' \to S'$ of simplicial V-algebras, such that both β and γ are quasi-isomorphisms, and $\psi' = \gamma \circ \beta^{-1}$ in $\mathsf{D}(s.V\text{-}\mathbf{Alg})$. Moreover, there exists morphisms $\varphi'' : R'' \to S''$ and $\psi'' : R'' \to R$ such that ψ'' is a quasi-isomorphism, and the resulting diagram

$$\begin{array}{ccc} R'' & \xrightarrow{\varphi''} & S'' \\ {\scriptstyle\psi''}\downarrow & & \downarrow{\scriptstyle\beta} \\ R & \xrightarrow{\varphi} & S \end{array}$$

commutes in $\mathsf{Hot}(V\text{-}\mathbf{Alg})$. By claim 8.1.72, we may then replace φ by φ'' and ψ' by β, after which we may assume that ψ' is a morphism of simplicial V-algebras.

Likewise, we may find morphisms $\beta' : R'' \to R$ and $\gamma' : R'' \to R'$ of simplicial V-algebras that are quasi-isomorphisms, and such that $\psi = \gamma' \circ \beta'^{-1}$ in $\mathsf{D}(s.V\text{-}\mathbf{Alg})$; again by lemma 8.1.69, we see that φ^a is weakly étale if and only if the same holds for $\varphi^a \circ \beta^a$, so we may replace φ by $\varphi \circ \beta$, ψ by $\psi \circ \gamma$, and assume from start that also ψ is a map of simplicial V-algebras.

In this situation, by applying once again lemma 8.1.69 we see that φ^a (resp. φ'^a) is weakly étale if and only if the same holds for $\psi'^a \circ \varphi^a$ (resp. for $\varphi'^a \circ \psi^a$). Therefore, we may assume from start that $R = R'$, $S = S'$, and $\varphi, \varphi' : R \to S$ are two maps of simplicial V-algebras that represent the same morphism in $\mathsf{D}(s.V\text{-}\mathbf{Alg})$. In this case, we may find a morphism $\vartheta : R''' \to R$ of simplicial V-algebras, such that ϑ is a quasi-isomorphism, and $\varphi\circ\vartheta, \varphi'\circ\vartheta : R''' \to S$ are homotopic maps; then the assertion follows from claim 8.1.72 (and again, from lemma 8.1.69: details left to the reader). □

Corollary 8.1.73. *Let $\varphi : R \to S$ and $\psi : S \to T$ be any two morphisms of simplicial V^a-algebras. We have :*

(i) *If φ and ψ are weakly étale, the same holds for $\psi\circ\varphi$.*
(ii) *If φ and $\psi\circ\varphi$ are weakly étale, the same holds for ψ.*
(iii) *If φ is weakly étale, and $R \to R'$ is any morphism of simplicial V^a-algebras, then $\varphi' := R' \overset{\ell}{\otimes}_R \varphi$ is weakly étale.*
(iv) *φ is weakly étale if and only if the same holds for $\Delta(\varphi)$.*

Proof. (iv) is immediate from the discussion of (8.1.63).

(iii): Endow $S' := R' \overset{\ell}{\otimes}_R S$ with the R'-algebra structure deduced from its left tensor factor; then the assertion is an immediate consequence of the following more general

Claim 8.1.74. There exists a commutative diagram in $\mathsf{D}(R'\text{-}\mathbf{Alg})$

$$\begin{tikzcd} S' \overset{\ell}{\otimes}_{R'} S' \arrow[rr] \arrow[dr, "\Delta(\varphi')"'] & & R' \overset{\ell}{\otimes}_R (S \overset{\ell}{\otimes}_R S) \arrow[dl, "R'\overset{\ell}{\otimes}_R\Delta(\varphi)"] \\ & S' & \end{tikzcd}$$

whose horizontal arrow is an isomorphism.

Proof of the claim. Pick any resolution $\rho : P \to S$ with P a flat R-algebra; then $R' \otimes_R P$ represents $R' \overset{\ell}{\otimes}_R S$, and we have a natural isomorphism of R'-algebras

$$(R' \otimes_R P) \otimes_{R'} (R' \otimes_R P) \overset{\sim}{\to} R' \otimes_R (P \otimes_R P).$$

In view of remark 4.5.21(i), the source of this map represents $S' \overset{\ell}{\otimes}_{R'} S'$, and the target represents $R' \overset{\ell}{\otimes}_R (S \overset{\ell}{\otimes}_R S)$. Under these identifications, the morphism $R' \overset{\ell}{\otimes}_R \Delta(\varphi)$ becomes the map $\mathbf{1}_{R'} \otimes_R (\mu_S \circ (\rho \otimes_R \rho))$ (notation of (8.1.63)), whereas $\Delta(\varphi')$ is represented by the multiplication map $\mu_{R'\otimes_R P} = \mathbf{1}_{R'} \otimes_R \mu_P$ of $R' \otimes_R P$. The claim follows straightforwardly. ◇

(i): Pick a resolution $P \to S_{!!}$ with P a flat $R_{!!}$-algebra, and a resolution $Q \to T_{!!}$ with Q a flat P-algebra; in light of proposition 8.1.70 it suffices to show that the resulting morphism $R \to Q^a$ is weakly étale, so we may replace S by P^a and T by Q^a, and assume from start that both φ and ψ are flat morphisms. Due to (iv), it then suffices to check that $\Delta(\psi\circ\varphi) : T \otimes_R T \to T$ is weakly étale; however, the latter can be factored as the composition

$$T \otimes_R T \overset{\sim}{\to} T \otimes_S (S \otimes_R S) \otimes_S T \xrightarrow{T\otimes_S\Delta(\varphi)\otimes_S T} T \otimes_S T \xrightarrow{\Delta(\psi)} T \tag{8.1.75}$$

and by (iv) the maps $\Delta(\psi)$ and $\Delta(\varphi)$ are weakly étale, so the same holds for $T \otimes_S \Delta(\varphi) \otimes_S T$, in view of (iii). We may therefore replace ψ by $\Delta(\psi)$ and φ by $T \otimes_S \Delta(\varphi) \otimes_S T$, after which, we may assume that $\Delta(\psi)$ is an isomorphism in $\mathsf{D}(s.A\text{-}\mathbf{Alg})$. In this case, the factorization (8.1.75) makes it clear that $\Delta(\psi\circ\varphi)$ is weakly étale, as stated.

(ii): Set $S' := S \overset{\ell}{\otimes}_R S$ and $T' := S \overset{\ell}{\otimes}_R T$; we may factor ψ as the composition

$$S \xrightarrow{S\overset{\ell}{\otimes}_R(\psi\circ\varphi)} T' \xrightarrow{T'\overset{\ell}{\otimes}_{S'}\Delta(\varphi)} T$$

so the assertion follows from (i), (iii) and (iv). □

Theorem 8.1.76. *Let $\varphi : R \to S$ be a morphism of simplicial V-algebras, $c, p \in \mathbb{N}$ two integers, with p a prime, and suppose that φ^a is weakly étale. We have :*

(i) *If the ideal $\mathfrak{m} \cdot H_0(S)$ of $H_0(S)$ satisfies condition $(\mathbf{B})$ of* [36, §2.1.6], *then $\mathbb{L}^a_{S/R} = 0$ in $\mathsf{D}(S^a\text{-}\mathbf{Mod})$.*

(ii) *If S is a $\mathbb{Z}_{(p)}$-algebra and $H_i\mathbb{L}_{S/R} = 0$ for every $i < c$, then $H_i\mathbb{L}^a_{S/R} = 0$ for every $i < c + 2p - 1$.*

Proof. We begin with the following :

Claim 8.1.77. In order to prove the theorem, we may assume that both $\Delta(\varphi^a)$ and $H_0(\varphi)$ are isomorphisms.

Proof of the claim. To ease notation, set $S' := S \overset{\ell}{\otimes}_R S$, and consider the sequence of morphisms

$$S \xrightarrow{\mathbf{1}_S \overset{\ell}{\otimes}_R \varphi} S' \xrightarrow{\mu_B} S \tag{8.1.78}$$

where μ_S is the composition of the multiplication map $S \otimes_R S \to S$ and the natural map $S' \to S \otimes_R S$; the transitivity triangle ([56, III.2.1.2]) relative to (8.1.78) yields a natural isomorphism

$$\mathbb{L}_{S'/S} \stackrel{\sim}{\to} \sigma S \otimes_{S'} \mathbb{L}_{S'/S} \stackrel{\sim}{\to} \sigma S \otimes_{S'} (S' \otimes_S \mathbb{L}_{S/R}) \qquad \text{in } \mathsf{D}(S\text{-}\mathbf{Mod})$$

where the last isomorphism follows from the base change theorem of [56, III.2.2.1]. Thus, $\mathbb{L}_{S'/S} \simeq \sigma\mathbb{L}_{S/R}$ in $\mathsf{D}(S\text{-}\mathbf{Mod})$, so assertion (i) (resp. (ii)) holds for φ, provided it holds for $\Delta(\varphi)$, and then the claim follows from corollary 8.1.73(iv). ◊

Henceforth, we assume that both $\Delta(\varphi^a)$ and $H_0(\varphi)$ are isomorphisms. Let $P \to S$ be a resolution, with P an R-algebra, such that $P[n]$ is a free $R[n]$-algebra for every $n \in \mathbb{N}$; set $P' := S \otimes_R P$, and recall that there are natural isomorphisms

$$\mathbb{L}_{S/R} \stackrel{\sim}{\to} S \otimes_P \Omega^1_{P/R} \stackrel{\sim}{\to} S \otimes_{P'} \Omega^1_{P'/S}$$

where $\Omega^1_{P/R}$ denotes the flat simplicial P-module such that $\Omega^1_{P/R}[n] := \Omega_{P[n]/R[n]}$ (and likewise for $\Omega^1_{P'/S}$), with faces and degeneracies deduced from those of P and R (resp. those of P' and S), in the obvious way. In other words, set

$$J := \mathrm{Ker}(\mu_S : P' \to S)$$

(where μ_S is as in the proof of claim 8.1.77); then $\mathbb{L}_{S/R} \simeq J/J^2$ in $\mathsf{D}(S\text{-}\mathbf{Mod})$, and notice that

$$J^a = 0 \qquad \text{in } \mathsf{D}(S^a\text{-}\mathbf{Mod}) \tag{8.1.79}$$

since $\Delta(\varphi^a)$ is an isomorphism.

Claim 8.1.80. J is a quasi-regular ideal, and $H_0J = 0$.

Proof of the claim. Since $H_0(\varphi)$ is an isomorphism, it is clear that $H_0J = 0$. Next, for every $n \in \mathbb{N}$, the $S[n]$-algebra $P'[n]$ is free, hence we reduce to showing the following. Let B be any ring, $C := B[X_i \mid i \in I]$ any free B-algebra (for any set I), and $f : C \to B$ any morphism of B-algebras; then $\mathrm{Ker}\, f$ is a quasi-regular ideal of C. However, for every $i \in I$, set $b_i := f(X_i)$, and let $g : C \stackrel{\sim}{\to} C$ be the isomorphism of B-algebras such that $g(X_i) = X_i - a_i$ for every $i \in I$; clearly $\mathrm{Ker}\, f$ is a quasi-regular ideal if and only if the same holds for $g^{-1}\mathrm{Ker}\, f$. But the latter is the ideal $(X_i \mid i \in I)$, so the assertion follows from remark 4.5.33(ii). ◊

(i): We need to show that $H_n(J/J^2)^a = 0$ for every $n \in \mathbb{N}$, and we shall argue by induction on n. The assertion for $n = 0$ is clear from (8.1.79), in view of the exact sequence

$$H_nJ \to H_n(J/J^2) \to H_{n-1}J^2 \qquad \text{for every } n \in \mathbb{N}.$$

Hence, suppose that $n > 0$, and the assertion is already known for every degree $< n$. The same exact sequence reduces to checking that $H_{n-1}(J^2)^a = 0$. Now, from claim 8.1.80 and theorem 4.5.37, we know that $H_{n-1}J^n = 0$. Therefore, by an easy induction, we are further reduced to showing that $H_{n-1}(J^i/J^{i+1})^a = 0$ for every $i = 2, \dots, n-1$. However, on the one hand, proposition 4.5.35(ii.a) says that the natural map

$$L\mathrm{Sym}^i_S(J/J^2) \to J^i/J^{i+1}$$

is an isomorphism in $\mathsf{D}(S\text{-}\mathbf{Mod})$, for every $i \in \mathbb{N}$; on the other hand, the inductive assumption and theorem 8.1.52(i) imply that $H_j(L\mathrm{Sym}^i J/J^2)^a = 0$ for every $j \leq n$, whence the contention.

(ii): We show, by induction on n, that $H_n(J/J^2)^a = 0$ for every $n < c + 2p - 1$. Arguing as in the foregoing, we may assume that $c + 2p - 1 > n > 0$, and the sought vanishing is already known in degrees $< n$, in which case we reduce to checking that $H_{n-1}(L\mathrm{Sym}^i_S J/J^2)^a = 0$ for every $i = 2, \dots, n-1$. Set $M := \mathrm{cosk}_n(s.\mathrm{trunc}_n J/J^2)$; then $H_j M = 0$ for every $j < c$, and the inductive assumption implies that $M^a = 0$ in $\mathsf{D}(S^a\text{-}\mathbf{Mod})$, so $H_{n-1}(L\mathrm{Sym}^i_S M)^a = 0$, by theorem 8.1.58(iii). Lastly, by corollary 4.5.24(i), we see that

$$H_{n-1}(L\mathrm{Sym}^i_S M) = H_{n-1}(L\mathrm{Sym}^i_S J/J^2) \qquad \text{for every } i \in \mathbb{N}$$

whence the contention. □

Remark 8.1.81. Let $A \to B$ be any morphism of simplicial rings such that the multiplication map of B is an isomorphism $\mu_B : B \otimes_A B \xrightarrow{\sim} B$. Pick a resolution $\varphi : P \to B$ such that $P[n]$ is a free $A[n]$-algebra for every $n \in \mathbb{N}$, and set $P' := B \otimes_A P$, $J := \mathrm{Ker}\,(\mu_B \circ (B \otimes_A \varphi) : P' \to B)$. The assumption on B implies that $H_0 J = 0$. On the other hand, arguing as in the proof of claim 8.1.80 we see that J is a quasi-regular ideal. Taking into account proposition 4.5.35(ii) the spectral sequence associated to the J-adic filtration of P' can be written as

$$E^2_{pq} := H_{p+q}(L\mathrm{Sym}^q_B(\mathbb{L}_{B/A})) \Rightarrow \mathrm{Tor}^A_{p+q}(B \otimes_A B).$$

Though the filtration is not finite, one can show that this spectral sequence is convergent. We will not use this result.

Corollary 8.1.82. *Let $\varphi : A \to B$ be a morphism of V^a-algebras, $p \in \mathbb{N}$ a prime integer, and suppose that*

(a) *B is a $V^a \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$-algebra*
(b) *the induced morphism $s.\varphi : s.A \to s.B$ of simplicial V^a-algebras is weakly étale.*

Then $H_i \mathbb{L}^a_{B/A} = 0$ for every $i \leq 2p$.

Proof. Letting $c := 0$ in theorem 8.1.76(ii), we see that $H_i \mathbb{L}^a_{B/A} = 0$ for $i = 0, 1$. But in view of claim 8.1.43, the latter implies that $H_i \mathbb{L}_{B_{!!}/A_{!!}} = 0$ for $i = 0, 1$. So, actually the morphism φ fulfills the conditions of theorem 8.1.76(ii), with $c = 2$, whence the assertion. □

8.1.83. We conclude this section with a few words on the cohomology of sheaves of almost modules. Namely, let $(V, \mathfrak{m})$ again be an arbitrary basic setup, which we view as a basic setup relative to the one-point topos $\{\mathrm{pt}\}$, in the sense of [36, §3.3]. Let $(X, \mathscr{O}_X)$ be a ringed topos, $\mathfrak{m}_X \subset \mathscr{O}_X$ an ideal, such that $(\mathscr{O}_X, \mathfrak{m}_X)$ is a basic setup relative to the topos X, and suppose also that we are given a morphism of ringed topoi :

$$\pi : (X, \mathscr{O}_X) \to (\{\mathrm{pt}\}, V) \quad \text{such that} \quad (\pi^*\mathfrak{m}) \cdot \mathscr{O}_X \subset \mathfrak{m}_X.$$

We deduce a natural morphism :

$$R\pi_* : \mathsf{D}^+((\mathscr{O}_X, \mathfrak{m}_X)^a\text{-}\mathbf{Mod}) \to \mathsf{D}^+((V, \mathfrak{m})^a\text{-}\mathbf{Mod})$$

which can be constructed as usual, by taking injective resolutions. In case $\widetilde{\mathfrak{m}}$ is a flat V-module, this is the same as setting

$$R\pi_*(K^\bullet) := (R\pi_* K^\bullet_!)^a.$$

Using [36, Cor.2.2.24] one may verify that the two definitions coincide : the details shall be left to the reader. In many cases, both statements and proofs of results concerning the cohomology of $\mathscr{O}_X$-modules carry over *verbatim* to $\mathscr{O}_X^a$-modules. One sets, as customary :

$$H^\bullet(X, K^\bullet) := H^\bullet R\pi_* K^\bullet \qquad \text{for every object } K^\bullet \text{ of } \mathsf{D}^+((\mathscr{O}_X, \mathfrak{m}_X)^a\text{-}\mathbf{Mod}).$$

8.1.84. As an illustration, we consider the following situation, which shall occur in section 9.2. Suppose that :

- $f : A \to A'$ is a map of V-algebras, and set :
$$\varphi := \operatorname{Spec} f : X' := \operatorname{Spec} A' \to X := \operatorname{Spec} A.$$
- $t \in A$ is an element which is regular both in A and in A', and such that the induced map $A/tA \to A'/tA'$ is an isomorphism.
- $U \subset X$ is any open subset containing $D(t) := \{\mathfrak{p} \in X \mid t \notin \mathfrak{p}\}$, and set $U' := \varphi^{-1}U$.
- $\mathscr{F}$ is a quasi-coherent $\mathscr{O}_U^a$-module, and set $\mathscr{F}' := \varphi^*_{|U'}\mathscr{F}$, which is a quasi-coherent $\mathscr{O}_{U'}^a$-module.

Then we have natural morphisms of A^a-modules :

$$H^q(U, \mathscr{F}) \to H^q(U', \mathscr{F}') \qquad \text{for every } q \in \mathbb{N}. \tag{8.1.85}$$

Lemma 8.1.86. *In the situation of* (8.1.84)*, suppose moreover that t is $\mathscr{F}$-regular. Then* (8.1.85) *is an isomorphism for every $q > 0$, and induces an isomorphism of A'^a-modules :*

$$H^0(U, \mathscr{F}) \otimes_A A' \xrightarrow{\sim} H^0(U', \mathscr{F}').$$

Proof. To ease notation, we shall write φ instead of $\varphi_{|U'}$. To start out, we remark :

Claim 8.1.87. (i) $\mathscr{T}or_1^{\mathscr{O}_U^a}(\mathscr{F}, \mathscr{O}_U^a/t\mathscr{O}_U^a) = 0$ and $\operatorname{Tor}_1^{A^a}(H^0(U, \mathscr{F}), A/tA) = 0$.

(ii) The $\mathscr{O}_U^a$-module $\varphi_*\mathscr{F}' = \mathscr{F} \otimes_{\mathscr{O}_U^a} \varphi_*\mathscr{O}_{U'}^a$ (resp. the A'^a-module $H^0(U, \mathscr{F}) \otimes_{A^a} A'^a$) is t-torsion-free.

Proof of the claim. (i): Since t is regular on $\mathscr{O}_U$, we have a short exact sequence : $\mathscr{E} := (0 \to \mathscr{O}_U^a \to \mathscr{O}_U^a \to \mathscr{O}_U^a/t\mathscr{O}_U^a \to 0)$, and since t is regular on $\mathscr{F}$, the sequence $\mathscr{E} \otimes_{\mathscr{O}_U^a} \mathscr{F}$ is still exact; the vanishing of $\mathscr{T}or_1^{\mathscr{O}_U^a}(\mathscr{F}, \mathscr{O}_U^a/t\mathscr{O}_U^a)$ is an easy consequence. For the second stated vanishing one argues similarly, using the exact sequence $0 \to A \to A \to A/tA \to 0$.

(ii): Under the current assumptions, the natural map $\mathscr{O}_U/t\mathscr{O}_U \to \varphi_*(\mathscr{O}_{U'}/t\mathscr{O}_{U'})$ is an isomorphism, whence a short exact sequence : $\mathscr{E}' := (0 \to \varphi_*\mathscr{O}_{U'}^a \to \varphi_*\mathscr{O}_{U'}^a \to \mathscr{O}_U^a/t\mathscr{O}_U^a \to 0)$. In view of (i), the sequence $\mathscr{E}' \otimes_{\mathscr{O}_U^a} \mathscr{F}$ is still exact, so $\mathscr{F} \otimes_{\mathscr{O}_U^a} \varphi_*\mathscr{O}_{U'}^a$ is t-torsion-free. An analogous argument works as well for $H^0(U, \mathscr{F}) \otimes_{A^a} A'^a$. $\Diamond$

Claim 8.1.88. For every $n > 0$, the map $A/t^nA \to A'/t^nA'$ induced by f is an isomorphism.

Proof of the claim. Indeed, since t is regular on both A and A', and $A/tA \xrightarrow{\sim} A'/tA'$, the map of graded rings $\oplus_{n\in\mathbb{N}} t^nA/t^{n+1}A \to \oplus_{n\in\mathbb{N}} t^nA'/t^{n+1}A'$ is bijective. Then the claim follows from [14, Ch.III, §2, n.8, Cor.3]. $\Diamond$

Let $j : D(t) \to U$ be the natural open immersion, and set :

$$\mathscr{F}[t^{-1}] := j_*j^*\mathscr{F} \simeq \mathscr{F} \otimes_{\mathscr{O}_U^a} j_*\mathscr{O}_{D(t)}^a.$$

Notice the natural isomorphisms :

$$j_*j^*\varphi_*\mathscr{F}' \simeq (\mathscr{F} \otimes_{\mathscr{O}_U^a} \varphi_*\mathscr{O}_{U'}^a) \otimes_{\mathscr{O}_U^a} j_*\mathscr{O}_{D(t)}^a \simeq \mathscr{F}[t^{-1}] \otimes_{\mathscr{O}_U^a} \varphi_*\mathscr{O}_{U'}^a. \tag{8.1.89}$$

Since t is $\mathscr{F}$-regular, the natural map $\mathscr{F} \to \mathscr{F}[t^{-1}]$ is a monomorphism, and the same holds for the corresponding map $\varphi_*\mathscr{F}' \to j_*j^*\varphi_*\mathscr{F}'$, in view of claim 8.1.87(ii). Moreover, $\mathscr{G} := \mathscr{F}[t^{-1}]/\mathscr{F}$ can be written as the increasing union of its subsheaves $\operatorname{Ann}_{\mathscr{G}}(t^n)$ (for all $n \in \mathbb{N}$);

hence claim 8.1.88 implies that the natural map $\mathscr{G} \to \mathscr{G} \otimes_{\mathscr{O}_U^a} \varphi_*\mathscr{O}_{U'}^a$ is an isomorphism. There follows a ladder of short exact sequences :

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{F} \arrow[r] \arrow[d] & \mathscr{F}[t^{-1}] \arrow[r] \arrow[d] & \mathscr{G} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \varphi_*\mathscr{F}' \arrow[r] & \mathscr{F}[t^{-1}] \otimes_{\mathscr{O}_U^a} \varphi_*\mathscr{O}_{U'}^a \arrow[r] & \mathscr{G} \arrow[r] & 0. \end{tikzcd} \tag{8.1.90}$$

On the other hand, since j is an affine morphism, we may compute :

$$H^q(U, \mathscr{F}[t^{-1}]) \simeq H^q(U, Rj_*j^*\mathscr{F}) \simeq H^q(D(t), j^*\mathscr{F}) \simeq 0 \qquad \text{for every } q > 0.$$

Likewise, from (8.1.89) we get :

$$H^q(U, \mathscr{F}[t^{-1}] \otimes_{\mathscr{O}_U^a} \varphi_*\mathscr{O}_{U'}^a) \simeq 0 \qquad \text{for every } q > 0.$$

Thus, in the commutative diagram :

$$\begin{tikzcd} H^{q-1}(U, \mathscr{G}) \arrow[r, "\partial"] \arrow[d, "\partial'"'] & H^q(U, \mathscr{F}) \arrow[d] \\ H^q(U, \varphi_*\mathscr{F}) \arrow[r, "\sim"] & H^q(U', \mathscr{F}') \end{tikzcd}$$

the boundary maps ∂ and ∂' are isomorphisms whenever $q > 1$, and the right vertical arrow is (8.1.85), so the assertion follows already for every $q > 1$. To deal with the remaining cases with $q = 0$ or 1, we look at the ladder of exact cohomology sequences:

$$\begin{tikzcd} 0 \arrow[r] & H^0(U, \mathscr{F}) \arrow[r, "\alpha"] \arrow[d] & H^0(D(t), \mathscr{F}) \arrow[r, "\beta"] \arrow[d] & H^0(U, \mathscr{G}) \arrow[r] \arrow[d, equal] & H^1(U, \mathscr{F}) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & H^0(U', \mathscr{F}') \arrow[r] & H^0(D(t), \mathscr{F}) \otimes_{A^a} A'^a \arrow[r, "\beta'"] & H^0(U, \mathscr{G}) \arrow[r] & H^1(U', \mathscr{F}') \arrow[r] & 0 \end{tikzcd}$$

deduced from (8.1.90). On the one hand, we remark that the natural inclusion $\operatorname{Im}\beta \subset \operatorname{Im}\beta'$ factors as a composition

$$\operatorname{Im}\beta \xrightarrow{\iota} M := A'^a \otimes_{A^a} \operatorname{Im}\beta \xrightarrow{\pi} \operatorname{Im}\beta'$$

where ι is given by the rule : $x \mapsto 1 \otimes x$, for all $x \in A^a_*$, and π is an epimorphism.

On the other hand, the image of β is isomorphic to $N := \operatorname{Coker}\alpha$, and the latter is the increasing union of its submodules $\operatorname{Ann}_N(t^n)$ (for all $n \in \mathbb{N}$). Again from claim 8.1.88 we deduce that $M = \operatorname{Im}\beta$, hence $\operatorname{Im}\beta' = \operatorname{Im}\beta$, which shows that (8.1.85) is an isomorphism also for $q = 1$. By the same token, $\operatorname{Ker}(\alpha \otimes_{A^a} \mathbf{1}_{A'^a})$ is the increasing union of its t^n-torsion submodules (for all $n \in \mathbb{N}$), since it is a quotient of $\operatorname{Tor}_1^{A^a}(\operatorname{Im}\beta, A'^a)$; hence $\operatorname{Ker}(\alpha \otimes_{A^a} \mathbf{1}_{A'^a}) = 0$, in view of claim 8.1.87(ii). This easily implies the last assertion for $q = 0$. $\square$

8.2. **Almost pure pairs.** Throughout this section, we fix a basic setup $(V, \mathfrak{m})$ (see [36, §2.1.1]) and we set $S := \operatorname{Spec} V$; for every S-scheme X we may consider the sheaf $\mathscr{O}_X^a$ of almost algebras on X, and we refer to [36, §5.5] for the definition of quasi-coherent $\mathscr{O}_X^a$-modules and algebras. However, whereas in [36] it was assumed that $\widetilde{\mathfrak{m}} := \mathfrak{m} \otimes_V \mathfrak{m}$ is a flat V-module, in this section the basic setup can be arbitrary, except where it is explicitly said otherwise. This of course requires some care when quoting from [36]; however, it turns out that – thanks to the work done in section 8.1 – most of the results in [36] do extend *verbatim* to the case of a general setup. The main exception is the theory of the finite and almost finite rank of almost projective modules, that relies on the existence of a well behaved exterior power functor, which is available only if the basic setup satisfies some minimal conditions (see (8.1.53)). In any case, whenever we need to import some theorem from [36], we shall comment on its range of validity.

We begin with a few complements on sheaves of almost algebras.

Definition 8.2.1. Let X be an S-scheme, $\mathscr{A}$ a quasi-coherent $\mathscr{O}_X^a$-algebra, and $\mathscr{F}$ an $\mathscr{A}$-module which is quasi-coherent as an $\mathscr{O}_X^a$-module.

(i) $\mathscr{F}$ is said to be an $\mathscr{A}$-module *of almost finite type* (resp. *of almost finite presentation*) if, for every affine open subset $U \subset X$, the $\mathscr{A}(U)$-module $\mathscr{F}(U)$ is almost finitely generated (resp. almost finitely presented).

(ii) $\mathscr{F}$ is said to be an *almost coherent* $\mathscr{A}$-module if it is an $\mathscr{A}$-module of almost finite type, and for every open subset $U \subset X$, every quasi-coherent $\mathscr{A}_{|U}$-submodule of $\mathscr{F}_{|U}$ of almost finite type, is almost finitely presented.

(iii) We say that $\mathscr{F}$ is a *torsion-free* $\mathscr{O}_X^a$-module if we have $\mathrm{Ker}\, b \cdot \mathbf{1}_{\mathscr{F}(U)} = 0$, for every affine open subset $U \subset X$ and every regular element $b \in \mathscr{O}_X(U)$.

(iv) Suppose that the ideal $\mathfrak{m}$ satisfies condition $(\mathbf{B})$ of [36, §2.1.6]. Then we say that $\mathscr{F}$ is an $\mathscr{A}$-module *of almost finite rank* (resp. *of finite rank*) if, for every affine open subset $U \subset X$, the $\mathscr{A}(U)$-module $\mathscr{F}(U)$ is almost finitely generated projective of almost finite (resp. of finite) rank.

Remark 8.2.2. In the situation of definition 8.2.1 :

(i) We denote by $\mathscr{E}nd_{\mathscr{A}}(\mathscr{F})$ the $\mathscr{O}_X$-module of *$\mathscr{A}$-linear endomorphisms of $\mathscr{F}$*, defined by the rule : $U \mapsto \mathrm{End}_{\mathscr{A}(U)}(\mathscr{F}(U))$ for every open subset $U \subset X$. It is easily seen that $\mathscr{E}nd_{\mathscr{A}}(\mathscr{F})^a$ is an $\mathscr{A}$-module.

(ii) Suppose that $\mathscr{F}$ is a flat and almost finitely presented $\mathscr{A}$-module. Then there exists a *trace morphism*

$$\mathrm{tr}_{\mathscr{F}/\mathscr{A}} : \mathscr{E}nd_{\mathscr{A}}(\mathscr{F}) \to \mathscr{A}$$

that, on every affine open subset $U \subset X$, induces the trace morphism $\mathrm{tr}_{\mathscr{F}(U)\mathscr{A}(U)}$ of the almost finitely generated projective $\mathscr{A}(U)$-module $\mathscr{F}(U)$ (details left to the reader : see [36, §4.1], which does not depend on any assumption on the basic setup).

Lemma 8.2.3. *Let $f : Y \to X$ a faithfully flat and quasi-compact morphism of S-schemes, $\mathscr{A}$ a quasi-coherent $\mathscr{O}_X^a$-algebra, $\mathscr{F}$ a quasi-coherent $\mathscr{A}$-module. Then :*

(i) *The $\mathscr{A}$-module $\mathscr{F}$ is of almost finite type (resp. of almost finite presentation) if and only if the same holds for the $f^*\mathscr{A}$-module $f^*\mathscr{F}$.*

(ii) *Suppose that $\mathfrak{m}$ fulfills condition $(\mathbf{B})$. Then the $\mathscr{A}$-module $\mathscr{F}$ is of almost finite rank (resp. of finite rank) if and only if the same holds for the $f^*\mathscr{A}$-module $f^*\mathscr{F}$.*

(iii) *If the $f^*\mathscr{A}$-module $f^*\mathscr{F}$ is almost coherent, then the same holds for the $\mathscr{A}$-module $\mathscr{F}$.*

(iv) *Suppose that $\mathfrak{m}$ fulfills condition $(\mathbf{B})$, that X is integral, and that $\mathscr{F}$ is a flat $\mathscr{O}_X^a$-module of almost finite type. Then $\mathscr{F}$ is an $\mathscr{O}_X^a$-module of finite rank.*

Proof. (i): One reduces easily to the case where X and Y are affine, in which case the assertion follows from [36, Rem.3.2.26(ii)], which holds for any basic setup.

(ii): It suffices to apply [36, Rem.3.2.26(iii)], and recall that exterior powers commute with any base changes : the details shall be left to the reader.

(iii) follows easily from (i).

(iv): Let η be the generic point of X; notice that $\mathscr{F}_\eta$ is a free $\kappa(\eta)^a$-module of finite rank, and let r be this rank. From [36, Prop.2.4.19], it follows easily that $\mathscr{F}$ is almost finitely presented. Moreover, the exterior powers of $\mathscr{F}$ are still flat $\mathscr{O}_X^a$-modules; then the r-th exterior power vanishes, since it vanishes at the generic point of X. $\square$

Lemma 8.2.4. *Let X be any S-scheme.*

(i) *The full subcategory $\mathscr{O}_X^a\text{-}\mathbf{Mod}_{\mathrm{acoh}}$ of $\mathscr{O}_X^a\text{-}\mathbf{Mod}_{\mathrm{qcoh}}$ consisting of all almost coherent modules is abelian and closed under extensions. (More precisely, the embedding $\mathscr{O}_X^a\text{-}\mathbf{Mod}_{\mathrm{acoh}} \to \mathscr{O}_X^a\text{-}\mathbf{Mod}_{\mathrm{qcoh}}$ is an exact functor.)*

(ii) *If X is a coherent scheme, then every quasi-coherent $\mathscr{O}_X^a$-module of almost finite presentation is almost coherent.*

Proof. Both assertions are local on X, hence we may assume that X is affine, say $X = \operatorname{Spec} R$.

(i): Let $f : M \to N$ be a morphism of almost coherent R^a-modules. It is clear that $\operatorname{Coker} f$ is again almost coherent. Moreover, $f(M) \subset N$ is almost finitely generated, hence almost finitely presented; then by [36, Lemma 2.3.18], $\operatorname{Ker} f$ is almost finitely generated, and so it is almost coherent as well. Similarly, using [36, Lemma 2.3.18] one sees that $\mathscr{O}_X^a\text{-}\mathbf{Mod}_{\mathrm{acoh}}$ is closed under extensions.

(ii): We sketch the argument and leave the details to the reader. Let M be an almost finitely presented R^a-module; according to [36, Cor.2.3.13], M can be approximated arbitrarily closely by a module of the form N^a, where N is a finitely presented R-module. Likewise, any given almost finitely generated submodule $M' \subset M$ can be approximated by a finitely generated submodule $N' \subset N$. Then N' is finitely presented, hence M' is almost finitely presented, as claimed. $\square$

Definition 8.2.5. Let X be an S-scheme, and $\varphi : \mathscr{A} \to \mathscr{B}$ a morphism of quasi-coherent $\mathscr{O}_X^a$-algebras.

(i) $\mathscr{B}$ is said to be an *almost finite* (resp. *flat*, resp. *weakly unramified*, resp. *weakly étale*, resp. *unramified*, resp. *étale*) $\mathscr{A}$-algebra if, for every affine open subset $U \subset X$, $\mathscr{B}(U)$ is an almost finite (resp. flat, resp. weakly unramified, resp. weakly étale, resp. unramified, resp. étale) $\mathscr{A}(U)$-algebra.

(ii) We define the $\mathscr{O}_X$-algebra $\mathscr{A}_{!!}$ by the short exact sequence :

$$\widetilde{\mathfrak{m}} \otimes_V \mathscr{O}_X \to \mathscr{O}_X \oplus \mathscr{A}_! \to \mathscr{A}_{!!} \to 0$$

analogous to [36, §2.2.25]. This agrees with the definition of [36, §3.3.2], corresponding to the basic setup $(\mathscr{O}_X, \mathfrak{m}\mathscr{O}_X)$ relative to the Zariski topos of X; in general, this is *not* the same as forming the algebra $\mathscr{A}_{!!}$ relative to the basic setup $(V, \mathfrak{m})$. The reason why we prefer the foregoing version of $\mathscr{A}_{!!}$, is explained by the following lemma 8.2.18(i).

(iii) For any monomorphism $\mathscr{R} \subset \mathscr{S}$ of quasi-coherent $\mathscr{O}_X$-algebras, let $\mathrm{i.c.}(\mathscr{R}, \mathscr{S})$ denote the integral closure of $\mathscr{R}$ in $\mathscr{S}$, which is a quasi-coherent $\mathscr{O}_X$-subalgebra of $\mathscr{S}$.

Suppose that φ is a monomorphism. The *integral closure of $\mathscr{A}$ in $\mathscr{B}$*, is the $\mathscr{A}$-algebra $\mathrm{i.c.}(\mathscr{A}, \mathscr{B}) := \mathrm{i.c.}(\mathscr{A}_{!!}, \mathscr{B}_{!!})^a$ (see also [36, Lemma 8.2.28]). In view of lemma 8.2.18(i), this is a well defined quasi-coherent $\mathscr{O}_X^a$-algebra. It is characterized as the unique $\mathscr{A}$-subalgebra of $\mathscr{B}$ such that :

$$\mathrm{i.c.}(\mathscr{A}, \mathscr{B})(U) := \mathrm{i.c.}(\mathscr{A}(U), \mathscr{B}(U))$$

for every affine open subset $U \subset X$ (notation of [36, Def.8.2.27]). We say that $\mathscr{A}$ is *integrally closed in $\mathscr{B}$*, if $\mathscr{A} = \mathrm{i.c.}(\mathscr{A}, \mathscr{B})$. We say that $\mathscr{B}$ is an *integral $\mathscr{A}$-algebra* if $\mathrm{i.c.}(\mathscr{A}, \mathscr{B}) = \mathscr{B}$.

Remark 8.2.6. Let X be an S-scheme, $\mathscr{A} \to \mathscr{B}$ be a morphism of quasi-coherent $\mathscr{O}_X^a$-algebras.

(i) $\mathscr{B}$ is an unramified $\mathscr{A}$-algebra if and only if, for every affine open subset $U \subset X$, there exists an idempotent element $e_{\mathscr{B}(U)/\mathscr{A}(U)} \in (\mathscr{B} \otimes_{\mathscr{A}} \mathscr{B})(U)_*$, uniquely characterized by the following conditions :

(a) $\mu_{\mathscr{B}/\mathscr{A}}(e_{\mathscr{B}(U)/\mathscr{A}(U)}) = 1$, where $\mu_{\mathscr{B}/\mathscr{A}} : \mathscr{B} \otimes_{\mathscr{A}} \mathscr{B} \to \mathscr{B}$ is the multiplication morphism of the $\mathscr{A}$-algebra $\mathscr{B}$.

(b) $e_{\mathscr{B}(U)/\mathscr{A}(U)} \cdot \operatorname{Ker} \mu_{\mathscr{B}/\mathscr{A}}(U) = 0$

([36, Lemma 3.1.4]). It is easily seen that, on $U' \subset U$, the element $e_{\mathscr{B}(U)/\mathscr{A}(U)}$ restricts to $e_{\mathscr{B}(U')/\mathscr{A}(U')}$; hence, the system $(e_{\mathscr{B}(U)/\mathscr{A}(U)} \mid U \subset X)$, for U ranging over all the affine open subsets of X, determines a global section

$$e_{\mathscr{B}/\mathscr{A}} \in \Gamma(X, \mathscr{B} \otimes_{\mathscr{A}} \mathscr{B})_*$$

which we call the *diagonal idempotent* of the $\mathscr{A}$-algebra $\mathscr{B}$ (see [36, §5.5.4]).

(ii) If $\mathscr{B}$ is a flat and almost finitely presented $\mathscr{A}$-algebra, there exists a *trace form*

$$t_{\mathscr{B}/\mathscr{A}} : \mathscr{B} \otimes_{\mathscr{A}} \mathscr{B} \to \mathscr{A}$$

that, on each open affine subset $U \subset X$, induces the trace form $t_{\mathscr{B}(U)/\mathscr{A}(U)}$ of [36, §4.1.12] (details left to the reader).

(iii) Suppose that $\mathscr{B}$ is a flat, unramified and almost finitely presented $\mathscr{A}$-algebra. Then the diagonal idempotent and the trace form of $\mathscr{B}$ are related by the identity

$$t_{\mathscr{B}/\mathscr{A}}(e_{\mathscr{B}/\mathscr{A}} \cdot (1 \otimes b)) = b = t_{\mathscr{B}/\mathscr{A}}(e_{\mathscr{B}/\mathscr{A}} \cdot (b \otimes 1))$$

for every affine open subset $U \subset X$ and every $b \in \mathscr{B}(U)_*$ ([36, Rem.4.1.17]).

(iv) Suppose that $\mathscr{B}$ is an integral $\mathscr{A}$-algebra, and let also $\mathscr{R} \to \mathscr{S}$ be a morphism of quasi-coherent $\mathscr{O}_X$-algebra, such that $\mathscr{S}$ is an integral $\mathscr{R}$-algebra. Then :

(a) $\mathscr{S}^a$ is an integral $\mathscr{R}^a$-algebra. Indeed, we may assume that X is affine, in which case the assertion follows from [36, Lemma 8.2.28].

(b) $\mathscr{B}_{!!}$ is an integral $\mathscr{A}_{!!}$-algebra. Indeed, by assumption we have $\mathscr{B} = \mathrm{i.c.}(\mathscr{A}_{!!}, \mathscr{B}_{!!})^a$, whence by adjunction, a morphism of $\mathscr{A}_{!!}$-algebras $\mathscr{B}_{!!} \to \mathrm{i.c.}(\mathscr{A}_{!!}, \mathscr{B}_{!!})$, which must be the identity map.

Thus, $\mathscr{B}$ is an integral $\mathscr{A}$-algebra if and only if $\mathscr{B}_{!!}$ is an integral $\mathscr{A}_{!!}$-algebra.

Proposition 8.2.7. *Suppose that the ideal $\mathfrak{m}$ fulfills condition $(\mathbf{B})$ of* [36, §2.1.6]. *Let X be a quasi-compact S-scheme, $\mathscr{A}$ a quasi-coherent $\mathscr{O}_X^a$-algebra, and $\mathscr{B}$ a flat $\mathscr{A}$-algebra that is almost finitely presented as an $\mathscr{A}$-module. Then $\mathscr{B}(X)$ is an integral $\mathscr{A}(X)$-algebra.*

Proof. For every $b \in \mathscr{B}(X)_*$ and every $i \in \mathbb{N}$, set :

$$c_i(b) := \mathrm{tr}_{\Lambda^i_{\mathscr{A}}\mathscr{B}/\mathscr{A}}(\Lambda^i_{\mathscr{A}}(b\mathbf{1}_{\mathscr{B}})) \in \mathscr{A}(X)$$

(see remark 8.2.2(ii)). In view of [36, Lemma 8.2.28], it suffices to show

Claim 8.2.8. For every $b \in \mathscr{B}(X)_*$ and $a \in \mathfrak{m}$, there exists $n \in \mathbb{N}$ such that :

(i) $c_i(ab) = 0$ for every $i > n$.

(ii) $\sum_{i=0}^n (ab)^{i+1} \cdot c_{n-i}(ab) = 0$.

Proof of the claim. Since X admits a finite affine covering, and since the trace morphism commutes with arbitrary base changes, we are easily reduced to the case where X is affine. In this case, set $A := \mathscr{A}(X)$ and $B := \mathscr{B}(X)$; then B is an almost finitely generated projective A-algebra, and there exists $n \in \mathbb{N}$ such that $a\mathbf{1}_B$ factors through A-linear maps $u : B \to A^{\oplus n}$ and $v : A^{\oplus n} \to B$ ([36, Lemma 2.4.15]). Thus, $ab\mathbf{1}_B = v \circ (u \circ b\mathbf{1}_B)$, and we get :

$$c_i(ab) = \mathrm{tr}_{\Lambda^i_A A^{\oplus n}/A}(\Lambda^i_A(u \circ b\mathbf{1}_B \circ v)) \qquad \text{for every } i \in \mathbb{N}$$

([36, Lemma 4.1.2(i)]), from which (i) already follows. By the same token, we also obtain :

$$\chi := \sum_{i=0}^n (u \circ b\mathbf{1}_B \circ v)^i \cdot c_i(ab) = 0$$

([36, Prop.4.4.30]). However, the left-hand side of the identity of (ii) is none else than $v \circ \chi \circ (u \circ b\mathbf{1}_B)$; whence the claim. □

The following lemma generalizes [36, Cor. 4.4.31].

Lemma 8.2.9. *Let A be a V^a-algebra, $B \subset A$ a V^a-subalgebra, P an almost finitely generated projective A-module, and φ an A-linear endomorphism of P. Suppose that :*

(a) B *is integrally closed in* A.
(b) φ *is integral over* B_*.

Then we have :

(i) $\mathrm{tr}_{P/A}(\varphi) \in B_*$.
(ii) *If* $\mathfrak{m}$ *fulfills condition* $(\mathbf{B})$ *of* [36, §2.1.6], *then* $\det(\mathbf{1}_P + T\varphi) \in B_*[[T]]$.

Proof. For the meaning of assumption (b), see the proof of [36, Cor. 4.4.31].

(i): For every $\varepsilon \in \mathfrak{m}$ we may find a free A-module L of finite rank, and A-linear morphisms $u : P \to L$, $v : L \to P$ such that $v \circ u = \varepsilon \mathbf{1}_P$ ([36, Lemma 2.4.15]). It follows that

$$\mathrm{tr}_{L/A}(u \circ \varphi \circ v) = \mathrm{tr}_{P/A}(\varphi \circ v \circ u) = \varepsilon \cdot \mathrm{tr}_{P/A}(\varphi)$$

([36, Lemma 4.1.2]). On the other hand, say that the polynomial $T^n + \sum_{j=0}^{n-1} b_j T^j \in B_*[T]$ annihilates φ; it is easily seen that $T^n + \sum_{j=0}^{n-1} b_j \varepsilon^{n-j} T^j$ annihilates $u \circ \varphi \circ v$, so the latter is integral over B_* as well, and therefore its trace lies in B_* ([36, Cor.4.4.31 and Rem.8.2.30(i)]). Summing up, we have shown that $\varepsilon \cdot \mathrm{tr}_{P/A}(\varphi) \in B_*$ for every $\varepsilon \in \mathfrak{m}$, whence the contention.

(ii): Recall that $\det(\mathbf{1}_P + T\varphi)$ is the power series in the variable T, whose coefficient in degree i is the trace of $\Lambda^i_A \varphi$ on $\Lambda^i_A P$: see [36, §4.3.1, §4.3.3].

Claim 8.2.10. Let Q be another almost finitely generated projective A-module, ψ be an endomorphism of P that is also integral over B_*. Then the endomorphism $\varphi \otimes_A \psi$ of $P \otimes_A Q$ is integral over B_*.

Proof of the claim. The tensor product defines a map of unitary associative B_*-algebras

$$\mathrm{End}_A(P) \otimes_{B_*} \mathrm{End}_A(Q) \to \mathrm{End}_A(P \otimes_A Q).$$

By assumption, the subalgebra $B_*[\varphi] \subset \mathrm{End}_A(P)$ is finite over B_*, and similarly for $B_*[\psi] \subset \mathrm{End}_A(Q)$. Hence, the same holds for the image of $B_*[\varphi] \otimes_{B_*} B_*[\psi]$ in $\mathrm{End}_A(P \otimes_A Q)$, whence the claim. $\diamondsuit$

It follows easily from claim 8.2.10, that the endomorphism $\Lambda^i_A \varphi$ of $\Lambda^i_A P$ is integral over B_*. Hence, we may replace P by $\Lambda^i_A P$, and reduce to showing that the trace of φ lies in B_*, which is known, by (i). $\square$

Lemma 8.2.11. *Let* $j : U \to X$ *be an open quasi-compact immersion of* S*-schemes,* $\mathscr{B}$ *a quasi-coherent* $\mathscr{O}^a_U$*-algebra,* $\mathscr{A}$ *a quasi-coherent* $\mathscr{O}^a_X$*-algebra, and* $j^*\mathscr{A} \to \mathscr{B}$ *an étale morphism. Then the induced morphism*

$$\mathscr{A} \to j_* j^* \mathscr{A} \to j_* \mathscr{B}$$

is flat if and only if it is étale.

Proof. (i): It suffices to show that $j_*\mathscr{B}$ admits a diagonal idempotent as in remark 8.2.6(i). To this aim, let us remark more generally :

Claim 8.2.12. Let $f : Y \to X$ be a quasi-compact and quasi-separated morphism, $\mathscr{F}$ a flat quasi-coherent $\mathscr{A}$-module, and $\mathscr{G}$ an $f^*\mathscr{A}$-module, quasi-coherent as an $\mathscr{O}^a_Y$-module. Then the natural map

$$\mathscr{F} \otimes_{\mathscr{A}} f_*\mathscr{G} \to f_*(f^*\mathscr{F} \otimes_{f^*\mathscr{A}} \mathscr{G})$$

is an isomorphism.

Proof of the claim. This is proved just as for $\mathscr{O}_X$-modules. Namely, one reduces easily to the case where X is affine, and one needs to check that the natural map

$$\mathscr{F}(X) \otimes_{\mathscr{A}(X)} \mathscr{G}(Y) \to (f^*\mathscr{F} \otimes_{f^*\mathscr{A}} \mathscr{G})(Y)$$

is an isomorphism. However, by assumption Y can be covered by finitely many affine open subsets $U_1, \dots, U_n$, and the intersection $U_{ij} := U_i \cap U_j$ is still affine, for every $i, j = 1, \dots, n$. Since $f^*\mathscr{F}$, $\mathscr{G}$ and $f^*\mathscr{A}$ are quasi-coherent $\mathscr{O}^a_Y$-modules, the natural map

$$\mathscr{F}(X) \otimes_{\mathscr{A}(X)} \mathscr{G}(U_{ij}) \to (f^*\mathscr{F} \otimes_{f^*\mathscr{A}} \mathscr{G})(U_{ij})$$

is an isomorphism, for every $i, j = 1, \dots, n$. Then we get :

$$(f^*\mathscr{F} \otimes_{\mathscr{A}} \mathscr{G})(Y) = \mathrm{Ker}(\prod_{i=1}^n \mathscr{F}(X) \otimes_{\mathscr{A}(X)} \mathscr{G}(U_i) \rightrightarrows \prod_{i,j=1}^n \mathscr{F}(X) \otimes_{\mathscr{A}(X)} \mathscr{G}(U_{ij}))$$
$$= \mathscr{F}(X) \otimes_{\mathscr{A}(X)} \mathrm{Ker}(\prod_{i=1}^n \mathscr{G}(U_i) \rightrightarrows \prod_{i,j=1}^n \mathscr{G}(U_{ij}))$$

(since $\mathscr{F}(X)$ is a flat $\mathscr{A}(X)$-module) whence the claim. $\diamondsuit$

Now, set $\mathscr{R} := \mathscr{B} \otimes_{j^*\mathscr{A}} \mathscr{B}$; since j is quasi-compact and $j_*\mathscr{B}$ is a flat $\mathscr{A}$-algebra, claim 8.2.12 implies that the natural morphism :

$$j_*\mathscr{B} \otimes_{\mathscr{A}} j_*\mathscr{B} \to j_*\mathscr{R}$$

is an isomorphism. Especially, the diagonal idempotent of $\mathscr{B}$ extends to a global section of $j_*\mathscr{B} \otimes_{\mathscr{A}} j_*\mathscr{B}$, and the assertion follows. $\square$

The criterion of lemma 8.2.11 has limited usefulness, since it is not usually known a priori that $j_*\mathscr{B}$ is a flat $\mathscr{A}$-algebra. In several situations, one can however apply the following variant.

Proposition 8.2.13. *Let X be an S-scheme, $j : U \to X$ a quasi-compact open immersion, $\mathscr{A} \to \mathscr{B}$ a morphism of quasi-coherent $\mathscr{O}^a_X$-algebras, and suppose that :*

(a) *The units of adjunction $\mathscr{A} \to j_*j^*\mathscr{A}$ and $\mathscr{B} \to j_*j^*\mathscr{B}$ are monomorphisms.*
(b) *$\mathscr{A}$ is integrally closed in $j_*j^*\mathscr{A}$, and $\mathscr{B}$ is an integral $\mathscr{A}$-algebra.*
(c) *$j^*\mathscr{B}$ is an étale $j^*\mathscr{A}$-algebra and an almost finitely presented $j^*\mathscr{A}$-module.*
(d) *The diagonal idempotent $e_{j^*\mathscr{B}/j^*\mathscr{A}}$ is a global section of the subalgebra*

$$\mathrm{Im}(\mathscr{B} \otimes_{\mathscr{A}} \mathscr{B} \to j_*j^*(\mathscr{B} \otimes_{\mathscr{A}} \mathscr{B})).$$

Then $\mathscr{B}$ is an étale $\mathscr{A}$-algebra and an almost finitely presented $\mathscr{A}$-module.

Proof. We may assume that X is affine. Under our assumptions, the restriction map $\mathscr{A}(X)_* \to \mathscr{A}(U)_*$ is injective, and its image is integrally closed in $\mathscr{A}(U)_*$ ([36, Rem.8.2.30]); moreover, $\mathscr{B}(X)_{!!}$ is an integral $\mathscr{A}(X)_{!!}$-algebra (remark 8.2.6(iv)). In view of lemma 8.2.9(i), we deduce a commutative diagram

(8.2.14)
$$\begin{tikzcd} \mathscr{B} \otimes_{\mathscr{A}} \mathscr{B} \arrow[r, "t"] \arrow[d] & \mathscr{A} \arrow[d] \\ j_*j^*(\mathscr{B} \otimes_{\mathscr{A}} \mathscr{B}) \arrow[r, "j_*(t_{j^*\mathscr{B}/j^*\mathscr{A}})"] & j_*j^*\mathscr{A} \end{tikzcd}$$

whose vertical arrows are the units of adjunctions, and where t is a uniquely determined $\mathscr{A}$-bilinear form. Then we are reduced to showing :

Claim 8.2.15. Let $j : U \to X$ be as in the proposition, and $\mathscr{A} \to \mathscr{B}$ a morphism of quasi-coherent $\mathscr{O}^a_X$-algebras fulfilling conditions (a), (c), (d) of the proposition, and such that there exists a bilinear form t making commute diagram 8.2.14. Then $\mathscr{B}$ is an étale $\mathscr{A}$-algebra and an almost finitely presented $\mathscr{A}$-module.

Proof of the claim. The proof proceeds by a familiar argument (see *e.g.* the proof of [36, Claim 3.5.33]). Namely, for a given $\varepsilon \in \mathfrak{m}$, we can write

$$\varepsilon \cdot e_{j^*\mathscr{B}/j^*\mathscr{A}} = \sum_{j=1}^m a_j \otimes b_j \qquad \text{where } a_j, b_j \in \mathscr{B}(X)_* \text{ for every } j = 1, \dots, m.$$

In light of remark 8.2.6(iii), we deduce

$$\varepsilon \cdot a = \sum_{j=1}^{m} t(a \otimes a_j) \cdot b_j \qquad \text{for every } a \in \mathscr{B}(X)_*.$$

Indeed, the identity holds after restriction to U, and assumption (a) implies that the restriction map $\mathscr{B}(X)_* \to \mathscr{B}(U)_*$ is injective. We may then define $\mathscr{A}$-linear maps : $\mathscr{B} \xrightarrow{\varphi} \mathscr{A}^{\oplus m} \xrightarrow{\psi} \mathscr{B}$ by the rules :

$$\varphi(a) := (t(a \otimes a_1), \dots, t(a \otimes a_m)) \qquad \psi(s_1, \dots, s_m) := \sum_{j=1}^{m} s_j b_j$$

for every open subset $U \subset X$, every $a \in \mathscr{B}(U)_*$, and every $s_1, \dots, s_m \in \mathscr{A}(U)_*$. Thus we have $\psi \circ \varphi = \varepsilon \mathbf{1}_{\mathscr{B}}$, and since ε is arbitrary, this already proves that $\mathscr{B}$ is an $\mathscr{A}$-module of almost finite type, so we may apply [36, Lemma 2.4.15 and Prop.2.4.18(i)] to deduce that $\mathscr{B}$ is a flat and almost finitely presented $\mathscr{A}$-module. Now, assumption (a) and claim 8.2.12 imply that the horizontal arrows of the induced diagram

$$\begin{tikzcd} \mathscr{B} \otimes_{\mathscr{A}} \mathscr{B} \arrow[r] \arrow[d, "\mu_{\mathscr{B}/\mathscr{A}}"'] & j_*j^*(\mathscr{B} \otimes_{\mathscr{A}} \mathscr{B}) \arrow[d, "j_*j^*\mu_{\mathscr{B}/\mathscr{A}}"] \\ \mathscr{B} \arrow[r] & j_*j^*\mathscr{B} \end{tikzcd}$$

are monomorphisms. From the characterization of remark 8.2.6(i), it follows easily that the section $e_{j^*\mathscr{B}/j^*\mathscr{A}}$ is the diagonal idempotent for the morphism $\mathscr{A} \to \mathscr{B}$, so $\mathscr{B}$ is an unramified $\mathscr{A}$-algebra, and the proof is concluded. $\square$

Corollary 8.2.16. *Let X be an S-scheme, $j : U \to X$ a quasi-compact open immersion, $\mathscr{A} \to \mathscr{B}$ a morphism of quasi-coherent $\mathscr{O}_U^a$-algebras, and suppose that :*

(a) *$\mathscr{B}$ is an étale $\mathscr{A}$-algebra and an almost finitely presented $\mathscr{A}$-module.*
(b) *The diagonal idempotent $e_{\mathscr{B}/\mathscr{A}}$ is a global section of the subalgebra*

$$\mathrm{Im}((j_*\mathscr{B}) \otimes_{j_*\mathscr{A}} (j_*\mathscr{B}) \to j_*(\mathscr{B} \otimes_{\mathscr{A}} \mathscr{B})).$$

Then $j_*\mathscr{B}$ is an étale $j_*\mathscr{A}$-algebra and an almost finitely presented $j_*\mathscr{A}$-module.

Proof. Indeed, the induced morphism $j_*\mathscr{A} \to j_*\mathscr{B}$ fulfills conditions (a), (c) and (d) of proposition 8.2.13, and diagram 8.2.14 trivially commutes with $t := j_*(t_{\mathscr{B}/\mathscr{A}})$, so the corollary follow from claim 8.2.15. $\square$

Using lemma 8.2.9 we can also relax one assumption in [36, Prop.8.2.31(i)]; namely, we have the following :

Proposition 8.2.17. *Let $A \subset B$ be a pair of V^a-algebras, such that $A = \mathrm{i.c.}(A, B)$. Then, for every étale almost finite projective A-algebra A_1 we have $A_1 = \mathrm{i.c.}(A_1, A_1 \otimes_A B)$.*

Proof. Set $B_1 := A_1 \otimes_A B$, and suppose that $x \in B_{1*}$ is integral over A_{1*}. Let $e \in (A_1 \otimes_A A_1)_*$ be the diagonal idempotent of the unramified A-algebra A_1; for given $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \mathfrak{m}$ we write $\varepsilon_1 \cdot e = \sum_{j=1}^k c_i \otimes d_i$ for some $c_i, d_i \in A_{1*}$. According to remark 8.2.6(iii) and [36, Prop.4.1.8(ii)], we have $\sum_{j=1}^k c_i \cdot \mathrm{Tr}_{B_1/B}(xd_i) = \varepsilon \cdot x$. On the other hand, $\varepsilon_2 \cdot x$ and $\varepsilon_3 \cdot d_i$ are integral over A_* (by [36, Lemma 5.1.13(i)], which holds for arbitrary basic setups); then lemma 8.2.9(i) implies that $\mathrm{Tr}_{B_1/B}(\varepsilon_2\varepsilon_3 \cdot xd_i) \in A_*$ for every $i = 1, \dots, k$. Hence $\varepsilon_1\varepsilon_2\varepsilon_3 \cdot x \in A_{1*}$, and since $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are arbitrary, the assertion follows. $\square$

Lemma 8.2.18. *Let X be an S-scheme, $\mathscr{A}$ a quasi-coherent $\mathscr{O}_X^a$-algebra. We have :*

(i) $\mathscr{A}_{!!}$ *is a quasi-coherent* $\mathscr{O}_X$*-algebra, and if* $\mathscr{A}$ *is almost finite, then* $\mathscr{A}_{!!}$ *is an integral* $\mathscr{O}_X$*-algebra (see definition* 8.2.5(i,ii)*).*

(ii) *Let* $f : Y \to X$ *a faithfully flat and quasi-compact morphism. Then* $\mathscr{A}$ *is an étale (resp. weakly étale, resp. weakly unramified)* $\mathscr{O}_X^a$*-algebra if and only if* $f^*\mathscr{A}$ *is an étale (resp. weakly étale, resp. weakly unramified)* $\mathscr{O}_Y^a$*-algebra.*

(iii) *Suppose that* X *is integral and* $\mathscr{A}$ *is a weakly étale* $\mathscr{O}_X^a$*-algebra. Let* $\eta \in X$ *be the generic point; then* $\mathscr{A}$ *is an étale* $\mathscr{O}_X^a$*-algebra if and only if* $\mathscr{A}_\eta$ *is an étale* $\mathscr{O}_{X,\eta}^a$*-algebra.*

(iv) *Suppose that* X *is normal and irreducible, and* $\mathscr{A}$ *is integral, torsion-free and unramified over* $\mathscr{O}_X^a$*. Then* $\mathscr{A}$ *is étale over* $\mathscr{O}_X^a$*, and almost finitely presented as an* $\mathscr{O}_X^a$*-module.*

(v) *Let* $j : X \to Y$ *be an open quasi-compact immersion of* S*-schemes, with* Y *normal and irreducible, and such that* $j_*\mathscr{O}_X = \mathscr{O}_Y$*, and suppose that* $\mathscr{A}$ *is integral and torsion-free over* $\mathscr{O}_X^a$*. Then* $j_*\mathscr{A}$ *is an integral* $\mathscr{O}_Y^a$*-algebra.*

Proof. (i): First of all, $\mathscr{A}_!$ is a quasi-coherent $\mathscr{O}_X$-module ([36, §5.5.4]), hence the same holds for $\mathscr{A}_{!!}$. It remains to show that, if $\mathscr{A}$ is almost finite, then $\mathscr{A}_{!!}(U) = \mathscr{A}(U)_{!!}$ is an integral $\mathscr{O}_X(U)$-algebra for every affine open subset $U \subset X$. Since $\mathscr{A}(U)_{!!} = \mathscr{O}_X(U) + \mathfrak{m}\mathscr{A}(U)_{!!}$, we need only show that every element of $\mathfrak{m}\mathscr{A}(U)_{!!}$ is integral over $\mathscr{O}_X(U)$. However, by adjunction we deduce a map :

$$\mathscr{A}(U)_{!!} \to A_U := \mathscr{O}_X^a(U)_* + \mathfrak{m}\mathscr{A}(U)_*$$

whose kernel is annihilated by $\mathfrak{m}$, and according to [36, Lemma 5.1.13(i)], A_U is integral over $\mathscr{O}_X^a(U)_*$. Let $a \in \mathfrak{m}\mathscr{A}(U)_{!!}$ be any element, and $\overline{a}$ its image in $\mathfrak{m}\mathscr{A}(U)_*$; we can then find $b_0, \dots, b_n \in \mathscr{O}_X(U)_*$ such that $\overline{a}^{n+1} + \sum_{i=0}^n b_i\overline{a}^i = 0$ in A_U. It follows that $(\varepsilon a)^{n+1} + \sum_{i=0}^n \varepsilon^{n+1-i} b_i(\varepsilon a)^i = 0$ in $\mathscr{A}(U)_{!!}$, for every $\varepsilon \in \mathfrak{m}$. Since $\varepsilon^{n+1-i}b_i$ lies in the image of $\mathscr{O}_X(U)$ for every $i \le n$, the claim follows easily.

(ii): One reduces easily to the case where X and Y are affine, in which case the assertion follows from [36, §3.4.1].

(iii): Let $U \subset X$ be any affine open subset; by assumption $\mathscr{A}(U)$ is flat over the almost ring $B := \mathscr{A}(U) \otimes_{\mathscr{O}_X^a(U)} \mathscr{A}(U)$, and $\mathscr{A}(U) \otimes_{\mathscr{O}_X^a(U)} \kappa(\eta)^a$ is almost projective over $B \otimes_{\mathscr{O}_X^a(U)} \kappa(\eta)^a$. Then the assertion follows from [36, Prop.2.4.19].

(iv): Let $\eta \in X$ be the generic point. We begin with the following :

Claim 8.2.19. Suppose that $V = \mathfrak{m}$ (the "classical limit" case of [36, Ex.2.1.2(ii)]), and let $f : A \to B$ be a local homomorphism of local rings. Then :

(i) f^a is weakly étale if and only if f extends to an isomorphism of strict henselizations $A^{\mathrm{sh}} \to B^{\mathrm{sh}}$.

(ii) Especially, if A is a field and f^a is weakly étale, then B is a separable algebraic extension of A.

Proof of the claim. (i): In the classical limit case, a weakly étale morphism is the same as an "absolutely flat" map as defined in [66]; then the assertion is none else than [66, Th.5.2].

(ii): If A is a field, A^{sh} is a separable closure of A; hence (i) implies that B is a subring of A^{sh}, hence it is a subfield of A^{sh}. ◊

Claim 8.2.20. Suppose that $V = \mathfrak{m}$. Let A be a field and $f : A \to B$ a ring homomorphism such that f^a is weakly étale. Then :

(i) Every finitely generated A-subalgebra of B is finite étale over A (that is, in the usual sense of [33, Ch.IV, §17]).

(ii) f^a is étale if and only if f is étale (in the usual sense of [33]).

Proof of the claim. (i): It follows from claim 8.2.19(ii) that B is reduced of Krull dimension 0, and all its residue fields are separable algebraic extensions of A. We consider first the case of a monogenic extension $C := A[b] \subset B$. For every prime ideal $\mathfrak{p} \subset B$, let $b_{\mathfrak{p}}$ be the image of b in

$B_\mathfrak{p}$; then for every such $\mathfrak{p}$ we may find an irreducible separable monic polynomial $P_\mathfrak{p}(T) \in A[T]$ with $P_\mathfrak{p}(b_\mathfrak{p}) = 0$. This identity persists in an open neighborhood $U_\mathfrak{p} \subset \operatorname{Spec} B$ of $\mathfrak{p}$, and finitely many such $U_\mathfrak{p}$ suffice to cover $\operatorname{Spec} B$. Thus, we find finitely many $P_{\mathfrak{p}_1}(T), \dots, P_{\mathfrak{p}_k}(T)$ such that $\prod_{i=1}^k P_{\mathfrak{p}_i}(b) = 0$ holds in B, and after omitting repetitions, we may assume that all these polynomials are distinct. Since C is a quotient of the separable A-algebra $A[T]/(\prod_{i=1}^k P_{\mathfrak{p}_i}(T))$, the claim follows in this case. In the general case, we may write $C = A[b_1, \dots, b_n]$ for certain $b_1, \dots, b_n \in B$. Then $\operatorname{Spec} C$ is a reduced closed subscheme of $\operatorname{Spec} A[b_1] \times_{\operatorname{Spec} A} \cdots \times_{\operatorname{Spec} A} \operatorname{Spec} A[b_n]$; by the foregoing, the latter is étale over $\operatorname{Spec} A$, hence the same holds for $\operatorname{Spec} C$.

(ii): We may assume that f^a is étale, and we have to show that f is étale, *i.e.* that B is a finitely generated A-module. Hence, let $e_{B/A} \in B \otimes_A B$ be the diagonal idempotent (see remark 8.2.6(i)); we choose a finitely generated A-subalgebra $C \subset B$ such that $e_{B/A}$ is the image of an element $e' \in C \otimes_A C$. Notice that $1 - e'$ lies in the kernel of the multiplication map $\mu_{C/A} : C \otimes_A C \to C$. By (i), the morphism $A \to C$ is étale, hence it admits as well a diagonal idempotent $e_{C/A} \in C \otimes_A C$; on the other hand, [36, Lemma 3.1.2(v)] says that B^a is étale over C^a, especially B is a flat C-algebra, hence the natural map $C \otimes_A C \to B \otimes_A B$ is injective. Since $1 - e_{C/A}$ lies in the kernel of the multiplication map $\mu_{B/A} : B \otimes_A B \to B$, we have :

$$e_{B/A}(1 - e_{C/A}) = 0 = e_{C/A}(1 - e') = e_{C/A}(1 - e_{B/A})$$

from which it follows easily that $e_{B/A} = e_{C/A}$. Moreover, the induced morphism $\operatorname{Spec} B \to \operatorname{Spec} C$ has dense image, so it must be surjective, since $\operatorname{Spec} C$ is a discrete finite set; therefore B is even a faithfully flat C-algebra. Let $J := \operatorname{Ker}(B \otimes_A B \to B \otimes_C B)$; then J is the ideal generated by all elements of the form $1 \otimes c - c \otimes 1$, where $c \in C$; clearly this is the same as the extension of the ideal $I_{C/A} := \operatorname{Ker} \mu_{C/A}$. However, $I_{C/A}$ is generated by the idempotent $1 - e_{C/A}$ ([36, Cor.3.1.9]), and consequently J is generated by $1 - e_{B/A}$, *i.e.* $J = \operatorname{Ker} \mu_{B/A}$. So finally, the multiplication map $\mu_{B/C} : B \otimes_C B \to B$ is an isomorphism, with inverse given by the map $B \to B \otimes_C B : b \mapsto b \otimes 1$. The latter is of the form $j \otimes_C \mathbf{1}_B$, where $j : C \to B$ is the natural inclusion map. By faithfully flat descent we conclude that $C = B$, whence the claim. ◊

Next, we remark that $\mathscr{A}_\eta$ is a finitely presented $\mathscr{O}^a_{X,\eta}$-module. Indeed, since $K := \mathscr{O}_{X,\eta}$ is a field, we have either $\mathfrak{m}K = 0$, in which case the category of K^a-modules is trivial and there is nothing to show, or else $\mathfrak{m}K = K$. So we may assume that we are in the “classical limit” case, and then the assertion follows from claim 8.2.20(ii).

Now, set $\mathscr{B} := \mathscr{A} \otimes_{\mathscr{O}^a_X} \mathscr{A}$, and let $j : X(\eta) \to X$ be the natural morphism; since X is normal and $\mathscr{A}$ is torsion-free, the units of adjunction $\mathscr{O}^a_X \to j_*j^*\mathscr{O}_{X(\eta)}$ and $\mathscr{A} \to j_*j^*\mathscr{A}$ are monomorphisms. Since $\mathscr{A}$ is unramified over $\mathscr{O}^a_X$, the diagonal idempotent of $j^*\mathscr{A}$ lies in the image of the restriction map $\mathscr{B}(X)_* \to \mathscr{B}_{\eta*}$. In view of these observations, an easy inspection shows that the proof of proposition 8.2.13 carries over *verbatim* to the current situation, and yields assertion (iv).

(v): Let $\mathscr{T} \subset \mathscr{A}_{!!}$ be the maximal torsion $\mathscr{O}_X$-subsheaf, and set $\mathscr{B} := \mathscr{A}_{!!}/\mathscr{T}$. Then $\mathscr{B}$ is an integral, torsion-free $\mathscr{O}_X$-algebra, by remark 8.2.6(iv), and $\mathscr{B}^a \simeq \mathscr{A}$, hence $(j_*\mathscr{B})^a \simeq j_*\mathscr{A}$. Let $\eta \in Y$ be the generic point; in light of remark 8.2.6(iv), it then suffices to show :

Claim 8.2.21. Under the assumptions of (v), let $\mathscr{R}$ be an integral quasi-coherent and torsion-free $\mathscr{O}_X$-algebra. Then $j_*\mathscr{R}$ is an integral $\mathscr{O}_Y$-algebra.

Proof of the claim. Let us write $\mathscr{R}_\eta$ as the filtered union of the family $(R_\lambda \mid \lambda \in \Lambda)$ of its finite $\mathscr{O}_{Y,\eta}$-subalgebras, and for every $\lambda \in \Lambda$, let $\mathscr{R}_\lambda \subset \mathscr{R}$ be the quasi-coherent $\mathscr{O}_X$-subalgebra such that $\mathscr{R}_\lambda(V) = R_\lambda \cap \mathscr{R}(V)$ for every non-empty open subset $V \subset X$. Then $\mathscr{R}$ is the filtered colimit of the family $(\mathscr{R}_\lambda \mid \lambda \in \Lambda)$, and clearly it suffices to show the claim for every $\mathscr{R}_\lambda$. We may thus assume from start that $\mathscr{R}_\eta$ is a finite $\mathscr{O}_{Y,\eta}$-algebra. Let $\mathscr{R}^\nu$ be the integral closure of $\mathscr{R}$, *i.e.* the quasi-coherent $\mathscr{O}_X$-algebra such that $\mathscr{R}^\nu(V)$ is the integral closure of $\mathscr{R}(V)$ in

$\mathscr{R}_\eta$, for every non empty affine open subset $V \subset X$. Clearly it suffices to show that $j_*\mathscr{R}^\nu$ is integral, hence we may replace $\mathscr{R}$ by $\mathscr{R}^\nu$ and assume from start that $\mathscr{R}$ is integrally closed. In this case, for every non-empty open subset $V \subset X$ the restriction map $\mathscr{R}(V) \to \mathscr{R}_\eta$ induces a bijection between the idempotents of $\mathscr{R}(V)$ and those of $\mathscr{R}_\eta$. Especially, $\mathscr{R}(X)$ admits finitely many idempotents, and moreover we have a natural decomposition $\mathscr{R} = \mathscr{R}_1 \times \cdots \times \mathscr{R}_k$, as a product of $\mathscr{O}_X$-algebras, such that $\mathscr{R}_{i,\eta}$ is a field for every $i \le k$. It then suffices to show the claim for every $j_*\mathscr{R}_i$, and then we may assume throughout that $\mathscr{R}_\eta$ is a field. Up to replacing $\mathscr{R}$ by its normalization in a finite extension of $\mathscr{R}_\eta$, we may even assume that $\mathscr{R}_\eta$ is a finite normal extension of $\mathscr{O}_{Y,\eta}$. Hence, let $V \subset Y$ be any non-empty affine open subset, and $a \in \mathscr{R}(X \cap V)$ any element. Let $P(T)$ be the minimal polynomial of a over the field $\mathscr{O}_{Y,\eta}$; we have to show that the coefficients of $P(T)$ lie in $\mathscr{O}_Y(V) = \mathscr{O}_X(X \cap V)$, and since Y is normal, it suffices to prove that these coefficients are integral over $\mathscr{O}_X(W)$, for every non-empty affine open subset $W \subset X \cap V$. However, since $\mathscr{R}_\eta$ is normal over $\mathscr{O}_{Y,\eta}$, such coefficients can be written as some elementary symmetric polynomials of the conjugates of a in $\mathscr{R}_\eta$. Hence, we come down to showing that the conjugates of a are still integral over $\mathscr{O}_X(W)$. The latter assertion is clear: indeed, if $Q(T) \in \mathscr{O}_X(W)[T]$ is a monic polynomial with $Q(a) = 0$, then we have also $Q(a') = 0$ for every conjugate a' of a. □

Proposition 8.2.22. *Let X be a quasi-compact and quasi-separated S-scheme, $j : U \to X$ a quasi-compact open immersion, and $\mathscr{A}$ a quasi-coherent $\mathscr{O}_U^a$-algebra, almost finitely presented as an $\mathscr{O}_U^a$-module. Then for every finitely generated subideal $\mathfrak{m}_0 \subset \mathfrak{m}$, there exists a quasi-coherent $\mathscr{O}_X$-algebra $\mathscr{B}$, finitely presented as an $\mathscr{O}_X$-module, and a morphism $\mathscr{B}^a_{|U} \to \mathscr{A}$ of $\mathscr{O}_U^a$-algebras, whose kernel and cokernel are annihilated by $\mathfrak{m}_0$.*

Proof. Set $\mathscr{C} := \mathrm{i.c.}(\mathscr{O}_X^a, j_*\mathscr{A})$; then $\mathscr{C}$ is a quasi-coherent $\mathscr{O}_X^a$-algebra, and $\mathscr{C}_{|U} = \mathscr{A}$, by lemma 8.2.18(i). According to proposition 5.2.19, we may write $\mathscr{C}_{!!}$ as the colimit of a filtered family $(\mathscr{F}_i \mid i \in I)$ of finitely presented quasi-coherent $\mathscr{O}_X$-modules. Pick a finitely generated subideal $\mathfrak{m}_1 \subset \mathfrak{m}$ such that $\mathfrak{m}_0 \subset \mathfrak{m}_1^2$; for every affine open subset $U' \subset U$, we may find $i \in I$ such that $\mathfrak{m}_1 \cdot \mathscr{A}_{!!}(U') \subset \mathrm{Im}(\mathscr{F}_i(U') \to \mathscr{A}_{!!}(U'))$, and since U is quasi-compact, finitely many such open subsets cover U; hence, we may find $i \in I$ such that $\mathfrak{m}_1 \cdot \mathscr{A}_{!!} \subset \mathrm{Im}(\mathscr{F}_{i|U} \to \mathscr{A}_{!!})$; we set $\mathscr{F} := \mathscr{F}_i$, and let $\varphi : \mathscr{G} := \mathrm{Sym}^\bullet_{\mathscr{O}_X}\mathscr{F} \to \mathscr{C}_{!!}$ be the induced morphism of quasi-coherent $\mathscr{O}_X$-algebras. Notice that $\mathscr{G}$ is finitely presented as an $\mathscr{O}_X$-algebra. Using again proposition 5.2.19 (or [26, Ch.I, Cor.9.4.9]) we may write $\mathrm{Ker}\,\varphi$ as the colimit of a filtered family $(\mathscr{K}_\lambda \mid \lambda \in \Lambda)$ of quasi-coherent $\mathscr{O}_X$-submodules of finite type. For any affine open subset $U' \subset X$, let $f_1, \dots, f_n$ be a finite set of generators of $\mathscr{F}(U')$; we may find monic polynomials $P_1(T), \dots, P_n(T)$ with coefficients in $\mathscr{O}_X(U')$ such that $P_i(\varphi(f_i)) = 0$ in $\mathscr{C}(U')_{!!}$, for every $i \le n$. Let $\lambda \in \Lambda$ be chosen large enough, so that $P_i(f_i) \in \mathscr{K}_\lambda(U')$ for every $i \le n$, and for every U' in a finite affine open covering of X. Let also $\mathscr{K}'_\lambda \subset \mathscr{G}$ be the ideal generated by $\mathscr{K}_\lambda$; then $\mathscr{G}' := \mathscr{G}/\mathscr{K}'_\lambda$ is an integral $\mathscr{O}_X$-algebra of finite presentation, hence it is finitely presented as an $\mathscr{O}_X$-module, in view of claim 5.7.8. Clearly $\varphi_{|U}$ descends to a map $\varphi' : \mathscr{G}'_{|U} \to \mathscr{A}_{!!}$, whose cokernel is annihilated by $\mathfrak{m}_1$, and by [36, Claim 2.3.12 and the proof of Cor.2.3.13] we may find, for every affine open subset $U' \subset U$, a finitely generated submodule $K_{U'} \subset \mathrm{Ker}\,(\varphi'_{U'} : \mathscr{G}'(U') \to \mathscr{A}(U')_{!!})$ such that $\mathfrak{m}_1^2 \cdot \mathrm{Ker}\,\varphi'_{U'} \subset K_{U'}$. Another invocation of proposition 5.2.19 ensures the existence of a quasi-coherent $\mathscr{O}_X$-submodule of finite type $\mathscr{J} \subset \mathrm{Ker}\,\varphi$ such that $K_{U'} \subset \mathscr{J}(U')$ for all the U' of a finite covering of U; the $\mathscr{O}_X$-algebra $\mathscr{B} := \mathscr{G}'/\mathscr{J} \cdot \mathscr{G}'$ fulfills the required conditions. □

8.2.23. Let $(X_i \mid i \in I)$ be a filtered system of quasi-compact and quasi-separated S-schemes, with affine transition morphisms $h_\varphi : X_j \to X_i$, for every morphism $\varphi : j \to i$ in I. Let also $U_i \subset X_i$ be a quasi-compact open subset, for every $i \in I$, such that $U_j = h_\varphi^{-1} U_i$ for every $\varphi : j \to i$ in I. Set

$$X := \lim_{i \in I} X_i \qquad U := \lim_{i \in I} U_i.$$

and denote by $h_i : U \to X_i$ the natural morphism, for every $i \in I$.

Corollary 8.2.24. *In the situation of* (8.2.23)*, let $\mathscr{A}$ be any flat almost finitely presented $\mathscr{O}_U^a$-algebra. Then, for every $b \in \mathfrak{m}$ there exists $i \in I$, a quasi-coherent $\mathscr{O}_{X_i}$-algebra $\mathscr{R}$, finitely presented as a $\mathscr{O}_{X_i}$-module, and a map of $\mathscr{O}_U^a$-algebras $f : h_i^*\mathscr{R}^a \to \mathscr{A}$ such that :*

(i) $\mathrm{Ker}\, f$ *and* $\mathrm{Coker}\, f$ *are annihilated by* b*.*
(ii) *For every $x \in U_i$, the map $b \cdot \mathbf{1}_{\mathscr{R},x} : \mathscr{R}_x \to \mathscr{R}_x$ factors through a free $\mathscr{O}_{U_i,x}$-module.*

Proof. From [32, Ch.IV, Th.8.3.11] it is easily seen that X is a quasi-compact and quasi-separated S-scheme, and U is a quasi-compact open subset of X. Pick a subideal $\mathfrak{m}_1$ such that $b \in \mathfrak{m}_1^4$, and suppose that $b_1, b_2, b_3, b_4 \in \mathfrak{m}_1$ are any four elements; according to proposition 8.2.22, we may find a quasi-coherent $\mathscr{O}_X$-algebra $\mathscr{B}$, finitely presented as an $\mathscr{O}_X$-module, and a morphism $g : \mathscr{B}^a_{|U} \to \mathscr{A}$ of $\mathscr{O}_U^a$-algebras whose kernel and cokernel are annihilated by $\mathfrak{m}_1$. By standard arguments we deduce that, for every $x \in U$, the map $b_1 b_2 \cdot \mathbf{1}^a_{\mathscr{B},x} : \mathscr{B}^a_x \to \mathscr{B}^a_x$ factors through $\mathscr{A}_x$. On the other hand, [36, Lemma 2.4.15] says that $b_3 \cdot \mathbf{1}_{\mathscr{A},x}$ factors through a free $\mathscr{O}^a_{U,x}$-module F, and therefore $b_1b_2b_3 \cdot \mathbf{1}^a_{\mathscr{B},x}$ factors through F as well. Finally, this implies that $b_1b_2b_3b_4 \cdot \mathbf{1}_{\mathscr{B},x}$ factors through $\mathscr{O}^{\oplus m}_{U,x}$ for some $m \in \mathbb{N}$. Now, we have $b = \sum_{j=1}^n c_j$ for some $c_1, \dots, c_n \in \mathfrak{m}$ that can be written as products of four elements of $\mathfrak{m}_1$; for each such summand c_j, the foregoing shows that there exists $m_j \in \mathbb{N}$ and maps $u_j : \mathscr{B}_x \to \mathscr{O}^{\oplus m_j}_{U,x}$ and $v_j : \mathscr{O}^{\oplus m_j}_{U,x} \to \mathscr{B}_x$, with $v_j \circ u_j = c_j \cdot \mathbf{1}_{\mathscr{B},x}$. Set $m := \sum_{j=1}^n m_j$, and define $v : \mathscr{B}_x \to \mathscr{O}^{\oplus m}_{U,x}$ and $u : \mathscr{O}^{\oplus m}_{U,x} \to \mathscr{B}_x$ by the rules :

$$t \mapsto (u_1(t), \dots, u_n(t)) \qquad (s_1, \dots, s_n) \mapsto \sum_{j=1}^n v_j(s_j)$$

for every $t \in \mathscr{B}_x$ and every $s_j \in \mathscr{O}^{\oplus m_j}_{U,x}$ ($j = 1, \dots, n$); it is easily seen that $v \circ u = b \cdot \mathbf{1}_{\mathscr{B}_x}$. Since $\mathscr{B}$ is finitely presented, such factorization extends to some affine open neighborhood V of x in U, and since U is quasi-compact, finitely many such open subsets suffice to cover U. It follows easily (see [32, Ch.IV, Th.8.5.2]) that, for $i \in I$ large enough, $\mathscr{B}$ descends to a quasi-coherent $\mathscr{O}_{X_i}$-algebra $\mathscr{R}$, finitely presented as an $\mathscr{O}_{X_i}$-module, and such that (ii) holds. ☐

Henceforth, and until the end of this section, *we assume that* $\mathfrak{m}$ *fulfills condition* $(\mathbf{B})$.

Definition 8.2.25. Let X be an S-scheme, $Z \subset X$ a closed subscheme such that $U := X \setminus Z$ is a dense subset of X, and denote by $j : U \to X$ the open immersion.

(i) We say that the pair (X, Z) is *almost pure* if the restriction functor

(8.2.26) $$\mathscr{O}_X^a\text{-}\mathbf{\acute{E}t}_{\mathrm{fr}} \to \mathscr{O}_U^a\text{-}\mathbf{\acute{E}t}_{\mathrm{fr}} \quad : \quad \mathscr{A} \mapsto \mathscr{A}_{|U}$$

from the category of étale $\mathscr{O}_X^a$-algebras of finite rank, to the category of étale $\mathscr{O}_U^a$-algebras of finite rank, is an equivalence.

(ii) We say that the pair (X, Z) is *normal* if Z is a constructible subset of X, and the natural map $\mathscr{O}_X^a \to j_*\mathscr{O}_U^a$ is a monomorphism, whose image is integrally closed in $j_*\mathscr{O}_U^a$.

Remark 8.2.27. Let (X, Z) be a pair as in definition 8.2.25, where Z is a constructible subset of X, and set $U := X \setminus Z$.

(i) Consider the following conditions:
 (a) $\mathscr{O}_{X,z}$ is a normal domain, for every $z \in Z$.
 (b) the natural map $\mathscr{O}_X \to j_*\mathscr{O}_U$ is a monomorphism, and $\mathscr{O}_X = \mathrm{i.c.}(\mathscr{O}_X, j_*\mathscr{O}_U)$.
 (c) The pair (X, Z) is normal.

Then (a)⇒(b), since $(j_*\mathscr{O}_U)_z = \mathscr{O}_{X(z)}(U \cap X(z))$ for every $z \in Z$. Also, (b)⇒(c), by virtue of [36, Lemma 8.2.28].

(ii) Let $\mathscr{A}$ be a quasi-coherent $\mathscr{O}_U^a$-algebra. Then $j_*\mathscr{A}_{!!}$ is a quasi-coherent $\mathscr{O}_X$-algebra ([26, Ch.I, Prop.9.4.2(i)] and lemma 8.2.18(i)), hence the integral closure $\mathscr{A}^\nu$ of the image of $\mathscr{O}_X^a$ in $j_*\mathscr{A}$ is a well defined quasi-coherent $\mathscr{O}_X^a$-algebra ([36, Lemma 8.2.28]). We call $\mathscr{A}^\nu$ the *normalization* of $\mathscr{A}$ over X.

(iii) Let (X, Z) be any normal pair, and $U \subset X$ any open subset. Directly from the definition, we see that $(U, Z \cap U)$ is still a normal pair.

Lemma 8.2.28. *Let (X, Z) be a normal pair as in definition* 8.2.25(ii)*, and set $U := X \setminus Z$. Let $\mathscr{A}$ be an étale $\mathscr{O}_U^a$-algebra and an almost finitely presented $\mathscr{O}_U^a$-module (resp. and an $\mathscr{O}_U^a$-module of finite rank). The following conditions are equivalent :*

(a) *The normalization $\mathscr{A}^\nu$ of $\mathscr{A}$ over X (see remark* 8.2.27(ii)*) is a weakly unramified $\mathscr{O}_X^a$-algebra.*

(b) *$\mathscr{A}^\nu$ is an étale $\mathscr{O}_X^a$-algebra, and an almost finitely presented $\mathscr{O}_X^a$-module (resp. and an $\mathscr{O}_X^a$-module of finite rank).*

Proof. Obviously (b)⇒(a), hence suppose that (a) holds, and let $\mathscr{B} := \mathscr{A}^\nu \otimes_{\mathscr{O}_X^a} \mathscr{A}^\nu$. We may assume that X is affine, and we set

$$A := \mathscr{A}^\nu(X) \qquad B := \mathscr{B}(X).$$

By assumption, the multiplication morphism $\mu : \mathscr{B} \to \mathscr{A}^\nu$ is flat, especially A is a flat B-module. Let also B' be the image of B in $\mathscr{B}(U)$; clearly $\mu(X) : B \to A$ factors through an epimorphism $\mu' : B' \to A$, therefore $A = A \otimes_B B'$, so A is also a flat B'-module. Pick a finite covering $U = U_1 \cup \cdots \cup U_k$ consisting of affine open subsets of U. The induced morphism

$$B' \to \mathscr{B}(U_1) \times \cdots \times \mathscr{B}(U_k)$$

is a monomorphism. On the other hand, notice that $A \otimes_{B'} \mathscr{B}(U_i) = \mathscr{A}(U_i)$ is an almost finitely presented $\mathscr{B}(U_i)$-module. It follows that A is an almost finitely generated projective B'-module ([36, Prop. 2.4.18 and 2.4.19]), and therefore the kernel of μ' is generated by an idempotent $e \in B'_*$ ([36, Rem. 3.1.8]). A simple inspection shows that e is necessarily the diagonal idempotent of the unramified $\mathscr{O}_U^a$-algebra $\mathscr{A}$. Then all the assumptions of proposition 8.2.13 are fulfilled, so $\mathscr{A}^\nu$ is an étale $\mathscr{O}_X^a$-algebra, and an almost finitely presented $\mathscr{O}_X^a$-module.

Lastly, suppose that $\mathscr{A}$ is an $\mathscr{O}_U^a$-module of finite rank, and pick $r \in \mathbb{N}$ such that, for every $i = 1, \dots, k$, the r-th exterior power of the $\mathscr{O}_U^a(U_i)$-module $\mathscr{A}(U) = A \otimes_{\mathscr{O}_X^a(X)} \mathscr{O}_U^a(U_i)$ vanishes. Since the induced morphism $\mathscr{O}_X^a \to \mathscr{O}_U^a(U_1) \times \cdots \times \mathscr{O}_U^a(U_1)$ is a monomorphism, and A is a flat $\mathscr{O}_X^a$-module, we deduce that the r-th exterior power of A vanishes as well. Especially, $\mathscr{A}^\nu$ is an $\mathscr{O}_X^a$-module of finite rank. □

Lemma 8.2.29. *Let (X, Z) be a normal pair, $j : X \setminus Z \to X$ the open immersion, $\mathscr{B}$ any étale almost finitely presented $\mathscr{O}_X^a$-algebra, and $Z' \subset Z$ any constructible closed subset. We have :*

(i) *The natural map $\mathscr{B} \to j_*j^*\mathscr{B}$ factors through an isomorphism $\mathscr{B} \xrightarrow{\sim} (j^*\mathscr{B})^\nu$.*

(ii) *The restriction functor* (8.2.26) *is fully faithful.*

(iii) *The pair (X, Z') is normal, and if the pair (X, Z) is almost pure, the same holds for (X, Z').*

Proof. (i): Set $U := X \setminus Z$ and $\mathscr{A} := \mathscr{B}_{|U}$; the natural map $\mathscr{B} \to j_*\mathscr{A}$ factors through a morphism $\varphi : \mathscr{B} \to \mathscr{A}^\nu$ of $\mathscr{O}_X^a$-algebras (lemma 8.2.18(i) and remark 8.2.6(iv)). Fix $z \in Z$, and set $R := \mathscr{O}_{X,z}^a$, $R' := (j_*\mathscr{O}_X^a)_z$. It follows that the stalk $\mathscr{B}_z$ is an étale R-algebra. Notice that $(j_*\mathscr{A})_z = R' \otimes_R \mathscr{B}_z$, therefore φ is a monomorphism. Moreover, since $\mathscr{B}$ is also an almost finitely presented $\mathscr{O}_X^a$-module, we have

$$\mathscr{B}_z = \text{i.c.}(\mathscr{B}_z, R' \otimes_R \mathscr{B}_z) \qquad \mathscr{A}_z^\nu = \text{i.c.}(R, R' \otimes_R \mathscr{B}_z)$$

(proposition 8.2.17). On the other hand, $\mathscr{B}_z$ is an integral R-algebra, therefore $\mathscr{B}_z = \mathscr{A}_z^\nu$, and since z is arbitrary, we conclude that φ is an isomorphism, as asserted.

(ii): Let $\mathscr{B}_1$ and $\mathscr{B}_2$ be two étale almost finitely presented $\mathscr{O}_X^a$-algebra, set $\mathscr{A}_i := \mathscr{B}_{i|U}$ for $i = 1, 2$, and let $\psi : \mathscr{A}_1 \to \mathscr{A}_2$ be any morphism of $\mathscr{O}_U^a$-algebras; by (i), ψ extends uniquely to a morphism $\psi^\nu : \mathscr{B}_1 = \mathscr{A}_1^\nu \to \mathscr{A}_2^\nu = \mathscr{B}_2$ of $\mathscr{O}_X^a$-algebras, whence the contention.

(iii): Set $U' := X \setminus Z'$ and let $j' : U' \to X$ be the open immersion; the natural morphism $\mathscr{O}_X^a \to j_*\mathscr{O}_U$ factors through the morphism $\mathscr{O}_X^a \to j'_*\mathscr{O}_{U'}$, so the latter is a monomorphism. In order to show that (X, Z') is normal, it then suffices to check that the image of $\mathscr{O}_X^a$ is integrally closed in $j'_*\mathscr{O}_{U'}^a$, and since $\mathscr{O}_X^a$ is integrally closed in $j_*\mathscr{O}_U^a$, we are reduced to proving that the natural morphism $j'_*\mathscr{O}_{U'}^a \to j_*\mathscr{O}_U^a$ is a monomorphism. However, let $V \subset X$ be any affine open subset, and write $U' \cap V = V_1 \cap \cdots \cap V_n$ for certain affine open subsets $V_1, \dots, V_n$ of X; by assumption, the restriction map $\mathscr{O}_X^a(V_i) \to \mathscr{O}_X^a(U \cap V_i)$ is a monomorphism for every $i = 1, \dots, n$, so the same holds for the restriction map $\mathscr{O}_X^a(U' \cap V) \to \mathscr{O}_X^a(U \cap V)$, whence the assertion.

Lastly, suppose that (X, Z) is almost pure; in view of (ii), in order to check that (X, Z') is almost pure, it suffices to show that every étale $\mathscr{O}_{U'}^a$-algebra $\mathscr{A}$ of finite rank extends to an étale $\mathscr{O}_X^a$-algebra of finite rank. However, the assumption implies that $\mathscr{A}_{|U}$ extends to an $\mathscr{O}_X^a$-algebra $\mathscr{B}$ as sought, and since $(X \setminus Z', Z \setminus Z')$ is normal (remark 8.2.27(iii)), (ii) says that the isomorphism $\mathscr{B}_{|U} \xrightarrow{\sim} \mathscr{A}$ extends to an isomorphism $\mathscr{B}_{|U'} \xrightarrow{\sim} \mathscr{A}$, *i.e.* $\mathscr{B}$ is an extension of $\mathscr{A}$, as required. □

Proposition 8.2.30. *Let (X, Z) be a normal pair, and set $U := X \setminus Z$. Then the following conditions are equivalent :*

(a) *The pair (X, Z) is almost pure.*
(b) *For every étale $\mathscr{O}_U^a$-algebra $\mathscr{A}$ of finite rank, the normalization $\mathscr{A}^\nu$ of $\mathscr{A}$ over X is an étale $\mathscr{O}_X^a$-algebra of finite rank (see definition* 8.2.5(iii) *and remark* 8.2.27(ii)*).*
(c) *For every étale $\mathscr{O}_U^a$-algebra $\mathscr{A}$ of finite rank, $\mathscr{A}^\nu$ is a weakly unramified $\mathscr{O}_X^a$-algebra.*
(d) *Every étale $\mathscr{O}_U^a$-algebra $\mathscr{A}$ of finite rank extends to an étale almost finite $\mathscr{O}_X^a$-algebra.*

Proof. The equivalence (a)⇔(b) is already clear from lemma 8.2.29.

Next, clearly (b)⇒(c). Conversely, suppose that (c) holds, and let $\mathscr{A}$ be any étale $\mathscr{O}_U^a$-algebra of finite rank, so $\mathscr{A}^\nu$ is a weakly unramified $\mathscr{O}_X^a$-algebra. Then $\mathscr{A}^\nu$ is actually an étale $\mathscr{O}_X^a$-algebra, and an almost finitely presented $\mathscr{O}_X^a$-module (lemma 8.2.28). Fix any affine open subset $V \subset X$, and pick a finite covering of $U \cap V$ consisting of affine open subsets $V_1, \dots, V_k$ of U. Let $r \in \mathbb{N}$ be an integer such that the r-th exterior power of the $\mathscr{O}_X^a(V_i)$-modules $\mathscr{A}(V_i)$ vanish. The induced map

$$B := \mathscr{O}_X^a(V) \to B' := \mathscr{O}_X^a(V_1) \times \cdots \times \mathscr{O}_X^a(V_k)$$

is a monomorphism, since (X, Z) is a normal pair. Moreover, the r-th exterior power of the B'-module $\mathscr{A}^\nu(V) \otimes_B B'$ vanishes; since $\mathscr{A}^\nu(V)$ is a flat B-module, we conclude that the r-th exterior power of $\mathscr{A}^\nu(V)$ vanishes as well. Especially, $\mathscr{A}^\nu$ is of finite rank, so (b) holds.

Lastly, since (b)⇒(d), we suppose that (d) holds, and deduce that (c) holds as well. Indeed, let $\mathscr{B}$ be an almost finite étale $\mathscr{O}_X^a$-algebra extending $\mathscr{A}$. In the foregoing, we have already remarked that $\mathscr{B} \subset \mathscr{A}^\nu$; especially, the diagonal idempotent of $\mathscr{A}$ lies in the image of the restriction map $\mathscr{A}^\nu \otimes_{\mathscr{O}_X^a} \mathscr{A}^\nu(X)_* \to \mathscr{A} \otimes_{\mathscr{O}_X^a} \mathscr{A}(U)_*$. Then proposition 8.2.13 implies that $\mathscr{A}^\nu$ is an étale almost finite $\mathscr{O}_X^a$-algebra, as sought. □

Corollary 8.2.31. *Let (X, Z) be a normal pair, and suppose that :*

(a) *The scheme $Z(z)$ is finite dimensional, for every $z \in Z$.*
(b) *The pair $(X(z), \{z\})$ is almost pure, for every $z \in Z$.*

Then the pair (X, Z) is almost pure.

Proof. Let $\mathscr{A}$ be any étale $\mathscr{O}_U^a$-algebra of finite rank, and $\mathscr{A}^\nu$ the normalization of $\mathscr{A}$ over X. In light of proposition 8.2.30, it suffices to show that $\mathscr{A}_z^\nu$ is a weakly unramified $\mathscr{O}_{X,z}^a$-algebra, for every $z \in Z$. Suppose, by way of contradiction, that the latter assertion fails; in view of condition (a), we may then find a point $z \in Z$, such that $\mathscr{A}_z^\nu$ is not weakly unramified over $\mathscr{O}_{X,z}^a$, but for every proper generization $w \in Z$ of z, the $\mathscr{O}_{X,w}^a$-algebra $\mathscr{A}_w^\nu$ is weakly unramified.

Now, set $U(z) := U \cap X(z)$, $V(z) := X(z)\setminus\{z\}$, and let $j : V(z) \to U$ be the induced morphism. Notice that $(V(z), Z(z))$ is a normal pair, and our assumption implies that $j^*\mathscr{A}^\nu$ is a weakly unramified $\mathscr{O}_{V(z)}^a$-algebra. By lemma 8.2.28, it follows that $j^*\mathscr{A}^\nu$ is actually an étale $\mathscr{O}_{V(z)}^a$-algebra of finite rank. Since by assumption, $(X(z), \{z\})$ is almost pure, proposition 8.2.30 says that $\mathscr{A}_z^\nu$ is an étale $\mathscr{O}_{X,z}^a$-module of finite rank, a contradiction. $\square$

8.2.32. Let $f : X' \to X$ be a morphism of schemes, ξ' a geometric point of X' localized at $x' \in X'$, and set $\xi := f(\xi')$. We shall say that f is *pro-smooth at the point* x' if the induced map of strictly henselian local rings

$$\mathscr{O}_{X(\xi),\xi} \to \mathscr{O}_{X'(\xi'),\xi'}$$

is ind-smooth (*i.e.* a filtered colimit of smooth ring homomorphisms).

Lemma 8.2.33. *Let $f : X' \to X$ be a morphism of schemes, ξ' a geometric point of X', $\mathscr{R} \subset \mathscr{S}$ a monomorphism of quasi-coherent $\mathscr{O}_X$-algebras, and suppose that f is pro-smooth at the support of ξ'. We have :*

(i) *f is flat at the support of ξ'.*

(ii) *Denote also*

$$\varphi : f^*\mathrm{i.c.}(\mathscr{R}, \mathscr{S}) \to \mathrm{i.c.}(f^*\mathscr{R}, f^*\mathscr{S})$$

the induced morphism. Then $\varphi_{\xi'}$ is an isomorphism.

Proof. (i): Let x' be the support of ξ' and set $x := f(x')$, $\xi := f(\xi')$. We have a commutative diagram of local S-schemes

$$\begin{tikzcd} X'(\xi') \arrow[r, "f_{(\xi')}"] \arrow[d, "\varphi'"'] & X(\xi) \arrow[d, "\varphi"] \\ X'(x') \arrow[r, "f_{(x')}"] & X(x) \end{tikzcd}$$

whose vertical arrows are ind-étale morphisms, and whose top horizontal arrow is ind-smooth by assumption. Especially, since φ' is faithfully flat, the same holds for $f_{(x')}$, *i.e.* f is flat.

(ii): We easily reduce to the case where X is affine, say $X = \operatorname{Spec} A$ for some ring A. Let $g : X'(\xi') \to X'$ be the natural morphism. It suffices to show that $g^*\varphi$ is an isomorphism. However, $g^*\mathrm{i.c.}(f^*\mathscr{R}, f^*\mathscr{S}) = \mathrm{i.c.}(g^*f^*\mathscr{R}, g^*f^*\mathscr{S})$ by [31, Ch.IV, Prop.6.14.4], and $g^*\varphi$ is the induced morphism

$$(f \circ g)^*\mathrm{i.c.}(\mathscr{R}, \mathscr{S}) \to \mathrm{i.c.}((f \circ g)^*\mathscr{R}, (f \circ g)^*\mathscr{S}).$$

Notice that $f \circ g$ is induced by an ind-smooth ring homomorphism $A \to \mathscr{O}_{X',\xi'}$, so we may invoke again [31, Ch.IV, Prop.6.14.4] to conclude. $\square$

Proposition 8.2.34. *Let (X, Z) be a normal pair, and $f : X' \to X$ a morphism of S-schemes, such that :*

(a) *$f(f^{-1}Z) = Z$ and f is pro-smooth at every point of $f^{-1}Z$.*

(b) *The pair $(X', f^{-1}Z)$ is almost pure.*

Then the pair (X, Z) is almost pure.

Proof. Set $Z' := f^{-1}Z$, $U := X\setminus Z$, $U' := X'\setminus Z'$, and let $f_{|U} : U' \to U$ be the restriction of f. Notice that the open immersion $j' : U' \to X'$ is quasi-compact ([26, Ch.I, Prop.6.6.4]), and since f is flat at every point of Z' (lemma 8.2.33(i)), U' is dense in X'. Moreover :

Claim 8.2.35. (i) The pair (X', Z') is normal.

(ii) More generally, let $\mathscr{A}$ be any quasi-coherent $\mathscr{O}_U^a$-algebra. Then the natural morphism

$$f^*(\mathscr{A}^\nu) \to (f^*_{|U}\mathscr{A})^\nu$$

is an isomorphism (notation of remark 8.2.27(ii)).

Proof of the claim. Notice first that, since f is flat at every point of Z (lemma 8.2.33(i)), and the unit of adjunction $\mathscr{O}_X^a \to j_*\mathscr{O}_U^a$ is a monomorphism, the induced morphism $\mathscr{O}_X^a \to f^*j_*\mathscr{O}_U^a = j'_*\mathscr{O}_{U'}^a$ is a monomorphism as well (corollary 5.1.19). Also, $(\mathscr{O}_U^a)^\nu = \mathscr{O}_X^a$, since (X, Z) is a normal pair; therefore, (i) follows from (ii).

(ii): Denote $\mathscr{A}_{!!}^\nu$ the integral closure in $j_*(\mathscr{A}_{!!})$ of the image of of $\mathscr{O}_X$; by corollary 5.1.19, the natural map

$$f^*j_*(\mathscr{A}_{!!}) \to j'_*f^*_{|U}(\mathscr{A}_{!!})$$

is an isomorphism of $\mathscr{O}_{X'}$-algebras. On the other hand, lemma 8.2.33(ii) implies that the integral closure of the image of $\mathscr{O}_{X'}$ in $f^*j_*(\mathscr{A}_{!!})$ equals $f^*(\mathscr{A}_{!!}^\nu)$. The assertion follows. ◊

Suppose now that $\mathscr{A}$ is an étale $\mathscr{O}_U^a$-algebra of finite rank. Since, by assumption, (X', Z') is almost pure, proposition 8.2.30 says that $(f^*_{|U}\mathscr{A})^\nu$ is an étale $\mathscr{O}_{X'}^a$-algebra of finite rank. By claim 8.2.35 and lemmata 8.2.33(ii), 8.2.18(ii), we deduce that $\mathscr{A}^\nu$ is a weakly étale $\mathscr{O}_X^a$-algebra. To conclude the proof, it suffices now to invoke proposition 8.2.30. □

Lemma 8.2.36. *Let $(A_i \mid i \in \mathbb{N})$ be a system of V^a-algebras, and set $A := \prod_{i\in\mathbb{N}} A_i$. Let also P be an A-module, B an A-algebra, and suppose that*

(a) $\lim_{i\to\infty} \mathrm{Ann}_{V^a}(A_i) = V^a$ *for the uniform structure of* [36, Def.2.3.1].

(b) *For every $i \in \mathbb{N}$, the A_i-modules $P_i := P \otimes_A A_i$, $B_i := B \otimes_A A_i$ are almost projective of finite constant rank equal to i, and B_i is an étale A_i-algebra.*

Then P is an almost projective A-module of almost finite rank, and B is an étale A-algebra.

Proof. For every $j \in \mathbb{N}$, the finite product $P_{\le j} := \prod_{i=1}^j P_i$ is an almost projective A-module of finite rank, and clearly the induced morphism $\pi_j : P \to P_{\le j}$ is an epimorphism. On the other hand, the $(j+1)$-th exterior power of P equals the $(j+1)$-th exterior power of $\mathrm{Ker}\,\pi_j$, and from condition (i) we see that $\lim_{j\to\infty} \mathrm{Ann}_{V^a}(\mathrm{Ker}\,\pi_j) = 0$, whence the assertion for P.

It follows already that B is an almost projective A-module. It remains to show that B is an unramified A-algebra, to which aim, we may apply the criterion of [36, Prop.3.1.4].

Claim 8.2.37. Under the assumptions of the lemma, the natural morphism

$$\varphi : B \otimes_A B \to C := \prod_{i\in\mathbb{N}} B_i \otimes_{A_i} B_i$$

is an isomorphism of A-algebras.

Proof of the claim. For every $j \in \mathbb{N}$, let $\pi_j : C \to \prod_{i\le j} B_i \otimes_A B_i$ be the natural morphism. Then

$$\lim_{j\to\infty} \mathrm{Ann}_{V^a}\mathrm{Ker}(\pi_j \circ \varphi) = V^a = \lim_{j\to\infty} \mathrm{Ann}_{V^a}\mathrm{Ker}\,\pi_j.$$

The first identity implies that φ is a monomorphism. Next, since $\pi_j \circ \varphi$ is an epimorphism, the natural morphism $\mathrm{Ker}\,\pi_j \to \mathrm{Coker}\,\varphi$ is an epimorphism; then the second identity implies that φ is also an epimorphism. ◊

Now, by [36, Prop.3.1.4], for every $i \in \mathbb{N}$ there exists an idempotent $e_i \in (B_i \otimes_{A_i} B_i)_*$ uniquely characterized by the conditions (i)–(iii) of *loc.cit.* In view of claim 8.2.37, the sequence $(e_i \mid i \in \mathbb{N})$ defines an idempotent in $(B \otimes_A B)_*$, which clearly fulfills the same conditions, whence the contention, again by [36, Prop.3.1.4]. □

Proposition 8.2.38. *Let (X, Z) be a normal pair, and set $U := X \setminus Z$. The following conditions are equivalent :*

(a) *(X, Z) is an almost pure pair.*
(b) *The restriction functor*

$$\mathscr{O}_X^a\text{-}\mathbf{\acute{E}t}_{\mathrm{afr}} \to \mathscr{O}_U^a\text{-}\mathbf{\acute{E}t}_{\mathrm{afr}} \quad : \quad \mathscr{A} \mapsto \mathscr{A}_{|U}$$

from the category of étale $\mathscr{O}_X^a$-algebras of almost finite rank, to the category of étale $\mathscr{O}_U^a$-algebras of almost finite rank, is an equivalence.

Proof. (b)⇒(a): Indeed, suppose that $\mathscr{A}$ is an étale $\mathscr{O}_U^a$-algebra of finite rank. By (b), $\mathscr{A}$ extends to an étale $\mathscr{O}_X^a$-algebra of almost finite rank. Then proposition 8.2.30 implies that (a) holds.

Now, suppose that (a) holds; let $\mathscr{A}$ be an étale $\mathscr{O}_U^a$-algebra of almost finite rank, $V \subset U$ an affine open subset, and set $A := \mathscr{O}_X^a(V)$, $B := \mathscr{A}(V)$. According to [36, Th.4.3.28], there exists a decomposition of A as an infinite product of a system of V^a-algebras $(A_i \mid i \in \mathbb{N})$ fulfilling condition (a) of lemma 8.2.36. Such a decomposition determines a system of idempotent elements $e_{V,i} \in A_*$, for every $i \in \mathbb{N}$, such that $e_{V,i} \cdot e_{V,j} = 0$ whenever $i \neq j$, and characterized by the identities $e_{V,i}A = A_i$ for every $i \in \mathbb{N}$. Since $\mathscr{O}_{X*}$ is a sheaf ([36, §5.5.4]), condition (ii) ensures that, for any fixed $i \in \mathbb{N}$, and V ranging over the affine open subsets of U, the idempotents $e_{V,i}$ glue to a global section $e_i \in \mathscr{O}_X^a(U)_*$, which will still be an idempotent; moreover, the direct factor $\mathscr{A}_i$ of $\mathscr{A}$ is an étale $\mathscr{O}_U^a$-algebra of finite rank, and the projection $\mathscr{A} \to \mathscr{A}_i$ is a morphism of $\mathscr{O}_U^a$-algebras. Especially, A and B fulfill as well condition (b) of lemma 8.2.36. By (a) and proposition 8.2.30, we deduce that the normalization $\mathscr{A}_i^\nu$ of $\mathscr{A}_i$ over X is an étale $\mathscr{O}_X^a$-algebra of finite rank, for every $i \in \mathbb{N}$. On the other hand, notice that the natural morphism

$$\mathscr{O}_{X*}^a \to j_*(\mathscr{O}_{U*}^a) = (j_*\mathscr{O}_U^a)_*$$

is a monomorphism, and its image is integrally closed in $j_*\mathscr{O}_{U*}^a$ ([36, Rem.8.2.30(i)]). It follows easily that $e_i \in \mathscr{O}_{X*}^a(X)$; hence

$$e_i\mathscr{A}_i^\nu = \mathscr{A}_i^\nu \qquad \text{for every } i \in \mathbb{N}$$

and the latter is the integral closure of $e_i\mathscr{O}_X^a$ in $j_*(\mathscr{A}_i)$. Consequently, $\mathscr{A}^\nu = \prod_{i\in\mathbb{N}} \mathscr{A}^\nu$ ([36, Rem.8.2.30(ii)]). By lemma 8.2.36, it follows that $\mathscr{A}^\nu$ is an étale $\mathscr{O}_X^a$-algebra of almost finite rank. Arguing as in the proof of proposition 8.2.30, we see that $\mathscr{A}^\nu$ is (up to unique isomorphism) the only étale almost finite $\mathscr{O}_X^a$-algebra extending $\mathscr{A}$, and (b) follows easily. □

Lemma 8.2.39. *Let (A, I) be a tight henselian pair (see* [36, Def.5.1.9]*). Then we have :*

(i) *The base change functor*

$$\mathbf{Cov}(\operatorname{Spec} A) \to \mathbf{Cov}(\operatorname{Spec} A/I)$$

is an equivalence (notation of [36, §8.2.22]*).*
(ii) *More generally, the base change functor A-$\mathbf{\acute{E}t} \to A/I$-$\mathbf{\acute{E}t}$ restricts to an equivalence*

$$A\text{-}\mathbf{\acute{E}t}_{\mathrm{afr}} \xrightarrow{\sim} A/I\text{-}\mathbf{\acute{E}t}_{\mathrm{afr}}$$

on the respective full subcategories of étale algebras of almost finite rank.

Proof. In view of [36, Th.5.5.7(iii)], it suffices to show :

Claim 8.2.40. Let P be an almost finitely generated projective A-module, such that P/IP is an A/I-module of finite rank (resp. of almost finite rank). Then P is an A-module of finite rank (resp. of almost finite rank).

Proof of the claim. Suppose first that P/IP is of finite rank, so there exists $i \in \mathbb{N}$ such that $\Lambda^i_{A/I}(P/IP) = 0$. Hence $(\Lambda^i_A P) \otimes_A P/IP = 0$, and then [36, Lemma 5.1.7] yields $\Lambda^i_A P = 0$, as sought. In the more general case where P/IP has almost finite rank, pick $n \in \mathbb{N}$ and a finitely generated subideal $\mathfrak{m}_0 \subset \mathfrak{m}$, such that $I^n \subset \mathfrak{m}_0$. Let also $\mathfrak{m}_1 \subset \mathfrak{m}$ be any finitely generated subideal such that $\mathfrak{m}_0 \subset \mathfrak{m}_1^{n+1}$; by assumption, there exists $i \in \mathbb{N}$ such that $\mathfrak{m}_1\Lambda^i_{A/I}(P/IP) = 0$, *i.e.* $\mathfrak{m}_1\Lambda^i_A P \subset I\Lambda^i_A P$. Therefore

$$\mathfrak{m}_0\Lambda^i_A P \subset \mathfrak{m}_1^{n+1}\Lambda^i_A P \subset I^{n+1}\Lambda^i_A P \subset \mathfrak{m}_0 I\Lambda^i_A P$$

whence $\mathfrak{m}_0\Lambda^i_A P = 0$, by [36, Lemma 5.1.7], and finally, $\mathfrak{m}_1^{n+1}\Lambda^i_A P = 0$. Since $\mathfrak{m}_1$ is arbitrary, we deduce that P has almost finite rank, as claimed. □

8.2.41. Let R be a V-algebra, $I \subset R$ a principal ideal generated by a regular element, $R^\wedge$ the I-adic completion of R. Set

$$X := \operatorname{Spec} R \qquad X^\wedge := \operatorname{Spec} R^\wedge \qquad Z := \operatorname{Spec} R/I \qquad Z^\wedge := \operatorname{Spec} R^\wedge/IR^\wedge.$$

Proposition 8.2.42. *In the situation of* (8.2.41), *suppose furthermore that :*

(i) *There exists $n \in \mathbb{N}$ and a finitely generated subideal $\mathfrak{m}_0 \subset \mathfrak{m}$ such that $I^n \subset \mathfrak{m}_0$.*
(ii) *The pair (R, I) is henselian.*

Then the pair (X, Z) is almost pure if and only if the same holds for the pair $(X^\wedge, Z^\wedge)$.

Proof. Under the current assumptions, the pair (R^a, I^a) is tight henselian ([36, §5.1.12]). Hence the assertion is a straightforward consequence of [36, Prop.5.4.53] and lemma 8.2.39(i) (details left to the reader). □

8.2.43. Let $p \in \mathbb{Z}$ be a prime integer with $pV \subset \mathfrak{m}$, and such that p is a regular element of V. Let A and B be two flat V-algebras, and for every $n \in \mathbb{N}$ set $V_n := V/p^{n+1}V$, $A_n := A/p^{n+1}A$, $B_n := B/p^{n+1}B$. Let also

$$A_0 \xrightarrow{\sim} B_0 \tag{8.2.44}$$

be a given isomorphism of V_0-algebras, and suppose that the Frobenius endomorphisms of V_0 and A_0 induce isomorphisms

$$V/p^{1/p}V \xrightarrow{\sim} V_0 \qquad A/p^{1/p}A \xrightarrow{\sim} A_0. \tag{8.2.45}$$

Set $X := \operatorname{Spec} A$, $X_0 := \operatorname{Spec} A_0$, $Y := \operatorname{Spec} B$ and $Y_0 := \operatorname{Spec} B_0$.

Corollary 8.2.46. *In the situation of* (8.2.43), *suppose moreover that the pairs (A, pA) and (B, pB) are henselian. Then the pair (X, X_0) is almost pure if and only if the same holds for the pair (Y, Y_0).*

Proof. In view of proposition 8.2.42, it suffices to show that (8.2.44) lifts to an isomorphism $A^\wedge \xrightarrow{\sim} B^\wedge$ between the p-adic completions of A and B. In turns, this reduces to exhibiting a system of isomorphisms $(\varphi_n : A_n \xrightarrow{\sim} B_n \mid n \in \mathbb{N})$ such that $\varphi_n \otimes_{V_n} V_{n_1} = \varphi_{n-1}$ for every $n > 0$. To this aim, it suffices to show that

$$\mathbb{L}_{A_n/V_n} = 0 \qquad \text{in } \mathsf{D}(s.A\text{-}\mathbf{Mod}) \text{ for every } n \in \mathbb{N}$$

([36, Prop.3.2.16]). However, in view of [36, Th.2.5.36], for every $n \in \mathbb{N}$, the short exact sequence $0 \to p^{n+1}V/p^{n+2}V \to V_{n+1} \to V_n \to 0$ induces a distinguished triangle in $\mathsf{D}(s.A\text{-}\mathbf{Mod})$

$$\mathbb{L}_{A_0/V_0} \to \mathbb{L}_{A_{n+1}/V_{n+1}} \to \mathbb{L}_{A_n/V_n} \to \sigma\mathbb{L}_{A_0/V_0}.$$

Hence, an easy induction further reduces to checking that $\mathbb{L}_{A_0/V_0} = 0$ in $\mathsf{D}(s.A\text{-}\mathbf{Mod})$. According to [36, Lemma 6.5.13], this will follow, once we have shown that the natural map

$$V_{0,(\Phi)} \overset{\mathbf{L}}{\otimes}_{V_0} A_0 \to A_{0,(\Phi)}$$

is an isomorphism in $\mathsf{D}(V_0\text{-}\mathbf{Mod})$ (notation of *loc.cit.*). Taking into account the isomorphisms (8.2.45), this holds if and only if the natural map

$$V/p^{1/p}V \overset{\mathbf{L}}{\otimes}_{V_0} A_0 \to A/p^{1/p}$$

is an isomorphism in $\mathsf{D}(V_0\text{-}\mathbf{Mod})$. The latter assertion is clear, since A is a flat V-algebra. $\square$

8.3. **Normalized lengths.** Let $(V, |\cdot|)$ be any valuation ring, with value group Γ_V; according to our general conventions, the composition law of Γ_V is denoted multiplicatively; however, sometimes it is convenient to switch to an additive notation. Hence, we adopt the notation :

$$(\log \Gamma_V, \leq)$$

to denote the ordered group Γ_V with additive composition law and whose ordering is the reverse of the original ordering of Γ_V. The unit of $\log \Gamma_V$ shall be naturally denoted by 0, and we shall extend the ordering of $\log \Gamma_V$ by adding a largest element $+\infty$, as customary. Also, we set : $\log \Gamma_V^+ := \{\gamma \in \log \Gamma_V \mid \gamma \geq 0\}$ and $\log |0| := +\infty$.

8.3.1. In this section, $(K, |\cdot|)$ denotes a valued field of rank one, with value group Γ. We let κ be the residue field of K^+. As usual, we set $S := \operatorname{Spec} K^+$, and denote by s (resp. η) the closed (resp. generic) point of S. Let $\mathfrak{m}_K \subset K^+$ be the maximal ideal, and set $\mathfrak{m} := \mathfrak{m}_K$ in case Γ is not discrete, or else $\mathfrak{m} := K^+$, in case $\Gamma \simeq \mathbb{Z}$; in the following, whenever we refer to almost rings or almost modules, we shall assume – unless otherwise stated – that the underlying almost ring theory is the one defined by the standard setup $(K^+, \mathfrak{m})$ (see [36, §6.1.15]).

Let A be any K^{+a}-algebra, and c a cardinal number; following [36, §2.3], we denote by $\mathscr{M}_c(A)$ the set of isomorphism classes of K^{+a}-modules which admit a set of generators of cardinality $\leq c$. The set $\mathscr{M}_c(A)$ carries a natural uniform structure (see [36, Def.2.3.1]), which admits the fundamental system of entourages

$$(E_\gamma \mid \gamma \in \log \Gamma^+ \setminus \{0\})$$

defined as follows. For any $b \in \mathfrak{m} \setminus \{0\}$, we let $E_{|b|}$ be the set of all pairs (M, M') of elements of $\mathscr{M}_c(A)$ such that there exist a third A-module N and A-linear morphisms $N \to M$, $N \to M'$ whose kernel and cokernel are annihilated by b.

8.3.2. The aim of this section is to define and study a well-behaved notion of *normalized length* for torsion modules M over K^+-algebras of a fairly general type. This shall be achieved in several steps. Let us first introduce the categories of algebras with which we will be working.

Definition 8.3.3. Let V be any valuation ring, with maximal ideal $\mathfrak{m}_V$.

(i) We let $V\text{-}\mathbf{m}.\mathbf{Alg}_0$ be the subcategory of $V\text{-}\mathbf{Alg}$ whose objects are the local and essentially finitely presented V-algebras A whose maximal ideal $\mathfrak{m}_A$ contains $\mathfrak{m}_V A$. The morphisms $\varphi : A \to B$ in $K^+\text{-}\mathbf{m}.\mathbf{Alg}_0$ are the local maps. Notice that every morphism in $V\text{-}\mathbf{m}.\mathbf{Alg}_0$ is an essentially finitely presented ring homomorphism ([30, Ch.IV, Prop.1.4.3(v)]).

Recall as well, that every object of $V\text{-}\mathbf{m}.\mathbf{Alg}_0$ is a coherent ring (see (5.7)). More generally, if $\underline{A} := (A_i \mid i \in I)$ is any filtered system of essentially finitely presented V-algebras with flat transition morphisms, then the colimit of $\underline{A}$ is still a coherent ring (lemma 5.6.6(ii.a)).

(ii) We say that a local V-algebra A is *measurable* if it admits an ind-étale local map of V-algebras $A_0 \to A$, from some object A_0 of $V\text{-}\mathbf{m}.\mathbf{Alg}_0$. The measurable V-algebras form a category $V\text{-}\mathbf{m}.\mathbf{Alg}$, whose morphisms are the local maps of V-algebras. As noted above, every measurable V-algebra is a coherent ring.

Lemma 8.3.4. *In the situation of definition* 8.3.3, *let* $(A, \mathfrak{m}_A)$ *be any measurable* V*-algebra. Then the following holds :*

(i) $A/\mathfrak{m}_V A$ *is a noetherian ring.*

(ii) *If the valuation of V has finite rank, every $\mathfrak{m}_A$-primary ideal of A contains a finitely generated $\mathfrak{m}_A$-primary subideal.*

(iii) *For every A-module M of finite type supported at the closed point of* $\mathrm{Spec}\,A$*, the A-module $M/\mathfrak{m}_V M$ has finite length.*

(iv) *For every finitely generated ideal $I \subset A$, the V-algebra A/I is measurable.*

Proof. (i): Let A^{sh} be the strict henselization of A at a geometric point localized at the closed point; then $A^{\mathrm{sh}}/\mathfrak{m}_V A^{\mathrm{sh}}$ is a strict henselization of $A/\mathfrak{m}_V A$ ([33, Ch.IV, Prop.18.8.10]), and it is therefore also a strict henselization of a $V/\mathfrak{m}_V$-algebra of finite type. Hence $A^{\mathrm{sh}}/\mathfrak{m}_V A^{\mathrm{sh}}$ is noetherian, and then the same holds for $A/\mathfrak{m}_V A$ ([33, Ch.IV, Prop.18.8.8(iv)]).

(iii) follows easily from (i) : the details shall be left to the reader.

(ii): If the valuation of V has finite rank, there exists an element $t_0 \in \mathfrak{m}_V$ that generates a $\mathfrak{m}_V$-primary ideal. Let $I \subset A$ be a $\mathfrak{m}_A$-primary ideal; then $t_0^N \in I$, for some integer $N > 0$; moreover, the image $\overline{I}$ of I in $A/\mathfrak{m}_V A$ is finitely generated, by (i). Pick elements $t_1, \dots, t_n \in I$ whose images in $A/\mathfrak{m}_V A$ form a system of generators of $\overline{I}$; it follows that $t_0^N, t_1, \dots, t_n$ form a $\mathfrak{m}_A$-primary ideal contained in I.

(iv): We may find an ind-étale local morphism $A_0 \to A$ from an object A_0 of V-$\mathbf{m}.\mathbf{Alg}_0$, and a finitely generated ideal $I_0 \subset A_0$ such that $I = I_0 A$. Then A/I_0 is an object of V-$\mathbf{m}.\mathbf{Alg}_0$ as well, and the induced map $A_0/I_0 \to A/I$ is ind-étale. □

8.3.5. Now, suppose that V is both a valuation ring and a flat, measurable K^+-algebra, and denote $\mathfrak{m}_V$ (resp. $\kappa(V)$, resp. $|\cdot|_V$) the maximal ideal (resp. the residue field, resp. the valuation) of V. We claim that V has rank one, and the ramification index $(\Gamma_V : \Gamma)$ is finite. Indeed, let us write V as the colimit of a filtered system $(V_i \mid i \in I)$ of objects of K^+-$\mathbf{m}.\mathbf{Alg}_0$ with essentially étale transition maps; it follows that each V_i is a valuation ring of rank one (claim 5.7.9). Also, the transition maps induce isomorphisms on the value groups : indeed, this is clear if K^+ is a discrete valuation ring, since in that case the same holds for the V_i, and the transition maps are unramified by assumption; in the case where Γ is not discrete, the assertion follows from corollary 5.7.25. Therefore, we are reduced to the case where V is an object of K^+-$\mathbf{m}.\mathbf{Alg}_0$, to which corollary 5.7.25 applies.

Suppose first that M is a finitely generated V-module; in this case the Fitting ideal $F_0(M) \subset V$ is well defined, and it is shown in [36, Lemma 6.3.1 and Rem.6.3.5] that the map $M \mapsto F_0(M)$ is additive on the set of isomorphism classes of finitely generated V-modules, *i.e.* for every short exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ of such modules, one has :

$$F_0(M_2) = F_0(M_1) \cdot F_0(M_3). \tag{8.3.6}$$

Next, the almost module $F_0(M)^a$ is an element of the group of fractional ideals $\mathrm{Div}(V^a)$ defined in [36, §6.1.16], and there is a natural isomorphism

$$\mathrm{Div}(V^a) \simeq \log \Gamma_V^\wedge \qquad I \mapsto |I| \tag{8.3.7}$$

where $\Gamma_V^\wedge$ is the completion of Γ_V for the uniform structure deduced from the ordering : see [36, Lemma 6.1.19]. Hence we may define :

$$\lambda_V(M) := (\Gamma_V : \Gamma) \cdot |F_0(M)^a| \in \log \Gamma_V^\wedge.$$

In view of (8.3.6), we see that :

$$\lambda_V(M') \leq \lambda_V(M) \qquad \text{whenever } M' \subset M \text{ are finitely generated.} \tag{8.3.8}$$

More generally, if M is any torsion V-module, we let :

$$\lambda_V(M) := \sup \{\lambda_V(M') \mid M' \subset M,\ M' \text{ finitely generated}\} \in \log \Gamma_V^\wedge \cup \{+\infty\} \tag{8.3.9}$$

which, in view of (8.3.8), agrees with the previous definition, in case M is finitely generated.

8.3.10. Suppose now that the V^a-module M^a is uniformly almost finitely generated. Then according to [36, Prop.2.3.23] one has a well defined Fitting ideal $F_0(M^a) \subset V^a$, which agrees with $F_0(M)^a$ in case M is finitely generated.

Lemma 8.3.11. *In the situation of* (8.3.10), *we have :*

(i) $\lambda_V(M) = (\Gamma_V : \Gamma) \cdot |F_0(M^a)|$.

(ii) *If* $0 \to N_1 \to N_2 \to N_3 \to 0$ *is any short exact sequence of uniformly almost finitely generated* V^a*-modules, then*

$$|F_0(N_2)| = |F_0(N_1)| + |F_0(N_3)|.$$

Proof. (ii): It is a translation of [36, Lemma 6.3.1 and Rem.6.3.5(ii)].

(i): Let k be a uniform bound for M^a, and denote by $\mathscr{I}_k(M)$ the set of all submodules of M generated by at most k elements. Set $e := (\Gamma_V : \Gamma)$; if $N \in \mathscr{I}_k(M)$, then $\lambda_V(N) \leq e \cdot |F_0(M^a)|$, due to (ii). By inspecting the definitions, it then follows that

$$e \cdot |F_0(M^a)| = \sup \{\lambda_V(N) \mid N \in \mathscr{I}_k(M)\} \leq \lambda_V(M).$$

Now, suppose $M' \subset M$ is any submodule generated by, say r elements; for every $\varepsilon \in \mathfrak{m}$ we can find $N \in \mathscr{I}_k(M)$ such that $\varepsilon M \subset N$, hence $\varepsilon M' \subset N$, therefore $\lambda_V(\varepsilon M') \leq \lambda_V(N) \leq e \cdot |F_0(M^a)|$. However, $\lambda_V(M') - \lambda_V(\varepsilon M') = \lambda_V(M'/\varepsilon M') \leq \lambda_V(K^{+\oplus r}/\varepsilon K^{+\oplus r}) = r \cdot \lambda_V(K^+/\varepsilon K^+)$. We deduce easily that $\lambda_V(M') \leq e \cdot |F_0(M^a)|$, whence the claim. □

Proposition 8.3.12. (i) *Let* $0 \to M_1 \to M_2 \to M_3 \to 0$ *be a short exact sequence of torsion* V*-modules. Then :*

$$\lambda_V(M_2) = \lambda_V(M_1) + \lambda_V(M_3).$$

(ii) $\lambda_V(M) = 0$ *if and only if* $M^a = 0$.

(iii) *Let* M *be any torsion* V*-module. Then :*

(a) *If* M *is finitely presented,* $\lambda_V(N) > 0$ *for every non-zero submodule* $N \subset M$.

(b) *If* $(M_i \mid i \in I)$ *is a filtered system of submodules of* M*, then :*

$$\lambda_V(\operatorname*{colim}_{i \in I} M_i) = \lim_{i \in I} \lambda_V(M_i).$$

Proof. (ii): By lemma 8.3.11(i) it is clear that $\lambda_V(M) = 0$ whenever $M^a = 0$. Conversely, suppose that $\lambda_V(M) = 0$, and let $m \in M$ be any element; then necessarily $\lambda_V(Vm) = 0$, and then it follows easily that $\mathfrak{m} \subset \mathrm{Ann}_V(m)$, so $M^a = 0$.

(iii.a): It suffices to show that M does not contain non-zero elements that are annihilated by $\mathfrak{m}$, which follows straightforwardly from [36, Lemma 6.1.14].

(iii.b): Set $M := \operatorname*{colim}_{i \in I} M_i$. By inspecting the definitions we see easily that $\lambda_V(M) \geq \sup\{\lambda_V(M_i) \mid i \in I\}$. To show the converse inequality, let $N \subset M$ be any finitely generated submodule; we may find $i \in I$ such that $N \subset M_i$, hence $\lambda_V(N) \leq \lambda_V(M_i)$, and the assertion follows.

(i): Set $e := (\Gamma_V : \Gamma)$. We shall use the following :

Claim 8.3.13. Every submodule of a uniformly almost finitely generated V^a-module is uniformly almost finitely generated.

Proof of the claim. Let $N' \subset N$, with N uniformly almost finitely generated, and let k be a uniform bound for N; for every $\varepsilon \in \mathfrak{m}$ we can find $N'' \subset N$ such that N'' is generated by at most k almost elements and $\varepsilon N \subset N''$. Clearly, it suffices to show that $N' \cap N''$ is uniformly almost finitely generated and admits k as a uniform bound, so we may replace N by N'' and N' by $N' \cap N''$, and assume from start that N is finitely generated. Let us pick an epimorphism $\varphi : (V^a)^{\oplus k} \to N$; it suffices to show that $\varphi^{-1}(N')$ is uniformly almost finitely generated with k as a uniform bound, so we are further reduced to the case where N is free of rank k.

Then we can write $N' = L^a$ for some submodule $L \subset V^{\oplus k}$; notice that $(V^{\oplus k}/L)^a$ is almost finitely presented, since it is finitely generated ([36, Prop.6.3.6(i)]), hence N' is almost finitely generated ([36, Lemma 2.3.18(iii)]). Furthermore, L is the colimit of the family $(L_i \mid i \in I)$ of its finitely generated submodules, and each L_i is a free V-module ([14, Ch.VI, §3, n.6, Lemma 1]). Necessarily the rank of L_i is $\leq k$ for every $i \in I$, hence $\Lambda_V^{k+1} L \simeq \operatorname{colim}_{i \in I} \Lambda_V^{k+1} L_i = 0$, and then the claim follows from [36, Prop.6.3.6(ii)]. ◊

Now, let $N \subset M_2$ be any finitely generated submodule, $\overline{N} \subset M_3$ the image of N; by claim 8.3.13, $M_1 \cap N$ is uniformly almost finitely generated, hence lemma 8.3.11(i,ii) shows that :

$$\lambda_V(N) = e \cdot |F_0(N^a)| = e \cdot |F_0(M_1^a \cap N^a)| + e \cdot |F_0(\overline{N}^a)| = \lambda_V(M_1 \cap N) + \lambda_V(\overline{N}).$$

Taking the supremum over the family $(N_i \mid i \in I)$ of all finitely generated submodules of M_2 yields the identity :

$$\lambda_V(M_2) = \lambda_V(M_3) + \sup\{\lambda_V(M_1 \cap N_i) \mid i \in I\}.$$

By definition, $\lambda_V(M_1 \cap N_i) \leq \lambda_V(M_1)$ for every $i \in I$; conversely, every finitely generated submodule of M_1 is of the form N_i for some $i \in I$, whence the contention. □

Remark 8.3.14. Suppose that $\Gamma \simeq \mathbb{Z}$, and let $\gamma_0 \in \log \Gamma^+$ be the positive generator. Then by a direct inspection of the definition one finds the identity :

$$\lambda(M) = (\Gamma_V : \Gamma) \cdot \mathrm{length}_V(M) \cdot \gamma_0$$

for every torsion V-module M. The verification shall be left to the reader.

8.3.15. Let M be a torsion V-module, such that M^a is almost finitely generated; we wish now to explain that $\lambda_V(M)$ can also be computed in terms of a suitable sequence of elementary divisors for M^a. Indeed, suppose first that N is a finitely presented torsion V-module; then we have a decomposition

$$N = (V/a_0V) \oplus \cdots \oplus (V/a_nV) \qquad \text{where } n := \dim_{\kappa(V)} N/\mathfrak{m}_V N - 1 \tag{8.3.16}$$

for certain $a_0, \dots, a_n \in \mathfrak{m}_V$ ([36, Lemma 6.1.14]). Clearly $\gamma_i := \log |a_i|_V > 0$ for every $i = 0, \dots, n$, and after reordering we may assume that $\gamma_0 \geq \cdots \geq \gamma_n$; then we may set $\gamma_i := 0$ for every $i > n$, and the resulting sequence $(\gamma_i \mid i \in \mathbb{N})$ of *elementary divisors* of N is independent of the chosen decomposition (8.3.16). Moreover, a simple inspection yields the identity

$$\lambda_V(M) = (\Gamma_V : \Gamma) \cdot (\gamma_0 + \cdots + \gamma_n). \tag{8.3.17}$$

We regard the sequence $(\gamma_i \mid i \in \mathbb{N})$ as an element of the $\log \Gamma_V^\wedge$-normed space $L^1(\Gamma_V^+)$ of *bounded* sequences of elements of $\log \Gamma_V^{\wedge +}$, *i.e.* the set of all sequences $\underline{\delta} := (\delta_i \mid i \in \mathbb{N})$ with

$$\|\underline{\delta}\| := \sup(\delta_i \mid i \in \mathbb{N}) < +\infty.$$

Lemma 8.3.18. *Let $\varphi : N \to N'$ be a map of finitely presented torsion V-modules, and denote by $(\gamma_i \mid i \in \mathbb{N})$ (resp. $(\gamma'_i \mid i \in \mathbb{N})$) the sequence of elementary divisors of N (resp. N'). Then :*

(i) *If φ is injective, we have $\gamma_i \leq \gamma'_i$ for every $i \in \mathbb{N}$.*
(ii) *If φ is surjective, we have $\gamma_i \geq \gamma'_i$ for every $i \in \mathbb{N}$.*

Proof. Denote by K_V the field of fractions of V; notice first that N and $\mathrm{Hom}_V(N, K_V/V)$ are isomorphic V-modules, hence their sequences of elementary divisors coincide. Now, if φ is injective, $\mathrm{Hom}_V(\varphi, K_V/V)$ is surjective (lemma 5.8.9(i)); hence (ii)⇒(i), and it remains only to show that (ii) holds.

By way of contradiction, suppose that φ is a surjection such that (ii) fails, and let $i_0 \in \mathbb{N}$ be the smallest integer such that $\gamma_{i_0} < \gamma'_{i_0}$; pick $b \in V$ with $|b|_V = \gamma'_{i_0}$, set $M := N/bN$, $M' := N'/bN'$, and let $(\delta_i \mid i \in \mathbb{N})$ (resp. $(\delta'_i \mid i \in \mathbb{N})$) be the sequence of elementary divisors of M (resp. of M'). Clearly $\delta_i = \gamma'_{i_0}$ for every $i < i_0$, and $\delta_{i_0} = \gamma_{i_0}$. On the other hand, we have $\delta'_i = \gamma'_{i_0}$ for every $i \leq i_0$; especially, M' admits a direct summand L which is a free V/bV-module of rank $i_0 + 1$. Now, φ induces a surjection $M \to M'$, and therefore M also surjects onto L; hence L is a direct summand of M, which is absurd, since $\delta_{i_0} < \gamma'_{i_0}$. □

8.3.19. We fix now a large cardinal number ω, and write just $\mathscr{M}(V^a)$ instead of $\mathscr{M}_\omega(V^a)$. Also, for any $\gamma \in \log \Gamma_V^+$, set $[-\gamma, \gamma] := \{\delta \in \log \Gamma_V \mid -\gamma \leq \delta \leq \gamma\}$. Suppose that N and N' are two finitely presented V-modules such that $(N^a, N'^a) \in E_\delta$ for some $\delta \in \log \Gamma_V^+ \setminus \{0\}$. Say that $\delta = \log |b|_V$ for some $b \in \mathfrak{m}_V$; by standard arguments, we obtain maps $\varphi : N \to N'$ and $\varphi' : N' \to N$ such that $\varphi' \circ \varphi = b^4 \cdot \mathbf{1}_N$ and $\varphi' \circ \varphi = b^4 \cdot \mathbf{1}_{N'}$. Let now $(\gamma_i \mid i \in \mathbb{N})$ (resp. $(\gamma'_i \mid i \in \mathbb{N})$) be the sequence of elementary divisors for N (resp. for N'). Then the sequence of elementary divisors for $b^4 N$ is $(\max(0, \gamma_i - 4\delta) \mid i \in \mathbb{N})$, and likewise for $b^4 N'$. In view of lemma 8.3.18, we deduce easily that

$$\gamma_i - \gamma'_i \in [-4\delta, 4\delta] \qquad \text{for every } i \in \mathbb{N}.$$

Consider now an almost finitely generated V^a-module M, and recall that M is almost finitely presented ([36, Prop.6.3.6(i)]); we then may attach to M a net of elements of $L^1(\log \Gamma_V^+)$, as follows. For every $\delta \in \log \Gamma_V^+ \setminus \{0\}$, pick a finitely presented V-module N_δ such that $(M, N_\delta^a) \in E_\delta$, and denote by $\underline{\gamma}_\delta$ the sequence of elementary divisors of N_δ. The foregoing easily implies that the system $(\underline{\gamma}_\delta \mid \delta \in \log \Gamma_V^+ \setminus \{0\})$ is a net for the uniform structure of $L^1(\log \Gamma_V^+)$ induced by the norm $\| \cdot \|$. However, $(L^1(\log \Gamma_V^+), \| \cdot \|)$ is a complete normed space, hence this net converges to a well defined sequence $\underline{\gamma}_M := (\gamma_i \mid i \in \mathbb{N}) \in L^1(\log \Gamma_V^+)$. It is easily seen that $\underline{\gamma}_M$ is independent of the chosen net, and defines an invariant which we call the *sequence of elementary divisors* of M. A simple inspection of the construction shows that the sequence $\underline{\gamma}_M$ is monotonically decreasing, and

$$\lim_{i \to +\infty} \gamma_i = 0.$$

Lastly, taking into account (8.3.17), we arrive at the identity

$$\lambda_V(M) = (\Gamma_V : \Gamma) \cdot \sum_{i \in \mathbb{N}} \gamma_i.$$

For future refernce, let us also point out the following :

Proposition 8.3.20. *Let $b \in \mathfrak{m}_V$ be any non-zero element, M an almost finitely presented V^a/bV^a-module, and $(\gamma_i \mid i \in \mathbb{N})$ the sequence of elementary divisors of M. The following conditions are equivalent :*

(a) *M is a flat V^a/bV^a-module.*
(b) *There exists $n \in \mathbb{N}$ such that $\gamma_i = \log |b|_V$ for every $i \leq n$, and $\gamma_i = 0$ for every $i > n$.*

Proof. (a)⇒(b): Suppose that (a) holds, let $n \in \mathbb{N}$ be the smallest integer such that $\gamma_n < \log |b|$, and suppose that (ii) fails, so $\gamma_n > 0$. For every non-zero $c \in \mathfrak{m}$ pick a finitely presented V/bV-module M_c and V/bV-linear maps $M_c^a \to M$ and $M \to M_c^a$ whose kernel and cokernel are annihilated by c, in which case a standard argument shows that

(8.3.21) $$c \cdot \mathrm{Tor}_1^{V/bV}(M_c, N) = 0 \qquad \text{for every } V/bV\text{-module } N.$$

Let $\delta \in \log \Gamma_K^+$ be small enough, so that $2\delta < \gamma_n < \log |b|_V - 2\delta$. Let $(\gamma_{c,i} \mid i \in \mathbb{N})$ denote the sequence of elementary divisors of M_c; if $\log |c|$ is sufficiently close to $0 \in \log \Gamma_K$, we have $\gamma_{c,n} - \gamma_n \in [-\delta, \delta]$ (notation of (8.3.19)), whence

(8.3.22) $$\min(\gamma_{c,n}, \log |b|_V - \gamma_{c,n}) > \delta.$$

On the other hand, a direct computation shows that

$$\mathrm{Tor}_1^{V/bV}(V/eV, V/eV) \simeq V/(eV + e^{-1}bV) \qquad \text{for every } e \in V \text{ such that } \log |e|_V \leq \log |b|_V.$$

Especially, if we take $\log|e|_V = \gamma_{c,n}$, we see that $\mathrm{Tor}_1^{V/bV}(M_c, M_c)$ contains a direct summand isomorphic to $V/(eV + e^{-1}bV)$, and (8.3.22) means that the latter module is not annihilated by any $x \in K^+$ with $\log|x| \leq \delta$. If $\log|c|$ is small enough, this contradicts (8.3.21).

(b)⇒(a): Suppose that (b) holds; then, for every non-zero $c \in \mathfrak{m}$ we may find a finitely presented V-module M_c, with sequence of elementary divisors $(\gamma_{c,i} \mid i \in \mathbb{N})$, and a V^a-linear morphism $\varphi : M_c^a \to M$ whose kernel and cokernel are annihilated by c and such that, moreover, $\gamma_i - \gamma_{c,i} \in [-\log|c|, \log|c|]$ for every $i \in \mathbb{N}$. Especially, $\gamma_{c,i} \leq \log|c|$ for every $i > n$, and $\gamma_{c,i} \geq \log|c^{-1}b|$ for $i \leq n$. Hence, we may replace M_c by cM_c, φ by its restriction to cM_c, and c by c^2, after which we may assume that $M_c = (V/b'_0V) \oplus \cdots \oplus (V/b'_nV)$, where $\log|b'_i| \in [\log|c^{-1}b|, \log|b|]$ for every $i \in \mathbb{N}$. A simple computation shows that (8.3.21) holds in this case, and then a standard argument yields

$$c^3 \cdot \mathrm{Tor}_1^{V^a/bV^a}(M, N) = 0 \qquad \text{for every } V^a/bV^a\text{-module } N$$

whence (a). □

8.3.23. In order to deal with general measurable K^+-algebras, we introduce hereafter some further notation which shall be standing throughout this section.

- To begin with, any ring homomorphism $\varphi : A \to B$ induces functors

$$\text{(8.3.24)} \qquad \varphi_* : B\text{-}\mathbf{Mod} \to A\text{-}\mathbf{Mod} \qquad \text{and} \qquad \varphi^* : A\text{-}\mathbf{Mod} \to B\text{-}\mathbf{Mod}.$$

Namely, φ_* assigns to any B-module M the A-module φ_*M obtained by restriction of scalars, and $\varphi^*(M) := B \otimes_A M$.

- For any local ring $(A, \mathfrak{m}_A)$, we let $\kappa(A) := A/\mathfrak{m}_A$, and we denote by $s(A)$ the closed point of $\mathrm{Spec}\, A$. If A is also a coherent ring, we denote by $A\text{-}\mathbf{Mod}_{\mathrm{coh},\{s\}}$ the full subcategory of $A\text{-}\mathbf{Mod}$ consisting of all the finitely presented A-modules M such that $\mathrm{Supp}\, M \subset \{s(A)\}$. Notice that the coherence of A implies that $A\text{-}\mathbf{Mod}_{\mathrm{coh},\{s\}}$ is an abelian category.
- Lastly, let $\mathscr{A}$ be any small abelian category; recall that $K_0(\mathscr{A})$ is the abelian group defined by generators and relations as follows. The generators are the isomorphism classes $[T]$ of objects T of $\mathscr{A}$, and the relations are generated by the elements of the form $[T_1] - [T_2] + [T_3]$, for every short exact sequence $0 \to T_1 \to T_2 \to T_3 \to 0$ of objects of $\mathscr{A}$. One denotes by $K_0^+(\mathscr{A}) \subset K_0(\mathscr{A})$ the submonoid generated by the classes $[T]$ of all objects of $\mathscr{A}$. We shall use the following well known *dévissage* lemma :

Lemma 8.3.25. *Let $\iota : \mathscr{B} \subset \mathscr{A}$ be an additive exact and fully faithful inclusion of abelian categories, and suppose that :*

(a) *If T is an object of $\mathscr{B}$ and T' is a subquotient of $\iota(T)$, then T' is in the essential image of ι.*
(b) *Every object T of $\mathscr{A}$ admits a finite filtration $\mathrm{Fil}^\bullet T$ such that the associated graded object $\mathrm{gr}^\bullet T$ is in the essential image of ι.*

Then ι induces an isomorphism :

$$K_0(\mathscr{B}) \xrightarrow{\sim} K_0(\mathscr{A}).$$

Proof. Left to the reader. □

Proposition 8.3.26. *Let $\varphi : A \to B$ be a morphism of measurable K^+-algebras. We have :*

(i) *If φ induces a finite field extension $\kappa(A) \to \kappa(B)$, then the functor φ_* of* (8.3.24) *restricts to a functor*

$$\varphi_* : B\text{-}\mathbf{Mod}_{\mathrm{coh},\{s\}} \to A\text{-}\mathbf{Mod}_{\mathrm{coh},\{s\}}$$

which induces a group homomorphism of the respective K_0-groups :

$$\varphi_* : K_0(B\text{-}\mathbf{Mod}_{\mathrm{coh},\{s\}}) \to K_0(A\text{-}\mathbf{Mod}_{\mathrm{coh},\{s\}}).$$

(ii) *If φ induces an integral morphism $\kappa(A) \to B/\mathfrak{m}_A B$, then* $\mathrm{length}_B(B/\mathfrak{m}_A B)$ *is finite, and the functor φ^* of* (8.3.24) *restricts to a functor*

$$\varphi^* : A\text{-}\mathbf{Mod}_{\mathrm{coh},\{s\}} \to B\text{-}\mathbf{Mod}_{\mathrm{coh},\{s\}}$$

and if φ is also a flat morphism, φ^* induces a group homomorphism :

$$\varphi^* : K_0(A\text{-}\mathbf{Mod}_{\mathrm{coh},\{s\}}) \to K_0(B\text{-}\mathbf{Mod}_{\mathrm{coh},\{s\}}).$$

Proof. Write A (resp. B) as the colimit of a filtered system $\underline{A} := (A_i \mid i \in I)$ (resp. $\underline{B} := (B_j \mid j \in J)$) of objects of $K^+\text{-}\mathbf{m}.\mathbf{Alg}_0$, with local and essentially étale transition maps. After replacing J (resp. I) by a cofinal subsets, we may assume that the indexing set admits an initial element $0 \in J$ (resp. $0 \in I$). Furthermore, we may assume that the induced map $A_0 \to B$ factors through a morphism $A_0 \to B_0$ in $K^+\text{-}\mathbf{m}.\mathbf{Alg}_0$.

(i): Let M be any object of $B\text{-}\mathbf{Mod}_{\mathrm{coh},\{s\}}$. We need to show that $\varphi_* M$ is finitely presented. We may find $j \in J$ and a finitely presented B_j-module M_j, with an isomorphism $M \xrightarrow{\sim} M_j \otimes_{B_j} B$ of B-modules. After replacing J by J/j, we may assume that $j = 0$ is the initial index. Since the natural map $B_0 \to B$ is local and ind-étale, it is easily seen that M_0 is an object of $B_0\text{-}\mathbf{Mod}_{\mathrm{coh},\{s\}}$. Especially, there exists a finitely generated $\mathfrak{m}_{B_0}$-primary ideal $I \subset \mathrm{Ann}_{B_0} M_0$. We may then replace the system $\underline{B}$ by $(B_j/IB_j \mid j \in J)$ and assume that each B_j has Krull dimension zero.

Claim 8.3.27. Let $\varphi : A \to B$ be a morphism of measurable K^+-algebras inducing a finite residue field extension $\kappa(A) \to \kappa(B)$, and such that B has dimension zero. Let also M be any finitely presented B-module. Then we may find :

(a) a cocartesian diagram of local maps of K^+-algebras

$$\begin{array}{ccc} A_l & \xrightarrow{\varphi_l} & C_l \\ \downarrow & & \downarrow \\ A & \xrightarrow{\varphi} & B \end{array}$$

whose vertical arrows are ind-étale, and where φ_l is a morphism in $K^+\text{-}\mathbf{m}.\mathbf{Alg}_0$

(b) and an object M_l of $C_l\text{-}\mathbf{Mod}_{\mathrm{coh},\{s\}}$, with an isomorphism $M_l \otimes_{C_l} B \xrightarrow{\sim} M$.

Proof of the claim. Define $\underline{A}$ and $\underline{B}$ as in the foregoing. Notice that – under the current assumptions – B_j is a henselian ring, for every $j \in J$. Moreover, we may find $j \in J$ such that $\kappa(B)$ is generated by the image of $\kappa(A) \otimes_{\kappa(B_0)} \kappa(B_j)$; after replacing again J by J/j, we may then also assume that $\kappa(B) = \kappa(A) \cdot \kappa(B_0)$.

For every $i \in I$, set $B'_i := A_i \otimes_{A_0} B_0$; the natural map $B_0 \to B$ factors through a map $B'_i \to B$, and we let $\mathfrak{p}_i \subset B'_i$ be the preimage of $\mathfrak{m}_B$. Set also $C_i := B'_{i,\mathfrak{p}_i}$, so we deduce a filtered system of local maps $(C_i \to B \mid i \in I)$, whose limit is a local map $\psi : C \to B$ of local ind-étale B_0-algebras, which – by construction – induces an isomorphism $\kappa(C) \xrightarrow{\sim} \kappa(B)$ on residue fields. It follows easily that ψ is itself ind-étale, so say that ψ is the colimit of a filtered system $(\psi_\lambda : C \to D_\lambda \mid \lambda \in \Lambda)$ of étale C-algebras. Notice that C is a henselian local ring; in light of [33, Ch.IV, Th.18.5.11] we may then assume that D_λ is a local ring and ψ_λ is a finite étale map, for every $\lambda \in \Lambda$. Clearly the induced residue field extension $\kappa(C) \to \kappa(D_\lambda)$ is an isomorphism; in view of [33, Ch.IV, Prop.18.5.15] it follows that ψ_λ is an isomorphism, for every $\lambda \in \Lambda$, so the same holds for ψ.

Notice that the sequence of residue degrees $d_i := [\kappa(C_i) : \kappa(A_i)]$ is non-increasing, hence there exists $l \in I$ such that $d_i = d := d_l$ for every index $i \geq l$. Notice as well that, for $i \geq l$, the local algebra C_i is also a localization of $C'_i := A_i \otimes_{A_l} C_l$, and the latter is an essentially finitely presented K^+-algebras of Krull dimension zero, hence its spectrum is finite and discrete (lemma 5.7.3). Moreover, since the image of the map $\mathrm{Spec}\, C_l \to \mathrm{Spec}\, A_l$ is the closed point, it is clear that the same holds for the image of the induced map $\mathrm{Spec}\, C'_i \to \mathrm{Spec}\, A_i$. Since the extension $\kappa(A_l) \to \kappa(A_i)$ is finite and separable, we conclude that

$$\kappa(A_i) \otimes_{\kappa(A_l)} \kappa(C_l) = \prod_{\mathfrak{p} \in \mathrm{Spec}\, C'_i} \kappa(C'_{i,\mathfrak{p}}). \qquad (8.3.28)$$

However, clearly the left-hand side of (8.3.28) is a $\kappa(A_i)$-algebra of degree d, whereas one of factors of the right-hand side – namely $\kappa(C_i)$ – is already of degree d over $\kappa(A_i)$. Hence $\mathrm{Spec}\, C'_i$ contains a single element, *i.e.* $C'_i = C_i$ is a local ring, and $C = A \otimes_{A_l} C_l$. Summing up, we have obtained the sought cocartesian diagram, and the claim holds with $M_l := M_0 \otimes_{B_0} C_l$. $\diamondsuit$

Let M_l and φ_l be as in claim 8.3.27; then $\varphi_* M$ is isomorphic to $A \otimes_{A_l} \varphi_{l*} M_l$, so may replace from start φ by φ_l, and assume that φ is a morphism in $K^+\text{-}\mathbf{m}.\mathbf{Alg}_0$, with B of Krull dimension zero. In such situation, one sees easily that φ is integral, hence B/I is a finitely presented A-module, by proposition 5.7.7(i), therefore $\varphi_* M$ is a finitely presented A-module, as required. Lastly, since the functor φ_* is exact, it is clear that it induces a map on K_0-groups as stated.

(ii): Under the current assumptions, the induced map $\kappa(A_0) \to B_0/\mathfrak{m}_{A_0} B_0$ is integral and essentially finitely presented, hence it is finite, so $B_0/\mathfrak{m}_{A_0} B_0$ is a B_0-module of finite length; but this is also the length of the B-module $B/\mathfrak{m}_A B$, whence the first assertion. Next, let M be an object of $A\text{-}\mathbf{Mod}_{\mathrm{coh},\{s\}}$; then $\varphi^* M$ is a finitely presented B-module; moreover, we may find a $\mathfrak{m}_A$-primary ideal $I \subset A$ such that M is a A/I-module, hence $\varphi^* M$ is a B/IB-module. Notice that the induced map $\mathrm{Spec}\, B/\mathfrak{m}_A B \to \mathrm{Spec}\, B/IB$ is bijective, and its target is a local scheme of dimension zero (since $B/\mathfrak{m}_A B$ is integral over a field). It follows easily that IB is a $\mathfrak{m}_B$-primary ideal, so $\varphi^* M$ is an object of $B\text{-}\mathbf{Mod}_{\mathrm{coh},\{s\}}$. The last assertion is then a trivial consequence of the exactness of the functor φ^*, when φ is flat. $\square$

Lemma 8.3.29. *Let A be any measurable K^+-algebra. Then there exists a morphism*

$$V \xrightarrow{\varphi} A/I$$

of measurable K^+-algebras, where :

(a) *$I \subset A$ is a finitely generated $\mathfrak{m}_A$-primary ideal.*
(b) *V is a valuation ring, φ is a finitely presented surjection and the natural map $K^+ \to V$ induces an isomorphism of value groups $\Gamma \xrightarrow{\sim} \Gamma_V$.*

Proof. Suppose first that A is an object of $K^+\text{-}\mathbf{m}.\mathbf{Alg}_0$. In this case, choose an affine finitely presented S-scheme X and a point $x \in X$ such that $A = \mathscr{O}_{X,x}$; next, take a finitely presented closed immersion $h : X \to Y := \mathbb{A}^n_{K^+}$ of S-schemes; set $\overline{Y} := \mathbb{A}^n_\kappa \subset Y$, pick elements $f_1, \dots, f_d \in B := \mathscr{O}_{Y,h(x)}$ whose images in the regular local ring $\mathscr{O}_{\overline{Y},h(x)}$ form a regular system of parameters (*i.e.* a regular sequence that generates the maximal ideal), and let $J \subset B$ be the ideal generated by the f_i, $i = 1, \dots, d$. Let $I \subset A$ be any finitely generated $\mathfrak{m}_A$-primary ideal containing the image of J. We deduce a surjection $\varphi : V := B/J \to A/I$, and by construction $V/\mathfrak{m}_K V$ is a field; moreover, the induced map $K^+ \to V$ is flat by virtue of [32, Ch.IV, Th.11.3.8]. It follows that V is a valuation ring with the sought properties, by proposition 5.7.7(ii). Also, φ is finitely presented, by proposition 5.7.7(i).

Next, let A be a general measurable K^+-algebra, and write A as the colimit of a filtered system $(A_j \mid j \in J)$ of objects of $K^+\text{-}\mathbf{m}.\mathbf{Alg}_0$. We may assume that $0 \in J$ is an initial index, and the foregoing case yields an $\mathfrak{m}_{A_0}$-primary ideal $I_0 \subset A_0$, and a surjective finitely presented morphism $\varphi_0 : V_0 \to A_0/I_0$ in $K^+\text{-}\mathbf{m}.\mathbf{Alg}_0$ from a valuation ring V_0, such that $\Gamma_V = \Gamma$. Notice that A/I_0 is a henselian ring, hence φ_0 extends to a ring homomorphism $\varphi_0^{\mathrm{h}} : V_0^{\mathrm{h}} \to A_0/I_0$ from the henselization V_0^{h} of V_0; more precisely, φ_0 induces an isomorphism of V_0^{h}-algebras :

$$V_0^{\mathrm{h}} \otimes_{V_0} (A_0/I_0) \xrightarrow{\sim} A_0/I_0$$

so φ_0^{h} is still finitely presented. On the one hand, φ_0^{h} induces an identification

$$\kappa(V_0^{\mathrm{h}}) = \kappa(A_0).$$

On the other hand, we have the filtered system of separable field extensions $(\kappa(A_j) \mid j \in J)$, whose colimit is $\kappa(A)$. There follows a corresponding filtered system $(V_j^{\mathrm{h}} \mid j \in J)$ of finite étale V_0^{h}-algebras, whose colimit we denote V ([33, Ch.IV, Prop.18.5.15]). Then V is a valuation ring, and the map $V_0^{\mathrm{h}} \to V$ induces an isomorphism on value groups. Moreover, the induced isomorphisms $\kappa(V_j^{\mathrm{h}}) \xrightarrow{\sim} \kappa(A_j)$ lift uniquely to morphisms of A_0-algebras $\varphi_j^{\mathrm{h}} : V_j^{\mathrm{h}} \to A_j/I_0A_j$, for every $j \in J$ ([33, Ch.IV, Cor.18.5.12]). Due to the uniqueness of φ_j^{h}, we see that the resulting system $(\varphi_j^{\mathrm{h}} \mid j \in J)$ is filtered, and its colimit is a morphism $\varphi : V \to A/I_0A$. Moreover, φ_j^{h} induces an isomorphism $V_j^{\mathrm{h}} \otimes_{V_0^{\mathrm{h}}} A_0/IA_0 \xrightarrow{\sim} A_j/I_0A_j$, especially φ_j^{h} is surjective for every $j \in J$, so the same holds for φ. More precisely, φ_0^{h} induces an isomorphism $V \otimes_{V_0^{\mathrm{h}}} (A_0/I_0) \xrightarrow{\sim} A/I_0A$, hence φ is still finitely presented. $\square$

Theorem 8.3.30. *With the notation of* (8.3.23)*, the following holds :*

(i) *For every measurable K^+-algebra A there is a natural group isomorphism :*

$$\boldsymbol{\lambda}_A : K_0(A\text{-}\mathbf{Mod}_{\mathrm{coh},\{s\}}) \xrightarrow{\sim} \log \Gamma$$

which induces an isomorphism $K_0^+(A\text{-}\mathbf{Mod}_{\mathrm{coh},\{s\}}) \xrightarrow{\sim} \log \Gamma^+$.

(ii) *The family of isomorphisms $\boldsymbol{\lambda}_A$ (for A ranging over the measurable K^+-algebras) is characterized uniquely by the following two properties.*

(a) *If V is a valuation ring and a flat measurable K^+-algebra, then*

$$\boldsymbol{\lambda}_V([M]) = \lambda_V(M) \qquad \text{for every object } M \text{ of } V\text{-}\mathbf{Mod}_{\mathrm{coh},\{s\}}$$

where $\lambda_V(M)$ is defined as in (8.3.2)*.*

(b) *Let $\psi : A \to B$ be a morphism of measurable K^+-algebras inducing a finite residue field extension $\kappa(A) \to \kappa(B)$. Then*

$$\boldsymbol{\lambda}_A(\psi_*[M]) = [\kappa(B) : \kappa(A)] \cdot \boldsymbol{\lambda}_B([M]) \qquad \text{for every } [M] \in K_0(B\text{-}\mathbf{Mod}_{\mathrm{coh},\{s\}}).$$

(iii) *For every $a \in K^+\setminus\{0\}$ and any object M of A-$\mathbf{Mod}_{\mathrm{coh},\{s\}}$ we have :*

(a) *$[M] = 0$ in $K_0(A\text{-}\mathbf{Mod}_{\mathrm{coh},\{s\}})$ if and only if $M = 0$.*

(b) *If M is flat over K^+/aK^+, then :*

$$\boldsymbol{\lambda}_A([M]) = |a| \cdot \mathrm{length}_A(M \otimes_{K^+} \kappa). \tag{8.3.31}$$

(iv) *Let $\psi : A \to B$ be a flat morphism of measurable K^+-algebras inducing an integral map $\kappa(A) \to B/\mathfrak{m}_A B$. Then :*

$$\boldsymbol{\lambda}_B(\psi^*[M]) = \mathrm{length}_B(B/\mathfrak{m}_A B) \cdot \boldsymbol{\lambda}_A([M]) \qquad \text{for every } [M] \in K_0(A\text{-}\mathbf{Mod}_{\mathrm{coh},\{s\}}).$$

Proof. We start out with the following :

Claim 8.3.32. Let $(K, |\cdot|) \to (E, |\cdot|)$ be an extension of valued fields of rank one inducing an isomorphism of value groups, $a \in K^+ \setminus \{0\}$ any element, M an E^+/aE^+ module. Then :

(i) M is a flat E^+/aE^+-module if and only if it is a flat K^+/aK^+-module.

(ii) $M \otimes_{K^+} \kappa = M \otimes_{E^+} \kappa(E)$.

Proof of the claim. According to [61, Th.7.8], in order to show (i) it suffices to prove that

$$\mathrm{Tor}_1^{E^+/aE^+}(E^+/bE^+, M) = \mathrm{Tor}_1^{K^+/aK^+}(K^+/bK^+, M)$$

for every $b \in K^+$ such that $|b| \geq |a|$. The latter assertion is an easy consequence of the faithful flatness of the extension $K^+ \to E^+$. (ii) follows from the identity : $\mathfrak{m}_E = \mathfrak{m}_K E^+$, which holds since $\Gamma_E = \Gamma$. $\Diamond$

Claim 8.3.33. Let $f : K^+ \to V$ be a morphism of measurable K^+-algebras, where V is a valuation ring and f induces an isomorphism on value groups. Then $\boldsymbol{\lambda}_V$ (defined by (ii.a)) is an isomorphism and assertion (iii) holds for $A = V$.

Proof of the claim. According to (ii.a), $\boldsymbol{\lambda}_V([M]) = \lambda_V(M)$ for every finitely presented torsion V-module. However, every such module M admits a decomposition of the form $M \simeq (V/a_1V) \oplus \cdots \oplus (V/a_kV)$, with $a_1, \dots, a_k \in \mathfrak{m}_K$ ([36, Lemma 6.1.14]). Then by claim 8.3.32(i), M is flat over K^+/aK^+ if and only if M is flat over V/aV, if and only if $|a| = |a_i|$ for every $i \leq k$. In this case, an explicit calculation shows that $\lambda_V(M) = |a| \cdot \mathrm{length}(M \otimes_V \kappa(V))$, which is equivalent to (8.3.31), in view of claim 8.3.32(ii). Next, we consider the map :

$$\boldsymbol{\mu} : \log \Gamma^+ \to K_0(V\text{-}\mathbf{Mod}_{\mathrm{coh},\{s\}}) \quad : \quad |a| \mapsto [V/aV] \quad \text{for every } a \in K^+ \setminus \{0\}.$$

We leave to the reader the verification that $\boldsymbol{\mu}$ extends to a group homomorphism well-defined on the whole of Γ, that provides an inverse to $\boldsymbol{\lambda}_V$. Finally, it is clear that $\lambda_V(M) = 0$ if and only if $M = 0$, so also (iii.a) holds. ◊

Claim 8.3.34. Let A be any measurable K^+-algebra. For every object N of $A\text{-}\mathbf{Mod}_{\mathrm{coh},\{s\}}$ there exists a finite filtration $0 = N_0 \subset \cdots \subset N_k = N$ by finitely presented A-submodules, and elements $a_1, \dots, a_k \in \mathfrak{m}$ such that N_i/N_{i-1} is a flat K^+/a_iK^+-module for every $1 \leq i \leq k$.

Proof of the claim. Let us find $I \subset A$ and $\varphi : V \to A/I$ as in lemma 8.3.29. It suffices to show the claim for the finitely presented A-modules $I^nN/I^{n+1}N$ (for every $n \in \mathbb{N}$), hence we may assume that N is an A/I-module. Then φ_*N is a finitely presented V-module, hence of the form $(V/a_1V) \oplus \cdots \oplus (V/a_kV)$ for some $a_i \in \mathfrak{m}$; we may order the summands so that $|a_i| \geq |a_{i+1}|$ for all $i < k$. We argue by induction on $d(N) := \dim_\kappa(N \otimes_V \kappa(V))$. If $d(N) = 0$, then $N = 0$ by Nakayama's lemma. Suppose $d > 0$; we remark that N/a_1N is a flat V/a_1V-module, hence a flat K^+/a_1K^+-module (claim 8.3.32), and $d(a_1N) < d(N)$; the claim follows. ◊

Claim 8.3.35. Let A be any measurable K^+-algebra, $I \subset A$ a finitely generated $\mathfrak{m}_A$-primary ideal, and $\pi_I : A \to A/I$ the natural projection. Then the map :

$$\pi_{I*} : K_0(A/I\text{-}\mathbf{Mod}_{\mathrm{coh}}) \to K_0(A\text{-}\mathbf{Mod}_{\mathrm{coh},\{s\}})$$

is an isomorphism.

Proof of the claim. On the one hand we have :

$$A\text{-}\mathbf{Mod}_{\mathrm{coh},\{s\}} = \bigcup_{n \in \mathbb{N}} A/I^n\text{-}\mathbf{Mod}_{\mathrm{coh}}$$

and on the other hand, in view of lemma 8.3.25, we see that the projections $A/I^{n+1} \to A/I^n$ induce isomorphisms $K_0(A/I^n\text{-}\mathbf{Mod}_{\mathrm{coh}}) \to K_0(A/I^{n+1}\text{-}\mathbf{Mod}_{\mathrm{coh}})$ for every $n > 0$, whence the claim. ◊

Let A, I and $\varphi : V \to A/I$ be as in lemma 8.3.29, and $\pi_I : A \to A/I$ the natural surjection; taking into account claim 8.3.35, we may let :

$$\boldsymbol{\lambda}_A := \boldsymbol{\lambda}_V \circ \varphi_* \circ \pi_{I*}^{-1} \tag{8.3.36}$$

where $\boldsymbol{\lambda}_V$ is given by the rule of (ii.a). In view of claim 8.3.33 we see that $\boldsymbol{\lambda}_A([M]) = 0$ if and only if $M = 0$, so (iii.a) follows already.

Claim 8.3.37. The isomorphism $\boldsymbol{\lambda}_A$ is independent of the choice of I, V and φ.

Proof of the claim. Indeed, suppose that $J \subset A$ is another ideal and $\psi : W \to A/J$ is another surjection from a valuation ring W, fulfilling the foregoing conditions. We consider the commutative diagram

$$\begin{tikzcd}
 & A/I \arrow[dr, "\overline{\pi}_J"] & V \arrow[l, "\varphi"'] \arrow[d, "\varphi'"] \\
A \arrow[ur, "\pi_I"] \arrow[rr, "\pi_{I+J}"] \arrow[dr, "\pi_J"'] & & A/(I+J) \\
 & A/J \arrow[ur, "\overline{\pi}_I"'] & W. \arrow[l, "\psi"] \arrow[u, "\psi'"']
\end{tikzcd}$$

We compute : $\varphi_* \circ \pi^{-1}_{I*} = \varphi_* \circ \pi_{J*} \circ \overline{\pi}^{-1}_{J*} \circ \overline{\pi}^{-1}_{I*} = \varphi'_* \circ \pi^{-1}_{I+J,*}$, and a similar calculation shows that $\psi_* \circ \pi^{-1}_{J*} = \psi'_* \circ \pi^{-1}_{I+J,*}$. We are thus reduced to showing that $\boldsymbol{\lambda}_V \circ \varphi'_*([N]) = \boldsymbol{\lambda}_W \circ \psi'_*([N])$ for every $A/(I+J)$-module N. In view of claim 8.3.34 we may assume that N is flat over K^+/aK^+, for some $a \in \mathfrak{m}$, in which case the assertion follows from claim 8.3.33. $\diamondsuit$

Claims 8.3.33 and 8.3.37 show already that (i) holds. Next, let $\psi : A \to B$ be as in (ii.b). Choose an ideal $I \subset A$ and a surjection $\varphi : V \to A/I$ as in lemma 8.3.29. Let also $J \subset B$ be a finitely generated $\mathfrak{m}_B$-primary ideal containing $\psi(I)$, and $\overline{\psi} : A/I \to B/J$ the induced map.

By inspecting the definitions we see that (ii.b) amounts to the identity : $\boldsymbol{\lambda}_V((\overline{\psi} \circ \varphi)_*[M]) = [\kappa(B) : \kappa(A)] \cdot \boldsymbol{\lambda}_{B/J}([M])$ for every finitely presented B/J-module M. Furthermore, up to enlarging J, we may assume that there is a finitely presented surjection $\xi : W \to B/J$ of K^+-algebras, where W is a valuation ring with value group Γ, and then we come down to showing:

$$\boldsymbol{\lambda}_V((\overline{\psi} \circ \varphi)_*[M]) = [\kappa(B) : \kappa(A)] \cdot \boldsymbol{\lambda}_W(\xi_*[M]).$$

In view of claim 8.3.34 we may also assume that M is a flat K^+/aK^+-module for some $a \in \mathfrak{m}$, in which case the identity becomes :

$$|a| \cdot \mathrm{length}_V(M \otimes_{K^+} \kappa) = [\kappa(B) : \kappa(A)] \cdot |a| \cdot \mathrm{length}_W(M \otimes_{K^+} \kappa)$$

thanks to claim 8.3.33. However, the latter is an easy consequence of the identities : $\kappa(A) = \kappa(V)$ and $\kappa(B) = \kappa(W)$. Next, we show that (iii.b) holds for a general measurable K^+-algebra A. Indeed, let M be any object of A-$\mathbf{Mod}_{\mathrm{coh},\{s\}}$; as usual, we may find a local ind-étale map $A_0 \to A$ from some object A_0 of K^+-$\mathbf{m}.\mathbf{Alg}_0$, and an object M_0 of A_0-$\mathbf{Mod}_{\mathrm{coh},\{s\}}$ with an isomorphism of A-modules $A \otimes_{A_0} M_0 \xrightarrow{\sim} M$ (cp. the proof of proposition 8.3.26(i)). Since (ii.b) is already proved, we have

$$\boldsymbol{\lambda}_A([M]) = \boldsymbol{\lambda}_{A_0}([M_0]) \quad \text{and} \quad \mathrm{length}_A(M \otimes_{K^+} \kappa) = \mathrm{length}_{A_0}(M \otimes_{K^+} \kappa).$$

Therefore, we may replace A by A_0, and assume that A is an object of K^+-$\mathbf{m}.\mathbf{Alg}_0$. In this case, by lemma 5.7.5 we may find morphisms $f : K^+ \to V$ and $g : V \to A$ in K^+-$\mathbf{m}.\mathbf{Alg}_0$ such that V is a valuation ring with value group Γ and g induces a finite extension of residue fields $\kappa(V) \to \kappa(A)$. Let N be an A-module supported at $s(A)$ and flat over K^+/aK^+; we may find a finitely generated $\mathfrak{m}_A$-primary ideal $I \subset A$ such that $I \subset \mathrm{Ann}_A(N)$, and since (ii.b) is already known in general, we reduce to showing that (iii.b) holds for the A/I-module N. However, the induced map $\overline{g} : V \to A/I$ is finite and finitely presented (proposition 5.7.7(i)), so another application of (ii.b) reduces to showing that (8.3.31) holds for $A := V$ and $M := \overline{g}_*N$, in which case the assertion is already known by claim 8.3.33.

(ii.a): Suppose that A is a valuation ring, and let M be any object of A-$\mathbf{Mod}_{\mathrm{coh},\{s\}}$. The sought assertion is obvious when $\Gamma_A = \Gamma$, since in that case we can choose $A = V$ and $\varphi = \pi_I$ in (8.3.36). However, we know already that the rank of A equals one, and $e := (\Gamma_A : \Gamma)$ is finite (see (8.3.5)); we can then assume that $e > 1$, in which case corollary 5.7.25(ii) implies that

$\Gamma \simeq \mathbb{Z}$. Then it suffices to check (ii.a) for $M = \kappa(A)$, which is a (flat) κ-module, so that – by assertion (iii) – one has $\boldsymbol{\lambda}_A([M]) = e \cdot \gamma_0$, where $\gamma_0 \in \log \Gamma_A^+$ is the positive generator. In view of remark 8.3.14, we see that this value agrees with $\lambda_V(M)$, as stated.

(iv): In view of claim 8.3.34, we may assume that M is a flat K^+/aK^+-module, for some $a \in \mathfrak{m}$, and then the same holds for ψ^*M, since ψ is flat. In view of (iii.b), it then suffices to show that :

$$\mathrm{length}_B(B \otimes_A M) = \mathrm{length}_A(M) \cdot \mathrm{length}_B(B/\mathfrak{m}_A B)$$

for any A-module M of finite length. In turn, this is easily reduced to the case where $M = \kappa(A)$, for which the identity is obvious. $\square$

8.3.38. Let A be a measurable K^+-algebra. The next step consists in extending the definition of $\boldsymbol{\lambda}_A$ to the category A-$\mathbf{Mod}_{\{s\}}$ of arbitrary A-modules M supported at $s(A)$. First of all, suppose that M is finitely generated. Let $\mathscr{C}_M$ be the set of isomorphism classes of objects M' of A-$\mathbf{Mod}_{\mathrm{coh},\{s\}}$ that admit a surjection $M' \to M$. Then we set :

$$\lambda^*_A(M) := \inf \{\boldsymbol{\lambda}_A([M']) \mid M' \in \mathscr{C}_M\} \in \log \Gamma^\wedge.$$

Notice that – by the positivity property of theorem 8.3.30(i) – we have $\lambda^*_A(M) = \boldsymbol{\lambda}_A([M])$ whenever M is finitely presented. Next, for a general object of A-$\mathbf{Mod}_{\{s\}}$ we let :

(8.3.39) $\lambda_A(M) := \sup \{\lambda^*_A(M') \mid M' \subset M$ and M' is finitely generated$\} \in \log \Gamma^\wedge \cup \{+\infty\}$.

Lemma 8.3.40. *If M is finitely generated, then $\lambda^*_A(M) = \lambda_A(M)$.*

Proof. Let $M' \subset M$ be a finitely generated submodule; we have to show that $\lambda^*(M') \leq \lambda^*(M)$. To this aim, let $f : N \to M$ and $g : N' \to M'$ be two surjections of A-modules with $N \in \mathscr{C}_M$ and $N' \in \mathscr{C}_{M'}$; by filtering the kernel of f by the system of its finitely generated submodules, we obtain a filtered system $(N_i \mid i \in I)$ of finitely presented quotients of N, with surjective transition maps, such that $\operatorname*{colim}_{i \in I} N_i = M$. By [36, Prop.2.3.16(ii)] the induced map $N' \to M$ lifts to a map $h : N' \to N_i$ for some $i \in I$. Since A is coherent, $h(N')$ is a finitely presented A-module with a surjection $h(N') \to M'$, hence $\lambda^*_A(M') \leq \boldsymbol{\lambda}_A([h(N')]) \leq \boldsymbol{\lambda}_A([N_i]) \leq \boldsymbol{\lambda}_A(N)$. Since N is arbitrary, the claim follows. $\square$

Proposition 8.3.41. (i) *If A is a valuation ring,* (8.3.39) *agrees with* (8.3.9).

(ii) *If $(M_i \mid i \in I)$ is a filtered system of objects of A-$\mathbf{Mod}_{\{s\}}$ with injective (resp. surjective) transition maps, then :*

$$\lambda_A(\operatorname*{colim}_{i \in I} M_i) = \lim_{i \in I} \lambda_A(M_i)$$

(resp. provided there exists $i \in I$ such that $\lambda_A(M_i) < +\infty$).

(iii) *If $\psi : A \to B$ is a morphism of measurable K^+-algebras inducing a finite extension $\kappa(A) \to \kappa(B)$ of residue fields, then :*

$$\lambda_B(M) = \frac{\lambda_A(\psi_* M)}{[\kappa(B) : \kappa(A)]} \qquad \textit{for every object } M \textit{ of } B\text{-}\mathbf{Mod}_{\{s\}}.$$

(iv) *Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a short exact sequence in A-$\mathbf{Mod}_{\{s\}}$. Then :*

$$\lambda_A(M_2) = \lambda_A(M_1) + \lambda_A(M_3).$$

(v) *Let ψ be a flat morphism of measurable K^+-algebras inducing an integral map $\kappa(A) \to B/\mathfrak{m}_A B$. Then :*

$$\lambda_B(\psi^* M) = \mathrm{length}_B(B/\mathfrak{m}_A B) \cdot \lambda_A(M) \qquad \textit{for every object } M \textit{ of } A\text{-}\mathbf{Mod}_{\{s\}}.$$

Proof. (i): Set $e := (\Gamma_V : \Gamma)$. It suffices to check the assertion for a finitely generated A-module M, in which case one has to show the identity :

$$e \cdot |F_0(M)^a| = \lambda^*_A(M).$$

By proposition 8.3.12(i) we have $|F_0(M')^a| \geq |F_0(M)^a|$ for every $M' \in \mathscr{C}_M$, hence $\lambda^*_A(M) \geq e \cdot |F_0(M)^a|$. On the other hand, let us fix a surjection $V^{\oplus n} \to M$, and let us write its kernel in the form $K = \bigcup_{i\in I} K_i$, for a filtered system $(K_i \mid i \in I)$ of a finitely generated V-submodules; it follows that :

$$F_0(M) = \bigcup_{i\in I} F_0(V^{\oplus n}/K_i).$$

In view of proposition 8.3.12(iii.b) we deduce : $e\cdot|F_0(M)^a| = \lim_{i\in I} e\cdot|F_0(V^{\oplus n}/K_i)^a| \geq \lambda^*_A(M)$.

The proof of (ii) in the case where the transition maps are injective, is the same as that of proposition 8.3.12(iii.b).

(iii): Let us write $M = \bigcup_{i\in I} M_i$ for a filtered family $(M_i \mid i \in I)$ of finitely generated B-submodules. By the case already known of (ii) we have : $\lambda_B(M) = \lim_{i\in I} \lambda_B(M_i)$, and likewise for $\lambda_A(\psi_*M)$, hence we may assume from start that M is a finitely generated B-module, in which case the annihilator of M contains a finitely generated $\mathfrak{m}_B$-primary ideal $J' \subset B$. Choose a finitely generated $\mathfrak{m}_A$-primary ideal $J \subset A$ contained in the kernel of the induced map $A \to B/J'$.

Claim 8.3.42. $\lambda_B(M) = \lambda_{B/J'}(M)$ and $\lambda_A(\psi_*M) = \lambda_{A/J}(\overline{\psi}_*M)$.

Proof of the claim. Directly on the definition (and by applying theorem 8.3.30(ii.b) to the surjection $B \to B/J'$) we see that $\lambda_B(M) \leq \lambda_{B/J'}(M)$. On the other hand, any surjection of B-modules $M' \to M$ with M' finitely presented, factors through the natural map $M' \to M'/J'M'$, and by theorem 8.3.30(i,ii.b) we have $\lambda_B(M') - \lambda_{B/J'}(M'/J'M') = \lambda_B(J'M') \geq 0$, whence the first stated identity. The proof of the second identity is analogous. ◊

In view of claim 8.3.42 we are reduced to proving the assertion for the morphism $\overline{\psi}$ and the B/J'-module M, hence we may replace ψ by $\overline{\psi}$, and assume from start that A and B have Krull dimension zero.

Claim 8.3.43. Let A be a measurable K^+-algebra of Krull dimension zero. Then there exists a morphism $\varphi : V \to A$ of measurable K^+-algebras, with V a valuation ring flat over K^+, such that the residue field extension $\kappa(V) \to \kappa(A)$ is finite, and the induced map of value groups $\Gamma \to \Gamma_V$ is an isomorphism.

Proof of the claim. Let $A_0 \to A$ be a local ind-étale map, from an object A_0 of $K^+\textbf{-m.Alg}_0$. By lemma 5.7.5 we may find a K^+-flat valuation ring V_0 in $K^+\textbf{-m.Alg}_0$ and a local map $\varphi_0 : V \to A_0$, inducing a finite residue field extension $\kappa(V_0) \to \kappa(A_0)$ and an isomorphism on value groups $\Gamma \xrightarrow{\sim} \Gamma_V$. Then A_0 has also Krull dimension zero; especially, it is henselian, hence φ_0 factors through a morphism $\varphi_0^{\mathrm{h}} : V_0^{\mathrm{h}} \to A_0$, from the henselization V_0^{h} of V_0. Let $E \subset \kappa(A)$ be the largest separable subextension of $\kappa(V_0) = \kappa(V_0^{\mathrm{h}})$ contained in $\kappa(A)$; there exists a unique (up to unique isomorphism) finite étale morphism $V_0^{\mathrm{h}} \to V'$ with an isomorphism $\kappa(V') \xrightarrow{\sim} E$ of $\kappa(V_0)$-algebras, and φ_0^{h} factors through a morphism $V' \to A_0$. Notice that V' is still a henselian valuation ring and a measurable K^+-algebra, hence we may replace V_0 by V', and assume that V_0 is henselian, and the residue field extension $\overline{\varphi}_0 : \kappa(V_0) \to \kappa(A_0)$ is purely inseparable. Since A_0 is henselian, we may write A as the colimit of a filtered system $(A_i \mid i \in I)$ of finite étale A_0-algebras. Now, on the one hand, $\overline{\varphi}_0$ induces an equivalence from the category of finite étale $\kappa(V_0)$-algebras, to the category of finite étale $\kappa(A_0)$-algebras (lemma 7.1.7(i)). On the other hand, the category of finite étale V_0-algebras is equivalent to the category of finite étale $\kappa(V_0)$-algebras, and likewise for A_0. Therefore, for every $i \in I$ we may find a finite étale morphism

$V_0 \to V_i$, unique up to unique isomorphism, inducing an isomorphism of $\kappa(A_0)$-algebras :

$$\kappa(V_i) \otimes_{\kappa(V_0)} \kappa(A_0) \xrightarrow{\sim} \kappa(A_i) \tag{8.3.44}$$

and the transition maps of residue fields $\kappa(A_i) \to \kappa(A_j)$ induce unique maps $V_i \to V_j$ of V_0-algebras, compatible with the isomorphisms (8.3.44). Hence, the resulting system $(V_i \mid i \in I)$ is filtered, and its colimit is a valuation ring V, which is still a measurable K^+-algebra. Moreover, the field extensions $\kappa(V_i) \to \kappa(A_i)$ deduced from (8.3.44) lift uniquely to maps of V_0-algebras $V_i \to A_i$ ([33, Ch.IV, Cor.18.5.12]); taking colimits, we get finally a map $V \to A$ as sought. $\diamondsuit$

Let φ be as in claim 8.3.43; clearly it suffices to prove the sought identity for the two morphisms $\psi \circ \varphi$ and φ, so we may replace A by V, and assume from start that A is a valuation ring. Let us set $d := [\kappa(B) : \kappa(A)]$; we deduce :

$$\lambda_B(M) = \inf \{d^{-1} \cdot \lambda_A(\psi_* M') \mid M' \in \mathscr{C}_M\} \geq d^{-1} \cdot \lambda_A(\psi_* M)$$

by theorem 8.3.30(ii.b). Furthermore, let us choose a surjection $B^{\oplus k} \to M$, whose kernel we write in the form $K := \bigcup_{i \in I} K_i$ where $(K_i \mid i \in I)$ is a filtered family of finitely generated B-submodules of K. Next, by applying (i), proposition 8.3.12(i,iii.b), and theorem 8.3.30(ii.b) we derive :

$$\begin{aligned}
\lambda_B(M) &\leq \inf \{\lambda_B(B^{\oplus k}/K_i) \mid i \in I\} = \inf \{d^{-1} \cdot \lambda_A(\psi_*(B^{\oplus k}/K_i)) \mid i \in I\} \\
&= d^{-1} \cdot (\lambda_A(\psi_* B^{\oplus k}) - \sup \{\lambda_A(\psi_* K_i) \mid i \in I\}) \\
&= d^{-1} \cdot (\lambda_A(\psi_* B^{\oplus k}) - \lambda_A(\psi_* K)) \\
&= d^{-1} \cdot \lambda_A(\psi_* M)
\end{aligned}$$

whence the claim.

(iv): Let $(N_i \mid i \in I)$ be the filtered system of finitely generated submodules of M_2. For every $i \in I$ we have short exact sequences : $0 \to M_1 \cap N_i \to N_i \to \overline{N}_i \to 0$, where $\overline{N}_i$ is the image of N_i in M_3. In view of the case of (ii) already known, we may then replace M_2 by N_i, and thus assume from start that M_2 is finitely generated, so that we may find a finitely generated $\mathfrak{m}_A$-primary ideal $J \subset A$ that annihilates M_2. By (iii) we have $\lambda_A(M_2) = \lambda_{A/J}(M_2)$, and likewise for M_1 and M_3, hence we may replace A by A/J. By claim 8.3.43, we may then find a morphism $V \to A$ of measurable K^+-algebras with V a valuation ring, inducing a finite residue field extension $\kappa(V) \to \kappa(A)$; then again (iii) reduces to the case where $A = V$, to which one may apply (i) and proposition 8.3.12(i) to conclude the proof.

Next we consider assertion (ii) for the case where the transition maps are surjective. We may assume that I admits a smallest element i_0; for every $i \in I$ let K_i denote the kernel of the transition map $M_{i_0} \to M_i$. We deduce a short exact sequence :

$$0 \to \bigcup_{i \in I} K_i \to M_{i_0} \to \operatorname*{colim}_{i \in I} M_i \to 0$$

and we may then compute using (iv) and the previous case of (ii) :

$$\lambda_A(\operatorname*{colim}_{i \in I} M_i) = \lambda_A(M_{i_0}) - \lambda_A(\bigcup_{i \in I} K_i) = \lim_{i \in I}(\lambda_A(M_{i_0}) - \lambda_A(K_i)) = \lim_{i \in I} \lambda_A(M_{i_0}/K_i)$$

whence the claim.

(v): Since ψ_* is an exact functor which commutes with colimits, we may use (ii) to reduce to the case where M is finitely presented, for which the assertion is already known, in view of theorem 8.3.30(iv). $\square$

Remark 8.3.45. Suppose that the valuation of K is discrete; then one sees easily that theorem 8.3.30(i,iii) still holds (with simpler proof) when A is replaced by any local noetherian K^+-algebra, and by inspecting the definition, the resulting map λ_A is none else than the standard length function for modules supported on $\{s(A)\}$.

Lemma 8.3.46. *Let A be a measurable K^+-algebra, and M an A-module supported at $s(A)$ with $\lambda_A(M) < \infty$. Then*

$$\{a \in K^+ \mid \log |a| > \lambda_A(M)\} \subset \mathrm{Ann}_{K^+} M.$$

Proof. Using proposition 8.3.41(ii), we easily reduce, first, to the case where M is finitely generated, and second, to the case where M is finitely presented. Pick an ideal $I \subset A$ and a valuation ring V mapping onto A/I, as in lemma 8.3.29; by considering the I-adic filtration of M, the additivity properties of λ_A allow to further reduce to the case where M is an A/I-module. Next, by theorem 8.3.30(ii.b) we may replace A by V, and therefore assume that A is a valuation ring whose valuation group equals Γ. In this case, λ_A is computed by Fitting ideals, so the assertion follows easily from [36, Prop.6.3.6(iii)]. □

8.3.47. Let us consider a K^+-algebra R_∞ that is the colimit of an inductive system

$$R_0 \to R_1 \to R_2 \to \cdots \tag{8.3.48}$$

of morphisms of measurable K^+-algebras inducing integral ring homomorphisms $\kappa(R_i) \to R_{i+1}/\mathfrak{m}_{R_i} R_i$ for every $i \in \mathbb{N}$. The final step consists in generalizing the definition of normalized length to the category R_∞-$\mathbf{Mod}_{\{s\}}$ of R_∞-modules supported at the closed point $s(R_\infty)$ of $\mathrm{Spec}\, R_\infty$. To this purpose, we shall axiomatize the general situation in which we can solve this problem. Later we shall see that our axioms are satisfied in many interesting cases.

8.3.49. Hence, let R_∞ be as in (8.3.47). After fixing an order-preserving isomorphism

$$(\mathbb{Q} \otimes_{\mathbb{Z}} \log \Gamma)^\wedge \xrightarrow{\sim} \mathbb{R} \tag{8.3.50}$$

we may regard the mappings λ_A (for any measurable K^+-algebra A) as real-valued functions on A-modules. To ease notation, for every R_n-module N supported on $\{s(R_n)\}$ we shall write $\lambda_n(N)$ instead of $\lambda_{R_n}(N)$. Notice that, for every such N, and every $m \geq n$, the R_m-module $R_m \otimes_{R_n} N$ is supported at $\{s(R_m)\}$, since by assumption the map $R_n \to R_m/\mathfrak{m}_{R_n} R_m$ is integral.

Definition 8.3.51. In the situation of (8.3.49), we say that R_∞ is an *ind-measurable* K^+-algebra, if there exists a sequence of real *normalizing factors* $(d_n > 0 \mid n \in \mathbb{N})$ such that :

(a) For every $n \in \mathbb{N}$ and every object N of R_n-$\mathbf{Mod}_{\mathrm{coh},\{s\}}$, the sequence :

$$m \mapsto d_m^{-1} \cdot \lambda_m(R_m \otimes_{R_n} N)$$

converges to an element $\lambda_\infty(R_\infty \otimes_{R_n} N) \in \mathbb{R}$.

(b) For every $m \in \mathbb{N}$, every finitely generated $\mathfrak{m}_{R_0}$-primary ideal $I \subset R_0$, and every $\varepsilon > 0$ there exists $\delta(m, \varepsilon, I) > 0$ such that the following holds. For every $n \in \mathbb{N}$ and every surjection $N \to N'$ of finitely presented R_n/IR_n-modules generated by m elements, such that

$$|\lambda_\infty(R_\infty \otimes_{R_n} N) - \lambda_\infty(R_\infty \otimes_{R_n} N')| \leq \delta(m, \varepsilon, I)$$

we have :

$$d_n^{-1} \cdot |\lambda_n(N) - \lambda_n(N')| \leq \varepsilon.$$

8.3.52. Assume now that R_∞ is ind-measurable, and let N be a finitely presented R_n-module supported on $\{s(R_n)\}$. The first observation is that $\lambda_\infty(R_\infty \otimes_{R_n} N)$ only depends on the R_∞-module $R_\infty \otimes_{R_n} N$. Indeed, suppose that $R_\infty \otimes_{R_m} M \simeq R_\infty \otimes_{R_n} N$ for some $m \in \mathbb{N}$ and some finitely presented R_m-module M; then there exists $p \geq m, n$ such that $R_p \otimes_{R_m} M \simeq R_p \otimes_{R_n} N$, and then the assertion is clear.

The second observation – contained in the following lemma 8.3.53 – will show that the conditions of definition 8.3.51 impose some non-trivial restrictions on the inductive system $(R_n \mid n \in \mathbb{N})$.

Lemma 8.3.53. *Let* $(R_n; d_n \mid n \in \mathbb{N})$ *be the datum of an inductive system of measurable* K^+*-algebras and a sequence of positive reals, fulfilling conditions* (a) *and* (b) *of definition* 8.3.51. *Then:*

(i) *The natural map*
$$M \to R_m \otimes_{R_n} M$$
is injective for every $n, m \in \mathbb{N}$ *with* $m \geq n$ *and every* R_n*-module* M.

(ii) *Especially, the transition maps* $R_n \to R_{n+1}$ *are injective for every* $n \in \mathbb{N}$.

(iii) *Suppose that* $(d'_n \mid n \in \mathbb{N})$ *is another sequence of positive reals such that conditions* (a) *and* (b) *hold for the datum* $(R_n; d'_n \mid n \in \mathbb{N})$. *Then the sequence* $(d_n/d'_n \mid n \in \mathbb{N})$ *converges to a non-zero real number.*

Proof. (i): We reduce easily to the case where M is finitely presented over R_n. Let $N \subset \mathrm{Ker}\,(M \to R_m \otimes_{R_n} M)$ be a finitely generated R_n-module; since R_n is coherent, N is a finitely presented R_n-module. We suppose first that M is in $R_n\text{-}\mathbf{Mod}_{\mathrm{coh},\{s\}}$.

Claim 8.3.54. The natural map $R_m \otimes_{R_n} M \to R_m \otimes_{R_n} (M/N)$ is an isomorphism.

Proof of the claim. On the one hand, the R_m-module $R_m \otimes_{R_n} M$ represents the functor
$$R_m\text{-}\mathbf{Mod} \to \mathbf{Set} \quad : \quad Q \mapsto \mathrm{Hom}_{R_n}(M, Q).$$
On the other hand, the assumption on N implies that $\mathrm{Hom}_{R_n}(M, Q) = \mathrm{Hom}_{R_n}(M/N, Q)$ for every R_m-module Q, so the claim follows easily. $\diamondsuit$

From claim 8.3.54 we deduce that the natural map $R_\infty \otimes_{R_n} M \to R_\infty \otimes_{R_n} (M/N)$ is an isomorphism, and then condition (b) says that $\lambda_n(M) = \lambda_n(M/N)$, hence $\lambda_n(N) = 0$ and finally $N = 0$, as stated. Next, suppose M is any finitely presented R_n-module, and pick a finitely generated $\mathfrak{m}_{R_n}$-primary ideal $I \subset R_n$.

Claim 8.3.55. There exists $c \in \mathbb{N}$ such that $N \cap I^{k+c}M = I^k(N \cap I^cM)$ for every $k \geq 0$.

Proof of the claim. We may find a local ind-étale map $A \to R_n$ of K^+-algebras, where A is an object of $K^+\text{-}\mathbf{m}.\mathbf{Alg}_0$, and such that I, M and N descend respectively to a finitely generated ideal $I_0 \subset A$, a finitely presented A-module M_0, and a finitely generated submodule $N_0 \subset M_0$. Then theorem 5.7.29 ensures the existence of $c \in \mathbb{N}$ such that $N_0 \cap I_0^{k+c}M_0 = I_0^k(N_0 \cap I_0^cM_0)$ for every $k \geq 0$. Since R_n is a faithfully flat A-algebra, the claim follows. $\diamondsuit$

Pick $c \in \mathbb{N}$ as in claim 8.3.55; then $N/(N \cap I^{k+c}M)$ is in the kernel of the natural map $M/I^{k+c}M \to R_m \otimes_{R_n} (M/I^{k+c}M)$, hence $N = N \cap I^{k+c}M$ by the foregoing, so that $N \subset I^kN$ for every $k \geq 0$, and finally $N = 0$ by Nakayama's lemma.

(ii) is a special case of (i). To show (iii), let us denote by $\lambda'_\infty(M)$ the normalized length of any object M of $R_\infty\text{-}\mathbf{Mod}_{\mathrm{coh},\{s\}}$, defined using the sequence $(d'_n \mid n \in \mathbb{N})$. From (b) it is clear that $\lambda_\infty(M), \lambda'_\infty(M) \neq 0$ whenever $M \neq 0$. Then, for any such non-zero M, the quotient $\lambda'_\infty(M)/\lambda_\infty(M)$ is the limit of the sequence $(d_n/d'_n \mid n \in \mathbb{N})$. $\square$

Remark 8.3.56. (i) There is another situation of interest which leads to a well-behaved notion of normalized length. Namely, suppose that $R_\bullet := (R_n \mid n \in \mathbb{N})$ is an inductive system of local homomorphisms of local noetherian rings, such that the fibres of the induced morphisms $\mathrm{Spec}\, R_{n+1} \to \mathrm{Spec}\, R_n$ have dimension zero, and denote by R_∞ the inductive limit of the system $R_\bullet$. For every $n \in \mathbb{N}$, let also λ_n be the usual length function on the set of isomorphism classes of finitely generated R_n-modules supported on $s(R_n)$. Then for every $m, n \in \mathbb{N}$ with $m \geq n$, and every R_n-module M of finite length, the R_m-module $R_m \otimes_{R_n} M$ has again finite length, so the analogues of conditions (a) and (b) of (8.3.49) can be formulated (cp. remark 8.3.45), and if these conditions hold for $R_\bullet$, we shall say that R_∞ is an *ind-measurable ring*. In such situation, lemma 8.3.53 – as well as the forthcoming lemma 8.3.57 and theorem 8.3.62 – still hold, with simpler proofs : we leave the details to the reader.

(ii) In spite of the uniqueness properties expressed by lemma 8.3.53, we do not know to which extent the normalized length of an ind-measurable K^+-algebra depends on the chosen tower of measurable algebras. Namely, suppose that $(R_n \mid n \in \mathbb{N})$ and $(R'_n \mid n \in \mathbb{N})$ are two such towers, with isomorphic colimit R_∞, and suppose that we have found normalizing factors $(d_n \mid n \in \mathbb{N})$ (resp. $(d'_n \mid n \in \mathbb{N})$) for the first (resp. second) tower, whence a normalized length λ_∞ (resp. λ'_∞) for R_∞-modules. Then we do not know whether the ratio of λ_∞ and λ'_∞ is a constant.

Next, for a given finitely generated R_∞-module M supported on $\{s(R_\infty)\}$, we shall proceed as in (8.3.38) : we denote by $\mathscr{C}_M$ the set of isomorphism classes of finitely presented R_∞-modules supported on $\{s(R_\infty)\}$ that admit a surjection $M' \to M$, and we set

$$\lambda^*_\infty(M) := \inf \{\lambda_\infty(M') \mid M' \in \mathscr{C}_M\} \in \mathbb{R}.$$

Directly on the definitions, one checks that $\lambda^*_\infty(M) = \lambda_\infty(M)$ if M is finitely presented.

Lemma 8.3.57. *Let M be a finitely generated R_∞-module, $\Sigma \subset M$ any finite set of generators, $(N_i \mid i \in I)$ any filtered system of objects of R_∞-$\mathbf{Mod}_{\{s\}}$, with surjective transition maps, such that $\operatorname*{colim}_{i\in I} N_i \simeq M$. Then :*

(i) $\lambda^*_\infty(M) = \lim_{n\to\infty} d_n^{-1} \cdot \lambda_n(\Sigma R_n)$.

(ii) *If every N_i is finitely generated, we have :*

$$\inf \{\lambda^*_\infty(N_i) \mid i \in I\} = \lambda^*_\infty(M).$$

Proof. To start out, we show assertion (ii) in the special case where all the modules N_i are finitely presented. Indeed, let us pick any surjection $\varphi : M' \to M$ with $M' \in \mathscr{C}_M$. We may find $i \in I$ such that φ lifts to a map $\varphi_i : M' \to N_i$ ([36, Prop.2.3.16(ii)]), and up to replacing I by a cofinal subset, we may assume that φ_i is defined for every $i \in I$. Moreover

$$\operatorname*{colim}_{i\in I} \operatorname{Coker} \varphi_i = \operatorname{Coker} \varphi = 0$$

hence there exists $i \in I$ such that φ_i is surjective. It follows easily that $\lambda_\infty(N_i) \leq \lambda_\infty(M')$, whence (ii) in this case.

(i): Let k be the cardinality of Σ, $\varepsilon > 0$ any real number, $Q \subset \operatorname{Ann}_{R_0}\Sigma$ a finitely generated $\mathfrak{m}_{R_0}$-primary ideal, $S := R_\infty/QR_\infty$ and $\beta : S^{\oplus k} \to M$ a surjection that sends the standard basis onto Σ. By filtering $\operatorname{Ker}\beta$ by the system $(K_j \mid j \in J)$ of its finitely generated submodules, we obtain a filtered system $(N_j := S^{\oplus k}/K_j \mid j \in J)$ of finitely presented R_∞-modules supported on $\{s(R_\infty)\}$, with surjective transition maps and colimit isomorphic to M. Let $\psi_j : N_j \to M$ be the natural map; by the foregoing, we may find $j_0 \in J$ such that

$$0 \leq \lambda_\infty(N_{j_0}) - \lambda^*_\infty(M) \leq \min(\delta(k, \varepsilon, Q), \varepsilon) \tag{8.3.58}$$

(notation of definition 8.3.51(b)). We can then find $n \in \mathbb{N}$ and a finitely presented R_n-module M'_n supported on $\{s(R_n)\}$ and generated by at most k elements, such that $N_{j_0} = R_\infty \otimes_{R_n} M'_n$; we set $M'_m := R_m \otimes_{R_n} M_n$ for every $m \geq n$ and let M_m be the image of M'_m in M. Up to replacing n by a larger integer, we may assume that $M_m = \Sigma R_m$ for every $m \geq n$. Choose $m \in \mathbb{N}$ so that :

$$|\lambda_\infty(N_{j_0}) - d_m^{-1} \cdot \lambda_m(M'_m)| \leq \varepsilon. \tag{8.3.59}$$

Let $(M_{m,i} \mid i \in I)$ be a filtered system of finitely presented R_m-modules supported at $\{s(R_m)\}$, with surjective transition maps, such that $\operatorname*{colim}_{i\in I} M_{m,i} = M_m$. Arguing as in the foregoing, we show that there exists $i \in I$ such that the natural map $\varphi : M'_m \to M_m$ factors through a surjection $\varphi_i : M'_m \to M_{m,i}$, and up to replacing I by a cofinal subset, we may assume that such

a surjection φ_i exists for every $i \in I$. We obtain therefore a compatible system of surjections of R_∞-modules :
$$N_{j_0} \simeq R_\infty \otimes_{R_m} M'_m \to R_\infty \otimes_{R_m} M_{m,i} \to M$$
and combining with (8.3.58) we find :
$$|\lambda_\infty(N_{j_0}) - \lambda_\infty(R_\infty \otimes_{R_m} M_{m,i})| \leq \delta(k, \varepsilon, Q) \qquad \text{for every } i \in I.$$
In such situation, condition (b) of definition 8.3.51 ensures that :
$$d_m^{-1} \cdot |\lambda_m(M'_m) - \lambda_m(M_{m,i})| \leq \varepsilon \qquad \text{for every } i \in I.$$
Therefore : $d_m^{-1} \cdot |\lambda_m(M'_m) - \lambda_m(M_m)| \leq \varepsilon$, by proposition 8.3.41(ii). Combining with (8.3.59) and again (8.3.58) we obtain :
$$|\lambda^*_\infty(M) - d_m^{-1} \cdot \lambda_m(M_m)| \leq 3\varepsilon$$
which implies (i).

(ii): With no loss of generality, we may assume that I admits a smallest element i_0. Let us fix a surjection $F := R_\infty^{\oplus k} \to N_{i_0}$, and for every $i \in I$, let C_i denote the kernel of the induced surjection $F \to N_i$. We consider the filtered system $(D_j \mid j \in J)$ consisting of all finitely generated submodules $D_j \subset F$ such that $D_j \subset C_i$ for some $i \in I$. It is clear that $\operatorname*{colim}_{j \in J} F/D_j \simeq M$, hence $\lambda^*_\infty(M) = \inf\{\lambda_\infty(F/D_j) \mid j \in J\}$, by (i). On the other, by construction, for every $j \in J$ we may find $i \in I$ such that $\lambda^*_\infty(N_i) \leq \lambda_\infty(F/D_j)$; since clearly $\lambda^*_\infty(N_i) \geq \lambda^*_\infty(M)$ for every $i \in I$, the assertion follows. □

8.3.60. Let now M be an arbitrary object of the category $R_\infty\text{-}\mathbf{Mod}_{\{s\}}$. We let :
$$\lambda_\infty(M) := \sup\{\lambda^*_\infty(M') \mid M' \subset M \text{ and } M' \text{ is finitely generated}\} \in \mathbb{R} \cup \{+\infty\}.$$

Lemma 8.3.61. *If M is finitely generated, then $\lambda_\infty(M) = \lambda^*_\infty(M)$.*

Proof. Let $N \subset M$ be any finitely generated submodule. We choose finite sets of generators $\Sigma \subset N$ and $\Sigma' \subset M$ with $\Sigma \subset \Sigma'$. In view of proposition 8.3.41(iv) we have $\lambda_n(\Sigma R_n) \leq \lambda_n(\Sigma' R_n)$ for every $n \in \mathbb{N}$, hence $\lambda^*_\infty(N) \leq \lambda^*_\infty(M)$, by lemma 8.3.57(i). The contention follows easily. □

Theorem 8.3.62. (i) *Let $(M_i \mid i \in I)$ be a filtered system of objects of $R_\infty\text{-}\mathbf{Mod}_{\{s\}}$, and suppose that either :*

(a) *all the transition maps of the system are injections, or*
(b) *all the transition maps are surjections and $\lambda_\infty(M_i) < +\infty$ for every $i \in I$.*

Then :
$$\lambda_\infty(\operatorname*{colim}_{i \in I} M_i) = \lim_{i \in I} \lambda_\infty(M_i).$$

(ii) *Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence in $R_\infty\text{-}\mathbf{Mod}_{\{s\}}$. Then :*
$$\lambda_\infty(M) = \lambda_\infty(M') + \lambda_\infty(M'').$$

(iii) *Let M be a finitely presented R_∞-module, $N \subset M$ a submodule supported at $s(R_\infty)$. Then $\lambda_\infty(N) = 0$ if and only if $N = 0$.*

Proof. The proof of (i) in case (a) is the same as that of proposition 8.3.41(iii.b).

(ii): We proceed in several steps :

• Suppose first that M and M'' are finitely presented, hence M' is finitely generated ([36, Lemma 2.3.18(ii)]). We may then find an integer $n \in \mathbb{N}$ and finitely presented R_n-modules M_n and M''_n such that $M \simeq R_\infty \otimes_{R_n} M_n$ and $M'' \simeq R_\infty \otimes_{R_n} M''_n$. For every $m \geq n$ we set $M_m := R_m \otimes_{R_n} M_n$ and likewise we define M''_m. Up to replacing n by a larger integer, we may assume that the given map $\varphi : M \to M''$ descends to a surjection $\varphi_n : M_n \to M''_n$, and then φ is the colimit of the induced maps $\varphi_m := \mathbf{1}_{R_m} \otimes_{R_n} \varphi_n$, for every $m \geq n$. Moreover,

$M' \simeq \operatorname*{colim}_{m\geq n} \operatorname{Ker}\varphi_m$, and by the right exactness of the tensor product, the image of $\operatorname{Ker}\varphi_m$ generates M' for every $m \geq n$. Furthermore, since the natural maps $M_m \to M_p$ are injective for every $p \geq m \geq n$ (lemma 8.3.53), the same holds for the induced maps $\operatorname{Ker}\varphi_m \to \operatorname{Ker}\varphi_p$. The latter factors as a composition :

$$\operatorname{Ker}\varphi_m \xrightarrow{\alpha} R_p \otimes_{R_m} \operatorname{Ker}\varphi_m \xrightarrow{\beta} \operatorname{Ker}\varphi_p$$

where α is injective (lemma 8.3.53) and β is surjective. In other words, $R_p \cdot \operatorname{Ker}\varphi_m = \operatorname{Ker}\varphi_p$ for every $p \geq m \geq n$. In such situation, lemma 8.3.57(i) ensures that :

$$\lambda_\infty(M') = \lim_{m\geq n} d_m^{-1} \cdot \lambda_m(\operatorname{Ker}\varphi_m)$$

and likewise :

$$\lambda_\infty(M) = \lim_{m\geq n} d_m^{-1} \cdot \lambda_m(M_m) \qquad \lambda_\infty(M'') = \lim_{m\geq n} d_m^{-1} \cdot \lambda_m(M''_m).$$

To conclude the proof of (ii) in this case, it suffices then to apply proposition 8.3.41(iv).

• Suppose next that M, M' (and hence M'') are finitely generated. We choose a filtered system $(M_i \mid i \in I)$ of finitely presented R_∞-modules, with surjective transition maps, such that $M \simeq \operatorname*{colim}_{i\in I} M_i$. After replacing I by a cofinal subset, we may assume that M' is generated by a finitely generated submodule M'_i of M_i, for every $i \in I$, and that $(M'_i \mid i \in I)$ forms a filtered system with surjective transition maps, whose colimit is necessarily M'; set also $M''_i := M_i/M'_i$ for every $i \in I$, so that the colimit of the filtered system $(M''_i \mid i \in I)$ is M''. In view of lemmata 8.3.57(ii) and 8.3.61, we are then reduced to showing the identity :

$$\inf\{\lambda^*_\infty(M_i) \mid i \in I\} = \inf\{\lambda^*_\infty(M'_i) \mid i \in I\} + \inf\{\lambda^*_\infty(M''_i) \mid i \in I\}$$

which follows easily from the previous case.

• Suppose now that M is finitely generated. We let $(M'_i \mid i \in I)$ be the filtered family of finitely generated submodules of M'. Then :

$$M' \simeq \operatorname*{colim}_{i\in I} M'_i \quad \text{and} \quad M'' \simeq \operatorname*{colim}_{i\in I} M/M'_i.$$

Hence :

$$\lambda_\infty(M') = \lim_{i\in I} \lambda_\infty(M'_i) \quad \text{(resp.} \quad \lambda_\infty(M'') = \lim_{i\in I} \lambda_\infty(M/M'_i))$$

by (i.a) (resp. by lemmata 8.3.57(ii) and 8.3.61). However, the foregoing case shows that $\lambda_\infty(M) = \lambda_\infty(M'_i) + \lambda_\infty(M/M'_i)$ for every $i \in I$, so assertion (ii) holds also in this case.

• Finally we deal with the general case. Let $(M_i \mid i \in I)$ be the filtered system of finitely generated submodules of M; we denote by M''_i the image of M_i in M'', and set $M'_i := M' \cap M_i$ for every $i \in I$. By (i.a) we have :

$$\lambda_\infty(M) = \lim_{i\in I} \lambda_\infty(M_i)$$

and likewise for M' and M''. Since we already know that $\lambda_\infty(M_i) = \lambda_\infty(M'_i) + \lambda_\infty(M''_i)$ for every $i \in I$, we are done.

(iii): Let $f \in N$ be any element; in view of (ii) we see that $\lambda_\infty(fR_\infty) = 0$, and it suffices to show that $f = 0$. However, we may find $n \in \mathbb{N}$ and a finitely presented R_n-module M_n such that $M \simeq R_\infty \otimes_{R_n} M_n$; notice that the natural map $M_n \to M$ is injective, by lemma 8.3.53(i). We may also assume that f is in the image of M_n. Let $I \subset \operatorname{Ann}_{R_n}(f)$ be a finitely generated $\mathfrak{m}_{R_n}$-primary ideal; after replacing M by M/IM, we may assume that M_n is supported at $s(R_n)$. In light of (ii), we see that $\lambda_\infty(M) = \lambda_\infty(M/fR_\infty)$, hence $\lambda_n(M_n) = \lambda_n(M_n/fR_n)$, due to condition (b) of definition 8.3.51. Hence $\lambda_n(fR_n) = 0$ by proposition 8.3.41(iv), and finally $f = 0$ by theorem 8.3.30(i,iii.a).

To conclude, we consider assertion (i) in case (b) : set $M := \operatorname{colim}_{i\in I} M_i$; it is clear that $\lambda_\infty(M) \leq \lambda_\infty(M_i) \leq \lambda_\infty(M_j)$ whenever $i \geq j$, hence

$$\lim_{i\in I} \lambda_\infty(M_i) = \inf\{\lambda_\infty(M_i) \mid i \in I\} \geq \lambda_\infty(M).$$

For the converse inequality, fix $\varepsilon > 0$; without loss of generality, we may assume that I admits a smallest element i_0, and we can find a finitely generated submodule $N_{i_0} \subset M_{i_0}$ such that $\lambda_\infty(M_{i_0}) - \lambda_\infty(N_{i_0}) < \varepsilon$. For every $i \in I$, let $N_i \subset M_i$ be the image of N_{i_0}, and let $N \subset M$ be the colimit of the filtered system $(N_i \mid i \in I)$; then M_i/N_i is a quotient of M_{i_0}/N_{i_0}, and the additivity assertion (ii) implies that $\lambda_\infty(M_i) - \lambda_\infty(N_i) < \varepsilon$ for every $i \in I$. According to lemma 8.3.57(ii) (and lemma 8.3.61) we have :

$$\lambda_\infty(M) \geq \lambda_\infty(N) = \inf\{\lambda_\infty(N_i) \mid i \in I\} \geq \inf\{\lambda_\infty(M_i) \mid i \in I\} - \varepsilon$$

whence the claim. □

8.3.63. We wish now to show that the definition of normalized length descends to almost modules (see (8.3)). Namely, we have the following :

Proposition 8.3.64. *Let M, N be two objects of R_∞-$\mathbf{Mod}_{\{s\}}$ such that $M^a \simeq N^a$. Then $\lambda_\infty(M) = \lambda_\infty(N)$.*

Proof. Using additivity (theorem 8.3.62(ii)), we easily reduce to the case where $M^a = 0$, in which case we need to show that $\lambda_\infty(M) = 0$. Using theorem 8.3.62(i) we may further assume that M is finitely generated. Then, in view of lemma 8.3.57(i), we are reduced to showing the following :

Claim 8.3.65. Let A be any measurable K^+-algebra, and M any object of A-$\mathbf{Mod}_{\{s\}}$ such that $M^a = 0$. Then $\lambda_A(M) = 0$.

Proof of the claim. Arguing as in the foregoing we reduce to the case where M is finitely generated. Then, let us pick $I \subset A$ and $\varphi : V \to A/I$ as in lemma 8.3.29. It suffices to show the assertion for the finitely generated module $\bigoplus_{n\in\mathbb{N}} I^nM/I^{n+1}M$, hence we may assume that $I \subset \mathrm{Ann}_A M$, in which case, by proposition 8.3.41(iii) we may replace A by V and assume throughout that A is a valuation ring. Then the claim follows from propositions 8.3.41(i) and 8.3.12(ii). □

8.3.66. Proposition 8.3.64 suggests the following definition. We let R^a_∞-$\mathbf{Mod}_{\{s\}}$ be the full subcategory of R^a_∞-$\mathbf{Mod}$ consisting of all the R^a_∞-modules M such that $M_!$ is supported at $s(R_\infty)$, in which case we say that M *is supported at* $s(R_\infty)$. Then, for every such M we set :

$$\lambda_\infty(M) := \lambda_\infty(M_!).$$

With this definition, it is clear that theorem 8.3.62(i,ii) extends *mutatis mutandis* to almost modules. For future reference we point out :

Lemma 8.3.67. *Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of R^a_∞-modules. We have :*

(i) *If M lies in R^a_∞-$\mathbf{Mod}_{\{s\}}$, then $\lambda_\infty(abM) \leq \lambda_\infty(aM') + \lambda_\infty(bM'')$ for every $a, b \in \mathfrak{m}$.*
(ii) *If M is almost finitely presented, and M' is supported at $s(R_\infty)$, then $\lambda_\infty(M') = 0$ if and only if $M' = 0$.*

Proof. (i): To start out, we deduce a short exact sequence : $0 \to bM \cap M' \to bM \to bM'' \to 0$. Next, let $N := \mathrm{Ker}(a : bM \to bM)$, and denote by N' the image of N in bM''; there follows a short exact sequence : $0 \to a(bM \cap M') \to abM \to bM''/N' \to 0$. The claim follows.

(ii): We reduce easily to the case where M' is a cyclic R^a_∞-module, say $M' = R^a_\infty x$ for some $x \in M_!$. In this case, let $I \subset \mathrm{Ann}_{R_0}(x)$ be a finitely generated $\mathfrak{m}_{R_0}$-primary ideal; we may then replace M by M/IM, and assume that M lies in $R^a_\infty\text{-}\mathbf{Mod}_{\{s\}}$ as well. We remark :

Claim 8.3.68. M is almost finitely presented if and only if, for every $b \in \mathfrak{m}$ there exists a finitely presented R_∞-module N and a morphism $M \to N^a$ whose kernel and cokernel are annihilated by b. Moreover if M is supported at $s(R_\infty)$, one can choose N to be supported at $s(R_\infty)$.

Proof of the claim. The "if" direction is clear. For the "only if" part, we use [36, Cor.2.3.13], which provides us with a morphism $\varphi : N \to M_!$ with N finitely presented over R_∞, such that $b \cdot \mathrm{Ker}\,\varphi = b \cdot \mathrm{Coker}\,\varphi = 0$. Then $b \cdot \mathbf{1}^a_N$ factors through a morphism $\varphi' : \mathrm{Im}\,\varphi^a \to N^a$, and $b \cdot \mathbf{1}_M$ factors through a morphism $\varphi'' : M \to \mathrm{Im}\,\varphi^a$; the kernel and cokernel of $\varphi' \circ \varphi''$ are annihilated by b^2. Finally, suppose that M is supported on $s(R_\infty)$, and let $I \subset R_\infty$ be any finitely generated ideal such that $V(I) = \{s(R_\infty)\}$. By assumption, for every $f \in I$ and every $m \in M_!$ there exists $n \in \mathbb{N}$ such that $f^n m = 0$; it follows that I^n annihilates $\mathrm{Im}\,\varphi$ for every sufficiently large $n \in \mathbb{N}$, and we may then replace N by N/I^nN. ◊

Let M and M' be as in (ii), and choose a morphism $\varphi : M \to N^a$ as in claim 8.3.68; by adjunction we get a map $\psi : M'_! \to M_! \to N$ with $b \cdot \mathfrak{m} \cdot \mathrm{Ker}\,\psi = 0$. It follows that $\lambda_\infty(\mathrm{Im}\,\psi) = 0$, hence $\mathrm{Im}\,\psi = 0$, by theorem 8.3.62(iii). Hence $bM' = 0$; since b is arbitrary, the assertion follows. □

Simple examples show that an almost finitely generated (or even almost finitely presented) R^a_∞-module may fail to have finite normalized length. The useful finiteness condition for almost modules is contained in the following :

Definition 8.3.69. Let M be a R^a_∞-module supported at $s(R_\infty)$. We say that M has *almost finite length* if $\lambda_\infty(bM) < +\infty$ for every $b \in \mathfrak{m}$.

Lemma 8.3.70. (i) *The set of isomorphism classes of R^a_∞-modules of almost finite length forms a closed subset of the uniform space $\mathscr{M}(A)$ (notation of* [36, §2.3]*).*

(ii) *Especially, every almost finitely generated R^a_∞-module supported at $s(R_\infty)$ has almost finite length.*

Proof. Assertion (i) boils down to the following :

Claim 8.3.71. Let $a, b \in \mathfrak{m}$, and $f : N \to M$, $g : N \to M'$ morphisms of R^a_∞-modules, such that the kernel and cokernel of f and g are annihilated by $a \in \mathfrak{m}$, and such that $\lambda_\infty(bM) < +\infty$. Then $\lambda_\infty(a^2bM') < +\infty$.

Proof of the claim. By assumption, $\mathrm{Ker}\,f \subset \mathrm{Ker}\,a \cdot \mathbf{1}_N$, whence an epimorphism $f(N) \to aN$; likewise, we have an epimorphism $g(N) \to aM'$. It follows that

$$\lambda_\infty(a^2bM') \leq \lambda_\infty(ab \cdot g(M)) \leq \lambda_\infty(abN) \leq \lambda_\infty(b \cdot f(N)) < +\infty$$

as stated. ◊

(ii) follows from (i) and the obvious fact that every finitely generated R_∞-module supported at $s(R_\infty)$ has finite normalized length. □

8.3.72. For the case of a measurable K^+-algebra A of dimension zero, we can show a further *Lipschitz type* uniform estimate for the normalized length. Namely, for any integer $k > 0$ define the set $\mathscr{M}_k(A^a)$ with its uniform structure as in (8.3.1); *i.e.* we have the fundamental system of entourages

$$(E_r \mid r \in \mathbb{R}_{>0})$$

where each E_r consists of the pairs (N, N') such that there exists a third A^a-module N'' and A^a-linear maps $N'' \to N$, $N'' \to N'$ whose kernel and cokernels are annihilated by any $b \in K^+$ such that $\log |b| \geq r$.

Lemma 8.3.73. *In the situation of* (8.3.72), *we have :*

$$|\lambda_{A^a}(N) - \lambda_{A^a}(N')| \leq 4k \cdot \mathrm{length}_A(A/\mathfrak{m}A) \cdot r$$

for every $r \in \mathbb{R}_{>0}$ *and every* $(N, N') \in E_r$.

Proof. We begin with the following

Claim 8.3.74. For any finitely generated A-module N, and every $b \in K^+ \setminus \{0\}$ we have

$$\lambda_A(N/bN) \leq \mathrm{length}_A(N/\mathfrak{m}N) \cdot \log |b|.$$

Proof of the claim. Choose a map $\psi : V \to A$ of measurable K^+-algebras, inducing a finite residue field extension $\kappa(V) \to \kappa(A)$, where V is a K^+-flat valuation ring, and the induced map of value groups $\Gamma \to \Gamma_V$ is an isomorphism (claim 8.3.43). Let $d := \dim_{\kappa(V)} N/\mathfrak{m}_V N$; applying Nakayama's lemma, we get a surjection $(V/bV)^{\oplus d} \to N/bN$ of V-modules, for every $b \in K^+$. Hence

$$\lambda_A(N/bN) = \frac{\lambda_V(\psi_*(N/bN))}{[\kappa(V) : \kappa(A)]} \leq \frac{d \log |b|}{[\kappa(V) : \kappa(A)]} = \mathrm{length}_A(N/\mathfrak{m}N) \cdot \log |b|$$

as stated. ◊

Now, suppose that $(N, N') \in E_{\log |b|}$ for some $b \in K^+$, and pick maps $\varphi : N'' \to N$, $\psi : N'' \to N'$ whose kernel and cokernel are annihilated by b. Especially, if $n_1, \dots, n_k \in N_*$ (resp. $n'_1, \dots, n'_k \in N'_*$) is a system of generators for N (resp. for N'), we may find $n''_1, \dots, n''_{2k} \in N''_*$ such that $\varphi(n''_i) = bn_i$ for $i = 1, \dots, k$ and $\psi(n''_i) = n'_{i-k}$ for $i = k+1, \dots, 2k$. After replacing N'' by its submodule generated by $n''_1, \dots, n''_{2k}$, we may assume that $N'' \in \mathscr{M}_{2k}(A)$, $\mathrm{Ker}\,\varphi$ is still annihilated by b, but $\mathrm{Coker}\,\varphi$ is only annihilated by b^2. On the one hand, we deduce that

$$\lambda_{A^a}(N'') \geq \lambda_{A^a}(\varphi(N'')) \geq \lambda_{A^a}(b^2 N) = \lambda_{A^a}(N) - \lambda_{A^a}(N/b^2 N)$$

therefore

$$\lambda_{A^a}(N) - \lambda_{A^a}(N'') \leq \lambda_{A^a}(N/b^2 N).$$

On the other hand, the map $N'' \to N''$ given by the rule : $n'' \mapsto bn''$ for every $n'' \in N''_*$, factors through $\varphi(N'')$, therefore $\lambda_{A^a}(N) \geq \lambda_{A^a}(bN'')$ so the same calculation yields the inequality

$$\lambda_{A^a}(N'') - \lambda_{A^a}(N) \leq \lambda_{A^a}(N''/bN'').$$

Taking into account claim 8.3.74 (and proposition 8.3.64) we see that

$$|\lambda_{A^a}(N'') - \lambda_{A^a}(N)| \leq 2k \cdot \mathrm{length}_A(A/\mathfrak{m}A) \cdot \log |b|.$$

Of course, the same holds with N replaced by N', and the lemma follows. □

We conclude this section with some basic examples of the situation contemplated in definition 8.3.51.

Example 8.3.75. Suppose that all the transition maps $R_n \to R_{n+1}$ of the inductive system in (8.3.49) are flat. Then, proposition 8.3.41(v) implies that conditions (a) and (b) hold with:

$$d_n := \mathrm{length}_{R_n}(R_n/\mathfrak{m}_{R_0} R_n) \qquad \text{for every } n \in \mathbb{N}.$$

Example 8.3.76. Suppose that $p := \mathrm{char}\,\kappa > 0$. Let $d \in \mathbb{N}$ be any integer, $f : X \to \mathbb{A}^d_{K^+} := \mathrm{Spec}\, K^+[T_1, \dots, T_d]$ an étale morphism. For every $r \in \mathbb{N}$ we consider the cartesian diagram of schemes

$$\begin{tikzcd} X_r \arrow[r, "f_r"] \arrow[d, "\psi_r"'] & \mathbb{A}^d_{K^+} \arrow[d, "\varphi_r"] \\ X \arrow[r, "f"] & \mathbb{A}^d_{K^+} \end{tikzcd}$$

where φ_r is the morphism corresponding to the K^+-algebra homomorphism

$$\varphi_r^\natural : K^+[T_1, \dots, T_d] \to K^+[T_1, \dots, T_d]$$

defined by the rule: $T_j \mapsto T_j^{p^r}$ for $j = 1, \dots, d$. For every $r, s \in \mathbb{N}$ with $r \geq s$, ψ_r factors through an obvious S-morphism $\psi_{rs} : X_r \to X_s$, and the collection of the schemes X_r and transition morphisms ψ_{sr} gives rise to an inverse system $\underline{X} := (X_r \mid r \in \mathbb{N})$, whose inverse limit is representable by an S-scheme X_∞ ([32, Ch.IV, Prop.8.2.3]). Let $g_\infty : X_\infty \to S$ (resp. $g_r : X_r \to S$ for every $r \in \mathbb{N}$) be the structure morphism, $x \in g_\infty^{-1}(s)$ any point, $x_r \in X_r$ the image of x and $R_r := \mathscr{O}_{X_r,x_r}$ for every $r \in \mathbb{N}$. Clearly the colimit R_∞ of the inductive system $(R_r \mid r \in \mathbb{N})$ is naturally isomorphic to $\mathscr{O}_{X_\infty,x}$. Moreover, notice that the restriction $g_{r+1}^{-1}(s) \to g_r^{-1}(s)$ is a radicial morphism for every $r \in \mathbb{N}$. It follows easily that the transition maps $R_r \to R_{r+1}$ are finite; furthermore, by inspection one sees that $\varphi_r^\natural$ is flat and finitely presented, so R_{r+1} is a free R_r-module of rank p^d, for every $r \in \mathbb{N}$. Hence, the present situation is a special case of example 8.3.75, and therefore conditions (a) and (b) hold if we choose the sequence of integers $(d_i \mid i \in \mathbb{N})$ with

$$d_i := p^{id}/[\kappa(x_i) : \kappa(x_0)] \qquad \text{for every } i \in \mathbb{N}.$$

The foregoing discussion then yields a well-behaved notion of normalized length for arbitrary R_∞-modules supported at $\{s(R_\infty)\}$.

Example 8.3.77. In the situation of example 8.3.76, it is easy to construct R_∞^a-modules $M \neq 0$ such that $\lambda_\infty(M) = 0$. For instance, for $d := 1$, let $x \in X_\infty$ be the point of the special fibre where $T_1 = 0$; then we may take $M := R_\infty/I$, where I is the ideal generated by a non-zero element of $\mathfrak{m}_K$ and by the radical of $T_1 R_\infty$. The verification shall be left to the reader.

8.4. **Formal schemes.** In this section we wish to define a category of topologically ringed spaces that generalize the usual formal schemes from [26]. These foundations will be used to complete the proof of the almost purity theorem, in the crucial case of dimension three.

8.4.1. Quite generally, we shall deal with topological rings whose topology is *linear*, *i.e.* such that $(A, +)$ is a topological group and $0 \in A$ admits a fundamental system of open neighborhoods $\mathscr{I} := \{I_\lambda \mid \lambda \in \Lambda\}$ consisting of ideals of A. Since we consider exclusively topological rings of this kind, whenever we introduce a ring, we shall omit mentioning that its topology is linear, except in cases where the omission might be a source of ambiguities.

8.4.2. We shall also consider *topological A-modules* over a topological ring A. By definition, such a module M is endowed with a *linear* topology, such that the scalar multiplication $A \times M \to M$ is continuous. Additionally, *we assume that, for every open submodule $N \subset M$, there exists an open ideal $I \subset A$ such that $IM \subset N$.* (Notice that [26, Ch.0, §7.7.1] is slightly ambiguous : it is not clear whether all the topological modules considered there, are supposed to satisfy the foregoing additional condition.) If M is any such topological A-module, the (separated) completion $M^\wedge$ is a topological $A^\wedge$-module, where $A^\wedge$ is the completion of A. If $M' \subset M$ is any submodule, we denote by $\overline{M}' \subset M$ the topological closure of M'. Let N be any other topological A-module; following [26, Ch.0, §7.7.2], we may endow $M \otimes_A N$ with a natural topology, and the completion of the resulting topological A-module is denoted $M \widehat{\otimes}_A N$. For any two topological A-modules M and N, we denote by :

$$\mathrm{top.Hom}_A(M, N)$$

the A-module of all continuous A-linear maps $M \to N$. Also, we let A-$\mathbf{Mod}_{\mathrm{top}}$ be the category of topological A-modules with continuous A-linear maps. Notice that there is a natural identification :

$$\mathrm{top.Hom}_A(M, N^\wedge) \simeq \lim_{N' \subset N} \operatorname*{colim}_{M' \subset M} \mathrm{Hom}_A(M/M', N/N') \tag{8.4.3}$$

where N' (resp. M') ranges over the family of open submodules of N (resp. of M).

8.4.4. The category $A\text{-}\mathbf{Mod}_{\mathrm{top}}$ is, generally, not abelian; indeed, if $f : M \to N$ is a continuous map of topological A-modules, $\mathrm{Coker}(\mathrm{Ker}\, f \to M)$ is not necessarily isomorphic to $\mathrm{Ker}(N \to \mathrm{Coker}\, f)$, since the quotient topology (induced from M) on $\mathrm{Im}\, f$ may be finer than the subspace topology (induced from N). The topological A-modules do form an *exact category* in the sense of [67]; namely, the admissible monomorphisms (resp. epimorphisms) are the continuous injections (resp. surjections) $f : M \to N$ that are kernels (resp. cokernels), *i.e.* that induce homeomorphisms $M \xrightarrow{\sim} f(M)$ (resp. $M/\mathrm{Ker}\, f \xrightarrow{\sim} N$), where $f(M)$ (resp. $M/\mathrm{Ker}\, f$) is endowed with the subspace (resp. quotient) topology induced from N (resp. from M). An admissible epimorphism is also called a *quotient map* of topological A-modules. Correspondingly there is a well defined class of *admissible short exact sequences* of topological A-modules. The following lemma provides a simple criterion to check whether a short exact sequence is admissible.

Lemma 8.4.5. *Let :*

$$(\underline{E}_n \mid n \in \mathbb{N}) \quad : \quad 0 \to (M'_n \mid n \in \mathbb{N}) \to (M_n \mid n \in \mathbb{N}) \to (M''_n \mid n \in \mathbb{N}) \to 0$$

be an inverse system, with surjective transition maps, of short exact sequences of discrete A-modules. Then the induced complex of inverse limits :

$$\lim_{n\in\mathbb{N}} \underline{E}_n \quad : \quad 0 \to M' := \lim_{n\in\mathbb{N}} M'_n \to M := \lim_{n\in\mathbb{N}} M_n \to M'' := \lim_{n\in\mathbb{N}} M''_n \to 0$$

is an admissible short exact sequence of topological A-modules.

Proof. For every pair of integers $i, j \in \mathbb{N}$ with $i \leq j$, let $\varphi_{ji} : M_j \to M_i$ be the transition map in the inverse system $(M_n \mid n \in \mathbb{N})$, and define likewise φ'_{ji} and φ''_{ji}; the assumption means that all these maps are onto. Set $K_{ji} := \mathrm{Ker}\, \varphi_{ji}$ and define likewise K'_{ji}, K''_{ji}. By the snake lemma we deduce, for every $i \in \mathbb{N}$, an inverse system of short exact sequences :

$$0 \to (K'_{ji} \mid j \geq i) \to (K_{ji} \mid j \geq i) \to (K''_{ji} \mid j \geq j) \to 0$$

where again, all the transition maps are surjective. However, by definition, the decreasing family of submodules $(K_i := \lim_{j\geq i} K_{ji} \mid i \in \mathbb{N})$ is a fundamental system of open neighborhoods of $0 \in M$ (and likewise one defines the topology on the other two inverse limits). It follows already that the surjection $M \to M''$ is a quotient map of topological A-modules. To conclude it suffices to remark :

Claim 8.4.6. The natural map $K'_i := \lim_{j\geq i} K'_{ji} \to K_i$ induces an identification :

$$K'_i = K_i \cap M' \qquad \text{for every } i \in \mathbb{N}.$$

Proof of the claim. Indeed, for every $j \geq i$ we have a left exact sequence :

$$\underline{F}_j \quad : \quad (0 \to K'_{ji} \to M'_j \oplus K_{ji} \xrightarrow{\alpha} M_j)$$

where $\alpha(m', m) = m' - m$ for every $m' \in M'_j$ and $m \in K_{ji}$. Therefore $\lim_{j\geq i} \underline{F}_j$ is the analogous left exact sequence $0 \to K'_i \to M' \oplus K_i \to M$, whence the claim. □

8.4.7. Let X be any topological space; recall ([26, Ch.0, §3.1]) that a *sheaf of topological spaces* on X – also called a *sheaf with values in the category* $\mathbf{Top}$ *of topological spaces* – is the datum of a presheaf $\mathscr{F}$ on X with values in $\mathbf{Top}$ (in the sense of [39, Ch.I, §1.9]), such that the following holds.

(TS) For every topological space T, the rule :

$$U \mapsto \mathrm{Hom}_{\mathbf{Top}}(T, \mathscr{F}(U)) \qquad \text{for every open subset } U \subset X$$

defines a sheaf of sets on X.

A *sheaf of topological rings* is a presheaf $\mathscr{A}$ with values in the category of topological rings, whose underlying presheaf of topological spaces is a sheaf with values in $\mathbf{Top}$ (and therefore $\mathscr{A}$ is a sheaf of rings as well). A *sheaf of topological $\mathscr{A}$-modules* – or briefly, a *topological $\mathscr{A}$-module* – is a presheaf $\mathscr{F}$ of topological $\mathscr{A}$-modules, whose underlying presheaf of topological spaces is a sheaf with values in $\mathbf{Top}$; then $\mathscr{F}$ is also a sheaf of $\mathscr{A}$-modules. An $\mathscr{A}$-linear map $\varphi : \mathscr{F} \to \mathscr{G}$ between topological $\mathscr{A}$-modules is said to be *continuous* if $\varphi_U : \mathscr{F}(U) \to \mathscr{G}(U)$ is a continuous map for every open subset $U \subset X$. We denote by :

$$\mathrm{top.Hom}_{\mathscr{A}}(\mathscr{F}, \mathscr{G})$$

the A-module of all continuous $\mathscr{A}$-linear morphisms $\mathscr{F} \to \mathscr{G}$. Consider the presheaf on X defined by the rule :

$$U \mapsto \mathrm{top.Hom}_{\mathscr{A}_{|U}}(\mathscr{F}_{|U}, \mathscr{G}_{|U}). \tag{8.4.8}$$

As a consequence of **(TS)** we deduce that (8.4.8) is a sheaf on X, which we shall denote by :

$$top.\mathscr{H}om_{\mathscr{A}}(\mathscr{F}, \mathscr{G}).$$

8.4.9. In the situation of (8.4.1), set $X_\lambda := \operatorname{Spec} A/I_\lambda$ for every $\lambda \in \Lambda$; we define the *formal spectrum* of A as the colimit of topological spaces :

$$\mathrm{Spf}\, A := \operatorname*{colim}_{\lambda\in\Lambda} X_\lambda.$$

Hence, the set underlying $\mathrm{Spf}\, A$ is the filtered union of the X_λ, and a subset $U \subset \mathrm{Spf}\, A$ is open (resp. closed) in $\mathrm{Spf}\, A$ if and only if $U \cap X_\lambda$ is open (resp. closed) in X_λ for every $\lambda \in \Lambda$.

Let $I \subset A$ be any open ideal of A; then I contains an ideal I_λ, and therefore $\operatorname{Spec} A/I$ is a closed subset of $\operatorname{Spec} A/I_\lambda$, hence also a closed subset of $\mathrm{Spf}\, A$.

8.4.10. We endow $\mathrm{Spf}\, A$ with a sheaf of topological rings as follows. To start out, for every $\lambda \in \Lambda$, the structure sheaf $\mathscr{O}_{X_\lambda}$ carries a natural *pseudo-discrete topology* defined as in [26, Ch.0, §3.8]. Let $j_\lambda : X_\lambda \to \mathrm{Spf}\, A$ be the natural continuous map; then we set :

$$\mathscr{O}_{\mathrm{Spf}\, A} := \lim_{\lambda\in\Lambda} j_{\lambda*}\mathscr{O}_{X_\lambda}$$

where the limit is taken in the category of sheaves of topological rings ([26, Ch.0, §3.2.6]). It follows that :

$$\mathscr{O}_{\mathrm{Spf}\, A}(U) = \lim_{\lambda\in\Lambda} \mathscr{O}_{X_\lambda}(U \cap X_\lambda) \tag{8.4.11}$$

for every open subset $U \subset \mathrm{Spf}\, A$. In this equality, the right-hand side is endowed with the topology of the (projective) limit, and the identification with the left-hand side is a homeomorphism. Set $X := \mathrm{Spf}\, A$; directly from the definitions we get natural maps of locally ringed spaces :

$$(X_\lambda, \mathscr{O}_{X_\lambda}) \xrightarrow{j_\lambda} (X, \mathscr{O}_X) \xrightarrow{i_X} (\operatorname{Spec} A^\wedge, \mathscr{O}_{\operatorname{Spec} A^\wedge}) \qquad \text{for every } \lambda \in \Lambda \tag{8.4.12}$$

where $A^\wedge := \mathscr{O}_X(X)$ is the (separated) completion of A, and i_X is given by the universal property [25, Ch.I, Prop.1.6.3] of the spectrum of a ring. Clearly the composition $i_X \circ j_\lambda$ is the map of affine schemes induced by the surjection $A^\wedge \to A/I_\lambda$. Therefore i_X identifies the set underlying X with the subset $\bigcup_{\lambda\in\Lambda} X_\lambda \subset \operatorname{Spec} A^\wedge$. However, the topology of X is usually strictly finer than the subspace topology on the image of i_X. Let $f \in A^\wedge$ be any element; we let:

$$\mathfrak{D}(f) := i_X^{-1} D(f)$$

where as usual, $D(f) \subset \operatorname{Spec} A^\wedge$ is the open subset consisting of all prime ideals that do not contain f.

8.4.13. Let $f : A \to B$ be a continuous map of topological rings. For every open ideal $J \subset B$, we have an induced map $A/f^{-1}J \to B/J$, and after taking colimits, a natural continuous map of topologically ringed spaces :

$$\mathrm{Spf}\, f : (\mathrm{Spf}\, B, \mathscr{O}_{\mathrm{Spf}\, B}) \to (\mathrm{Spf}\, A, \mathscr{O}_{\mathrm{Spf}\, A}).$$

Lemma 8.4.14. *Let $f : A \to B$ be a continuous map of topological rings. Then :*

(i) $\mathrm{Spf}\, A$ *is a locally and topologically ringed space.*

(ii) $\varphi := \mathrm{Spf}\, f : \mathrm{Spf}\, B \to \mathrm{Spf}\, A$ *is a morphism of locally and topologically ringed spaces; in particular, the induced map on stalks $\mathscr{O}_{\mathrm{Spf}\, A,\varphi(x)} \to \mathscr{O}_{\mathrm{Spf}\, B,x}$ is a local ring homomorphism, for every $x \in \mathrm{Spf}\, B$.*

(iii) *Let $I \subset A$ be any open ideal, $x \in \mathrm{Spec}\, A/I \subset \mathrm{Spf}\, A$ any point. The induced map :*

$$\mathscr{O}_{\mathrm{Spf}\, A,x} \to \mathscr{O}_{\mathrm{Spec}\, A/I,x} \qquad (8.4.15)$$

is a surjection.

Proof. (i): We need to show that the stalk $\mathscr{O}_{X,x}$ is a local ring, for every $x \in X := \mathrm{Spf}\, A$. Let $(I_\lambda \mid \lambda \in \Lambda)$ be a cofiltered fundamental system of open ideals. Then $x \in X_\mu := \mathrm{Spec}\, A/I_\mu$ for some $\mu \in \Lambda$; let $\kappa(x)$ be the residue field of the stalk $\mathscr{O}_{X_\mu,x}$; for every $\lambda \geq \mu$, the closed immersion $X_\mu \to X_\lambda$ induces an isomorphism of $\kappa(x)$ onto the residue field of $\mathscr{O}_{X_\lambda,x}$, whence a natural map

$$\mathscr{O}_{X,x} \to \kappa(x).$$

Suppose now that $g \in \mathscr{O}_{X,x}$ is mapped to a non-zero element in $\kappa(x)$; according to (8.4.11) we may find an open subset $U \subset X$, such that g is represented as a compatible system $(g_\lambda \mid \lambda \geq \mu)$ of sections $g_\lambda \in \mathscr{O}_{X_\lambda}(U \cap X_\lambda)$. For every $\lambda \geq \mu$, let $V_\lambda \subset X_\lambda$ denote the open subset of all $y \in X_\lambda$ such that $g_\lambda(y) \neq 0$ in $\kappa(y)$. Then $V_\eta \cap X_\lambda = V_\lambda$ whenever $\eta \geq \lambda \geq \mu$. It follows that $V := \bigcup_{\lambda \geq \mu} V_\lambda$ is an open subset of X, and clearly g is invertible at every point of V, hence g is invertible in $\mathscr{O}_{X,x}$, which implies the contention.

(ii): The assertion is easily reduced to the corresponding statement for the induced morphisms of schemes : $\mathrm{Spec}\, B/J \to \mathrm{Spec}\, A/f^{-1}J$, for any open ideal $J \subset B$. The details shall be left to the reader.

(iii): Indeed, using (8.4.12) with $X := \mathrm{Spf}\, A$ and $I_\lambda := I$, we see that the natural surjection $\mathscr{O}_{\mathrm{Spec}\, A^\wedge,x} \to \mathscr{O}_{\mathrm{Spec}\, A/I,x}$ factors through (8.4.15). $\square$

8.4.16. Let A and M be as in (8.4.2), and fix a (cofiltered) fundamental system $(M_\lambda \mid \lambda \in \Lambda)$ of open submodules $M_\lambda \subset M$. For every $\lambda \in \Lambda$ we may find an open ideal $I_\lambda \subset A$ such that M/M_λ is an A/I_λ-module. Let $j_\lambda : X_\lambda := \mathrm{Spec}\, A/I_\lambda \to X := \mathrm{Spf}\, A$ be the natural closed immersion (of ringed spaces). We define the topological $\mathscr{O}_X$-module :

$$M^\sim := \lim_{\lambda \in \Lambda} j_{\lambda *}(M/M_\lambda)^\sim$$

where, as usual, $(M/M_\lambda)^\sim$ denotes the quasi-coherent pseudo-discrete $\mathscr{O}_{X_\lambda}$-module associated to M/M_λ, and the limit is formed in the category of sheaves of topological $\mathscr{O}_X$-modules ([26, Ch.0, §3.2.6]). Thus, for every open subset $U \subset X$ one has the identity of topological modules:

$$M^\sim(U) = \lim_{\lambda \in \Lambda} (M/M_\lambda)^\sim(U \cap X_\lambda) \qquad (8.4.17)$$

that generalizes (8.4.11).

To proceed beyond these simple generalities, we need to add further assumptions. The following definition covers all the situations that we shall find in the sequel.

Definition 8.4.18. Let A be a topological ring (whose topology is, as always, linear).

(i) We say that A is *ω-admissible* if A is separated and complete, and $0 \in A$ admits a countable fundamental system of open neighborhoods.

(ii) We say that an open subset $U \subset \mathrm{Spf}\, A$ is *affine* if there exists an ω-admissible topological ring B and an isomorphism $(\mathrm{Spf}\, B, \mathscr{O}_{\mathrm{Spf}\, B}) \xrightarrow{\sim} (U, \mathscr{O}_{\mathrm{Spf}\, A|U})$ of topologically ringed spaces.

(iii) We say that an open subset $U \subset \mathrm{Spf}\, A$ is *truly affine* if $U \cap \mathrm{Spec}\, A/I$ is an affine open subset of $\mathrm{Spec}\, A/I$ for every open ideal $I \subset A$.

(iv) An *affine ω-formal scheme* is a topologically and locally ringed space $(X, \mathscr{O}_X)$ that is isomorphic to the formal spectrum of an ω-admissible topological ring.

(v) An *ω-formal scheme* is a topologically and locally ringed space $(X, \mathscr{O}_X)$ that admits an open covering $X = \bigcup_{i\in I} U_i$ such that, for every $i \in I$, the restriction $(U_i, \mathscr{O}_{X|U_i})$ is an affine ω-formal scheme.

(vi) A *morphism of ω-formal schemes* $f : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is a map of topologically and locally ringed spaces, *i.e.* a morphism of locally ringed spaces such that the corresponding map $\mathscr{O}_Y \to f_*\mathscr{O}_X$ induces continuous ring homomorphisms $\mathscr{O}_Y(U) \to \mathscr{O}_X(f^{-1}U)$, for every open subset $U \subset Y$.

Remark 8.4.19. Let X be any ω-formal scheme, and $U \subset X$ any open subset. Using axiom **(TS)**, it is easily seen that :

(i) $\mathscr{O}_X(U)$ is a complete and separated topological ring.

(ii) If additionally, $U = \bigcup_{i\in I} U_i$ for a countable family $(U_i \mid i \in I)$ of open subsets, such that $(U_i, \mathscr{O}_{X|U_i})$ is an affine ω-formal scheme for every $i \in I$ (briefly, each U_i is an *affine open subset* of X), then $\mathscr{O}_X(U)$ is ω-admissible.

Proposition 8.4.20. *Suppose that A is an ω-admissible topological ring. Then :*

(i) *The truly affine open subsets form a basis for the topology of $X := \mathrm{Spf}\, A$.*

(ii) *Let $I \subset A$ be an open ideal, $U \subset X$ a truly affine open subset. Then :*

  (a) *The natural map $\rho_U : A \to A_U := \mathscr{O}_X(U)$ induces an isomorphism :*

$$(\mathrm{Spf}\, A_U, \mathscr{O}_{\mathrm{Spf}\, A_U}) \xrightarrow{\sim} (U, \mathscr{O}_{X|U}).$$

  (b) *The topological closure I_U of IA_U in A_U is an open ideal, and the natural map : $\mathrm{Spec}\, A_U/I_U \to \mathrm{Spec}\, A/I$ is an open immersion.*

  (c) *U is affine.*

  (d) *Every open covering of U admits a countable subcovering.*

Proof. (i): By assumption, we may find a countable fundamental system $(I_n \mid n \in \mathbb{N})$ of open ideals of A, and clearly we may assume that this system is ordered under inclusion (so that $I_n \subset I_m$ whenever $n \geq m$). Let $x \in \mathrm{Spf}\, A$ be any point, and $U \subset \mathrm{Spf}\, A$ an open neighborhood of x. Then $x \in X_n := \mathrm{Spec}\, A/I_n$ for sufficiently large $n \in \mathbb{N}$. We are going to exhibit, by induction on $m \in \mathbb{N}$, a sequence $(f_m \mid m \geq n)$ of elements of A, such that :

$$x \in \mathfrak{D}(f_p) \cap X_m = \mathfrak{D}(f_m) \cap X_m \subset U \qquad \text{whenever } p \geq m \geq n \tag{8.4.21}$$

To start out, we may find $f_n \in A/I_n$ such that $x \in \mathfrak{D}(f_n) \cap X_n \subset U \cap X_n$. Next, let $m \geq n$ and suppose that f_m has already been found; we may write

$$U \cap X_{m+1} = X_{m+1} \setminus V(J) \qquad \text{for some ideal } J \subset A/I_{m+1}.$$

Let $\overline{J} \subset A/I_m$ and $\overline{f}_m \in A/I_m$ be the images of J and f_m; then $V(\overline{J}) \subset V(\overline{f}_m)$, hence there exists $k \in \mathbb{N}$ such that $\overline{f}{}_m^k \in \overline{J}$. Pick $\overline{f}_{m+1} \in J$ such that the image of $\overline{f}_{m+1}$ in A/I_m agrees with $\overline{f}{}_m^k$, and let $f_{m+1} \in A$ be any lifting of $\overline{f}_{m+1}$; with this choice, one verifies easily that (8.4.21) holds for $p := m + 1$. Finally, the subset :

$$U' := \bigcup_{m\geq n} \mathfrak{D}(f_m) \cap X_m$$

is an admissible affine open neighborhood of x contained in U.

(ii.a): For every $n, m \in \mathbb{N}$ with $n \geq m$, we have a closed immersion of affine schemes: $U \cap X_m \subset U \cap X_n$; whence induced surjections :

$$A_{U,n} := \mathscr{O}_{X_n}(U \cap X_n) \to A_{U,m} := \mathscr{O}_{X_m}(U \cap X_m).$$

By [26, Ch.0, §3.8.1], $A_{U,n}$ is a discrete topological ring, for every $n \in \mathbb{N}$, hence $A_U \simeq \lim_{n\in\mathbb{N}} A_{U,n}$ carries the linear topology that admits the fundamental system of open ideals $(\mathrm{Ker}\,(A_U \to A_{U,n}) \mid n \in \mathbb{N})$, especially A_U is ω-admissible. It follows that the topological space underlying $\mathrm{Spf}\,A_U$ is $\mathop{\mathrm{colim}}_{n\in\mathbb{N}} (U \cap X_n)$, which is naturally identified with U, under $\mathrm{Spf}\,\rho_U$. Likewise, let $i : U \to X$ be the open immersion; one verifies easily (*e.g.* using (8.4.11)) that the natural map :

$$i^* \lim_{n\in\mathbb{N}} j_{n*}\mathscr{O}_{X_n} \to \lim_{n\in\mathbb{N}} i^* j_{n*}\mathscr{O}_{X_n}$$

is an isomorphism of topological sheaves, which implies the assertion.

(ii.b): We may assume that $I_0 = I$. Since $U \cap X_n$ is affine for every $n \in \mathbb{N}$, we deduce short exact sequences :

$$\mathscr{E}_n := (0 \to I \cdot \mathscr{O}_{X_n}(X_n \cap U) \to \mathscr{O}_{X_n}(U \cap X_n) \to \mathscr{O}_{X_0}(U \cap X_0) \to 0)$$

and $\lim_{n\in\mathbb{N}} \mathscr{E}_n$ is the exact sequence :

$$0 \to I_U \to A_U \to \mathscr{O}_{X_0}(U \cap X_0) \to 0.$$

Since $U \cap X_0$ is an open subset of X_0, both assertions follow easily.

(ii.c): It has already been remarked that A_U is ω-admissible, and indeed, the proof of (ii.b) shows that the family of ideals $(\overline{I_n A_U} \mid n \in \mathbb{N})$ is a fundamental system of open neighborhoods of $0 \in A_U$. Hence the assertion follows from (ii.a).

(ii.d): By definition, U is a countable union of quasi-compact subsets, so the assertion is immediate. □

Corollary 8.4.22. *Let X be an ω-formal scheme, $U \subset X$ any open subset. Then $(U, \mathscr{O}_{X|U})$ is an ω-formal scheme.* □

Remark 8.4.23. Let A be an ω-admissible topological ring. Then $\mathrm{Spf}\,A$ may well contain affine subsets that are not truly affine. As an example, consider the one point compactification $X := \mathbb{N} \cup \{\infty\}$ of the discrete topological space $\mathbb{N}$ (this is the space which induces the discrete topology on its subset $\mathbb{N}$, and such that the open neighborhoods of ∞ are the complements of the finite subsets of $\mathbb{N}$). We choose any field F, which we endow with the discrete topology, and let A be the ring of all continuous functions $X \to F$. We endow A with the discrete topology, in which case $\mathrm{Spf}\,A = \mathrm{Spec}\,A$, and one can exhibit a natural homeomorphism $\mathrm{Spec}\,A \xrightarrow{\sim} X$ (exercise for the reader). On the other hand, we have an isomorphism of topological rings :

$$\mathscr{O}_{\mathrm{Spf}\,A}(\mathbb{N}) = \lim_{b\in\mathbb{N}} \mathscr{O}_{\mathrm{Spec}\,A}(\{0, \dots, b\}) \simeq k^{\mathbb{N}}$$

where $k^{\mathbb{N}}$ is endowed with the product topology. A verification that we leave to the reader, shows that the natural injective ring homomorphism $A \to k^{\mathbb{N}}$ induces an isomorphism of ringed spaces

$$\mathrm{Spf}\,k^{\mathbb{N}} \xrightarrow{\sim} (\mathbb{N}, \mathscr{O}_{\mathrm{Spf}\,A|\mathbb{N}})$$

hence $\mathbb{N} \subset \mathrm{Spf}\,A$ is an affine subset. However, $\mathbb{N}$ is not an affine subset of $\mathrm{Spec}\,A$, hence $\mathbb{N}$ is not a truly affine open subset of $\mathrm{Spf}\,A$.

Proposition 8.4.24. *Let X be an ω-formal scheme, A an ω-admissible topological ring. Then the rule :*

$$(f : X \to \mathrm{Spf}\,A) \mapsto (f^\natural : A \to \Gamma(X, \mathscr{O}_X)) \qquad (8.4.25)$$

establishes a natural bijection between the set of morphisms of ω-formal schemes $X \to \mathrm{Spf}\,A$ and the set of continuous ring homomorphisms $A \to \Gamma(X, \mathscr{O}_X)$.

Proof. We reduce easily to the case where $X = \mathrm{Spf}\, B$ for some ω-admissible topological ring B. Let $f : X \to Y := \mathrm{Spf}\, A$ be a morphism of ω-formal schemes; we have to show that $f = \mathrm{Spf}\, f^\natural$, where $f^\natural : A \to B$ is the map on global sections induced by the morphism of sheaves $\mathscr{O}_Y \to f^*\mathscr{O}_X$ that defines f. Let $U \subset X$ and $V \subset Y$ be two truly affine open subsets, such that $f(U) \subset V$, and let likewise :

$$f^\natural_{U,V} : A_V := \mathscr{O}_Y(V) \to B_U := \mathscr{O}_X(U)$$

be the map induced by $f_{|U}$. Using the universal property of [25, Ch.I, Prop.1.6.3], we obtain a commutative diagram of morphisms of locally ringed spaces:

(8.4.26)
$$\begin{tikzcd}
\mathrm{Spec}\, B_U \arrow[rrr, "\mathrm{Spec}\, f^\natural_{U,V}"] \arrow[ddd] & & & \mathrm{Spec}\, A_V \arrow[ddd] \\
& U \arrow[ul, "i_U"'] \arrow[r, "f_{|U}"] \arrow[d] & V \arrow[ur, "i_V"'] \arrow[d] & \\
& X \arrow[r, "f"] \arrow[dl, "i_X"] & Y \arrow[dr, "i_Y"'] & \\
\mathrm{Spec}\, B \arrow[rrr, "\mathrm{Spec}\, f^\natural"] & & & \mathrm{Spec}\, A
\end{tikzcd}$$

where i_X, i_Y, i_U and i_V are the morphisms of (8.4.12). Since i_X and i_Y are injective on the underlying sets, it follows already that f and $\mathrm{Spf}\, f^\natural$ induce the same continuous map of topological spaces. Let now $J \subset B$ be any open ideal, $I \subset f^{\natural-1}(J)$ an open ideal of A, and let J_U (resp. I_V) be the topological closure of JB_U in B_U (resp. of IA_V in A_V). Since the map $f^\natural_{U,V}$ is continuous, we derive a commutative diagram of schemes :

$$\begin{tikzcd}
\mathrm{Spec}\, B_U/J_U \arrow[r, "\varphi"] \arrow[d, "\alpha"'] & \mathrm{Spec}\, A_V/I_V \arrow[d, "\beta"] \\
\mathrm{Spec}\, B/J \arrow[r, "\psi"] & \mathrm{Spec}\, A/I
\end{tikzcd}$$

where α and β are open immersions, by proposition 8.4.20(ii.b), and φ (resp. ψ) is induced by $f^\natural_{U,V}$ (resp. by $f^\natural$). Let $\mathscr{I}$ be a fundamental system of open neighborhoods of $0 \in A$ consisting of ideals; from proposition 8.4.20(ii.b) (and its proof) it follows as well that $A_V \simeq \lim_{I\in\mathscr{I}} A_V/\overline{IA_V}$. Summing up, this shows that $f^\natural_{U,V}$ is determined by $f^\natural$, whence the contention. □

Corollary 8.4.27. *Let A be an ω-admissible topological ring, $U \subset \mathrm{Spf}\, A$ an affine open subset, and V a truly affine open subset of X with $V \subset U$. Then V is a truly affine open subset of U.*

Proof. Say that $U = \mathrm{Spf}\, B$, for some ω-admissible topological ring B; by proposition 8.4.24, the immersion $j : U \to X$ is of the form $j = \mathrm{Spf}\, \varphi$ for a unique map $\varphi : A \to B$. Let now $J \subset B$ be any open ideal, and set $I := \varphi^{-1}J$; there follows a commutative diagram of locally ringed spaces :

$$\begin{tikzcd}
U_0 := \mathrm{Spec}\, B/J \arrow[r] \arrow[d] & X_0 := \mathrm{Spec}\, A/I \arrow[d] \\
U \arrow[r, "j"] & X.
\end{tikzcd}$$

By assumption, $V \cap X_0 = \mathrm{Spec}\, C$ for some A/I-algebra C; it follows that :

$$V \cap U_0 = \mathrm{Spec}\, B/J \otimes_{A/I} C$$

and since J is arbitrary, the claim follows. □

8.4.28. Let A be an ω-admissible topological ring, and B, C two topological A-algebras (so the topologies of B and C are linear, and the structure maps $A \to B$, $A \to C$ are continuous). We denote by $B\widehat{\otimes}_A C$ the completed tensor product of B and C, defined as in [26, Ch.0, §7.7.5]. Then $B\widehat{\otimes}_A C$ is the coproduct of the A-algebras B and C, in the category of topological A-algebras ([26, Ch.0, §7.7.6]). By standard arguments, we deduce that the category of ω-formal schemes admits arbitrary fibre products.

8.4.29. We shall say that a topological A-module M is *ω-admissible* if M is complete and separated, and admits a countable fundamental system of open neighborhoods of $0 \in M$. Notice that the completion functor $N \mapsto N^\wedge$ on topological A-modules is not always "right exact", in the following sense. Suppose $N \to N'$ is a quotient map; then the induced map $N^\wedge \to N'^\wedge$ is not necessarily onto. However, this is the case if $N^\wedge$ is ω-admissible (see [61, Th.8.1]). For such A-modules, we have moreover the following :

Lemma 8.4.30. *Let A be a topological ring, M, M', N three ω-admissible topological A-modules, and $f : M \to M'$ a quotient map. Then :*

(i) *The homomorphism $f \,\widehat{\otimes}_A \mathbf{1}_N : M \,\widehat{\otimes}_A N \to M' \,\widehat{\otimes}_A N$ is a quotient map.*

(ii) *Let us endow $\mathrm{Ker}\, f$ with the subspace topology induced from M, and suppose additionally that, for every open ideal $I \subset A$:*

(a) *$\overline{IN}$ is an open submodule of N.*

(b) *$N/\overline{IN}$ is a flat A/I-module.*

Then the complex :

$$0 \to (\mathrm{Ker}\, f) \,\widehat{\otimes}_A N \to M \,\widehat{\otimes}_A N \to M' \,\widehat{\otimes}_A N \to 0$$

is an admissible short exact sequence of topological $A^\wedge$-modules.

Proof. By assumption, we may find an inverse system of discrete A-modules, with surjective transition maps $(M_n \mid n \in \mathbb{N})$ (resp. $(N_n \mid n \in \mathbb{N})$), and an isomorphism of topological A-modules : $M \simeq \lim_{n\in\mathbb{N}} M_n$ (resp. $N \simeq \lim_{n\in\mathbb{N}} N_n$); let us define $M'_n := M' \amalg_M M_n$ for every $n \in \mathbb{N}$ (the cofibred sum of M' and M_n over M). Since f is a quotient map, M'_n is a discrete A-module for every $n \in \mathbb{N}$, and the natural map : $M' \to \lim_{n\in\mathbb{N}} M'_n$ is a topological isomorphism. We deduce an inverse system of short exact sequences of discrete A-modules :

$$0 \to K_n \to M_n \otimes_A N_n \to M'_n \otimes_A N_n \to 0$$

where K_n is naturally a quotient of $\mathrm{Ker}(M_n \to M'_n) \otimes_A N_n$, for every $n \in \mathbb{N}$; especially, the transition maps $K_j \to K_i$ are surjective whenever $j \geq i$. Then assertion (i) follows from lemma 8.4.5.

(ii): We may find open ideals $I_n \subset A$ such that $I_n N_n = I_n M_n = 0$ for every $n \in \mathbb{N}$, hence $\overline{I_n N} \subset \mathrm{Ker}(N \to N_n)$, and from (a) we deduce that $(\overline{I_n N} \mid n \in \mathbb{N})$ is a fundamental system of open neighborhoods of $0 \in N$. Due to (b), we obtain short exact sequences :

$$0 \to \mathrm{Ker}(M_n \to M'_n) \otimes_A N/\overline{I_n N} \to M_n \otimes_A N/\overline{I_n N} \to M'_n \otimes_A N/\overline{I_n N} \to 0$$

for every $n \in \mathbb{N}$. Then it suffices to invoke again lemma 8.4.5. $\square$

Lemma 8.4.31. *Let A be an ω-admissible topological ring, M a topological A-module, and $U \subset X := \mathrm{Spf}\, A$ any affine open subset. Then the following holds :*

(i) *There is a natural isomorphism of topological $\mathscr{O}_X(U)$-modules :*

$$M^\sim(U) \simeq M\widehat{\otimes}_A \mathscr{O}_X(U).$$

(ii) *If L is any other topological A-module, the natural map :*

$$\mathrm{top.Hom}_A(L, M^\wedge) \to \mathrm{top.Hom}_{\mathscr{O}_X}(L^\sim, M^\sim) \quad \varphi \mapsto \varphi^\sim$$

is an isomorphism.

(iii) *The functor $M \mapsto M^\sim$ on topological A-modules, is left adjoint to the global sections functor $\mathscr{F} \mapsto \mathscr{F}(X)$, defined on the category of complete and separated topological $\mathscr{O}_X$-modules and continuous maps.*

Proof. (i): First we remark that the assertion holds whenever U is a truly affine open subset of X : the easy verification shall be left to the reader. For a general U, set $M_U := M\widehat{\otimes}_A\mathscr{O}_X(U)$, and denote by $M_U^\sim$ the associated $\mathscr{O}_U$-module. Let $V \subset U$ be any open subset which is truly affine in X; by corollary 8.4.27, V is truly affine in U as well. We deduce natural isomorphisms:

$$M^\sim(V) \simeq M_U\widehat{\otimes}_{\mathscr{O}_X(U)}\mathscr{O}_X(V) \simeq M_U^\sim(V)$$

which – in view of proposition 8.4.20(i) – amount to a natural isomorphism of $\mathscr{O}_U$-modules : $(M^\sim)_{|U} \xrightarrow{\sim} M_U^\sim$. Assertion (i) follows easily.

(iii): Given a continuous map $M^\sim \to \mathscr{F}$, we get a map of global sections $M = M^\sim(X) \to \mathscr{F}(X)$. Conversely, suppose $f : M \to \mathscr{F}(X)$ is a given continuous map; let $U \subset X$ be any affine open subset, and $f_U : M \to \mathscr{F}(U)$ the composition of f and the restriction map $\mathscr{F}(X) \to \mathscr{F}(U)$. Then f_U extends first – by linearity – to a map $M \otimes_A \mathscr{O}_X(U) \to \mathscr{F}(U)$, and second – by continuity – to a map $M\widehat{\otimes}_A\mathscr{O}_X(U) \to \mathscr{F}(U)$; the latter, in view of (i), is the same as a map $f_U^\sim : M^\sim(U) \to \mathscr{F}(U)$. Clearly the rule $U \mapsto f_U^\sim$ thus defined is functorial for inclusion of open subsets $U \subset U'$, whence (iii).

(ii) is a straightforward consequence of (iii). □

Lemma 8.4.32. *Let A be an ω-admissible topological ring, M an ω-admissible topological A-module, $N \subset M$ any submodule, and $U \subset X := \mathrm{Spf}\, A$ any affine open subset. Then :*

(i) *If we endow M/N with the quotient topology, the sequence of $\mathscr{O}_X$-modules :*

$$0 \to N^\sim \to M^\sim \to (M/N)^\sim \to 0$$

is short exact.

(ii) *The induced sequence*

$$0 \to N^\sim(U) \to M^\sim(U) \to (M/N)^\sim(U) \to 0$$

is short exact and admissible in the sense of (8.4.4).

Proof. (i): We recall the following :

Claim 8.4.33. Let $j : Z' := \mathrm{Spec}\, R' \to Z := \mathrm{Spec}\, R$ be an open immersion of affine schemes. Then R' is a flat R-algebra.

Proof of the claim. The assertion can be checked on the localizations at the prime ideals of R'; however, the induced maps $\mathscr{O}_{Z,j(z)} \to \mathscr{O}_{Z',z}$ are isomorphisms for every $z \in Z'$, so the claim is clear. ◊

Let now $V \subset X$ be any truly affine open subset. It follows easily from proposition 8.4.20(ii.b) and claim 8.4.33 that $\mathscr{O}_X(V)$ fulfills conditions (a) and (b) of lemma 8.4.30(ii). In light of lemma 8.4.31(i), we deduce that the sequence :

$$0 \to N^\sim(V) \to M^\sim(V) \to (M/N)^\sim(V) \to 0$$

is admissible short exact, whence the contention.

(ii): Clearly the sequence is left exact; using (ii) and lemma 8.4.30(i) we see that it is also right exact, and moreover $M^\sim(U) \to (M/N)^\sim(U)$ is a quotient map. It remains only to show that the topology on $N^\sim(U)$ is induced from $M^\sim(U)$, and to this aim we may assume – thanks to axiom **(TS)** of (8.4.7) – that U is a truly affine open subset of X, in which case the assertion has already been observed in the proof of (i). □

Definition 8.4.34. Let X be an ω-formal scheme. We say that a topological $\mathscr{O}_X$-module $\mathscr{F}$ is *quasi-coherent* if there exists a covering $\mathfrak{U} := (U_i \mid i \in I)$ of X consisting of affine open subsets, and for every $i \in I$, an ω-admissible topological $\mathscr{O}_X(U_i)$-module M_i with an isomorphism $\mathscr{F}_{|U_i} \simeq M_i^\sim$ of topological $\mathscr{O}_{U_i}$-modules (in the sense of (8.4.7)).

We denote by $\mathscr{O}_X\text{-}\mathbf{Mod}_{\mathrm{qcoh}}$ the category of quasi-coherent $\mathscr{O}_X$-modules and continuous $\mathscr{O}_X$-linear morphisms.

We say that a continuous morphism $f : \mathscr{F} \to \mathscr{G}$ of quasi-coherent $\mathscr{O}_X$-modules is a *quotient map* if the induced map $f(U) : \mathscr{F}(U) \to \mathscr{G}(U)$ is a quotient map of topological $\mathscr{O}_X(U)$-modules, for every affine open subset $U \subset X$.

8.4.35. The category $\mathscr{O}_X\text{-}\mathbf{Mod}_{\mathrm{qcoh}}$ is usually not abelian (see (8.4.4)); more than that, the kernel (in the category of abelian sheaves) of a continuous map $f : \mathscr{F} \to \mathscr{G}$ of quasi-coherent $\mathscr{O}_X$-modules may fail to be quasi-coherent. However, using lemma 8.4.32 one may show that $\mathrm{Ker}\, f$ is quasi-coherent whenever f is a quotient map, and in this case $\mathrm{Ker}\, f$ is also the kernel of f in the category $\mathscr{O}_X\text{-}\mathbf{Mod}_{\mathrm{qcoh}}$. Furthermore, any (continuous) morphism f in $\mathscr{O}_X\text{-}\mathbf{Mod}_{\mathrm{qcoh}}$ admits a cokernel. This can be exhibited as follows. To start out, let us define presheaves $\mathscr{I}$ and $\mathscr{L}$ by declaring that $\mathscr{I}(U) \subset \mathscr{G}(U)$ is the topological closure of the $\mathscr{O}_X(U)$-submodule $\mathrm{Im}(f(U) : \mathscr{F}(U) \to \mathscr{G}(U))$, and $\mathscr{L}(U) := \mathscr{G}(U)/\mathscr{I}(U)$, which we endow with its natural quotient topology, for every open subset $U \subset X$. Now, suppose that $V \subset U$ is an inclusion of sufficiently small affine open subsets of X (so that $\mathscr{F}_{|U}$ and $\mathscr{G}_{|U}$ are of the form $M^\sim$ for some topological $\mathscr{O}_X(U)$-module M); since $\mathscr{F}(U)\widehat{\otimes}_{\mathscr{O}_X(U)}\mathscr{O}_X(V) = \mathscr{F}(V)$, we see that the image of $\mathscr{I}(U)$ in $\mathscr{I}(V)$ generates a dense $\mathscr{O}_X(V)$-submodule. On the other hand, by construction the exact sequence $\mathscr{E} := (0 \to \mathscr{I}(U) \to \mathscr{G}(U) \to \mathscr{L}(U) \to 0)$ is admissible, hence the same holds for the sequence $\mathscr{E}\widehat{\otimes}_{\mathscr{O}_X(U)}\mathscr{O}_X(V)$ (lemmata 8.4.31(i) and 8.4.32(ii)). It follows that $\mathscr{I}(V) = \mathscr{I}(U)\widehat{\otimes}_{\mathscr{O}_X(U)}\mathscr{O}_X(V)$ and $\mathscr{L}(V) = \mathscr{L}(U)\widehat{\otimes}_{\mathscr{O}_X(U)}\mathscr{O}_X(V)$. Thus $\mathscr{I}$ and $\mathscr{L}$ are sheaves of topological $\mathscr{O}_X$-modules on the site C of all sufficiently small affine open subsets of X, and their sheafifications $\mathscr{I}'$ and $\mathscr{L}'$ are the topological $\mathscr{O}_X$-modules obtained as in [26, Ch.0, §3.2.1], by extension of $\mathscr{I}$ and $\mathscr{L}$ from the site C to the whole topology of X. It follow that $\mathscr{I}'$ and $\mathscr{L}'$ are quasi-coherent $\mathscr{O}_X$-modules; then it is easy to check that $\mathscr{L}'$ is the cokernel of f in the category $\mathscr{O}_X\text{-}\mathbf{Mod}_{\mathrm{qcoh}}$ (briefly : the *topological cokernel of* f), and shall be denoted $\mathrm{top.Coker}\, f$. The sheaf $\mathscr{I}'$ shall be called *the topological closure of the image of* f, and denoted $\overline{\mathrm{Im}}(f)$. A morphism f such that $\mathrm{top.Coker}\, f = 0$ shall be called a *topological epimorphism*. Notice also that the natural map $\mathrm{Coker}\, f \to \mathrm{top.Coker}\, f$ is an epimorphism of abelian sheaves, hence a continuous morphism of quasi-coherent $\mathscr{O}_X$-modules which is an epimorphism of $\mathscr{O}_X$-modules, is also a topological epimorphism.

Proposition 8.4.36. *Let X be an affine ω-formal scheme, $\mathscr{F}$ a quasi-coherent $\mathscr{O}_X$-module. Then $\mathscr{F}(X)$ is an ω-admissible $\mathscr{O}_X(X)$-module, and the natural map*

$$\mathscr{F}(X)^\sim \to \mathscr{F}$$

is an isomorphism of topological $\mathscr{O}_X$-modules.

Proof. By assumption we may find an affine open covering $\mathfrak{U} := (U_i \mid i \in I)$ of X such that, for every $i \in I$, $\mathscr{F}_{|U_i} \simeq M_i^\sim$ for some $\mathscr{O}_X(U_i)$-module M_i. In view of lemma 8.4.31(i) we may assume – up to replacing $\mathfrak{U}$ by a refinement – that U_i is a truly affine subset of X for every $i \in I$. Furthermore, we may write $X = \bigcup_{n\in\mathbb{N}} X_n$ for an increasing countable family of quasi-compact subsets; for each $n \in \mathbb{N}$ we may then find a finite subset $I(n) \subset I$ such that $X_n \subset \bigcup_{i\in I(n)} U_i$, and therefore we may replace I by $\bigcup_{n\in\mathbb{N}} I_n$, which allows to assume that I is countable.

Next, for every $i \in I$ we may find a countable fundamental system of open submodules $(M_{i,n} \mid n \in \mathbb{N})$ of M_i, and for every $n \in \mathbb{N}$ an open ideal $J_{i,n} \subset A_i := \mathscr{O}_X(U_i)$ such that $N_{i,n} := M_i/M_{i,n}$ is an $A_i/J_{i,n}$-module. Let $j_{i,n} : U_{i,n} := \mathrm{Spec}\, A_i/J_{i,n} \to U_i$ be the natural closed immersion; we may write :

$$\mathscr{F}_{|U_i} \simeq \lim_{n\in\mathbb{N}} j_{i,n*}N^{\sim}_{i,n}. \tag{8.4.37}$$

Let also $\iota_{i,n} : U_{i,n} \to X$ be the locally closed immersion obtained as the composition of $j_{i,n}$ and the open immersion $j_i : U_i \to X$; we deduce natural maps of $\mathscr{O}_X$-modules :

$$\varphi_{i,n} : \mathscr{F} \to j_{i*}\mathscr{F}_{|U_i} \to \mathscr{G}_{i,n} := \iota_{i,n*}N^{\sim}_{i,n}.$$

Claim 8.4.38. (i) There exists $m \in \mathbb{N}$ such that $\mathscr{G}_{i,n}$ is the extension by zero of a quasi-coherent $\mathscr{O}_{X_m}$-module.

(ii) $\varphi_{i,n}$ is continuous for the pseudo-discrete topology on $\mathscr{G}_{i,n}$.

Proof of the claim. (i): It is easy to see that $U_{i,n} \subset X_m$ for $m \in \mathbb{N}$ large enough; then the assertion follows from ([26, Ch.I, Cor.9.2.2]).

(ii): We need to check that the map $\varphi_{i,n,V} : \mathscr{F}(V) \to \mathscr{G}_{i,n}(V)$ is continuous for every open subset $V \subset X$. However, property **(TS)** of (8.4.7) implies that the assertion is local on X, hence we may assume that V is a truly affine open subset of X, in which case $\mathscr{G}_{i,n}(V)$ is a discrete space. We may factor $\varphi_{i,n,V}$ as a composition :

$$\mathscr{F}(V) \xrightarrow{\alpha} \mathscr{F}(V \cap U_i) \xrightarrow{\beta} N^{\sim}_{i,n}(U_{i,n} \cap V)$$

where the restriction map α is continuous, and β is continuous for the pseudo-discrete topology on $N^{\sim}_{i,n}(U_{i,n} \cap V)$. However, $U_i \cap V$ is a truly affine open subset of U_i; therefore $U_{i,n} \cap V$ is quasi-compact, and the pseudo-discrete topology on $N^{\sim}_{i,n}$ induces the discrete topology on $N^{\sim}_{i,n}(U_{i,n} \cap V)$. The claim follows. $\Diamond$

For every finite subset $S \subset I \times \mathbb{N}$, let $\varphi_S : \mathscr{F} \to \mathscr{G}_S := \prod_{(i,n)\in S} \mathscr{G}_{i,n}$ be the product of the maps $\varphi_{i,n}$; according to claim 8.4.38(ii), φ_S is continuous for the pseudo-discrete topology on $\mathscr{G}_S$. Hence, for every $i \in I$, the restriction $\varphi_{S|U_i} : \mathscr{F}_{|U_i} \to \mathscr{G}_{S|U_i}$ is of the form $f^{\sim}_{i,S}$, for some continuous map $f_{i,S} : M_i \to \mathscr{G}_S(U_i)$ (lemma 8.4.31(ii)). It follows easily that $(\mathrm{Im}\,\varphi_S)_{|U_i}$ is the quasi-coherent $\mathscr{O}_{U_i}$-module $(\mathrm{Im}\,f_{i,S})^{\sim}$, especially $\mathrm{Im}\,\varphi_S$ is quasi-coherent. Notice that $\mathscr{G}_S$ is already a quasi-coherent $\mathscr{O}_{X_m}$-module for $m \in \mathbb{N}$ large enough, hence the same holds for $\mathrm{Im}\,\varphi_S$; we may therefore find an $\mathscr{O}_{X_m}(X_m)$-module G_S such that $G^{\sim}_S \simeq \mathrm{Im}\,\varphi_S$. Furthermore, for every other finite subset $S' \subset I \times \mathbb{N}$ containing S, we deduce a natural $\mathscr{O}_X(X)$-linear surjection $G_{S'} \to G_S$, compatible with compositions of inclusions $S \subset S' \subset S''$. Let $(S_n \mid n \in \mathbb{N})$ be a countable increasing family of finite subsets, whose union is $I \times \mathbb{N}$; we endow $G := \lim_{n\in\mathbb{N}} G_{S_n}$ with the pro-discrete topology. It is then easy to check that G is an ω-admissible $\mathscr{O}_X(X)$-module; furthermore, by construction we get a unique continuous map $\varphi : \mathscr{F} \to G^{\sim}$ whose composition with the projection onto $G^{\sim}_{S_n}$ agrees with φ_{S_n} for every $n \in \mathbb{N}$. In light of (8.4.37) we see easily that $\varphi_{|U_i}$ is an isomorphism of topological $\mathscr{O}_{U_i}$-modules for every $i \in I$, hence the same holds for φ. Then φ necessarily induces an isomorphism $\mathscr{F}(X) \xrightarrow{\sim} G$. $\square$

Corollary 8.4.39. *Let X be an ω-formal scheme, $(\mathscr{F}_n \mid n \in \mathbb{N})$ an inverse system of quasi-coherent $\mathscr{O}_X$-modules, whose transition maps are topological epimorphisms. Then $\lim_{n\in\mathbb{N}} \mathscr{F}_n$ (with its inverse limit topology) is a quasi-coherent $\mathscr{O}_X$-module.*

Proof. We may assume that X is affine; then, for every $n \in \mathbb{N}$ we have $\mathscr{F}_n = M^{\sim}_n$ for some complete and separated $\mathscr{O}_X(X)$-module M_n (by proposition 8.4.36), and the transition maps $\mathscr{F}_{n+1} \to \mathscr{F}_n$ come from corresponding continuous linear maps $f_n : M_{n+1} \to M_n$. We choose inductively, for every $n \in \mathbb{N}$, a descending fundamental system of open submodules $(N_{n,k} \mid k \in \mathbb{N})$ of M_n, such that $f_n(N_{n+1,k}) \subset N_{n,k}$ for every $n, k \in \mathbb{N}$. Set $M_{n,k} := M_n/N_{n,k}$ for every $n, k \in \mathbb{N}$. By assumption, $\mathrm{top.Coker}\, f^{\sim}_n = 0$; hence the induced maps $M^{\sim}_{n+1,k} \to M^{\sim}_{n,k}$ are topological epimorphisms; since $M^{\sim}_{n,k}$ is pseudo-discrete (on the closure of its support), it follows that the latter maps are even epimorphisms (of abelian sheaves) so the corresponding maps $M_{n+1,k} \to M_{n,k}$ are onto for every $n, k \in \mathbb{N}$. Thus :

$$\lim_{n\in\mathbb{N}} \mathscr{F}_n \simeq \lim_{n\in\mathbb{N}} \lim_{k\in\mathbb{N}} M_{n,k}^{\sim} \simeq \lim_{n\in\mathbb{N}} M_{n,n}^{\sim} \simeq (\lim_{n\in\mathbb{N}} M_{n,n})^{\sim}$$

and it is clear that the $\mathscr{O}_X(X)$-module $\lim_{n\in\mathbb{N}} M_{n,n}$ is ω-admissible. □

8.4.40. Let X be any ω-formal scheme, $\mathfrak{U} := (U_i \mid i \in I)$ a covering of X by open subsets, and $\mathscr{F}$ any abelian sheaf on X. As usual we can form the Čech complex $C^\bullet(\mathfrak{U}, \mathscr{F})$, and its augmented version $C^\bullet_{\mathrm{aug}}(\mathfrak{U}, \mathscr{F})$, which are the cochain complexes associated to the cosimplicial complexes of (5.1.6). We denote by $H^\bullet(\mathfrak{U}, \mathscr{F})$ the cohomology of $C^\bullet(\mathfrak{U}, \mathscr{F})$.

Theorem 8.4.41. *Let X be a* semi-separated *ω-formal scheme,* i.e. *such that $U \cap V$ is an affine ω-formal scheme whenever U and V are open affine ω-formal subschemes of X. Then :*

(i) *For every covering $\mathfrak{U} := (U_i \mid i \in I)$ of X consisting of affine open subschemes, and every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, there is a natural isomorphism :*

$$H^\bullet(\mathfrak{U}, \mathscr{F}) \xrightarrow{\sim} H^\bullet(X, \mathscr{F}).$$

(ii) *Suppose furthermore that X is affine, say $X = \mathrm{Spf}\, A$ for an ω-admissible topological ring A. Then :*

$$H^i(X, \mathscr{F}) = 0 \qquad \textit{for every } i > 0.$$

Proof. (The statements refer to the cohomology of the abelian sheaf underlying $\mathscr{F}$, in other words, we forget the topology of the modules $\mathscr{F}(U)$.)

(ii): Let $U \subset X$ be any truly affine open subset, and $\mathfrak{U} := (U_i \mid i \in I)$ a covering of U consisting of truly affine open subsets. By proposition 8.4.36 we have $\mathscr{F}_{|U} \simeq M^{\sim}$, where $M := \mathscr{F}(U)$.

Claim 8.4.42. The augmented complex $C^\bullet_{\mathrm{aug}}(\mathfrak{U}, M^{\sim})$ is acyclic.

Proof of the claim. Let $(M_n \mid n \in \mathbb{N})$ be a fundamental system of neighborhoods of $0 \in M$, consisting of open submodules, and for every $n \in \mathbb{N}$, choose an open ideal $I_n \subset A$ such that $I_n M \subset M_n$. Set $X_n := \mathrm{Spec}\, A/I_n$ and $\mathfrak{U}_n := (U_i \cap X_n \mid i \in I)$ for every $n \in \mathbb{N}$. For every $n \in \mathbb{N}$ we may consider the augmented Cech complex $C^\bullet_{\mathrm{aug}}(\mathfrak{U}_n, (M/M_n)^{\sim})$, and in view of (8.4.17) we obtain a natural isomorphism of complexes :

$$C^\bullet_{\mathrm{aug}}(\mathfrak{U}, M^{\sim}) \xrightarrow{\sim} \lim_{n\in\mathbb{N}} C^\bullet_{\mathrm{aug}}(\mathfrak{U}_n, (M/M_n)^{\sim}).$$

We may view the double complex $C^\bullet_{\mathrm{aug}}(\mathfrak{U}_\bullet, (M/M_\bullet)^{\sim})$ also as a complex of inverse systems of modules, whose term in degree $i \in \mathbb{N}$ is $C^i_{\mathrm{aug}}(\mathfrak{U}_\bullet, (M/M_\bullet)^{\sim})$. Notice that, for every $i \in \mathbb{N}$, all the transition maps of this latter inverse system are surjective. Hence ([75, Lemma 3.5.3]) :

$$\lim_{n\in\mathbb{N}}{}^q\, C^i_{\mathrm{aug}}(\mathfrak{U}_n, (M/M_n)^{\sim}) = 0 \qquad \text{for every } q > 0.$$

In other words, these inverse systems are acyclic for the inverse limit functor. It follows (*e.g.* by means of [75, Th.10.5.9]) that :

$$R\lim_{n\in\mathbb{N}} C^\bullet_{\mathrm{aug}}(\mathfrak{U}_\bullet, (M/M_\bullet)^{\sim}) \simeq C^\bullet_{\mathrm{aug}}(\mathfrak{U}, M^{\sim}).$$

On the other hand, the complexes $C^\bullet_{\mathrm{aug}}(\mathfrak{U}_n, (M/M_n)^{\sim})$ are acyclic for every $n \in \mathbb{N}$ ([28, Ch.III, Th.1.3.1 and Prop.1.4.1]), hence $C^\bullet_{\mathrm{aug}}(\mathfrak{U}_\bullet, (M/M_\bullet)^{\sim})$ is acyclic, when viewed as a cochain complex of inverse systems of modules. The claim follows. ◊

Assertion (ii) follows from claim 8.4.42, proposition 8.4.20(i) and [39, Th.5.9.2].

Assertion (i) follows from (ii) and Leray's theorem [39, Ch.II, §5.4, Cor.]. □

Corollary 8.4.43. *In the situation of* (8.4.16), *suppose that* A *and* M *are* ω*-admissible. Then :*

$$\lim_{\lambda\in\Lambda}{}^q\, j_{\lambda*}(M/M_\lambda)^\sim = 0 \qquad \textit{for every } q>0.$$

Proof. For every truly affine open subset $U \subset X$, we have a topos U^Λ defined as in [36, §7.3.4], and the cofiltered system $\mathscr{M} := (j_{\lambda*}(M/M_\lambda)^\sim \mid \lambda \in \Lambda)$ defines an abelian sheaf on U^Λ. According to *loc.cit.* there are two spectral sequences :

$$E_2^{pq} := R^p\Gamma(U, \lim_{\lambda\in\Lambda}{}^q\, j_{\lambda*}(M/M_\lambda)^\sim) \Rightarrow H^{p+q}(U^\Lambda, \mathscr{M})$$
$$F_2^{pq} := \lim_{\lambda\in\Lambda}{}^q\, R^p\Gamma(U, j_{\lambda*}(M/M_\lambda)^\sim) \Rightarrow H^{p+q}(U^\Lambda, \mathscr{M})$$

and we notice that $F_2^{pq} = 0$ whenever $p > 0$ (since $U \cap X_\lambda$ is affine for every $\lambda \in \Lambda$) and whenever $q > 0$, since the cofiltered system $(\Gamma(U, j_{\lambda*}(M/M_\lambda)^\sim) \mid \lambda \in \Lambda)$ has surjective transition maps. One can then argue as in the proof of [36, Lemma 7.3.5] : the sheaf $L^q := \lim_{\lambda\in\Lambda}{}^q\, j_{\lambda*}(M/M_\lambda)^\sim$ is the sheafification of the presheaf : $U \mapsto H^q(U^\Lambda, \mathscr{M})$ and the latter vanishes by the foregoing. We supply an alternative argument. Since the truly affine open subsets form a basis of X, it suffices to show that $E_2^{0q} = 0$ whenever $q > 0$. We proceed by induction on q. For $q = 1$, we look at the differential $d_2^{01} : E_2^{0,1} \to E_2^{2,0}$; by theorem 8.4.41 we have $E_2^{2,0} = 0$, hence $E_2^{0,1} = E_\infty^{0,1}$, and the latter vanishes by the foregoing. Next, suppose that $q > 1$, and that we have shown the vanishing of L^j for $1 \leq j < q$. It follows that $E_2^{pj} = 0$ whenever $1 \leq j < q$, hence $E_r^{pj} = 0$ for every $r \geq 2$ and the same values of j. Consequently :

$$0 = E_\infty^{0q} \simeq \mathrm{Ker}(d_{q+1}^{0q} : E_2^{0q} = E_{q+1}^{0q} \to E_{q+1}^{q+1,0}).$$

However, theorem 8.4.41 implies that $E_r^{p0} = 0$ whenever $p > 0$ and $r \geq 2$, therefore $E_2^{0q} = E_\infty^{0q}$, *i.e.* $E_2^{0q} = 0$, which completes the inductive step. ☐

8.4.44. Let X be an ω-formal scheme, $\mathscr{F}$ a quasi-coherent $\mathscr{O}_X$-module. For every affine open subset $U \subset X$, we let $\mathrm{Cl}_{\mathscr{F}}(U)$ be the set consisting of all closed $\mathscr{O}_X(U)$-submodules of $\mathscr{F}(U)$. It follows from lemma 8.4.32(ii) that the rule

$$U \mapsto \mathrm{Cl}_{\mathscr{F}}(U)$$

defines a presheaf on the site of all affine open subsets of X. Namely, for an inclusion $U' \subset U$ of affine open subsets, the restriction map $\mathrm{Cl}_{\mathscr{F}}(U) \to \mathrm{Cl}_{\mathscr{F}}(U')$ assigns to $N \subset \mathscr{F}(U)$ the submodule $N^\sim(U') \subset \mathscr{F}(U')$ (here $N^\sim$ is a quasi-coherent $\mathscr{O}_U$-module).

Proposition 8.4.45. *With the notation of* (8.4.44), *the presheaf* $\mathrm{Cl}_{\mathscr{F}}$ *is a sheaf on the site of affine open subsets of* X.

Proof. Let $U \subset X$ be an affine open subset, and $U = \bigcup_{i\in I} U_i$ a covering of U by affine open subsets $U_i \subset X$. For every $i, j \in I$ we let $U_{ij} := U_i \cap U_j$. Suppose there is given, for every $i \in I$, a closed $\mathscr{O}_X(U_i)$-submodule $N_i \subset \mathscr{F}(U_i)$, with the property that :

$$N_{ij} := N_i^\sim(U_{ij}) = N_j^\sim(U_{ij}) \qquad \text{for every } i,j\in I$$

(an equality of topological $\mathscr{O}_X(U_{ij})$-submodules of $\mathscr{F}(U_{ij})$). Then $(N_i^\sim)_{|U_{ij}} = (N_j^\sim)_{|U_{ij}}$, by proposition 8.4.36, hence there exists a quasi-coherent $\mathscr{O}_U$-module $\mathscr{N}$, and isomorphisms $\mathscr{N}_{|U_i} \xrightarrow{\sim} N_i^\sim$, for every $i \in I$, such that the induced $\mathscr{O}_{U_i}$-linear maps $N_i^\sim \to \mathscr{F}_{|U_i}$ assemble into a continuous $\mathscr{O}_U$-linear morphism $\varphi : \mathscr{N} \to \mathscr{F}_{|U}$. By construction, we have a commutative diagram of continuous maps with exact rows :

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{N}(U) \arrow[r, "\rho_{\mathscr{N}}"] \arrow[d] & \prod_{i\in I} N_i \arrow[r] \arrow[d] & \prod_{i,j\in I} N_{ij} \arrow[d] \\ 0 \arrow[r] & \mathscr{F}(U) \arrow[r, "\rho_{\mathscr{F}}"] & \prod_{i\in I} \mathscr{F}(U_i) \arrow[r] & \prod_{i,j\in I} \mathscr{F}(U_{ij}) \end{tikzcd}$$

where the central vertical arrow is a closed imbedding. However, axiom **(TS)** implies that both $\rho_{\mathscr{N}}$ and $\rho_{\mathscr{F}}$ are admissible monomorphisms (in the sense of (8.4.4)), and moreover the image of $\rho_{\mathscr{N}}$ is a closed submodule, since each N_{ij} is a separated module. Hence also the left-most vertical arrow is a closed imbedding, and the assertion follows. □

Definition 8.4.46. Let $f : X \to Y$ be a morphism of ω-formal schemes. We say that f is *affine* (resp. a *closed immersion*) if there exists a covering $Y = \bigcup_{i\in I} U_i$ by affine open subsets, such that for every $i \in I$, the open subset $f^{-1}U_i \subset X$ is an affine ω-formal scheme (resp. is isomorphic, as a U_i-scheme, to a ω-formal scheme of the form $\mathrm{Spf}\, A_i/J_i$, where $A_i := \mathscr{O}_Y(U_i)$ and $J_i \subset A_i$ is a closed ideal).

Corollary 8.4.47. *Let $f : X \to Y$ be a morphism of ω-formal schemes. The following conditions are equivalent :*

(i) *f is an affine morphism (resp. a closed immersion).*
(ii) *For every affine open subset $U \subset Y$, the preimage $f^{-1}U$ is an affine ω-formal scheme (resp. is isomorphic, as a U-scheme, to a ω-formal scheme $\mathrm{Spf}\, A/J$, where $A := \mathscr{O}_Y(U)$, and $J \subset A$ is a closed ideal).*

Proof. Of course, it suffices to show that (i)⇒(ii). Hence, suppose that f is an affine morphism, and choose an affine open covering $Y = \bigcup_{i\in I} U_i$ such that $f^{-1}U_i$ is an affine ω-formal scheme for every $i \in I$. We may find an affine open covering $U = \bigcup_{j\in J} V_j$ such that, for every $j \in J$ there exists $i \in I$ with $V_j \subset I_i$; moreover, we may assume that J is countable, by proposition 8.4.20(ii.d). The assumption implies that $f^{-1}V_j$ is affine for every $j \in J$, and then remark 8.4.19(ii) says that $A := \mathscr{O}_X(f^{-1}U)$ is an ω-admissible topological ring, so there exists a unique morphism $g : f^{-1}U \to \mathrm{Spf}\, A$ such that $g^\natural : A \to \mathscr{O}_X(f^{-1}U)$ is the identity (proposition 8.4.24), and $f_{|f^{-1}U}$ factors as the composition of g and the morphism $h : \mathrm{Spf}\, A \to U$ induced by the natural map $B := \mathscr{O}_Y(U) \to A$. It remains to check that g is an isomorphism. To this aim, it suffices to verify that the restriction $g_{|V_j} : f^{-1}V_j \to h^{-1}V_j$ is an isomorphism for every $j \in J$. However, it is clear that $f_*\mathscr{O}_X$ is a quasi-coherent $\mathscr{O}_Y$-module, whence a natural isomorphism of A-algebras :

$$\mathscr{O}_X(f^{-1}V_j) = f_*\mathscr{O}_X(V_j) \xrightarrow{\sim} A_j := A\widehat{\otimes}_B\mathscr{O}_Y(V_j)$$

(proposition 8.4.36) as well as an isomorphism $f^{-1}V_j \xrightarrow{\sim} \mathrm{Spf}\, A_j$. On the other hand, $h^{-1}V_j = \mathrm{Spf}\, A_j$, and by construction $g_{|V_j}$ is the unique morphism such that $(g_{|V_j})^\natural$ is the identity map of A_j. The assertion follows.

Next, suppose that f is a closed immersion, and choose an affine open covering $Y = \bigcup_{i\in I} U_i$, and closed ideals $J_i \subset A_i := \mathscr{O}_Y(U_i)$ such that we have isomorphisms $f^{-1}U_i \xrightarrow{\sim} \mathrm{Spf}\, A_i/J_i$ for every $i \in I$. Set $U_{ij} := U_i \cap U_j$ for every $i, j \in I$. Clearly, $J_i^\sim(U_{ij}) = J_j^\sim(U_{ij})$ for every $i, j \in I$, hence there exists a unique closed ideal $J \subset A := \mathscr{O}_Y(U)$ such that :

$$J^\sim(U_i) = J_i \qquad \text{for every } i \in I \tag{8.4.48}$$

(proposition 8.4.45). Especially, we have $J\mathscr{O}_Y(U_i) \subset J_i$, whence a unique morphism of U-schemes : $g_i : f^{-1}U_i \to Z := \mathrm{Spf}\, A/J$, for every $i \in I$. The uniqueness of g_i implies in particular that $g_{i|U_{ij}} = g_{j|U_{ij}}$ for every $i, j \in I$, whence a unique morphism $g : f^{-1}U \to Z$ of U-schemes. It remains to verify that g is an isomorphism, and to this aim it suffices to check that the restriction $g^{-1}(U_i \cap Z) \to U_i \cap Z$ is an isomorphism for every $i \in I$. The latter assertion is clear, in view of (8.4.48). □

8.4.49. Finally, we wish to generalize the foregoing results to almost modules. Hence, we assume now that $(V, \mathfrak{m})$ is a given basic setup in the sense of [36, §2.1.1], and we shall consider only topological rings that are V-algebras. An *ω-formal scheme over V* is the datum of an ω-formal scheme X and a morphism of locally ringed spaces $X \to \operatorname{Spec} V$. Hence, for any ω-admissible topological V-algebra A, the spectrum $\operatorname{Spf} A$ is an ω-formal scheme over V in a natural way.

8.4.50. Let A be an ω-admissible topological V-algebra A, M an A^a-module. According to [36, Def.5.3.1], a linear topology on M is a collection $\mathscr{L}$ of submodules of M – the open submodules – that satisfies almost versions of the usual conditions. More generally, if X is an ω-formal scheme (resp. a scheme) over V, we may define a *topological $\mathscr{O}_X^a$-module* by repeating, *mutatis mutandis*, the definitions of (8.4.7) (including axiom **(TS)**). When X is a scheme, one has the class of *pseudo-discrete $\mathscr{O}_X^a$*-modules – defined by adapting [26, Ch.0, §3.8] to the almost case – and every $\mathscr{O}_X^a$-module can be endowed with a unique pseudo-discrete topology.

Similarly we have a well-defined notion of *continuous $\mathscr{O}_X^a$-linear morphism* of topological $\mathscr{O}_X^a$-modules. Clearly (8.4.3) holds also in the almost case. We denote by $A^a\text{-}\mathbf{Mod}_{\mathrm{top}}$ the category of topological A^a-modules and continuous A^a-linear morphisms. There is an obvious localization functor $A\text{-}\mathbf{Mod}_{\mathrm{top}} \to A^a\text{-}\mathbf{Mod}_{\mathrm{top}}$; namely, given a topological A-module M, the topology on M^a consists of those submodules of the form N^a, where $N \subset M$ is an open submodule. Notice that M^a is separated (resp. complete) whenever the same holds for M. Similarly, for any given topological A^a-module M, say with linear topology $\mathscr{L}$, the A-module $M_!$ carries a natural topology, namely the unique one which admits the fundamental system of open submodules $\mathscr{L}_! := (N_! \mid N \in \mathscr{L})$.

Lemma 8.4.51. *In the situation of* (8.4.50), *let $A\text{-}\mathbf{Mod}^{\wedge}_{\mathrm{top}}$ (resp. $A^a\text{-}\mathbf{Mod}^{\wedge}_{\mathrm{top}}$) denote the full subcategory of $A\text{-}\mathbf{Mod}_{\mathrm{top}}$ (resp. of $A^a\text{-}\mathbf{Mod}_{\mathrm{top}}$) consisting of complete and separated modules. Then the functor :*

$$A^a\text{-}\mathbf{Mod}^{\wedge}_{\mathrm{top}} \to A\text{-}\mathbf{Mod}^{\wedge}_{\mathrm{top}} \quad : \quad M \mapsto (M_!)^{\wedge}$$

is left adjoint to the localization functor $A\text{-}\mathbf{Mod}^{\wedge}_{\mathrm{top}} \to A^a\text{-}\mathbf{Mod}^{\wedge}_{\mathrm{top}}$.

Proof. It is an immediate consequence of (8.4.3) and its almost version. □

8.4.52. Keep the situation of (8.4.50). Naturally, we shall say that M is *ω-admissible* if it is complete and separated, admits a countable cofinal system $(M_\lambda \mid \lambda \in \Lambda)$ of open submodules, and for every $\lambda \in \Lambda$ there exists an open ideal $I_\lambda \subset A$ with $I_\lambda M \subset M_\lambda$. To such M we associate the topological $\mathscr{O}_X^a$-module

$$M^{\sim} := \lim_{\lambda \in \Lambda} j_{\lambda *}(M/M_\lambda)^{\sim}$$

where $j_\lambda : \operatorname{Spec} A/I_\lambda \to \operatorname{Spf} A$ is the natural closed immersion, and $(M/M_\lambda)^{\sim}$ is defined as in [36, §5.5.1]. With this notation, (8.4.17) still holds in the almost case. Likewise, the whole of lemmata 8.4.31 and 8.4.32 carry over to almost modules, with proof unchanged.

More generally, let X be an ω-formal scheme over V; then one says that a topological $\mathscr{O}_X^a$-module $\mathscr{F}$ is *quasi-coherent* if X admits an affine open covering $(U_i \mid i \in I)$ such that for every $i \in I$ we may find an ω-admissible $\mathscr{O}_X^a(U_i)$-module M_i and an isomorphism $\mathscr{F}_{|U_i} \simeq M_i^{\sim}$ of topological $\mathscr{O}_{U_i}$-modules. We denote by $\mathscr{O}_X^a\text{-}\mathbf{Mod}_{\mathrm{qcoh}}$ (resp. $\mathscr{O}_X^a\text{-}\mathbf{Alg}_{\mathrm{qcoh}}$) the category of quasi-coherent $\mathscr{O}_X^a$-modules (resp. $\mathscr{O}_X^a$-algebras) and continuous morphisms. Clearly, for any two topological $\mathscr{O}_X^a$-modules $\mathscr{F}$ and $\mathscr{G}$, the almost version of (8.4.8) (for $\mathscr{A} := \mathscr{O}_X^a$) yields a sheaf $top.\mathscr{H}om_{\mathscr{O}_X^a}(\mathscr{F}, \mathscr{G})$ on X. The notion of *quotient map* of quasi-coherent $\mathscr{O}_X^a$-modules is defined as in (8.4.35), and by repeating the arguments in *loc.cit.* one shows that every continuous map of quasi-coherent $\mathscr{O}_X^a$-modules admits a *topological cokernel*.

8.4.53. We have an obvious localization functor $\mathscr{O}_X\text{-}\mathbf{Mod} \to \mathscr{O}_X^a\text{-}\mathbf{Mod} : \mathscr{F} \mapsto \mathscr{F}^a$, where $\mathscr{F}^a(U) := \mathscr{F}(U)^a$ for every open subset $U \subset X$. In light of (8.4.17) and its almost version, this restricts to a functor :

$$\mathscr{O}_X\text{-}\mathbf{Mod}_{\mathrm{qcoh}} \to \mathscr{O}_X^a\text{-}\mathbf{Mod}_{\mathrm{qcoh}} \qquad \mathscr{F} \mapsto \mathscr{F}^a. \tag{8.4.54}$$

We may construct as follows a left adjoint to (8.4.54). For a given object $\mathscr{F}$ of $\mathscr{O}_X^a\text{-}\mathbf{Mod}_{\mathrm{qcoh}}$, let $\mathscr{G}$ be the presheaf of topological $\mathscr{O}_X$-modules given by the rule : $\mathscr{G}(U) := (\mathscr{F}(U)_!)^\wedge$ for every open subset $U \subset X$. Let also $\mathfrak{U} := (U_i \mid i \in I)$ be an affine covering of X such that the restrictions $\mathscr{F}_{U_i}$ are of the form $M_i^\sim$ for appropriate $\mathscr{O}_X^a(U_i)$-modules M_i. We notice that $\mathscr{G}$ is already a sheaf of topological modules on the site consisting of all affine open subsets V of X such that $V \subset U_i$ for some $i \in I$; indeed, this follows from the fact that the natural map :

$$\mathscr{G}(U_i)\widehat{\otimes}_{\mathscr{O}_X(U_i)}\mathscr{O}_X(V) \to \mathscr{G}(V)$$

is an isomorphism of topological modules, for every $i \in I$ and every affine open subset $V \subset U_i$; we leave the verification to the reader. It then follows from [26, Ch.0, §3.2.2] that the sheafification $\mathscr{F}_!^\wedge$ of $\mathscr{G}$ carries a unique structure of topological $\mathscr{O}_X$-module, such that the natural map $\mathscr{G}(V) \to \mathscr{F}_!^\wedge(V)$ is an isomorphism of topological modules for every V in the above site. Especially, $(\mathscr{F}_!^\wedge)_{|U_i} \simeq (M_{i!})^\sim$ for every $i \in I$ (isomorphism of topological $\mathscr{O}_{U_i}$-modules). One sees easily that $\mathscr{F}_!^\wedge$ is a quasi-coherent $\mathscr{O}_X$-module independent of the choice of the covering $\mathfrak{U}$, hence it yields a functor:

$$\mathscr{O}_X^a\text{-}\mathbf{Mod}_{\mathrm{qcoh}} \to \mathscr{O}_X\text{-}\mathbf{Mod}_{\mathrm{qcoh}} \qquad \mathscr{F} \mapsto \mathscr{F}_!^\wedge. \tag{8.4.55}$$

Furthermore, let $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X^a$-module, $\mathscr{G}$ a complete (separated) $\mathscr{O}_X$-module; it follows from lemma 8.4.31(iii) and its almost version, and from lemma 8.4.51, that for every sufficiently small affine open subset $U \subset X$ there are natural isomorphisms

$$\mathrm{top.Hom}_{\mathscr{O}_U}(\mathscr{F}_{!|U}^\wedge, \mathscr{G}_{|U}) \xrightarrow{\sim} \mathrm{top.Hom}_{\mathscr{O}_U^a}(\mathscr{F}_{|U}, \mathscr{G}_{|U}^a)$$

compatible with restrictions to smaller open subsets $V \subset U$. This system of isomorphisms amounts to an isomorphism of sheaves :

$$top.\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}_!^\wedge, \mathscr{G}) \xrightarrow{\sim} top.\mathscr{H}om_{\mathscr{O}_X^a}(\mathscr{F}, \mathscr{G}^a)$$

and by taking global sections we deduce that (8.4.55) is the sought left adjoint to (8.4.54). Notice that the functor $\mathscr{F} \mapsto \mathscr{F}_!^\wedge$ commutes with topological cokernels (since the latter are special colimits); in particular it preserves topological epimorphisms. This functor also preserves quotient maps, but does not preserve epimorphisms, in general.

Corollary 8.4.56. (i) *Proposition* 8.4.36 *and the whole of theorem* 8.4.41 *carry over* verbatim *to the case of an arbitrary quasi-coherent $\mathscr{O}_X^a$-module $\mathscr{F}$.*

(ii) *Likewise, corollary* 8.4.39 *carries over* verbatim *to the case of an inverse system of quasi-coherent $\mathscr{O}_X^a$-modules $(\mathscr{F}_n \mid n \in \mathbb{N})$ whose transition maps are topological epimorphisms.*

Proof. (i): By considering $\mathscr{F}_!^\wedge$, we reduce to the case of quasi-coherent $\mathscr{O}_X$-modules, which is proposition 8.4.36 (resp. theorem 8.4.41).

(ii): Likewise, the system of quasi-coherent $\mathscr{O}_X$-modules $(\mathscr{F}_{n!}^\wedge \mid n \in \mathbb{N})$ has topologically epimorphic transition maps, hence we are reduced to corollary 8.4.39. □

Remark 8.4.57. (i) Occasionally, we shall also need to consider the category of ω-formal $\mathscr{B}$-schemes, defined in the obvious way, as well as the category of $\mathscr{O}^a$-algebras, fibred over ω-formal $\mathscr{B}$-schemes, analogous to (8.5.4). We leave to the reader the task of spelling out these generalities. However, we remark that the pull back functor $f^* : \mathscr{O}^a_{X'}\text{-}\mathbf{Alg} \to \mathscr{O}^a_X\text{-}\mathbf{Alg}$ associated to a morphism $f : X \to X'$ of ω-formal $\mathscr{B}$-schemes, does not necessarily transform quasi-coherent $\mathscr{O}^a_{X'}$-algebras into quasi-coherent $\mathscr{O}^a_X$-algebras (this is the pull-back functor defined on the underlying ringed spaces).

(ii) It is also possible to define a (modified) pull-back functor that respects quasi-coherence, but for our limited purposes this is not needed.

8.5. **Algebras with an action of Frobenius.** We keep the notation of section 8.3, and we suppose furthermore that $\operatorname{char} K = 0$, $\operatorname{char} \kappa := p > 0$, and that there exists an element $\pi \in K^+$ such that $1 > |\pi^p| \geq |p|$. For any K^+-algebra R, we denote by

$$\overline{\Phi}_R : R/\pi R \to R/\pi^p R$$

the unique morphism induced by the Frobenius endomorphism

$$\Phi_R : R/\pi^p R \to R/\pi^p R \quad : \quad x \mapsto x^p.$$

Moreover we let $R_{(\Phi)}$ be the $R/\pi R$-algebra whose underlying ring is $R/\pi^p R$ and whose structure morphism $R/\pi R \to R_{(\Phi)}$ is $\overline{\Phi}_R$. The map $\overline{\Phi}_R$ induces a natural functor :

$$\overline{\Phi}{}^*_R : R/\pi R\text{-}\mathbf{Mod} \to R_{(\Phi)}\text{-}\mathbf{Mod} \quad : \quad M \mapsto M \otimes_R R_{(\Phi)}$$

as well as a functor $R/\pi R\text{-}\mathbf{Alg} \to R_{(\Phi)}\text{-}\mathbf{Alg}$, also denoted $\overline{\Phi}{}^*_R$. Notice that $\overline{\Phi}_R$ factors through a unique map of $K^+_{(\Phi)}$-algebras:

$$\overline{\Phi}_{R/K^+} : \overline{\Phi}{}^*_{K^+}(R/\pi R) \to R_{(\Phi)} \quad : \quad x \otimes y \mapsto xy^p \pmod{\pi^p}.$$

More precisely, we have the identity :

$$\overline{\Phi}_{R/K^+} \circ (\mathbf{1}_R \otimes_{K^+} \overline{\Phi}_{K^+}) = \overline{\Phi}_R. \tag{8.5.1}$$

8.5.2. Since the definition of the Frobenius endomorphism commutes with localizations, we can extend the above discussion to schemes. Namely, for any K^+-scheme X, the closed immersion

$$X_{/\pi} \xrightarrow{i_X} X_{(\Phi)} := X_{/\pi^p}$$

(notation of (5.7)) induces morphisms of schemes :

$$\Phi_X : X_{(\Phi)} \to X_{(\Phi)} \qquad \overline{\Phi}_X : X_{(\Phi)} \to X_{/\pi} \quad \text{such that} \quad \Phi_X = i_X \circ \overline{\Phi}_X$$

which are the identity maps on the underlying topological spaces, and induce the maps

$$\Phi_{\mathscr{O}_X(U)} : \mathscr{O}_{X_{/\pi}}(U_{(\Phi)}) \to \mathscr{O}_{X_{(\Phi)}}(U_{(\Phi)}) \qquad \overline{\Phi}_{\mathscr{O}_X(U)} : \mathscr{O}_{X_{/\pi}}(U_{/\pi}) \to \mathscr{O}_{X_{(\Phi)}}(U_{(\Phi)})$$

for every affine open subset $U \subset X$.

8.5.3. Proceeding as in [36, §3.5], the foregoing discussion can also be extended *verbatim* to almost modules and almost algebras. For instance, we have, for every K^+-algebra R, a functor

$$\overline{\Phi}{}^*_{R^a} : (R/\pi R)^a\text{-}\mathbf{Mod} \to R^a_{(\Phi)}\text{-}\mathbf{Mod}$$

with obvious notation. Likewise, the definitions of (8.5.2) can be extended to any $\mathscr{O}^a_X$-algebra, on any K^+-scheme X. Indeed, let $\mathscr{B}$ be the category of basic setups, defined as in *loc.cit.* We consider the fibred category of *$\mathscr{B}$-schemes* :

$$F : \mathscr{B}\text{-}\mathbf{Sch} \to \mathscr{B}^o.$$

Namely, the objects of $\mathscr{B}$-$\mathbf{Sch}$ are data of the form $(V, \mathfrak{m}, X)$, where $(V, \mathfrak{m})$ is a basic setup and X is any V-scheme; obviously we let $F(V, \mathfrak{m}, X) := (V, \mathfrak{m})$. The morphisms $(V, \mathfrak{m}, X) \to$

$(V', \mathfrak{m}', X')$ in $\mathscr{B}$-**Sch** are the pairs (f, φ) consisting of a morphism $f : (V', \mathfrak{m}') \to (V, \mathfrak{m})$ of basic setups and a morphism of schemes $\varphi : X \to X'$ such that the diagram of schemes:

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d] & X' \arrow[d] \\ \operatorname{Spec} V \arrow[r, "\operatorname{Spec} f"] & \operatorname{Spec} V' \end{tikzcd}$$

commutes. Such a morphism (f, φ) induces an adjoint pair of functors :

$$(\mathscr{O}_X, \mathfrak{m}\mathscr{O}_X)^a\text{-}\mathbf{Mod} \underset{(f,\varphi)^*}{\overset{(f,\varphi)_*}{\rightleftarrows}} (\mathscr{O}_{X'}, \mathfrak{m}'\mathscr{O}_{X'})^a\text{-}\mathbf{Mod}.$$

Namely, if M is any $\mathscr{O}_X^a$-module, and $U \subset X'$ is any open subset, one sets :

$$\Gamma(U, (f, \varphi)_* M) := f_U^* \Gamma(\varphi^{-1}U, M)$$

where $f_U^* : \mathscr{O}_X(f^{-1}U)\text{-}\mathbf{Mod} \to \mathscr{O}_{X'}(U)\text{-}\mathbf{Mod}$ is the pull-back functor defined by the fibred category $\mathscr{B}\text{-}\mathbf{Mod}$ of [36, 3.5.1]. We leave to the reader the task of spelling out the construction of $(f, \varphi)^*$. As usual, one defines an analogous adjoint pair for $\mathscr{O}^a$-algebras.

The morphisms (f, φ) such that f is the identity endomorphism of a basic setup $(V, \mathfrak{m})$, shall be usually denoted just by φ, unless the notation is ambiguous. Likewise, for such morphisms, we denote by φ_* and φ^* the corresponding functors on sheaves of almost modules.

8.5.4. We shall also consider the fibred category :

$$\mathscr{O}^a\text{-}\mathbf{\acute{E}t} \to \mathscr{B}\text{-}\mathbf{Sch}$$

of étale $\mathscr{O}^a$-algebras. An object of $\mathscr{O}^a$-**Ét** is a pair $(X, \mathscr{A})$, where X is a $\mathscr{B}$-scheme, and $\mathscr{A}$ is an étale $\mathscr{O}_X^a$-algebra. A morphism $(X, \mathscr{A}) \to (X', \mathscr{A}')$ is a pair (f, g) consisting of a morphism $f : X \to X'$ of $\mathscr{B}$-schemes, and a morphism $g : \mathscr{A}' \to f_*\mathscr{A}$ of $\mathscr{O}_{X'}^a$-algebras. Composition of morphism is given by the rule :

$$(f', g') \circ (f, g) := (f' \circ f, f'_*(g) \circ g').$$

For instance, if V is a $\mathbb{F}_p$-algebra and Z is any V-scheme, the Frobenius endomorphisms of V and Z induce a morphism of $\mathscr{B}$-schemes :

$$\Phi_Z := (\Phi_V, \Phi_Z) : (V, \mathfrak{m}, Z) \to (V, \mathfrak{m}, Z).$$

(Especially, to any K^+-scheme X we may attach the endomorphism (Φ_{K^+}, Φ_X) of the $\mathscr{B}$-scheme $(K^+_{(\Phi)}, \mathfrak{m}K^+_{(\Phi)}, X_{(\Phi)})$.) In this case, for every $\mathscr{O}_Z^a$-algebra $\mathscr{A}$ and every open subset $U \subset Z$, there is an endomorphism

$$\Phi_{\mathscr{A}(U)} : \mathscr{A}(U) \to \mathscr{A}(U) \quad : \quad b \mapsto b^p \text{ for every } b \in \mathscr{A}(U)_*.$$

(the Frobenius endomorphism : see [36, §3.5.6]), compatible with restriction maps for inclusion of open subsets, and the collection of such maps amounts to an endomorphism in $\mathscr{O}^a$-**Ét** :

$$\Phi_{\mathscr{A}} := (\Phi_Z, \Phi_{\mathscr{A}}) : ((V, \mathfrak{m}, Z), \mathscr{A}) \to ((V, \mathfrak{m}, Z), \mathscr{A}).$$

Lemma 8.5.5. *Let $Z := (V, \mathfrak{m}, Z)$ be a $\mathscr{B}$-scheme, with V a $\mathbb{F}_p$-algebra, and $\mathscr{A}$ a weakly étale quasi-coherent $\mathscr{O}_Z^a$-algebra. Then $\Phi_{\mathscr{A}} : (Z, \mathscr{A}) \to (Z, \mathscr{A})$ is a cartesian morphism of $\mathscr{O}^a$-***Ét**.

Proof. We may assume that Z is affine, in which case the assertion follows from [36, Th.3.5.13]. □

8.5.6. Consider now a K^+-algebra R fulfilling the conditions of proposition 4.6.33, *i.e.* such that there exist $\pi, a \in R$ where π is regular, a is invertible, $p = \pi^p \cdot a$, and the Frobenius map induces an isomorphism $\overline{\Phi}_R : R/\pi R \xrightarrow{\sim} R/pR$. We shall denote as usual by $R^\wedge$ the p-adic completion of R, and we shall deal with the topological rings $(\mathbf{E}(R)^+, \mathscr{T}_\mathbf{E})$ and $(\mathbf{A}(R)^+, \mathscr{T}_\mathbf{A})$ introduced in (4.6.25) and (4.6.38). To begin with, we let :

$$\begin{array}{ll} X := \mathrm{Spec}\, R & \mathbf{E}(X) := \mathrm{Spf}\, \mathbf{E}(R)^+ \\ X^\wedge := \mathrm{Spec}\, R^\wedge & \mathbf{A}(X) := \mathrm{Spf}\, \mathbf{A}(R)^+. \end{array}$$

(See definition 8.4.18.) By theorem 4.6.39(ii), the ghost component $\boldsymbol{\omega}_0$ induces a closed immersion :

$$\boldsymbol{\omega}_X := \mathrm{Spf}\, \boldsymbol{\omega}_0 : \mathbf{E}(X) \to \mathbf{A}(X).$$

We shall also need the maps $\overline{u}_R : \mathbf{E}(R)^+ \to R/pR$ and $u_R : \mathbf{A}(R)^+ \to R^\wedge$ defined as in (4.6.25), and respectively (4.6.29). Especially, recall that the kernel of u_R is a principal ideal whose generator is a regular element that we denote ϑ (proposition 4.6.33). For every $\mathbf{n} : \mathbb{Z} \to \mathbb{N}$ as in (4.6.38), we set :

$$X(\mathbf{n}) := \mathrm{Spec}\, \mathbf{A}(R)^+/\vartheta_\mathbf{n}\mathbf{A}(R)^+$$

where $\vartheta_\mathbf{n}$ is defined as in (4.6.38). The topological space underlying $\mathbf{A}(X)$ is the colimit of the filtered family of closed subsets $X(\mathbf{n}) \subset \mathrm{Spec}\, \mathbf{A}(R)^+$, for $\mathbf{n}$ ranging over all the mappings $\mathbb{Z} \to \mathbb{N}$ with finite support. Then the natural projections $\mathbf{A}(R)^+ \to \mathbf{A}(R)^+/\vartheta_\mathbf{n}\mathbf{A}(R)^+$ determine maps of locally ringed spaces :

$$X(\mathbf{n}) \to \mathbf{A}(X). \tag{8.5.7}$$

The schemes $X(\mathbf{n})$ and $X(\mathbf{n})_{/p}$ (notation of (5.7)) can be regarded as ω-formal schemes, by endowing their structure sheaves $\mathscr{O}_{X(\mathbf{n})}$ and $\mathscr{O}_{X(\mathbf{n})_{/p}}$ with the pseudo-discrete topologies. Then the morphisms (8.5.7) are closed immersions of ω-formal schemes.

8.5.8. Let $\mathbf{m} : \mathbb{Z} \to \mathbb{N}$ be any other mapping with finite support, such that $\mathbf{n} \geq \mathbf{m}$ (for the ordering of definition 4.6.37(iii)); recall that $\vartheta_\mathbf{n}\mathbf{A}(R)^+ \subset \vartheta_\mathbf{m}\mathbf{A}(R)^+$, whence a corresponding closed immersion :

$$X(\mathbf{m}) \to X(\mathbf{n}).$$

Suppose moreover that $k \in \mathbb{N}$ is any integer, and notice that :

$$\vartheta_{\mathbf{n}[k]}^{p^k} \equiv \vartheta_\mathbf{n} \pmod{p\mathbf{A}(R)^+}$$

(notation of definition 4.6.37(ii)) so that $(\vartheta_\mathbf{n}, p) \subset (\vartheta_{\mathbf{n}[k]}, p)$; especially, $X(\mathbf{n}[k])_{/p}$ is a closed subscheme of $X(\mathbf{n})_{/p}$ and we shall denote by

$$i_{(\mathbf{n}[k],\mathbf{n})} : X(\mathbf{n}[k])_{/p} \to X(\mathbf{n})_{/p}$$

the corresponding closed immersion.

8.5.9. For instance, for any fixed $k \in \mathbb{Z}$, consider the mapping $\boldsymbol{\delta}_k : \mathbb{Z} \to \mathbb{N}$ such that $\mathrm{Supp}\, \boldsymbol{\delta}_k = \{-k\}$ and $\boldsymbol{\delta}_k(-k) = 1$; for such mappings we shall write ϑ_k, $X(k)$ and $X(k)_{/p}$ instead of respectively $\vartheta_{\boldsymbol{\delta}_k}$, $X(\boldsymbol{\delta}_k)$ and $X(\boldsymbol{\delta}_k)_{/p}$. Notice that $\boldsymbol{\delta}_n[k] = \boldsymbol{\delta}_{n+k}$, hence $X(n+k)_{/p}$ is a closed subscheme of $X(n)_{/p}$, for every $n \in \mathbb{Z}$ and $k \in \mathbb{N}$; we shall therefore write likewise $i_{(n+k,n)}$ instead of $i_{(\boldsymbol{\delta}_n[k],\boldsymbol{\delta}_n)}$. In this special case, the maps introduced in (8.5.6) and (8.5.8) can be summarized by the following commutative diagram of ω-formal schemes :

$$\begin{tikzcd} X(n)_{/p} \arrow[r] \arrow[d] & X(m) \arrow[d] \\ X(n) \arrow[r] & \mathbf{A}(X) \end{tikzcd} \tag{8.5.10}$$

where all the arrows are the natural closed immersions, and $m, n \in \mathbb{Z}$ are any two integers with $n \geq m$.

8.5.11. Also, notice that the automorphism $\boldsymbol{\sigma}_R$ of (4.6.29) is a homeomorphism for the topology $\mathscr{T}_{\mathbf{A}}$, hence it induces an automorphism of ω-formal schemes:

$$\boldsymbol{\sigma}_X := \mathrm{Spf}\, \boldsymbol{\sigma}_R : \mathbf{A}(X) \xrightarrow{\sim} \mathbf{A}(X).$$

Clearly $\boldsymbol{\sigma}_X^t$ restricts to isomorphisms of schemes :

$$\boldsymbol{\sigma}^t_{X(\mathbf{n})} : X(\mathbf{n}) \xrightarrow{\sim} X(\mathbf{n}[t]) \qquad \boldsymbol{\sigma}^t_{X(\mathbf{n})_{/p}} : X(\mathbf{n})_{/p} \xrightarrow{\sim} X(\mathbf{n}[t])_{/p}$$

for every map $\mathbf{n} : \mathbb{Z} \to \mathbb{N}$ with finite support, and every $t \in \mathbb{Z}$. Again, for the special case where $\mathbf{n} = \boldsymbol{\delta}_k$, these isomorphisms shall be denoted respectively :

$$\boldsymbol{\sigma}^t_{X(k)} : X(k) \xrightarrow{\sim} X(k+t) \quad \text{and} \quad \boldsymbol{\sigma}^t_{X(k)_{/p}} : X(k)_{/p} \xrightarrow{\sim} X(k+t)_{/p}.$$

Notice that the morphism $\mathrm{Spec}\, u_R$ induces an isomorphism $X^\wedge \xrightarrow{\sim} X(0)$ (notation of (8.5.6)), and by composing with $\boldsymbol{\sigma}^k_{X(0)}$ we obtain natural identifications :

$$X^\wedge \xrightarrow{\sim} X(k) \qquad \text{for every } k \in \mathbb{Z}. \tag{8.5.12}$$

Furthermore, let $\Phi_{X(\mathbf{n})} : X(\mathbf{n})_{/p} \to X(\mathbf{n})_{/p}$ be the Frobenius endomorphism; since $\boldsymbol{\sigma}_R$ lifts the Frobenius endomorphism of $\mathbf{E}(R)^+$ (see (4.6.29)), we deduce a commutative diagram :

$$\begin{tikzcd} X(\mathbf{n})_{/p} \arrow[rr, "\Phi^k_{X(\mathbf{n})}"] \arrow[dr, "\boldsymbol{\sigma}^k_{X(\mathbf{n})_{/p}}"'] & & X(\mathbf{n})_{/p} \\ & X(\mathbf{n}[k])_{/p} \arrow[ur, "i_{(\mathbf{n}[k],\mathbf{n})}"'] & \end{tikzcd} \tag{8.5.13}$$

for every $k \in \mathbb{N}$ and every $\mathbf{n} : \mathbb{Z} \to \mathbb{N}$ with finite support.

Lemma 8.5.14. *Under the assumptions of* (8.5.6)*, diagram* (8.5.10) *is cartesian whenever* $n > m$.

Proof. We use the isomorphisms (8.5.12) to identify $X(m)$ and $X(n)$ with $X^\wedge$, and $X_{/p}(n)$ with $X_{/p}$, after which we come down to showing that the commutative diagram :

$$\mathscr{D}(m,n) \quad : \quad \begin{tikzcd} \mathbf{A}(R)^+ \arrow[r, "u_R \circ \boldsymbol{\sigma}^n_R"] \arrow[d, "u_R \circ \boldsymbol{\sigma}^m_R"'] & R^\wedge \arrow[d, "\mathrm{pr}"] \\ R^\wedge \arrow[r, "\Phi^{n-m}_R \circ \mathrm{pr}"] & R/pR \end{tikzcd}$$

is cocartesian, where $\mathrm{pr} : R^\wedge \to R/pR$ is the natural surjection. Since $\boldsymbol{\sigma}_R$ is an isomorphism we may further reduce to the case where $m = 0$. Hence, let $J := u_R(\mathrm{Ker}\, u_R \circ \boldsymbol{\sigma}^n_R) \subset R^\wedge$; the assertion is equivalent to the following :

Claim 8.5.15. $J = \mathrm{Ker}\,(\Phi^n_R \circ \mathrm{pr})$.

Proof of the claim. Under the current assumptions, we may find $\pi, a \in R$ where a is invertible in R, such that $p = a \cdot \pi^p$, and an element $\underline{\pi} := (\pi_k \mid k \in \mathbb{N}) \in \mathbf{E}(R)^+$ such that π_0 is the image of π in R/pR. For every $k \in \mathbb{N}$, let $f_k \in R$ be any lifting of π_k. From claim 4.6.34(i,ii) we deduce easily that

$$J \subset \mathrm{Ker}\,(\Phi^n_R \circ \mathrm{pr}) = f_{n-1} R^\wedge. \tag{8.5.16}$$

On the other hand, since $\boldsymbol{\sigma}_R$ lifts the Frobenius endomorphism $\Phi_{\mathbf{E}(R)^+}$ of $\mathbf{E}(R)^+$, the diagram $\mathscr{D}(0,n)\otimes_{\mathbb{Z}}\mathbb{F}_p$ is the same as the diagram :

$$\begin{tikzcd} \mathbf{E}(R)^+ \arrow[r, "\Phi^n_{\mathbf{E}(R)^+}\circ\overline{u}_R"] \arrow[d, "\overline{u}_R"'] & R/pR \arrow[d, equal] \\ R/pR \arrow[r, "\Phi^n_R"] & R/pR \end{tikzcd}$$

and a direct inspection shows that the latter is cocartesian, whence the identity :

$$\mathrm{Ker}\,(\Phi^n_R\circ\mathrm{pr}) = J + pR^\wedge. \tag{8.5.17}$$

Set $M := f_{n-1}R^\wedge/J$, and notice that $J+pR^\wedge\subset J+f^2_{n-1}R^\wedge$ for every $n>0$; combining with and (8.5.16) and (8.5.17) we finally get : $f_{n-1}M = M$, hence $M=0$ by Nakayama's lemma; *i.e.* $J=f_{n-1}R^\wedge$, as claimed. □

Remark 8.5.18. For every $n,m\in\mathbb{Z}$, let $I_{n,m}\subset\mathbf{A}(R)^+$ be the ideal generated by ϑ_n and ϑ_m.

(i) Lemma 8.5.14 comes down to the assertion that $I_{n,m}=(\vartheta_n,p)$, provided $n>m$. Especially, since $\vartheta^p_{k+1}-\vartheta_k\in p\mathbf{A}(R)^+$, we have a finite filtration of length p by subideals :

$$I_{k,k-1}\subset(\vartheta_k,p,\vartheta^{p-1}_{k+1})\subset(\vartheta_k,p,\vartheta^{p-2}_{k+1})\subset\cdots\subset(\vartheta_k,p,\vartheta_{k+1})=I_{k+1,k}\qquad\text{for every } k\in\mathbb{Z}$$

such that the resulting associated graded $\mathbf{A}(R)^+$-module is annihilated by $I_{k+1,k}$.

(ii) Moreover, the identifications (8.5.12) amount to natural isomorphisms

$$\mathbf{A}(R)^+/\vartheta_k\mathbf{A}(R)^+\xrightarrow{\sim}R^\wedge\qquad\text{for every } k\in\mathbb{Z}. \tag{8.5.19}$$

Since $\boldsymbol{\sigma}_R(I_{k+1,k})=I_{k,k-1}$, the latter in turn induce a commutative diagram

$$\begin{tikzcd} \mathbf{A}(R)^+ \arrow[r, "\boldsymbol{\sigma}_R"] \arrow[d] & \mathbf{A}(R)^+ \arrow[d] \\ R^\wedge/\vartheta_{k+1}R^\wedge \arrow[r, "\overline{\boldsymbol{\sigma}}_R"] & R^\wedge/\vartheta_{k-1}R^\wedge = R/pR \end{tikzcd}$$

whose vertical arrows are the surjections deduced from (8.5.19), and where $\overline{\boldsymbol{\sigma}}_R$ is an isomorphism. It then follows easily that $R^\wedge/\vartheta_{k+1}R^\wedge$ is naturally identified with $R/\pi R$, and under this identification, $\overline{\boldsymbol{\sigma}}_R$ becomes the isomorphism $\overline{\Phi}_R$.

(iii) The discussion of (ii) also shows, in particular, that the image of ϑ_{k+1} under (8.5.19) generates the ideal $\pi R^\wedge$.

8.5.20. The topological space underlying $\mathbf{E}(X)$ is the colimit of the filtered family of its closed subsets $X(\mathbf{n})_{/p}\subset\mathbf{E}(X)$. However, in view of claim 4.6.40(ii), the same space $\mathbf{E}(X)$ can also be described as the direct limit of the sequence of closed subsets $(X(-k)_{/p}\mid k\in\mathbb{N})$, with transition maps $i_{(1-k,-k)}:X(1-k)_{/p}\to X(-k)_{/p}$ for every $k\in\mathbb{N}$. But all such closed subschemes $X(-k)_{/p}$ share the same underlying topological space, and the transition maps induce the identity on the underlying spaces, so the closed immersion of ω-formal schemes :

$$\overline{u}_X := \mathrm{Spf}\,\overline{u}_R : X_{/p}\to\mathbf{E}(X)$$

induces a homeomorphism on the underlying topological spaces (here $\mathscr{O}_{X_{/p}}$ is endowed with its pseudo-discrete topology). Furthermore, the surjections $\mathbf{E}(R)^+\to\mathbf{E}(R)^+/\boldsymbol{\sigma}^n_R(\vartheta)\cdot\mathbf{E}(R)^+$ induce morphisms of ω-formal schemes

$$i_{(-n)}:X(-n)_{/p}\to\mathbf{E}(X)$$

and the structure sheaf $\mathscr{O}_{\mathbf{E}(X)}$ is naturally identified with the inverse limit of pseudo-discrete sheaves of rings :

$$\lim_{n\in\mathbb{N}} i_{(-n)*}\mathscr{O}_{X(-n)_{/p}} \tag{8.5.21}$$

where the transition maps $i_{(-n)*}\mathscr{O}_{X(-n)_{/p}} \to i_{(1-n)*}\mathscr{O}_{X(1-n)_{/p}}$ are induced by the morphisms :

$$i^\natural_{(1-n,-n)} : \mathscr{O}_{X(-n)_{/p}} \to i_{(1-n,-n)*}\mathscr{O}_{X(1-n)_{/p}}$$

that define the closed immersions $i_{(1-n,-n)}$. However, (8.5.13) yields a commutative diagram of $\mathscr{O}_{X(n)_{/p}}$-algebras:

$$\begin{tikzcd} \mathscr{O}_{X(n)_{/p}} \arrow[d, "i^\natural_{(1-n,-n)}"'] \arrow[dr, "\Phi_{\mathscr{O}_{X(n)}}"] & \\ i_{(1-n,-n)*}\mathscr{O}_{X(n+1)_{/p}} \arrow[r, "\sim"] & \Phi_{X(n)*}\mathscr{O}_{X(n)_{/p}} \end{tikzcd}$$

so that (8.5.21) is naturally identified with the inverse limit of the system of pseudo-discrete sheaves of rings on $X_{/p}$:

$$\cdots \longrightarrow \mathscr{O}_{X_{/p}} \xrightarrow{\Phi_{\mathscr{O}_X}} \mathscr{O}_{X_{/p}} \xrightarrow{\Phi_{\mathscr{O}_X}} \mathscr{O}_{X_{/p}}.$$

8.5.22. Suppose now that K^+ is a deeply ramified valuation ring of rank one, mixed characteristic $(0,p)$ and maximal ideal $\mathfrak{m}$; we let

$$\mathbf{E}^+ := \mathbf{E}(K^+)^+ \qquad \text{and} \qquad \mathbf{A}^+ := \mathbf{A}(K^+)^+.$$

According to [36, §6.6], the value group Γ of K^+ is p-divisible, hence the ring K^+ fulfills the conditions of proposition 4.6.33, so the maps $\overline{u}_{K^+} : \mathbf{E}^+ \to K^+/pK^+$ and $u_{K^+} : \mathbf{A}^+ \to K^{\wedge +}$ are surjective. Moreover, by lemma 4.6.27(ii), $\mathbf{E}^+$ is a valuation ring with value group Γ, whose maximal ideal we denote by $\mathfrak{m}_\mathbf{E}$. We may then consider the ideal of $\mathbf{A}^+$:

$$\mathfrak{m}_\mathbf{A} := \tau_{K^+}(\mathfrak{m}_\mathbf{E}) \cdot \mathbf{A}^+.$$

Notice that, since τ_{K^+} is multiplicative and sends regular elements of K^+ to regular elements of $\mathbf{A}^+$ (see (4.6.9)), $\mathfrak{m}_\mathbf{A}$ is a filtered union of invertible principal ideals, especially it is a flat $\mathbf{A}^+$-module and $\mathfrak{m}_\mathbf{A}^2 = \mathfrak{m}_\mathbf{A}$. In other words, condition **(A)** of [36, §2.1.6] holds for $\mathfrak{m}_\mathbf{A}$, and especially, the pair $(\mathbf{A}^+, \mathfrak{m}_\mathbf{A})$ is a basic setup in the sense of [36, §2.1.1]. It also follows easily that $\mathfrak{m}_\mathbf{A} = \widetilde{\mathfrak{m}}_\mathbf{A} := \mathfrak{m}_\mathbf{A} \otimes_{\mathbf{A}^+} \mathfrak{m}_\mathbf{A}$. We claim that the maps $\boldsymbol{\sigma}_{K^+}$ and u_{K^+} induce morphisms :

$$(\mathbf{A}^+, \mathfrak{m}_\mathbf{A}) \xrightarrow{\boldsymbol{\sigma}_{K^+}} (\mathbf{A}^+, \mathfrak{m}_\mathbf{A}) \qquad (\mathbf{A}^+, \mathfrak{m}_\mathbf{A}) \xrightarrow{u_{K^+}} (K^{\wedge +}, \mathfrak{m}K^{\wedge +})$$

in the category $\mathscr{B}$ of basic setups defined in [36, §3.5]. Indeed, the assertion for $\boldsymbol{\sigma}_{K^+}$ follows from the commutativity of (4.6.30) and from [36, Prop.2.1.7(ii)]. The assertion for u_{K^+} comes down to showing that $u_{K^+}(\mathfrak{m}_\mathbf{A}) = \mathfrak{m}K^{\wedge +}$, which is an easy consequence of (4.6.32) : we leave the details to the reader. Furthermore, since u_{K^+} is surjective, it is clear that the induced pull-back functors

$$u^*_{K^+} : (K^{\wedge +}, \mathfrak{m}K^{\wedge +})^a\text{-}\mathbf{Mod} \to (\mathbf{A}^+, \mathfrak{m}_\mathbf{A})^a\text{-}\mathbf{Mod}$$
$$u^*_{K^+} : (K^{\wedge +}, \mathfrak{m}K^{\wedge +})^a\text{-}\mathbf{Alg} \to (\mathbf{A}^+, \mathfrak{m}_\mathbf{A})^a\text{-}\mathbf{Alg}$$

are fully faithful. Therefore we may view an almost $K^{\wedge +}$-module as an almost $\mathbf{A}^+$-module in a natural way, and likewise for almost algebras. Now, if R is a K^+-algebra, $R^\wedge$ is a $K^{\wedge +}$-algebra, hence also a $\mathbf{A}^+$-algebra, via the map u_{K^+}. Especially, in the situation of (8.5.6) we have the $\mathscr{B}$-schemes :

$$(\mathbf{A}^+, \mathfrak{m}_\mathbf{A}, X^\wedge) \quad \text{and} \quad (\mathbf{A}^+, \mathfrak{m}_\mathbf{A}, \mathbf{A}(X))$$

and $\boldsymbol{\sigma}_X$ induces a morphism of $\mathscr{B}$-schemes :

$$(\boldsymbol{\sigma}_{K^+}, \boldsymbol{\sigma}_X) : (\mathbf{A}^+, \mathfrak{m}_\mathbf{A}, \mathbf{A}(X)) \to (\mathbf{A}^+, \mathfrak{m}_\mathbf{A}, \mathbf{A}(X)).$$

8.5.23. Keep the notation of (8.5.6) and (8.5.22); we specialize now to the case where R is a local K^+-algebra, whose residue field has characteristic p. Then the p-adic completion $R^\wedge$ is a local $K^{\wedge+}$-algebra with the same residue field. Let $x \in X$ be the closed point (notation of (8.5.6)); the only point of $X^\wedge$ lying over x is the closed point, so there is no harm in denoting the latter by x as well. Furthermore, let us set

$$U := X \setminus \{x\} \qquad U^\wedge := X^\wedge \setminus \{x\}$$

and we assume that $U_{/p}$ is not empty (notation of (5.7)). For every $k \in \mathbb{Z}$, the isomorphism (8.5.12) identifies x to a point $x_{(k)} \in X(k)$; however, the image of $x_{(k)}$ in $\mathbf{A}(X)$ does not depend on k, hence we may simply write x instead of $x_{(k)}$ for any and all of these points. Clearly $x \in X(\mathbf{n}) \subset \mathbf{A}(X)$ whenever $\operatorname{Supp} \mathbf{n} \neq \varnothing$, in which case we set

$$\mathbf{A}(U) := \mathbf{A}(X) \setminus \{x\} \qquad U(\mathbf{n}) := X(\mathbf{n}) \setminus \{x\}$$

and we write $U(k)$, $U(k)_{/p}$ instead of $U(\boldsymbol{\delta}_k)$, respectively $U(\boldsymbol{\delta}_k)_{/p}$ (see (8.5.9)). Since $\{x\}$ is a closed subset of $\mathbf{A}(X)$, corollary 8.4.22 implies that $\mathbf{A}(U)$ is an ω-formal scheme. Furthermore, the maps $\boldsymbol{\sigma}_X : \mathbf{A}(X) \to \mathbf{A}(X)$ and $\boldsymbol{\sigma}^t_{X(\mathbf{n})} : X(\mathbf{n}) \xrightarrow{\sim} X(\mathbf{n}[t])$ induce isomorphisms :

$$\boldsymbol{\sigma}_U : \mathbf{A}(U) \xrightarrow{\sim} \mathbf{A}(U) \qquad \boldsymbol{\sigma}^t_{U(\mathbf{n})} : U(\mathbf{n}) \xrightarrow{\sim} U(\mathbf{n}[t]) \qquad \boldsymbol{\sigma}^t_{U(\mathbf{n})_{/p}} : U(\mathbf{n})_{/p} \xrightarrow{\sim} U(\mathbf{n}[t])_{/p}.$$

8.5.24. In the situation of (8.5.23), let $\mathscr{A}^\wedge$ be a given étale $(\mathscr{O}_{U^\wedge}, \mathfrak{m}\mathscr{O}_{U^\wedge})^a$-algebra. We shall use $\mathscr{A}^\wedge$ to construct a compatible family of étale $\mathscr{O}^a_{U(\mathbf{n})}$-algebras, for every $\mathbf{n} : \mathbb{Z} \to \mathbb{N}$ with finite (non-empty) support. To this aim, we consider the following categories :

• The category $\mathscr{C}_0$, whose objects are the $\mathscr{B}$-schemes :

$$(\mathbf{A}^+, \mathfrak{m}_{\mathbf{A}}, U(\mathbf{n})) \quad \text{and} \quad (\mathbf{A}^+, \mathfrak{m}_{\mathbf{A}}, U(\mathbf{n})_{/p})$$

for all $\mathbf{n} : \mathbb{Z} \to \mathbb{N}$ with finite non-empty support. If $Z, Z' \in \operatorname{Ob}(\mathscr{C}_0)$, the set $\operatorname{Hom}_{\mathscr{C}_0}(Z, Z')$ consists of all the morphisms of $\mathscr{B}$-schemes $Z \to Z'$ which fit into a commutative diagram :

(8.5.25)
$$\begin{tikzcd} Z \arrow[rr] \arrow[d] & & Z' \arrow[d] \\ (\mathbf{A}^+, \mathfrak{m}_{\mathbf{A}}, \mathbf{A}(U)) \arrow[rr, "(\boldsymbol{\sigma}^n_{K^+}, \boldsymbol{\sigma}^n_U)"] & & (\mathbf{A}^+, \mathfrak{m}_{\mathbf{A}}, \mathbf{A}(U)) \end{tikzcd}$$

for some $n \in \mathbb{Z}$, where the vertical arrow are the natural closed immersions (and with composition of morphisms defined in the obvious way).

• The category $\mathscr{C}_1$, which is the full subcategory of $\mathscr{C}_0$ whose objects are the $\mathscr{B}$-schemes :

$$(\mathbf{A}^+, \mathfrak{m}_{\mathbf{A}}, U(n)) \quad \text{and} \quad (\mathbf{A}^+, \mathfrak{m}_{\mathbf{A}}, U(n)_{/p}) \qquad \text{for all } n \in \mathbb{Z}.$$

• The category $\mathscr{C}_2$, which is the subcategory of $\mathscr{C}_1$ which has the same objects, but whose morphisms $Z \to Z'$ are the morphisms of $\mathscr{B}$-schemes which fit into a commutative diagram (8.5.25) where $n = 0$ (so the bottom row is the identity, and therefore the top row is a closed immersion of $\mathbf{A}(U)$-subschemes).

• The category $\mathscr{C}_3$, whose objects are the $\mathscr{B}$-schemes $(\mathbf{A}^+, \mathfrak{m}_{\mathbf{A}}, U_{/p})$ and $(\mathbf{A}^+, \mathfrak{m}_{\mathbf{A}}, U^\wedge)$, which sometimes we denote simply $U_{/p}$ and $U^\wedge$, to ease notation. The only morphism $U^\wedge \to U^\wedge$ in $\mathscr{C}_3$ is the identity, and there are natural bijections of sets :

$$\mathbb{N} \xrightarrow{\sim} \operatorname{Hom}_{\mathscr{C}_3}(U_{/p}, U_{/p}) \xrightarrow{\sim} \operatorname{Hom}_{\mathscr{C}_3}(U_{/p}, U^\wedge)$$

given by the rules : $t \mapsto \Phi^t_U$, and $\Phi^t_U \mapsto \iota \circ \Phi^t_U$ for every $t \in \mathbb{N}$, where $\iota : U_{/p} \to U^\wedge$ is the natural closed immersion, and :

$$\Phi_U : (\mathbf{A}^+, \mathfrak{m}_{\mathbf{A}}, U_{/p}) \to (\mathbf{A}^+, \mathfrak{m}_{\mathbf{A}}, U_{/p})$$

is the Frobenius endomorphism of the $\mathscr{B}$-scheme $U_{/p}$, *i.e.*. the pair $(\boldsymbol{\sigma}_{K^+}, \Phi_U)$.

The categories $\mathscr{C}_i$ ($i = 0, \dots, 3$) are subcategories of $\mathscr{B}$-$\mathbf{Sch}$, and there is a functor :

$$\mathscr{C}_3 \to \mathscr{C}_1 \tag{8.5.26}$$

given on objects by the rule : $U^\wedge \mapsto U(0)$ and $U_{/p} \mapsto U(0)_{/p}$, and which is uniquely determined on morphism by letting $\Phi_U \mapsto (\boldsymbol{\sigma}_{K^+}, \Phi_{U(0)})$ and $\iota \mapsto \iota(0)$, where $\iota(0) : U(0)_{/p} \to U(0)$ is the natural closed immersion. It is easily seen that (8.5.26) is an equivalence.

8.5.27. Given an étale $\mathscr{O}^a_{U^\wedge}$-algebra $\mathscr{A}^\wedge$, we construct first a cartesian functor of categories over $\mathscr{B}$-$\mathbf{Sch}$ (see definition 1.4.3) :

$$F_{\mathscr{A}^\wedge} : \mathscr{C}_3 \to \mathscr{O}^a\text{-}\mathbf{\acute{E}t}$$

as follows. We let $U^\wedge \mapsto \mathscr{A}^\wedge$, and $U_{/p} \mapsto \mathscr{A}_{/p} := \mathscr{A}^\wedge \otimes_{\mathscr{O}^a_{U^\wedge}} \mathscr{O}^a_{U_{/p}}$. Of course, we let $F_{\mathscr{A}^\wedge}(\iota)$ be the tautological identification $\iota^*\mathscr{A}^\wedge \xrightarrow{\sim} \mathscr{A}_{/p}$. Next we let :

$$F_{\mathscr{A}^\wedge}(\Phi_U) := (\Phi_U, \Phi_{\mathscr{A}_{/p}})$$

(notation of (8.5.4)). Lemma 8.5.5 ensures that $F_{\mathscr{A}^\wedge}$ is indeed cartesian. Furthermore, the rule $\mathscr{A}^\wedge \mapsto F_{\mathscr{A}^\wedge}$ extends to a well defined functor:

$$F : \mathscr{O}^a_{U^\wedge}\text{-}\mathbf{\acute{E}t} \to \mathsf{Cart}_{\mathscr{B}\text{-}\mathbf{Sch}}(\mathscr{C}_3, \mathscr{O}^a\text{-}\mathbf{\acute{E}t})$$

(notation of definition 1.4.3); namely, to a given morphism $f : \mathscr{A}_1^\wedge \to \mathscr{A}_2^\wedge$ of étale $\mathscr{O}^a_{U^\wedge}$-algebras, one assigns the natural transformation $F_f : F_{\mathscr{A}_1^\wedge} \Rightarrow F_{\mathscr{A}_2^\wedge}$ such that

$$F_f(U^\wedge) := f \qquad \text{and} \qquad F_f(U_{/p}) := \iota^* f.$$

Next, we define a functor :

$$G : \mathscr{O}^a_{U^\wedge}\text{-}\mathbf{\acute{E}t} \to \mathsf{Cart}_{\mathscr{B}\text{-}\mathbf{Sch}}(\mathscr{C}_1, \mathscr{O}^a\text{-}\mathbf{\acute{E}t}) \qquad \mathscr{A}^\wedge \mapsto G_{\mathscr{A}^\wedge}$$

as the composition of the functor F with a quasi-inverse of the equivalence of categories :

$$\mathsf{Cart}_{\mathscr{B}\text{-}\mathbf{Sch}}(\mathscr{C}_1, \mathscr{O}^a\text{-}\mathbf{\acute{E}t}) \to \mathsf{Cart}_{\mathscr{B}\text{-}\mathbf{Sch}}(\mathscr{C}_3, \mathscr{O}^a\text{-}\mathbf{\acute{E}t})$$

deduced from (8.5.26).

Lemma 8.5.28. *The inclusion functor $\mathscr{C}_1 \to \mathscr{C}_0$ induces an equivalence of categories :*

$$\mathsf{Cart}_{\mathscr{B}\text{-}\mathbf{Sch}}(\mathscr{C}_0, \mathscr{O}^a\text{-}\mathbf{\acute{E}t}) \to \mathsf{Cart}_{\mathscr{B}\text{-}\mathbf{Sch}}(\mathscr{C}_1, \mathscr{O}^a\text{-}\mathbf{\acute{E}t}).$$

Proof. The criterion of proposition 1.4.21 reduces to showing that, for every $Z \in \mathrm{Ob}(\mathscr{C}_0)$, the induced functor $\mathsf{Cart}_{\mathscr{B}\text{-}\mathbf{Sch}}(\mathscr{C}_0/Z, \mathscr{O}^a\text{-}\mathbf{\acute{E}t}) \to \mathsf{Cart}_{\mathscr{B}\text{-}\mathbf{Sch}}(\mathscr{C}_1/Z, \mathscr{O}^a\text{-}\mathbf{\acute{E}t})$ is an equivalence. However, on the one hand, the inclusion functor $\mathscr{C}_2 \to \mathscr{C}_1$ induces an equivalence :

$$\mathsf{Cart}_{\mathscr{B}\text{-}\mathbf{Sch}}(\mathscr{C}_1/Z, \mathscr{O}^a\text{-}\mathbf{\acute{E}t}) \to \mathsf{Cart}_{\mathscr{B}\text{-}\mathbf{Sch}}(\mathscr{C}_2/Z, \mathscr{O}^a\text{-}\mathbf{\acute{E}t})$$

and on the other hand, the equivalence (see remark 1.4.20) :

$$\mathsf{ev}_Z : \mathsf{Cart}_{\mathscr{B}\text{-}\mathbf{Sch}}(\mathscr{C}_0/Z, \mathscr{O}^a\text{-}\mathbf{\acute{E}t}) \to \mathscr{O}^a_Z\text{-}\mathbf{\acute{E}t}^o$$

admits a natural quasi-inverse, that assigns to every $\mathscr{O}^a_Z$-algebra $\mathscr{R}$ the cartesian functor

$$\mathsf{c}(\mathscr{R}) : \mathscr{C}_0/Z \to \mathscr{O}^a\text{-}\mathbf{\acute{E}t} \qquad (f : Y \to Z) \mapsto f^*\mathscr{R}.$$

(This is the "standard cleavage" of the fibred category of $\mathscr{O}^a$-algebras.) Hence we are reduced to showing that the functor :

$$\mathscr{O}^a_Z\text{-}\mathbf{\acute{E}t}^o \to \mathsf{Cart}_{\mathscr{B}\text{-}\mathbf{Sch}}(\mathscr{S}_Z, \mathscr{O}^a\text{-}\mathbf{\acute{E}t}) \qquad \mathscr{R} \mapsto \mathsf{c}(\mathscr{R})_{|\mathscr{C}_2/Z}$$

is an equivalence for all $Z \in \mathrm{Ob}(\mathscr{C}_0)$. The restriction $\mathscr{C}_2/Z \to \mathscr{C}_2$ of the source functor s (see (1.1.17)) identifies $\mathscr{C}_2/Z$ with a sieve $\mathscr{S}_Z$ of the category $\mathscr{C}_2$. A generating family for $\mathscr{S}_Z$ can be exhibited explicitly as follows. For given $\mathbf{n} : \mathbb{Z} \to \mathbb{N}$ with finite non-empty support, denote by $[\mathbf{n}]$ the largest integer $\leq \log_p(\sum_{k\in\mathbb{Z}} \mathbf{n}(k) \cdot p^k)$. Now, if $Z = U(\mathbf{n})_{/p}$, set $\underline{Z} := \{U_{/p}(\mathbf{n})\}$; in case $Z = U(\mathbf{n})$, set $\underline{Z} := \{U(k) \mid k \in \mathrm{Supp}(\mathbf{n})\} \cup \{U(\mathbf{n})_{/p}\}$. By inspecting the definitions, it is easily seen that, in either case, $\underline{Z}$ is a generating family for $\mathscr{S}_Z$. Since fibre products are representable in $\mathscr{C}_2$, we are therefore further reduced to showing that the induced functor :

$$\rho_{\underline{Z}} : \mathscr{O}^a_Z\text{-}\mathbf{\acute{E}t}^o \to \mathsf{Desc}(\mathscr{O}^a\text{-}\mathbf{\acute{E}t}, \underline{Z})$$

is an equivalence (notation of (1.5.27)). Define :

$$\psi_Z : \coprod_{Y \in \underline{Z}} Y \to Z$$

as the morphism which restricts to the natural closed immersion $Y \to Z$, for every $Y \in \underline{Z}$; by inspecting (1.5.25), one verifies easily that there is a natural equivalence of categories :

$$\mathsf{Desc}(\mathscr{O}^a\text{-}\mathbf{\acute{E}t}, \underline{Z}) \to \mathsf{Desc}(\mathscr{O}^a\text{-}\mathbf{\acute{E}t}, \{\psi_Z\})$$

whose composition with $\rho_{\underline{Z}}$ is the corresponding functor

$$\rho_{\{\psi_Z\}} : (\mathscr{O}^a_Z\text{-}\mathbf{\acute{E}t})^o \to \mathsf{Desc}(\mathscr{O}^a\text{-}\mathbf{\acute{E}t}, \{\psi_Z\}).$$

So finally, the lemma boils down to the assertion that the morphism ψ_Z is of 2-descent for the fibration $\mathscr{O}^a\text{-}\mathbf{\acute{E}t} \to \mathscr{B}\text{-}\mathbf{Sch}$, which holds by the following more general :

Claim 8.5.29. Let $(V, \mathfrak{m})$ be any basic setup, $f : X \to Y$ a finite morphism of quasi-compact quasi-separated V-schemes, and denote also $f : (V, \mathfrak{m}, X) \to (V, \mathfrak{m}, Y)$ the induced morphism of $\mathscr{B}$-schemes. Suppose that kernel of the natural morphism $\mathscr{O}^a_Y \to f_*\mathscr{O}^a_X$ is a nilpotent ideal of $\mathscr{O}^a_Y$. Then f is a morphism of 2-descent for the fibred category of étale $\mathscr{O}^a$-algebras.

Proof of the claim. Suppose first that Y (and hence X) is separated, pick a covering $(U_i \mid i \in I)$ of Y consisting of affine open subsets, and let $U_{ij} := U_i \cap U_j$, $U_{ijk} := U_{ij} \cap U_k$ for every $i, j, k \in I$. In view of corollary 1.5.38, we are reduced to showing that the restrictions $f_i := f \times_Y U_i$, $f_{ij} := f \times_Y U_{ij}$, $f_{ijk} := f \times_Y U_{ijk}$ are of 2-descent for $\mathscr{O}^a\text{-}\mathbf{\acute{E}t}$, for every $i, j, k \in I$. Since all the U_{ij} and U_{ijk} are affine, the latter assertion follows from [36, Th.3.4.37(iii)].

Next, for a general quasi-separated scheme Y, we repeat the same procedure : choose an affine open covering $(U_i \mid i \in I)$ of Y, and notice that now all the intersections U_{ij} and U_{ijk} are separated schemes. Hence, by the previous case, all the restrictions f_i, f_{ij}, f_{ijk} are of 2-descent for $\mathscr{O}^a\text{-}\mathbf{\acute{E}t}$, and again we conclude by invoking corollary 1.5.38. □

8.5.30. We compose the functor G with a quasi-inverse of the equivalence of lemma 8.5.28, to obtain a functor :

$$\mathscr{O}^a_{U^\wedge}\text{-}\mathbf{\acute{E}t} \to \mathsf{Cart}_{\mathscr{B}\text{-}\mathbf{Sch}}(\mathscr{C}_0, \mathscr{O}^a\text{-}\mathbf{\acute{E}t}). \tag{8.5.31}$$

Hence, any given étale $\mathscr{O}^a_{U^\wedge}$-algebra $\mathscr{A}^\wedge$ defines a cartesian section $\mathscr{C}_0 \to \mathscr{O}^a\text{-}\mathbf{\acute{E}t}$ over $\mathscr{B}\text{-}\mathbf{Sch}$. Then, for every object $U(\mathbf{n})$ (resp. $U(\mathbf{n})_{/p}$) of $\mathscr{C}_0$, we write $\mathscr{A}(\mathbf{n})$ (resp. $\mathscr{A}(\mathbf{n})_{/p}$) for the étale $\mathscr{O}^a_{U(\mathbf{n})}$-algebra (resp. $\mathscr{O}^a_{U(\mathbf{n})_{/p}}$-algebra) supplied by this cartesian section. As usual, we also write $\mathscr{A}(k)$ (resp. $\mathscr{A}(k)_{/p}$) instead of $\mathscr{A}(\boldsymbol{\delta}_k)$ (resp. $\mathscr{A}(\boldsymbol{\delta}_k)_{/p}$), for every $k \in \mathbb{Z}$.

Since $U(\mathbf{n})$ is a closed subset of $\mathbf{A}(U)$, every quasi-coherent $\mathscr{O}^a_{U(\mathbf{n})}$-algebra can be regarded naturally as a quasi-coherent $\mathscr{O}^a_{\mathbf{A}(U)}$-algebra. Especially, for every given étale $\mathscr{O}^a_{U^\wedge}$-algebra $\mathscr{A}^\wedge$, the cartesian functor corresponding to $\mathscr{A}^\wedge$ under (8.5.31) can be viewed as a functor :

$$\mathscr{A}(\bullet) : \mathscr{C}^o_0 \to \mathscr{O}^a_{\mathbf{A}(U)}\text{-}\mathbf{Alg}_{\mathrm{qcoh}} \qquad U(\mathbf{n}) \mapsto \mathscr{A}(\mathbf{n}) \quad U(\mathbf{n})_{/p} \mapsto \mathscr{A}(\mathbf{n})_{/p}.$$

Let $\mathscr{C}_4$ be the subcategory of $\mathscr{C}_0$ whose objects are all the $\mathscr{B}$-schemes $U(\mathbf{n})$, and whose morphisms are the natural closed immersions $U(\mathbf{m}) \to U(\mathbf{n})$ for $\mathbf{n} \geq \mathbf{m}$; we endow each sheaf $\mathscr{A}(\mathbf{n})$ with its natural pseudo-discrete topology, and set :

$$\mathbf{A}(\mathscr{A})^+ := \lim_{\mathscr{C}^o_4} \mathscr{A}(\bullet)$$

where the limit is taken in the category of topological $\mathscr{O}^a_{\mathbf{A}(U)}$-algebras.

Next, since the functor $\mathscr{A}(\bullet)$ is actually defined on the whole of $\mathscr{C}_0$, we deduce an isomorphism of topological $\mathscr{O}^a_{\mathbf{A}(U)}$-algebras :

$$\boldsymbol{\sigma}_{\mathscr{A}} : \boldsymbol{\sigma}^*_U\mathbf{A}(\mathscr{A})^+ \xrightarrow{\sim} \mathbf{A}(\mathscr{A})^+. \tag{8.5.32}$$

Denote by $\mathscr{O}^a_{\mathbf{A}(U)}[\boldsymbol{\sigma}]\text{-}\mathbf{Alg}_{\mathrm{qcoh}}$ the category whose objects are the quasi-coherent $\mathscr{O}^a_{\mathbf{A}(U)}$-algebras $\mathscr{B}$ endowed with an isomorphism $\boldsymbol{\sigma} : \boldsymbol{\sigma}^*_U\mathscr{B} \xrightarrow{\sim} \mathscr{B}$; the morphisms in this category are the continuous $\boldsymbol{\sigma}$-equivariant morphisms.

Lemma 8.5.33. (i) $\mathbf{A}(\mathscr{A})^+$ *is a quasi-coherent* $(\mathscr{O}_{\mathbf{A}(U)}, \mathfrak{m}_{\mathbf{A}}\mathscr{O}_{\mathbf{A}(U)})^a$*-algebra.*

(ii) *The rule* $\mathscr{A}^\wedge \mapsto \mathbf{A}(\mathscr{A})^+$ *extends to a functor :*

$$\mathscr{O}^a_{U^\wedge}\text{-}\mathbf{\acute{E}t} \to \mathscr{O}^a_{\mathbf{A}(U)}[\boldsymbol{\sigma}]\text{-}\mathbf{Alg}_{\mathrm{qcoh}}.$$

Proof. (i) is a special case of corollary 8.4.56(ii), and the functoriality of $\mathbf{A}(\mathscr{A})^+$ is obvious, from the construction. The $\boldsymbol{\sigma}$-equivariance means that every morphism $f : \mathscr{A}^\wedge \to \mathscr{A}'^\wedge$ of étale $\mathscr{O}^a_{U^\wedge}$-algebras induces a commutative diagram :

$$\begin{tikzcd} \boldsymbol{\sigma}^*_U\mathbf{A}(\mathscr{A})^+ \arrow[r, "\boldsymbol{\sigma}_{\mathscr{A}}"] \arrow[d, "\boldsymbol{\sigma}^*_U\mathbf{A}(f)^+"'] & \mathbf{A}(\mathscr{A})^+ \arrow[d, "\mathbf{A}(f)^+"] \\ \boldsymbol{\sigma}^*_U\mathbf{A}(\mathscr{A}')^+ \arrow[r, "\boldsymbol{\sigma}_{\mathscr{A}'}"] & \mathbf{A}(\mathscr{A}')^+ \end{tikzcd}$$

which is also clear from the construction. □

Lemma 8.5.34. *For every mapping* $\mathbf{n} : \mathbb{Z} \to \mathbb{N}$ *with finite non-empty support, the sequence*

$$0 \to \mathbf{A}(\mathscr{A})^+ \xrightarrow{\vartheta_{\mathbf{n}}\cdot\mathbf{1}_{\mathbf{A}(\mathscr{A})^+}} \mathbf{A}(\mathscr{A})^+ \to \mathscr{A}(\mathbf{n}) \to 0 \tag{8.5.35}$$

is short exact (again, $\mathscr{A}(\mathbf{n})$ *is viewed as a* $\mathscr{O}^a_{\mathbf{A}(U)}$*-algebra).*

Proof. Since $\vartheta_{\mathbf{n}}$ is regular in $\mathbf{A}(R)^+$, the complex :

$$0 \to \mathbf{A}(R)^+/\vartheta_{\mathbf{m}}\mathbf{A}(R)^+ \xrightarrow{\vartheta_{\mathbf{n}}} \mathbf{A}(R)^+/\vartheta_{\mathbf{n}+\mathbf{m}}\mathbf{A}(R)^+ \to \mathbf{A}(R)^+/\vartheta_{\mathbf{n}}\mathbf{A}(R)^+ \to 0$$

is short exact for every $\mathbf{m} : \mathbb{Z} \to \mathbb{N}$ of finite support. After tensoring by the flat $\mathscr{O}^a_{X(\mathbf{n}+\mathbf{m})}$-algebra $\mathscr{A}(\mathbf{n}+\mathbf{m})$, there follows a compatible system of short exact sequences :

$$\mathscr{E}(\mathbf{m}) \quad : \quad 0 \to \mathscr{A}(\mathbf{m}) \xrightarrow{\vartheta_{\mathbf{n}}} \mathscr{A}(\mathbf{m}+\mathbf{n}) \to \mathscr{A}(\mathbf{n}) \to 0$$

and (8.5.35) is naturally identified with :

$$\mathscr{E} := \lim_{\mathscr{C}_3} \mathscr{E}(\bullet).$$

However, since the functor $\mathscr{A}(\bullet)$ is cartesian, for every morphism f in $\mathscr{C}_3$, the corresponding morphism $\Gamma(V, \mathscr{A}(f))$ is an epimorphism for every truly affine open subset V of $\mathbf{A}(X)$ contained in $\mathbf{A}(U)$, hence $\mathscr{E}(V)$ is still exact (say, by [36, Lemma 2.4.2(iii)]). □

8.5.36. Let $\mathscr{A}^\wedge$ be an étale $\mathscr{O}^a_{U^\wedge}$-algebra, and set $\mathscr{A}_{/p} := \iota^*\mathscr{A}^\wedge$, where $\iota : U_{/p} \to U^\wedge$ is the natural closed immersion. We let $\mathbf{E}(U) := \mathbf{E}(X) \setminus \{0\}$; the map $\boldsymbol{\omega}_X$ restricts to a closed immersion $\boldsymbol{\omega}_U : \mathbf{E}(U) \to \mathbf{A}(U)$, and according to (8.5.20), the restriction

$$\overline{u}_U : U_{/p} \to \mathbf{E}(U)$$

of $\overline{u}_X$ induces a homeomorphism on the underlying topological spaces. Thus, we may naturally identify $\mathscr{A}_{/p}$ to the $(\mathscr{O}_{\mathbf{E}(U)}, \mathfrak{m}_{\mathbf{A}(V)}\mathscr{O}_{\mathbf{E}(U)})^a$-algebra $\overline{u}_{U*}\mathscr{A}_{/p}$. Under the identification $\overline{u}_U$, the structure sheaf of $\mathbf{E}(U)$ can be described as the inverse limit of the system of pseudo-discrete sheaves of rings :

$$\cdots \to \mathscr{O}_{U_{/p}} \xrightarrow{\Phi_{\mathscr{O}_U}} \mathscr{O}_{U_{/p}} \xrightarrow{\Phi_{\mathscr{O}_U}} \mathscr{O}_{U_{/p}}.$$

We wish to provide an analogous description for the $\mathscr{O}^a_{\mathbf{E}(U)}$-algebra $\boldsymbol{\omega}^*_U\mathbf{A}(\mathscr{A})^+$. To this aim, recall that the Frobenius map is an automorphism

$$\Phi_{\mathbf{E}(U)} := (\boldsymbol{\sigma}_{K^+}, \Phi_{\mathbf{E}(U)}) : (\mathbf{A}^+, \mathfrak{m}_\mathbf{A}, \mathbf{E}(U)) \to (\mathbf{A}^+, \mathfrak{m}_\mathbf{A}, \mathbf{E}(U))$$

which induces the identity on the underlying space $\mathbf{E}(U)$. Therefore, for every $k \in \mathbb{Z}$ we have a natural map of $(\mathscr{O}_{\mathbf{E}(U)}, \mathfrak{m}_\mathbf{A}\mathscr{O}_{\mathbf{E}(U)})^a$-algebras :

$$\Phi^{-k}_{\mathbf{E}(U)*}(\Phi_{\mathscr{A}_{/p}}) : \Phi^{-k}_{\mathbf{E}(U)*}\mathscr{A}_{/p} \to \Phi^{1-k}_{\mathbf{E}(U)*}\mathscr{A}_{/p}. \tag{8.5.37}$$

(Recall that, for every $t \in \mathbb{Z}$, the push-forward functor $\Phi^t_{\mathbf{E}(U)*} : \mathscr{O}^a_{\mathbf{E}(U)}\text{-}\mathbf{Alg} \to \mathscr{O}^a_{\mathbf{E}(U)}\text{-}\mathbf{Alg}$ replaces the structure morphism $\underline{1}_\mathscr{R} : \mathscr{O}^a_{\mathbf{E}(U)} \to \mathscr{R}$ of any $\mathscr{O}^a_{\mathbf{E}(U)}$-algebra $\mathscr{R}$, with the composition $\underline{1}_\mathscr{R} \circ \Phi^t_{\mathscr{O}_{\mathbf{E}(U)}}$: this corresponds to the pull-back functor of [36, §3.5.7].) Thus, for k running over the natural numbers, we obtain an inverse system of $\mathscr{O}^a_{\mathbf{E}(U)}$-algebras, with transition maps given by the morphisms (8.5.37). We may therefore define the $\mathscr{O}^a_{\mathbf{E}(U)}$-algebra :

$$\mathbf{E}(\mathscr{A})^+ := \lim_{k\in\mathbb{N}} \Phi^{-k}_{\mathbf{E}(U)*}\mathscr{A}_{/p}.$$

Lemma 8.5.38. *There are natural isomorphisms of $\mathscr{O}^a_{\mathbf{E}(U)}$-algebras :*

$$\mathbf{E}(\mathscr{A})^+ \xrightarrow{\sim} \boldsymbol{\omega}^*_U\mathbf{A}(\mathscr{A})^+$$

that fit into a commutative diagram of $\mathscr{O}^a_{\mathbf{E}(U)}$-algebras :

$$\begin{tikzcd}
\mathbf{E}(\mathscr{A})^+ \arrow[rr, "\sim"] \arrow[d, "\Phi_{\mathbf{E}(\mathscr{A})^+}"'] & & \boldsymbol{\omega}^*_U\mathbf{A}(\mathscr{A})^+ \arrow[d, "\boldsymbol{\omega}^*_U\boldsymbol{\sigma}_\mathscr{A}"] \\
\Phi_{\mathbf{E}(U)_*}\mathbf{E}(\mathscr{A})^+ \arrow[r, "\sim"] & \Phi_{\mathbf{E}(U)_*} \circ \boldsymbol{\omega}^*_U\mathbf{A}(\mathscr{A})^+ \arrow[r, equal] & \boldsymbol{\omega}^*_U \circ \boldsymbol{\sigma}_{\mathbf{A}(U)*}\mathbf{A}(\mathscr{A})^+.
\end{tikzcd}$$

Proof. To begin with, we claim that the natural map

$$\boldsymbol{\omega}^*_U\mathbf{A}(\mathscr{A})^+ \to \lim_{\mathbf{n}:\mathbb{Z}\to\mathbb{N}} \mathscr{A}(\mathbf{n})_{/p}$$

is an isomorphism of $\mathscr{O}^a_{\mathbf{E}(U)}$-algebras. This can be established as in the proof of lemma 8.5.34. Namely, in view of claim 4.6.40(ii) we have short exact sequences :

$$0 \to \mathbf{A}(R)^+/\vartheta_\mathbf{n}\mathbf{A}(R)^+ \xrightarrow{p} \mathbf{A}(R)^+/\vartheta_\mathbf{n}\mathbf{A}(R)^+ \to \mathbf{A}(R)^+/(p, \vartheta_\mathbf{n}) \to 0$$

for every $\mathbf{n} : \mathbb{Z} \to \mathbb{N}$ with finite non-empty support. After tensoring with the flat $\mathscr{O}^a_{U(\mathbf{n})}$-algebra $\mathscr{A}(\mathbf{n})$ we deduce a short exact sequence $\mathscr{E}(\mathbf{n}) := (0 \to \mathscr{A}(\mathbf{n}) \to \mathscr{A}(\mathbf{n}) \to \mathscr{A}(\mathbf{n})_{/p} \to 0)$ of quasi-coherent $\mathscr{O}^a_{\mathbf{A}(U)}$-algebras, and the assertion follows after forming the limit over the cofiltered system of short exact sequences $(\mathscr{E}(\mathbf{n})(V) \mid \mathbf{n} : \mathbb{Z} \to \mathbb{N})$, which has epimorphic transition maps whenever $V \subset \mathbf{A}(U)$ is a truly affine open subset of $\mathbf{A}(X)$.

It follows likewise, that $\boldsymbol{\omega}^*_U\boldsymbol{\sigma}_\mathscr{A}$ is the limit of the cofiltered system of isomorphisms

$$\mathscr{A}(\boldsymbol{\sigma}_{U(\mathbf{n})_{/p}}) : \Phi^*_{\mathbf{E}(U)}\mathscr{A}(\mathbf{n}[1])_{/p} = \boldsymbol{\sigma}^*_{U(\mathbf{n})_{/p}}\mathscr{A}(\mathbf{n}[1])_{/p} \xrightarrow{\sim} \mathscr{A}(\mathbf{n})_{/p}.$$

Next, as explained in (8.5.20), $\mathbf{E}(U)$ is also the colimit of the direct system of closed subschemes $(U(-k)_{/p} \mid k \in \mathbb{N})$, with transition maps given by the restrictions of the morphisms $i_{(-h,-k)}$ for all $k > h$. Hence $\boldsymbol{\omega}^*_U\mathbf{A}(\mathscr{A})^+$ is also the inverse limit of a system of $\mathscr{O}^a_{\mathbf{E}(U)}$-algebras $\mathscr{A}(\bullet)_{/p} := (\mathscr{A}(-k)_{/p} \mid k \in \mathbb{N})$, and we need to make explicit the transition maps of this system.

By construction, the restriction to $\mathscr{C}_2$ of the functor $\mathscr{A}(\bullet)$ is isomorphic to the restriction of the functor $G_{\mathscr{A}^\wedge}$, so we are reduced to considering the system $(G_{\mathscr{A}^\wedge}(U(-k)_{/p}) \mid k \in \mathbb{N})$.

However, for every $k \in \mathbb{Z}$ we have a commutative diagram of morphisms of schemes :

$$\begin{tikzcd} U_{/p} \arrow[r, "\Phi_U"] \arrow[d] & U_{/p} \arrow[d] \\ U(k+1)_{/p} \arrow[r, "i_{(k+1,k)}"] & U(k)_{/p} \end{tikzcd}$$

whose vertical arrows are the restrictions of the isomorphisms (8.5.12). By construction, to this diagram there corresponds a diagram of cartesian morphisms of étale $\mathscr{O}^a$-algebras :

$$\begin{tikzcd} \mathscr{A}_{/p} \arrow[r, "\Phi_{\mathscr{A}_{/p}}"] \arrow[d] & \mathscr{A}_{/p} \arrow[d] \\ G_{\mathscr{A}^\wedge}(U(k+1)_{/p}) \arrow[r, "G_{\mathscr{A}^\wedge}(i_{(k+1,k)})"] & G_{\mathscr{A}^\wedge}(U(k)_{/p}). \end{tikzcd}$$

Especially, the vertical arrows are isomorphisms, whence the sought isomorphism. To show that the diagram of the lemma commutes, one argues analogously : it suffices to consider the restriction to $\mathscr{C}_1$ of the functor $\mathscr{A}(\bullet)$, in which case everything can be made explicit : the details shall be left to the reader. $\square$

8.6. **Finite group actions on almost algebras.** In this section we fix a basic setup $(V, \mathfrak{m})$ such that $\widetilde{\mathfrak{m}} := \mathfrak{m} \otimes_V \mathfrak{m}$ is a flat V-module (see [36, §2.1.1]), and we wish to consider some descent problems for V^a-algebras endowed with a finite group of automorphisms. Hence, the results below overlap with those of [36, §4.5].

8.6.1. Let G be a finite group, A a V^a-algebra, and let $S := \operatorname{Spec} V^a$, $X := \operatorname{Spec} A$. A *right action of* G *on* X is a group homomorphism :

$$\rho : G \to \operatorname{Aut}_{V^a\text{-}\mathbf{Alg}}(A)$$

from G to the group of automorphisms of A. Let G_S be the affine group S-scheme defined by G; hence every $g \in G$ determines a section $g_S : S \to G_S$ of the structure morphism $G_S \to S$, and the resulting morphism:

$$\coprod_{g \in G} g_S : S \amalg \cdots \amalg S \to G_S$$

is an isomorphism of S-schemes. Then ρ can be also regarded as a right action of G_S on X, as defined in [36, §3.3.6]. Especially, ρ induces morphisms of S-schemes :

$$\partial_i : X \times G := X \times_S G_S \to X \qquad i = 0, 1$$

as in *loc. cit.*, and we may define a *G-action on an A-module* M (covering the given action of G on X) as a morphism of quasi-coherent $\mathscr{O}_{X \times G}$-modules :

$$\beta : \partial_0^* M \to \partial_1^* M$$

fulfilling the conditions of [36, §3.3.7]. One also says that (M, β) is a *G-equivariant A-module*. We denote by $A[G]\text{-}\mathbf{Mod}$ the category of all G-equivariant A-modules and G-equivariant A-linear morphisms. Notice that $A[G]\text{-}\mathbf{Mod}$ is an abelian tensor category : indeed, for any two objects (M, β), (M, β') we may set $(M, \beta) \otimes_A (M', \beta') := (M \otimes_A M', \beta \otimes_{\mathscr{O}_{X \times G}} \beta')$.

8.6.2. Likewise, if B is any A-algebra, a G-action on B is a morphism $\beta : \partial_0^* B \to \partial_1^* B$ of quasi-coherent $\mathscr{O}_{X\times G}$-algebras, such that the pair (B, β) is a G-equivariant A-module. We say that (B, β) is a *G-equivariant A-algebra*, and we denote by $A[G]$-$\mathbf{Alg}$ the category of such pairs, with G-equivariant morphisms of A-algebras. One verifies easily that the datum (B, β) is the same as a morphism $\psi : A \to B$ of V^a-algebras, together with a G-action $\rho_B : G \to \mathrm{Aut}_{V^a\text{-}\mathbf{Alg}}(B)$ on the affine scheme $\mathrm{Spec}\, B$, such that the diagram :

$$\begin{tikzcd} A \arrow[r, "\psi"] \arrow[d, "\rho(g)"'] & B \arrow[d, "\rho_B(g)"] \\ A \arrow[r, "\psi"] & B \end{tikzcd}$$

commutes for every $g \in G$. We shall also consider the full subcategory $A[G]$-$\mathbf{Alg}_{\mathrm{fl}}$ (resp. $A[G]$-$\mathbf{w.\acute{E}t}$, resp. $A[G]$-$\mathbf{\acute{E}t}$, resp. $A[G]$-$\mathbf{\acute{E}t}_{\mathrm{afp}}$) of all such pairs, where B is a flat (resp. weakly étale, resp. étale, resp. étale and almost finitely presented) A-algebra.

8.6.3. The *trivial G-action* on X is the map ρ with $\rho(g) = \mathbf{1}_A$ for every $g \in G$; this is the same as saying that $\partial_0 = \partial_1$. If G acts trivially on X, a G-equivariant A-module (M, β) is the same as a group homomorphism $\overline{\beta} : G \to \mathrm{Aut}_A(M)$ from G to the group of A-linear automorphisms of M. Namely, for every $g \in G$, one lets :

$$\overline{\beta}(g) := (\mathbf{1}_X \times_S g_S)^* \beta \tag{8.6.4}$$

and conversely, to a given map $\overline{\beta}$ there corresponds a unique pair (M, β) such that (8.6.4) holds.

Under this correspondence, the *trivial G-action* $\overline{\beta}_0$ (such that $\overline{\beta}(g) = \mathbf{1}_M$ for every $g \in G$) corresponds to the identity morphism $\beta_0 : \partial_0^* M \xrightarrow{\sim} \partial_1^* M$.

More generally, let (M, β) be a G-action on an A-module M, covering the trivial G-action on X; for every A_*-valued character $\chi : G \to A_*^\times$ of G, we let

$$M_\chi := \bigcap_{g \in G} \mathrm{Ker}(\overline{\beta}(g) - \chi(g) \cdot \mathbf{1}_M).$$

The restriction of $\overline{\beta}$ defines a G-action on M_χ, such that the monomorphism $M_\chi \subset M$ is G-equivariant. In the special case where χ is the trivial character, we have $M_\chi = (M, \beta)^G$, the largest G-equivariant A-submodule of (M, β) on which β restricts to the trivial G-action (*i.e.* the submodule fixed by G). When the notation is not ambiguous, we shall often just write M^G instead of $(M, \beta)^G$.

8.6.5. If $H \subset G$ is a subgroup, any G-action ρ on X induces by restriction an H-action $\rho_{|H}$; then the morphisms $\partial_i : X \times H \to X$ are just the restrictions of the corresponding morphisms for G (under the natural closed immersion $X \times H \to X \times G$). Similarly, a G-action β on an A-module M induces by restriction an H-action $\beta_{|H}$ on the same module.

Let $Y := \mathrm{Spec}\, B$ be any affine S-scheme. The G-action ρ induces a G-action $Y \times_S \rho$ on $Y \times_S X$; namely, $Y \times_S \rho(g) := \mathbf{1}_Y \times_S \rho(g)$ for every $g \in G$. In terms of the group scheme G_S, this is the action given by the morphisms :

$$\partial_{Y,i} := \mathbf{1}_Y \times_S \partial_i : (Y \times_S X) \times G \to (Y \times_S X) \qquad i = 0, 1.$$

Let $\pi_X : Y \times_S X \to X$ be the natural morphism; then every G-equivariant A-module (M, β) induces a G-equivariant $B \otimes_{V^a} A$-module $\pi_X^*(M, \beta) := (\pi_X^* M, \pi_X^* \beta)$, whose action covers the G-action $Y \times_S \rho$ on $Y \times_S X$.

8.6.6. Let us set :

$$X/G := \mathrm{Coequal}(X \times G \overset{\partial_1}{\underset{\partial_1}{\rightrightarrows}} X) \quad \text{and} \quad X^{\langle g \rangle} := \mathrm{Equal}(X \overset{\rho(g)}{\underset{\mathbf{1}_X}{\rightrightarrows}} X) \quad \text{for every } g \in G.$$

In other words, $X/G = \mathrm{Spec}\, A^G$, the subalgebra fixed by G, and $X^{\langle g \rangle}$ is the closed subscheme fixed by the subgroup $\langle g \rangle \subset G$ generated by g. Thus, $X^{\langle g \rangle}$ is the spectrum of a quotient A/I_g of A, where $I_g \subset A$ is the ideal generated by the almost elements of the form $a - \rho(g)(a)$, for every $a \in A_*$.

Let $\pi : X \to X/G$ be the natural morphism, N any quasi-coherent $\mathscr{O}_{X/G}$-module; the pull-back $\pi^* N$ is the quasi-coherent $\mathscr{O}_X$-module $A \otimes_{A^G} N$. By construction, there is a natural isomorphism $\beta_N : \partial_0^*(\pi^* N) \xrightarrow{\sim} \partial_1^*(\pi^* N)$ (deduced from the natural isomorphisms of functors $\partial_i^* \circ \pi^* \simeq (\pi \circ \partial_i)^*$, for $i = 0, 1$), and one verifies easily that β_N is a G-action on $\pi^* N$.

Moreover, let $i_g : X^{\langle g \rangle} \to X$ be the natural closed immersion; if M is any A-module, then $i_g^* M$ is the A/I_g-module $M/I_g M$. Especially, take $M := \pi^* N$; we notice that the restriction $\rho_{|\langle g \rangle}$ of the given action ρ, induces the trivial $\langle g \rangle$-action on $X^{\langle g \rangle}$, and directly from the construction, we see that the natural action $\beta_{N|\langle g \rangle}$ restricts to the trivial $\langle g \rangle$-action on $i_g^* M$.

We are thus led to the :

Definition 8.6.7. Let G be a finite group, A a V^a-algebra, ρ a G-action on $X := \mathrm{Spec}\, A$.

(i) The category $A[G]\text{-}\mathbf{Mod}_{\mathrm{hor}}$ of *A-modules with horizontal G-action* is the full subcategory of $A[G]\text{-}\mathbf{Mod}$ consisting of all pairs (M, β) subject to the following condition. For every $g \in G$, the restriction $\beta_{|\langle g \rangle}$ induces the trivial action on $i_g^* M$.

(ii) We denote by $A[G]\text{-}\mathbf{Mod}_{\mathrm{hor.fl}}$ the full subcategory of $A[G]\text{-}\mathbf{Mod}_{\mathrm{hor}}$ consisting of the pairs (M, β) as above, such that M is a flat A-module.

(iii) Likewise, we denote by $A[G]\text{-}\mathbf{Alg}_{\mathrm{hor}}$ (resp. $A[G]\text{-}\mathbf{Alg}_{\mathrm{hor.fl}}$, resp. $A[G]\text{-}\mathbf{w.\acute{E}t}_{\mathrm{hor}}$, resp. $A[G]\text{-}\mathbf{\acute{E}t}_{\mathrm{hor}}$, resp. $A[G]\text{-}\mathbf{\acute{E}t}_{\mathrm{hor.afp}}$) the full subcategory of $A[G]\text{-}\mathbf{Alg}$ (resp. $A[G]\text{-}\mathbf{Alg}_{\mathrm{fl}}$, resp. $A[G]\text{-}\mathbf{w.\acute{E}t}$, resp. $A[G]\text{-}\mathbf{\acute{E}t}$, resp. $A[G]\text{-}\mathbf{\acute{E}t}_{\mathrm{afp}}$) consisting of all pairs (B, β) which are horizontal, when regarded as G-equivariant A-modules.

Notice that the tensor product of horizontal modules (resp. algebras) is again horizontal. By the foregoing, the rule $N \mapsto (\pi^* N, \beta_N)$ defines a functor :

$$A^G\text{-}\mathbf{Mod} \to A[G]\text{-}\mathbf{Mod}_{\mathrm{hor}}. \tag{8.6.8}$$

On the other hand, if (M, β) is any G-equivariant A-module, the pair $\pi_*(M, \beta) := (\pi_* M, \pi_* \beta)$ may be regarded as a G-action on $\pi_* M$, covering the trivial G-action on $\mathrm{Spec}\, A^G$, *i.e.* a group homomorphism $G \to \mathrm{Aut}_{A^G}(M)$ (see (8.6.3)). One verifies easily that the functor $N \mapsto (\pi^* N, \beta_N)$ is left adjoint to the functor $A[G]\text{-}\mathbf{Mod} \to A^G\text{-}\mathbf{Mod} : (M, \beta) \mapsto \pi_*(M, \beta)^G$ (details left to the reader). Hence, also (8.6.8) admits a right adjoint, given by the same rule. Similar assertions hold for the analogous functors :

$$A^G\text{-}\mathbf{Alg} \to A[G]\text{-}\mathbf{Alg}_{\mathrm{hor}} \tag{8.6.9}$$

and the variants considered in definition 8.6.7(iii).

Lemma 8.6.10. *In the situation of definition* 8.6.7*, suppose furthermore that the order $o(G)$ of G is invertible in A_*, and let (M, β) be any G-equivariant A-module. Then :*

(i) *For every A^G_*-valued character $\chi : G \to (A^G_*)^\times$, the natural A^G-linear monomorphism $\pi_* M_\chi \to \pi_* M$ admits a G-equivariant A^G-linear right inverse $\pi_* M \to \pi_* M_\chi$.*

(ii) *For every A^G-module N, the unit of adjunction :*

$$\varepsilon_N : N \to (A \otimes_{A^G} N)^G$$

is an isomorphism.

(iii) *Let* $Y := \mathrm{Spec}\, B$ *be any affine* S*-scheme; denote by* $\pi_X : Y \times_S X \to X$ *and* $\pi_{X/G} : Y \times_S (X/G) \to X/G$ *the natural projections. Then the natural morphism :*

$$\pi^*_{X/G}(M, \beta)^G \to (\pi^*_X M, \pi^*_X \beta)^G$$

is an isomorphism of $B \otimes_{V^a} A^G$*-modules.*

Proof. This is standard : for every χ as in (i), the ring algebra $A_*[G]$ admits the central idempotent :

$$e_\chi := \frac{1}{o(G)} \cdot \sum_{g \in G} \chi(g) \cdot g \tag{8.6.11}$$

and $M_\chi = e_\chi M$ for any G-equivariant A-module M. Especially, we may take $M := N \otimes_{A^G} A$, and e_0 the central idempotent associated to the trivial character, in which case $e_0 A = A^G$, and $M = N \oplus (N \otimes_{A^G} (1 - e_0)A)$, so all the claims follow easily. □

Definition 8.6.12. Let Γ be any finite abelian group with neutral element $0 \in \Gamma$.

(i) A Γ*-graded* V^a*-algebra* is a pair $\underline{A} := (A, \mathrm{gr}_\bullet A)$ consisting of a V^a-algebra A and a decomposition $A = \bigoplus_{\chi \in \Gamma} \mathrm{gr}_\chi A$ as a direct sum of V^a-modules, such that :

$$1 \in \mathrm{gr}_0 A_* \quad \text{and} \quad \mathrm{gr}_\chi A \cdot \mathrm{gr}_{\chi'} A \subset \mathrm{gr}_{\chi+\chi'} A \qquad \text{for every } \chi, \chi' \in \Gamma$$

(where as usual $\mathrm{gr}_\chi A \cdot \mathrm{gr}_{\chi'} A$ denotes the image of the restriction $\mathrm{gr}_\chi A \otimes_{V^a} \mathrm{gr}_{\chi'} A \to A$ of the multiplication morphism μ_A). Especially, $\mathrm{gr}_0 A$ is a V^a-subalgebra of A, and every submodule $\mathrm{gr}_\chi A$ is a $\mathrm{gr}_0 A$-module.

(ii) A Γ*-graded* $\underline{A}$*-module* is a pair $\underline{N} := (N, \mathrm{gr}_\bullet N)$ consisting of an A-module N and a decomposition $N = \bigoplus_{\chi \in \Gamma} \mathrm{gr}_\chi N$ as a direct sum of V^a-modules, such that :

$$\mathrm{gr}_\chi A \cdot \mathrm{gr}_{\chi'} N \subset \mathrm{gr}_{\chi+\chi'} N \qquad \text{for every } \chi, \chi' \in \Gamma.$$

Of course, a morphism of Γ-graded $\underline{A}$-modules $\underline{N} \to \underline{N}' := (N', \mathrm{gr}_\bullet N')$ is an A-linear morphism $N \to N'$ that respects the gradings.

(iii) For every subgroup $\Delta \subset \Gamma$, let $J_\Delta \subset A$ be the graded ideal generated by $\bigoplus_{\chi \notin \Delta} \mathrm{gr}_\chi A$. We say that $\underline{N}$ is *horizontal* if $\mathrm{gr}_\chi(N/J_\Delta N) := \mathrm{gr}_\chi N/(\mathrm{gr}_\chi N \cap J_\Delta N) = 0$ for every subgroup $\Delta \subset \Gamma$ and every $\chi \notin \Delta$.

(iv) If $\Delta \subset \Gamma$ is any subgroup, $p : \Gamma \to \Gamma/\Delta$ the natural projection, and $\rho \in \Gamma/\Delta$ any element, we let :

$$\mathrm{gr}^{\Gamma/\Delta}_\rho N := \bigoplus_{\chi \in p^{-1}(\rho)} \mathrm{gr}_\chi N$$

and set $\underline{N}_{\Gamma/\Delta} := (N, \mathrm{gr}^{\Gamma/\Delta}_\bullet N)$. Then $\underline{A}_{\Gamma/\Delta}$ is a Γ/Δ-graded V^a-algebra, and $\underline{N}_{\Gamma/\Delta}$ is a Γ/Δ-graded $\underline{A}_\Delta$-module. Moreover, let also $\underline{N}_{|\Delta}$ be the pair consisting of $N_{|\Delta} := \mathrm{gr}^{\Gamma/\Delta}_0 N$ together with its decomposition $N_{|\Delta} = \bigoplus_{\chi \in \Delta} \mathrm{gr}_\chi N$; then $\underline{A}_{|\Delta}$ is a Δ-graded V^a-algebra, and $\underline{N}_{|\Delta}$ is a Δ-graded $\underline{A}_{|\Delta}$-module.

Proposition 8.6.13. *In the situation of definition* 8.6.12*, let* $\underline{N}$ *be any* Γ*-graded* $\underline{A}$*-module. Then the following conditions are equivalent :*

(a) $\underline{N}$ *is horizontal.*
(b) *The natural morphism* $A \otimes_{\mathrm{gr}_0 A} \mathrm{gr}_0 N \to N$ *is an epimorphism.*

Proof. (b)⇒(a): The assertion is obvious for the Γ-graded $\underline{A}$-module consisting of $A \otimes_{\mathrm{gr}_0 A} \mathrm{gr}_0 N$ and its natural grading deduced from $\mathrm{gr}_\bullet A$; however any (graded) quotient of a horizontal module is horizontal, hence the assertion follows also for $\underline{N}$.

(a)⇒(b): We argue by induction on $o(\Gamma)$. The first case is covered by the following :

Claim 8.6.14. The proposition holds if $o(\Gamma)$ is a prime number.

Proof of the claim. Indeed, suppose that $\underline{N}$ is horizontal. We may replace N by $N/(A \cdot \mathrm{gr}_0 N)$, which is a horizontal Γ-graded $\underline{A}$-module, when endowed with the grading induced from $\mathrm{gr}_\bullet N$. Then $\mathrm{gr}_0 N = 0$, and we have to show that $N = 0$. By assumption, $N \subset J_{\{0\}} N$, *i.e.* :

$$\mathrm{gr}_\chi N \subset \sum_{\sigma \neq 0,\chi} \mathrm{gr}_\sigma A \cdot \mathrm{gr}_{\chi-\sigma} N \qquad \text{for every } \chi \neq 0.$$

For every $n > 0$, the symmetric group S_n acts on the set $(\Gamma \setminus \{0\})^n$ by permutations; we let $Q_n := (\Gamma \setminus \{0\})^n / S_n$, the set of equivalence classes under this action. For every $\underline{\sigma} := (\sigma_1, \dots, \sigma_n) \in Q_n$, let $|\underline{\sigma}| := \sum_{i=1}^n \sigma_i$. By an easy induction, it follows that :

$$\mathrm{gr}_\chi N \subset \sum_{\underline{\sigma} \in Q_n} \mathrm{gr}_{\sigma_1} A \cdots \mathrm{gr}_{\sigma_n} A \cdot \mathrm{gr}_{\chi - |\underline{\sigma}|} N.$$

Since every element of $\Gamma \setminus \{0\}$ generates Γ, it is also clear that there exists $n \in \mathbb{N}$ large enough such that every sequence $\underline{\sigma}$ in Q_n admits a subsequence, say $\underline{\tau} := (\tau_1, \dots, \tau_m)$ for some $m \leq n$, with $|\underline{\tau}| = \chi$ (details left to the reader). Up to a permutation, we may assume that $\underline{\tau}$ is the final segment of $\underline{\sigma}$; then we have :

$$\mathrm{gr}_{\sigma_1} A \cdots \mathrm{gr}_{\sigma_n} A \cdot \mathrm{gr}_{\chi - |\underline{\sigma}|} N \subset \mathrm{gr}_{\sigma_1} A \cdots \mathrm{gr}_{\sigma_{n-m}} A \cdot \mathrm{gr}_0 N = 0$$

whence the claim. ♢

Next, suppose that the assertion is already known for every subgroup $\Gamma' \subset \Gamma$, every graded Γ'-algebra $\underline{B}$, and every Γ'-graded $\underline{B}$-module $\underline{P}$. We choose a subgroup $\Gamma' \subset \Gamma$ such that $(\Gamma : \Gamma')$ is a prime number. We shall use the following :

Claim 8.6.15. Let G be a group, $0 \in G$ the neutral element, $H \subset G$ a subgroup. For every subgroup $L \subset G$ with $H \cap L = \{0\}$, choose an element $g_L \in G \setminus L$; denote by S the subgroup generated by all these elements g_L. Then $S \cap H \neq \{0\}$.

Proof of the claim. Indeed, if $S \cap H = \{0\}$, we would have $g_S \in S$, a contradiction. ♢

Claim 8.6.16. Suppose that $\underline{N}$ is horizontal. Then $\underline{N}_{|\Gamma'}$ is a horizontal Γ'-graded $\underline{A}_{|\Gamma'}$-module.

Proof of the claim. For any given subgroup $\Delta \subset \Gamma'$, let $J'_\Delta \subset A_{|\Gamma'}$ be the ideal generated by $\bigoplus_{\chi \in \Gamma' \setminus \Delta} \mathrm{gr}_\chi A$; have to show that $\mathrm{gr}_\chi(N/J'_\Delta N) = 0$ for every $\chi \in \Gamma' \setminus \Delta$. To this aim, we may replace Γ, Γ', $\underline{A}$, $\underline{A}_{|\Gamma'}$, $\underline{N}$ and $\underline{N}_{|\Gamma'}$, by respectively Γ/Δ, Γ'/Δ, $\underline{A}_{\Gamma/\Delta}$, $(\underline{A}_{|\Gamma'})_{\Gamma'/\Delta}$, $\underline{N}_{\Gamma/\Delta}$ and $(\underline{N}_{|\Gamma'})_{\Gamma'/\Delta}$, which allows to assume that $\Delta = \{0\}$. In this case, we have to show that :

$$N_{|\Gamma'} = \mathrm{gr}_0 N + J'_{\{0\}} \cdot N_{|\Gamma'}. \tag{8.6.17}$$

Notice that $A_{|\Gamma'} \cdot \mathrm{gr}_0 N = \mathrm{gr}_0 N + J'_{\{0\}} \cdot \mathrm{gr}_0 N$; the quotient $P := N/(A \cdot \mathrm{gr}_0 N)$ carries a unique Γ-grading $\mathrm{gr}_\bullet P$ such that the projection $\underline{N} \to \underline{P} := (P, \mathrm{gr}_\bullet P)$ is a morphism of Γ-graded $\underline{A}$-modules (namely, $\mathrm{gr}_\chi P := N_\chi / \mathrm{gr}_\chi A \cdot \mathrm{gr}_0 N$ for every $\chi \in \Gamma$). Moreover, P is horizontal, $\mathrm{gr}_0 P = 0$, and (8.6.17) is equivalent to : $P_{|\Gamma'} = J'_{\{0\}} \cdot P_{|\Gamma'}$. Therefore we may replace $\underline{N}$ by $\underline{P}$, and assume from start that $\mathrm{gr}_0 N = 0$. Similarly, notice that $J'_{\{0\}} \cdot N$ is a Γ-graded $\underline{A}$-submodule of $\underline{N}$, and the pair $\underline{Q}$ consisting of $Q := N/(J'_{\{0\}} \cdot N)$ and its quotient grading, is horizontal. Moreover, $\mathrm{gr}_\chi Q = \mathrm{gr}_\chi(N_{|\Gamma'}/J'_{\{0\}} N_{|\Gamma'})$, for every $\chi \in \Gamma'$. Hence, we may replace N by Q, which allows to assume as well that

$$J'_{\{0\}} N_{|\Gamma'} = 0 \tag{8.6.18}$$

in which case, we are reduced to showing that $N = 0$. Now, by assumption, for every subgroup $H \subset \Gamma$ we have :

$$N = N_{|H} + J_H N.$$

Let $\mathscr{C}$ be the set of all subgroups $H \subset \Gamma$ with $H \cap \Gamma' = \{0\}$; we deduce that :

$$\mathrm{gr}_\chi N \subset \sum_{\sigma \notin H} \mathrm{gr}_\sigma A \cdot \mathrm{gr}_{\chi-\sigma} N \qquad \text{for every } \chi \in \Gamma' \setminus \{0\} \text{ and every } H \in \mathscr{C}. \tag{8.6.19}$$

Moreover, in view of (8.6.18) we may omit from the sum in (8.6.19) all the elements σ that lie in Γ'; for the remaining elements we have $\chi - \sigma \notin \Gamma'$, hence we may apply claim 8.6.14 to the horizontal Γ/Γ'-graded $\underline{A}_{\Gamma/\Gamma'}$-module $\underline{N}_{\Gamma/\Gamma'}$, to deduce that :

$$\mathrm{gr}_{\chi-\sigma} N \subset \sum_{\delta \in \Gamma' \setminus \{0\}} \mathrm{gr}_{\chi-\sigma-\delta} A \cdot \mathrm{gr}_\delta N.$$

Therefore :

$$\mathrm{gr}_\chi N \subset \sum_{\sigma \notin H \cup \Gamma'} \sum_{\delta \in \Gamma' \setminus \{0\}} \mathrm{gr}_\sigma A \cdot \mathrm{gr}_{\chi-\sigma-\delta} A \cdot \mathrm{gr}_\delta N \qquad \text{for every } \chi \in \Gamma' \setminus \{0\} \text{ and every } H \in \mathscr{C}.$$

However – again due to (8.6.18) – we may omit from this sum all the terms corresponding to the pairs (σ, δ) with $\chi \neq \delta$, hence we conclude that :

$$\mathrm{gr}_\chi N \subset \sum_{\sigma \notin H \cup \Gamma'} \mathrm{gr}_\sigma A \cdot \mathrm{gr}_{-\sigma} A \cdot \mathrm{gr}_\chi N \qquad \text{for every } \chi \in \Gamma' \setminus \{0\} \text{ and every } H \in \mathscr{C}.$$

Denote by Σ the set of all mappings $\underline{\sigma} : \mathscr{C} \to \Gamma$ such that $\underline{\sigma}(H) \notin H \cup \Gamma'$ for every $H \in \mathscr{C}$. Moreover, for every $\underline{\sigma} \in \Sigma$, set :

$$B_{\underline{\sigma}} := \prod_{H \in \mathscr{C}} \mathrm{gr}_{\underline{\sigma}(H)} A \cdot \mathrm{gr}_{-\underline{\sigma}(H)} A$$

(this is an ideal of $\mathrm{gr}_0 A$) and let $S_{\underline{\sigma}} \subset \Gamma$ be the subgroup generated by the image of $\underline{\sigma}$. By an easy induction we deduce that :

$$\mathrm{gr}_\chi N \subset \sum_{\underline{\sigma} \in \Sigma} B^n_{\underline{\sigma}} \cdot \mathrm{gr}_\chi N \qquad \text{for every } n > 0.$$

By claim 8.6.15, for every $\underline{\sigma} \in \Sigma$ we may find $\gamma(\underline{\sigma}) \in S_{\underline{\sigma}} \cap \Gamma' \setminus \{0\}$. On the other hand, since Γ is finite and abelian, it is easy to verify that there exists $n \in \mathbb{N}$ large enough such that

$$B^n_{\underline{\sigma}} \subset \mathrm{gr}_{\gamma(\underline{\sigma})} A \cdot \mathrm{gr}_{-\gamma(\underline{\sigma})} A \qquad \text{for every } \underline{\sigma} \in \Sigma$$

(details left to the reader). But (8.6.18) implies that $\mathrm{gr}_{\gamma(\underline{\sigma})} A \cdot \mathrm{gr}_\chi N = 0$ whenever $\chi \in \Gamma' \setminus \{0\}$, so the claim follows. ◊

To conclude, we apply first claim 8.6.14 to the horizontal Γ/Γ'-graded $\underline{A}_{\Gamma/\Gamma'}$-module $\underline{N}_{\Gamma/\Gamma'}$, to see that $N_{|\Gamma/\Gamma'}$ generates the A-module N, and then claim 8.6.16 – together with our inductive assumption – to deduce that $\mathrm{gr}_0 N$ generates that $A_{|\Gamma/\Gamma'}$-module $N_{|\Gamma/\Gamma'}$. The proposition follows. □

8.6.20. Recall that a morphism $M \to N$ of A-modules is said to be *pure* if the natural morphism $Q \otimes_A M \to Q \otimes_A N$ is a monomorphism for every A-module Q. A morphism $A \to B$ of V^a-algebras is called *pure* if it is pure when regarded as a morphism of A-modules.

Lemma 8.6.21. *Let $f : A \to B$ be a pure morphism of V^a-algebras, M an A-module. Then :*

(i) *If $B \otimes_A M$ is a flat B-module, then M is a flat A-module (*i.e. *f descends flatness).*

(ii) *If $B \otimes_A M$ is almost finitely generated (resp. almost finitely presented) as a B-module, then M is almost finitely generated (resp. almost finitely presented) as an A-module.*

Proof. (i): To begin with, we remark :

Claim 8.6.22. Let R be any V-algebra such that $A = R^a$.

(i) A morphism $\varphi : M_1 \to M_2$ of A-modules is pure if and only if the same holds for the induced morphism $\varphi_! : M_{1!} \to M_{2!}$ of R-modules.
(ii) A morphism $g : A \to B'$ of V^a-algebras is pure if and only if the same holds for the induced morphism $g_{!!} : A_{!!} \to B'_{!!}$.

Proof of the claim. (i): Suppose that φ is pure, and let Q be any R-module. Then $Q \otimes_R M_{i!} \simeq (Q^a \otimes_A M_i)_!$ for $i = 1, 2$. Since the functor $M \mapsto M_!$ is exact ([36, Cor.2.2.24(i)]), we deduce that $\varphi_!$ is pure. The converse is easy, and shall be left to the reader.

(ii): From (i) we already see that g is pure whenever $g_{!!}$ is. Next, suppose that g is pure; we may assume that $R = A_{!!}$, and then (i) says that $g_!$ is a pure morphism of $A_{!!}$-modules. However, the natural diagram of $A_{!!}$-modules :

$$\begin{tikzcd} A_! \arrow[r] \arrow[d] & B'_! \arrow[d] \\ A_{!!} \arrow[r] & B'_{!!} \end{tikzcd}$$

is cofibred; since tensor products are right exact functors, the claim follows. ◊

The assertion now follows from claim 8.6.22(ii) and [45, Partie II, Lemme 1.2.1].

(ii): Suppose first that $B \otimes_A M$ is an almost finitely generated B-module. The assumption on f implies that $\mathrm{Ann}_A(B \otimes_A M) \subset \mathrm{Ann}_A M$; then [36, Rem.3.2.26(i)] shows that M is an almost finitely generated A-module.

Finally, we suppose that $B \otimes_A M$ is an almost finitely presented B-module, and we wish to show that M is an almost finitely presented A-module. To this aim, let $\varphi : N \to N'$ be a morphism of A-modules. The assumption on f implies that the natural morphism $\mathrm{Ker}\,\varphi \to \mathrm{Ker}(\mathbf{1}_B \otimes_A \varphi)$ is a monomorphism; especially :

$$\mathrm{Ann}_A(\mathrm{Ker}(\mathbf{1}_B \otimes_A \varphi)) \subset \mathrm{Ann}_A(\mathrm{Ker}\,\varphi).$$

Then one may easily adapt the proof of [36, Lemma 3.2.25(iii)], to derive the assertion. □

Theorem 8.6.23. *In the situation of definition* 8.6.7*, suppose furthermore that G is abelian and the order $o(G)$ of G is invertible in A_*. Then the following holds :*

(i) *Let M be any G-equivariant A-module. The G-action on M is horizontal if and only if the counit of adjunction :*

$$\eta_M : A \otimes_{A^G} M^G \to M$$

is an epimorphism (of A-modules).

(ii) *The functor* (8.6.8) *restricts to an equivalence on the full subcategory of flat A^G-modules :*

$$A^G\text{-}\mathbf{Mod}_{\mathrm{fl}} \xrightarrow{\sim} A[G]\text{-}\mathbf{Mod}_{\mathrm{hor.fl}}. \tag{8.6.24}$$

Proof. (i): For $m := o(G)$, let $\boldsymbol{\mu}_m \subset \overline{\mathbb{Q}}^\times$ be the group of m-th roots of 1, and set $B := V^a[\boldsymbol{\mu}_m] := (V[T]/(T^m - 1))^a$. Since B is a faithfully flat V^a-algebra, lemma 8.6.10(iii) allows to replace A by $B \otimes_{V^a} A$ and M by $B \otimes_{V^a} M$, and therefore we may assume from start that $\boldsymbol{\mu}_m \subset (A^G_*)^\times$. Set $\Gamma := \mathrm{Hom}_{\mathbb{Z}}(G, \boldsymbol{\mu}_m)$. For every $\chi \in \Gamma$, let $e_\chi \in A_*[G]$ be the central idempotent defined as in (8.6.11). A standard calculation shows that :

$$\sum_{\chi \in \Gamma} e_\chi = 1.$$

Hence, every G-equivariant A^G-module (N, β) admits the G-equivariant decomposition :

$$N \simeq \bigoplus_{\chi \in \Gamma} N_\chi. \tag{8.6.25}$$

Especially, $A = \bigoplus_{\chi\in\Gamma} A_\chi$, and clearly the datum $\underline{A}$ consisting of A and its decomposition, is a Γ-graded V^a-algebra. Furthermore, the datum $\underline{N}$ consisting of N and its decomposition (8.6.25) is a Γ-graded $\underline{A}$-module.

Claim 8.6.26. The functor $(N,\beta) \mapsto \underline{N}$ is an equivalence from $A[G]$-$\mathbf{Mod}$ to the category of Γ-graded $\underline{A}$-modules.

Proof of the claim. Indeed, if Q is a Γ-graded $\underline{A}$-module, we may define a G-action on Q by requiring that $Q_\chi = \mathrm{gr}_\chi Q$, the χ-graded direct summand of Q. This gives a quasi-inverse for the functor $(N,\beta) \mapsto \underline{N}$. (Details left to the reader.) $\Diamond$

Claim 8.6.27. Let (N,β) be any G-equivariant A-module, and $\underline{N}$ its associated Γ-graded $\underline{A}$-module. The following conditions are equivalent :

(a) (N,β) is horizontal.
(b) $\underline{N}$ is horizontal.

Proof of the claim. For every $g \in G$, let $\Delta(g) \subset \Gamma$ be the subgroup consisting of all $\chi \in \Gamma$ such that $\chi(g) = 1$; a direct inspection of the definitions shows that $J_{\Delta(g)}$ is the ideal I_g, as defined in (8.6.6), and (N,β) is horizontal if and only if $N_\chi/(N_\chi \cap I_g N) = 0$ for every $g \in G$ and every $\chi \notin \Delta(g)$. This already shows that (b)⇒(a); it also shows that condition (b) holds for the subgroups $\Delta(g)$, when (N,β) is horizontal. However, every subgroup of Γ can be written in the form $\Delta = \Delta(g_1) \cap \cdots \cap \Delta(g_n)$, for appropriate $g_1,\dots,g_n \in G$, and then $J_{\Delta(g_1)} + \cdots + J_{\Delta(g_n)} \subset J_\Delta$, hence (b) follows for all subgroups. $\Diamond$

Assertion (i) now follows from claim 8.6.27 and proposition 8.6.13.

(ii): Let (M,β) be any object of $A[G]$-$\mathbf{Mod}_{\mathrm{hor.fl}}$, and denote by L the kernel of the counit η_M. Since M is a flat A-module and η_M is an epimorphism by (i), it follows that $i_g^* L$ is the kernel of $i_g^*\eta_M$, for every $g \in G$ (notation of (8.6.6)). Since the category $A[G]$-$\mathbf{Mod}_{\mathrm{hor}}$ is abelian, we deduce that the natural G-action on L is horizontal, and then (i) says that L is generated by L^G. But lemma 8.6.10(ii) easily implies that $L^G = 0$, so η_M is an isomorphism. Next, letting $M := A$ in lemma 8.6.10(i), we deduce easily that the natural morphism $A^G \to A$ is pure, hence M^G is a flat A^G-module, by lemma 8.6.21(i). Now the assertion follows from lemma 8.6.10(ii) and [10, Prop.3.4.3]. $\square$

Corollary 8.6.28. *In the situation of theorem* 8.6.23*, the following holds :*

(i) *The functor* (8.6.24) *restricts to an equivalence from the subcategory of flat, almost finitely generated (resp. almost finitely presented) A^G-modules, onto the subcategory of G-equivariant, flat, horizontal and almost finitely generated (resp. almost finitely presented) $A[G]$-modules.*

(ii) *The functor* (8.6.9) *restricts to an equivalence :*

$$A^G\text{-}\mathbf{Alg}_{\mathrm{fl}} \to A[G]\text{-}\mathbf{Alg}_{\mathrm{hor.fl}}$$

and likewise for the subcategories of weakly étale (resp. étale, resp. étale and almost finitely presented) algebras.

Proof. (i) follows from theorem 8.6.23(ii), lemma 8.6.21(ii), and the fact that the morphism $A^G \to A$ is pure.

(ii): The assertion concerning A^G-$\mathbf{Alg}_{\mathrm{fl}}$ is an immediate consequence of theorem 8.6.23. Next, let (B,β) be an object of $A[G]$-$\mathbf{w.\acute{E}t}$; by the foregoing, B descends to a flat A^G-algebra B^G with a G-equivariant isomorphism : $B \simeq A \otimes_{A^G} B^G$. However, on the one hand, B is – by assumption – a flat $B \otimes_A B$-algebra, and on the other hand, $B \otimes_A B$ underlies a flat, horizontal $A[G]$-algebra with $(B \otimes_A B)^G \simeq B^G \otimes_{A^G} B^G$; theorem 8.6.23 then says that B^G is a flat $B^G \otimes_{A^G} B^G$-algebra, whence the assertion for A^G-$\mathbf{w.\acute{E}t}$. Next, since an almost finitely generated module is almost projective if and only if it is flat and almost finitely presented ([36, Prop.2.4.18]), the assertion for A^G-$\mathbf{\acute{E}t}$ follows from the same assertion for A^G-$\mathbf{w}.\mathbf{\acute{E}t}$ and lemma 8.6.21(ii). Likewise, the assertion for étale almost finitely presented A^G-algebras follows from the assertion for A^G-$\mathbf{\acute{E}t}$ and lemma 8.6.21(ii). $\square$

Remark 8.6.29. In case the G-action on X is *free, i.e.* when $(\partial_0, \partial_1) : X \times G \to X \times_S X$ is a monomorphism, corollary 8.6.28 also follows from [36, Prop.4.5.25].

8.7. **Complements : locally measurable algebras.** This section studies the global counterpart of the class of measurable algebras introduced in section 8.3. To begin with – and until (8.7.42) – we consider an arbitrary valued field $(K, |\cdot|)$, and we resume the notation of (5.7) and (8.3). Our first result is the following generalization of proposition 5.7.1 :

Proposition 8.7.1. *Let A be a measurable K^+-algebra, M a K^+-flat and finitely generated A-module. Then M is a finitely presented A-module.*

Proof. Let Σ be a finite system of generators for M; also let us write A as the colimit of a filtered system $(A_i \mid i \in I)$ of finitely presented K^+-algebras, with étale transition maps. For every $i \in I$, let M_i be the A_i-submodule of M generated by Σ; notice that M_i is still K^+-flat, hence it is a finitely presented A_i-module (proposition 5.7.1). We may then write M as the colimit of the filtered system $(M_i \otimes_{A_i} A \mid i \in I)$ of finitely presented A-modules, with surjective transition maps. Moreover, since $\overline{A} := A/\mathfrak{m}_K A$ is noetherian (lemma 8.3.4(i)), there exists $i \in I$ such that $M_j \otimes_{A_j} \overline{A} = M/\mathfrak{m}_K M$ for every $j \geq i$. Since the ring homomorphism $\overline{A}_j := A_j/\mathfrak{m}_K A_j \to \overline{A}$ is faithfully flat, we deduce that

$$M_i \otimes_{A_i} \overline{A}_j = M_j \otimes_{A_j} \overline{A}_j \qquad \text{for every } j \geq i. \tag{8.7.2}$$

Consider the short exact sequence

$$C \quad : \quad 0 \to N \to M_i \otimes_{A_i} A_j \to M_j \to 0.$$

Since M_i is K^+-flat, the same holds for N, and the latter is a finitely generated A_j-module, since both M_j and $M_i \otimes_{A_i} A_j$ are finitely presented; therefore, the sequence $C \otimes_{K^+} \kappa$ is still exact. Taking into account (8.7.2), we see that $N/\mathfrak{m}_K N = 0$. By Nakayama's lemma, it follows that $N = 0$, and finally, $M = M_i \otimes_{A_i} A$ is finitely presented, as stated. $\square$

Definition 8.7.3. Let A be a K^+-algebra. Set $X := \operatorname{Spec} A$, $S := \operatorname{Spec} K^+$, and denote by $f : X \to S$ the structure morphism. We say that A is *locally measurable*, if the following holds. For every $x \in X$ and every point ξ of X localized at x, the strict henselization of A at ξ is a measurable $\mathscr{O}_{S,f(x)}$-algebra.

Remark 8.7.4. Let A be a local K^+-algebra, A^{sh} the strict henselization of A at a geometric point localized at the closed point, and M an A-module.

(i) Clearly, A is locally measurable if and only if A^{sh} is measurable.

(ii) However, if A^{sh} is measurable, it does not necessarily follow that A is measurable.

(iii) On the other hand, if A is a normal local domain, one can show that A is measurable if and only if the same holds for A^{sh}.

(iv) Suppose that A is local and locally measurable. Since the natural map $A \to A^{\mathrm{sh}}$ is faithfully flat, and since every measurable K^+-algebra is a coherent ring, it is easily seen that A is coherent. If furthermore, the structure map $K^+ \to A$ is local, lemma 8.3.4(i) implies that $A/\mathfrak{m}_K A$ is a noetherian ring.

Remark 8.7.5. Let A be a locally measurable K^+-algebra. Then, for every finitely generated ideal $I \subset A$, the K^+-algebra A/I is also locally measurable. Indeed, for every geometric point ξ of $\operatorname{Spec} A$, let A^{sh}_ξ (resp. $(A/I)^{\mathrm{sh}}_\xi$) be the strict henselization of A (resp. of A/I) at ξ.

Then the natural map $A^{\mathrm{sh}}_\xi/IA^{\mathrm{sh}}_\xi \to (A/I)^{\mathrm{sh}}_\xi$ is an isomorphism for every such ξ ([33, Ch.IV, Prop.18.8.10]), so the assertion follows from lemma 8.3.4(iv).

Definition 8.7.6. Let A be a K^+-algebra, M an A-module, and $\gamma \in \log \Gamma^+$ any element.

(i) A K^+-*flattening sequence* for the A-module M is a finite sequence $\underline{b} := (b_0, \dots, b_n)$ of elements of K^+ such that
  (a) $\log |b_{i+1}| > \log |b_i|$ for every $i = 0, \dots, n-1$, $b_0 = 1$ and $b_n = 0$.
  (b) $b_i M/b_{i+1}M$ is a $K^+/b_i^{-1}b_{i+1}K^+$-flat module, for every $i = 0, \dots, n-1$.

We say that a K^+-flattening sequence $\underline{b}$ for M is *minimal*, if no proper subsequence of $\underline{b}$ is still flattening for M.

(ii) Say that $\gamma = \log |b|$ for some $b \in K^+$, and let $b_M : M \to bM$ be the map given by the rule : $m \mapsto bm$, for every $m \in M$. We say that γ *breaks* M if the κ-linear map

$$b_M \otimes_{K^+} \kappa : M \otimes_{K^+} \kappa \to bM \otimes_{K^+} \kappa$$

is not an isomorphism.

Remark 8.7.7. Let A be a K^+-algebra, M an A-module.

(i) Suppose that $\gamma \in \log \Gamma^+$ breaks M, and say that $\gamma = \log |c|$ for some $c \in K^+$. Clearly, for every $b \in cK^+$, the map b_M factors through c_M. We deduce that every $\gamma' \in \log \Gamma^+$ with $\gamma' \geq \gamma$ also breaks M.

(ii) For given $b \in K^+$, suppose that M is a flat K^+/bK^+-module. Then we claim that no $\gamma < \log |b|$ breaks M. Indeed, for such γ pick $c \in K^+$ with $\log |c| = \gamma$, and set $W := K^+/bK^+$; notice that the map $c_W : W \to cW$ induces an isomorphism $c_W \otimes_W \mathbf{1}_\kappa : \kappa \xrightarrow{\sim} cW \otimes_W \kappa$ (notation of definition 8.7.6(ii)), whence an isomorphism

$$\mathbf{1}_M \otimes_W c_W \otimes_W \mathbf{1}_\kappa : M \otimes_W \kappa \xrightarrow{\sim} M \otimes_W (cW \otimes_W \kappa).$$

Since M is a flat W-module, the natural map $M \otimes_W cW \to cM$ is an isomorphism, and the resulting isomorphism $M \otimes_W \kappa \xrightarrow{\sim} cM \otimes_W \kappa$ is naturally identified with c_M, whence the contention.

(iii) Let $\underline{b} := (b_0, \dots, b_n)$ be a sequence of elements of K^+ fulfilling condition (a) of definition 8.7.6(i), and suppose that $\underline{b}$ admits a subsequence that is K^+-flattening for M. Then $\underline{b}$ is K^+-flattening for M as well. Indeed, an easy induction reduces the contention to the following. Let $b, c \in K^+$ be any two elements such that $\log |c| < \log |b|$, and suppose that M is K^+/bK^+-flat; then M/cM is K^+/cK^+-flat, and cM is $K^+/c^{-1}bK^+$-flat. Of this two assertion, the first is trivial; to show the second, recall that M can be written as the colimit of a filtered system of free K^+/bK^+-modules ([57, Ch.I, Th.1.2]). Then we may assume that M is free, in which case the assertion is easily verified.

Lemma 8.7.8. *Let A be a K^+-algebra, M a coherent A-module that admits a K^+-flattening sequence, and suppose that the Jacobson radical of A contains $\mathfrak{m}_K A$. Then we have :*

(i) *M admits a minimal K^+-flattening sequence, unique up to units of K^+.*

(ii) *Let $(b_0, \dots, b_n)$ be a minimal K^+-flattening sequence for M. For every $\gamma \in \log \Gamma^+$ and every $i = 0, \dots, n-1$, the following conditions are equivalent :*
  (a) *γ breaks $b_i M$.*
  (b) *$\gamma \geq \log |b_i^{-1} b_{i+1}|$. (As usual, we set $\log |0| := +\infty$: see* (8.3).*)*

Proof. It is clear that M admits a minimal K^+-flattening sequence $(1, b_1, \dots, b_{n-1}, 0)$, and (i) asserts that every other such minimal sequence is of the type $(1, u_1 b_1, \dots, u_{n-1} b_{n-1}, 0)$ for some elements $u_1, \dots, u_{n-1} \in (K^+)^\times$. However, notice that the condition of (ii) characterize uniquely such a sequence, so we only have to show that (ii) holds. We may also assume that $n > 1$; indeed, when $n = 1$, the module M is K^+-flat, and the assertion is immediate. Moreover, since M is coherent, the same holds for bM, for every $b \in K^+$. Thus, an easy induction reduces to showing the equivalence of conditions (ii.a) and (ii.b) for $i = 0$.

However, notice that, if γ breaks M, then it obviously also breaks M/b_1M; in light of remark 8.7.7(ii), it follows already that (a)⇒(b). Therefore, taking into account remark 8.7.7(i), it remains only to show that $\log |b_1|$ breaks M.

If $n = 2$, then b_1M is a flat K^+-module; let $N := \mathrm{Ker}\, b_{1,M}$. We have already observed that b_1M is a finitely presented A-module, hence N is a finitely generated A-module, and $N \otimes_{K^+} \kappa$ is the kernel of $b_{1,M} \otimes_{K^+} \kappa$. Suppose that $\log |b_1|$ does not break M; then $N \otimes_{K^+} \kappa$ vanishes, and then $N = 0$, by Nakayama's lemma. It follows that M is K^+-flat, which contradicts the minimality of the sequence $(1, b_1, 0)$.

Next, we consider the case where $n > 2$ and set $M' := M/b_2M$, $b := b_1^{-1}b_2$; since $b_2M \subset b_1\mathfrak{m}M$, it suffices to show that $\log |b_1|$ breaks M', hence we may assume that $b_2M = 0$, b_1M is a flat K^+/bK^+-module, and the flattening sequence is $(1, b_1, b_2, 0)$. We remark :

Claim 8.7.9. If $\log |b_1|$ does not break M, the map $b_{1,M/bM} : M/bM \to b_1M$ is an isomorphism of K^+/bK^+-modules.

Proof of the claim. Indeed, let $N := \mathrm{Ker}\, b_{1,M/bM}$; since b_1M is K^+/bK^+-flat, $N \otimes_{K^+} \kappa$ is the kernel of $b_{1,M/bM} \otimes_{K^+} \kappa$, and the latter vanishes if $\log |b_1|$ does not break M; on the other hand, N is a finitely generated A-module, hence $N = 0$, by Nakayama's lemma. ◊

We shall use the following variant of the local flatness criterion :

Claim 8.7.10. Let R be a ring, $I_1, I_2 \subset A$ two ideals, and M an R-module such that :

(a) $I_1I_2 = 0$
(b) M/I_iM is R/I_i-flat for $i = 1, 2$
(c) the natural map $I_1 \otimes_R M/I_2M \to I_1M$ is an isomorphism.

Then M is a flat R-module.

Proof of the claim. Set $I_3 := I_1 \cap I_2$; to begin with, [36, Lemma 3.4.18] and (b) imply that M/I_3M is a flat R/I_3-module. We have an obviously commutative diagram

$$\begin{tikzcd} I_3 \otimes_R M/I_3M \arrow[rr, "\alpha"] \arrow[d, "\beta"'] & & I_3M \arrow[d, "\gamma"] \\ I_3 \otimes_{R/I_2} M/I_2M \arrow[r, "\delta"] & I_1 \otimes_{R/I_2} M/I_2M \arrow[r, "\tau"] & I_1M \end{tikzcd}$$

of R-linear maps. From (a) we see that $I_3^2 = 0$, and it follows easily that β is an isomorphism; γ is clearly injective, and the same holds for δ, since M/I_2M is a flat R/I_2-module; by the same token, τ is an isomorphism. We conclude that α is an isomorphism, and then the local flatness criterion ([61, Th.22.3]) yields the contention. ◊

We shall apply claim 8.7.10 with $R := K^+/b_2K^+$, $I_1 := b_1R$, $I_2 := bR$. Indeed, condition (a) obviously holds with these choices; by assumption, M/I_1M is a R/I_1-flat module, and if $\log |b_1|$ does not break M, claim 8.7.9 implies that M/I_2M is a flat R/I_2-module. Lastly, condition (c) is equivalent to claim 8.7.9. Summing up, we have shown that if $\log |b_1|$ does not break M, then M is a flat K^+/b_2-module, contradicting again the minimality of our flattening sequence. □

Proposition 8.7.11. *Let A be a K^+-algebra, M a finitely presented A-module, and suppose that either one of the following two conditions holds :*

(a) *A is an essentially finitely presented K^+-algebra.*
(b) *A is a local and locally measurable K^+-algebra.*

Then M admits a K^+-flattening sequence.

Proof. Suppose first that (b) holds, and let A^{sh} be the strict henselization of A at a geometric point localized at the closed point. Since A^{sh} is a faithfully flat A-algebra, it suffices to show that $M \otimes_A A^{\mathrm{sh}}$ admits a K^+-flattening sequence. Hence, we may assume that A is measurable, in which case we may find a finitely presented K^+-algebra A_0 with an ind-étale map $A_0 \to A$ and a finitely presented A_0-module M_0 with an isomorphism $M_0 \otimes_{A_0} A \xrightarrow{\sim} M$ of A-modules. We may then replace A by A_0, M by M_0, and therefore assume that A is finitely presented over K^+, especially, we are reduced to showing the assertion in the case where (a) holds. To this aim, let us remark :

Claim 8.7.12. Let $B := \bigoplus_{n\in\mathbb{N}} B_n$ be a $\mathbb{N}$-graded finitely presented K^+-algebra with $B_0 = K^+$, and $N := \bigoplus_{n\in\mathbb{N}} N_n$ a $\mathbb{N}$-graded finitely presented B-module. We have :

(i) The K^+-module N_n is finitely presented, for every $n \in \mathbb{N}$.
(ii) For every $n \in \mathbb{N}$, let $(\gamma_{n,i} \mid i \in \mathbb{N})$ be the sequence of elementary divisors of N_n (see (8.3.15)). Then $\Gamma(N) := \{\gamma_{n,i} \mid n, i \in \mathbb{N}\}$ is a finite set.
(iii) N admits a K^+-flattening sequence.

Proof of the claim. (i) is just a special case of proposition 4.4.16(iii).

(ii): Let $\mathscr{Q}$ be the set of all finitely presented graded quotients Q of the B-modules N, for which $\Gamma(Q)$ is infinite. We have to show that $\mathscr{Q} = \varnothing$. However, for every $Q \in \mathscr{Q}$, the $B\otimes_{K^+}\kappa$-module $\overline{Q} := Q \otimes_{K^+} \kappa$ is a quotient of $\overline{N} := N \otimes_{K^+} \kappa$; since $B \otimes_{K^+} \kappa$ is a noetherian ring, the set $\overline{\mathscr{Q}} := \{\overline{Q} \mid Q \in \mathscr{Q}\}$ admits minimal elements, if it is not empty. In the latter case, we may then replace N by any $Q_0 \in \mathscr{Q}$ such that $\overline{Q}_0$ is a minimal element of $\overline{\mathscr{Q}}$, and assume that, for every graded quotient Q of N, either $\overline{Q} = \overline{N}$, or else $\Gamma(Q)$ is a finite set. Now, for every $n \in \mathbb{N}$, let γ_n be the minimal non-zero elementary divisor of N_n, and pick $a_n \in \mathfrak{m}_K$ with $\log|a_n| = \gamma_n$ (if $N_n = 0$, set $a_n = 0$). Let $I \subset K^+$ be the ideal generated by $\{a_n \mid n \in \mathbb{N}\}$; it is easily seen that N_n/IN_n is a free K^+/I-module for every $n \in \mathbb{N}$, hence N/IN is a K^+/I-flat finitely presented B/IB-module. We may then find $a \in I$ such that N/aN is already a K^+/aK^+-flat B/aB-module ([32, Ch.IV, Cor.11.2.6.1]), so N_n/aN_n is a free K^+/aK^+-module for every $n \in \mathbb{N}$, and we easily deduce that $\log|a| \leq \gamma_n$ for every $n \in \mathbb{N}$, *i.e.* $I = aK^+$. Notice that aN is a finitely presented B-module (corollary 5.7.2); it follows that

$$\Gamma(N) = \{\gamma + \log|a| \mid \gamma \in \Gamma(aN)\} \cup \{0\}.$$

Especially, $\Gamma(aN)$ is an infinite set. On the other hand, there exists some $n \in \mathbb{N}$ such that $\dim_\kappa (aN_n) \otimes_{K^+} \kappa < \dim_\kappa N_n \otimes_{K^+} \kappa$, so $(aN) \otimes_{K^+} \kappa$ is a proper quotient of $\overline{N}$, a contradiction.

(iii): Let $|b_0|, \dots, |b_{n-1}|$ be the finitely many elements of $\Gamma(N)$ (for suitable $b_1, \dots, b_n \in K^+$), and set $b_n := 0$; after permutation, we may assume that $b_{i+1} \in b_i\mathfrak{m}_K$ for every $i = 0, \dots, n-1$, and $b_0 = 1$. Then we claim that $(b_0, \dots, b_n)$ is a K^+-flattening sequence for N, *i.e.* $b_iN_k/b_{i+1}N_k$ is $K^+/b_i^{-1}b_{i+1}K^+$-flat for every $i = 0, \dots, n$ and every $k \in \mathbb{N}$. However, say that $(\log|c_j| \mid j \in \mathbb{N})$ is the sequence of elementary divisors of N_k; we are reduced to checking that $(c_jK^+ + b_iK^+)/(c_jK^+ + b_{i+1}K^+)$ is a flat $K^+/b_i^{-1}b_{i+1}K^+$-module for every $j \in \mathbb{N}$. However, by construction we have either $c_jK^+ \subset b_{i+1}K^+$, or $b_iK^+ \subset c_jK^+$; in either case the assertion is clear. ◊

Let now A be an arbitrary essentially finitely presented K^+-algebra, and M an arbitrary finitely presented A-module. We easily reduce to the case where $A = K^+[T_1, , \dots, T_r]$ is a free polynomial K^+-algebra. In this case, we define a filtration $\mathrm{Fil}_\bullet A$ on A, by declaring that $\mathrm{Fil}_k A$ is the K^+-submodule of all polynomials of total degree $\leq k$, for every $k \in \mathbb{N}$; then $R := \mathsf{R}(A, \mathrm{Fil}_\bullet A)_\bullet$ is a free polynomial K^+-algebra as well (see example 4.4.28). Let

$$L_1 \xrightarrow{\varphi} L_0 \to M$$

be a presentation of M as quotient of free A-modules of finite rank. Let $\mathbf{e} := (e_1, \dots, e_n)$ (resp. $\mathbf{f} := (f_1, \dots, f_m)$) be a basis of L_0 (resp. of L_1); we endow L_0 with the good $(A, \mathrm{Fil}_\bullet A)$-filtration $\mathrm{Fil}_\bullet L_0$ associated to the pair $(\mathbf{e}, (1, \dots, 1))$ as in (4.4.29) (this means that $e_i \in \mathrm{Fil}_1 L_0$ for every $i = 1, \dots, n$). Also, for every $i = 1, \dots, m$, pick $j_i \in \mathbb{N}$ such that $\varphi(f_i) \in \mathrm{Fil}_{j_i} L_0$, and endow L_1 with the good $(A, \mathrm{Fil}_\bullet A)$-filtration $\mathrm{Fil}_\bullet L_1$ associated to the pair $(\mathbf{f}, (j_1, \dots, j_m))$. Set $L'_i := \mathsf{R}(L_i, \mathrm{Fil}_\bullet L_i)$, and notice that L'_i is a free R-module of finite rank, for $i = 0, 1$. With these choices, φ is a map of filtered A-modules, and there follows an R-linear map of $\mathbb{N}$-graded R-modules $\mathsf{R}(\varphi)_\bullet : L'_1 \to L'_0$, whose cokernel is a $\mathbb{N}$-graded finitely presented R-module $N_\bullet$. By inspecting the construction, it is easily seen that the inclusion $\mathrm{Fil}_k L_0 \subset \mathrm{Fil}_{k+1} L_0$ induces a K^+-linear map $N_k \to N_{k+1}$, for every $k \in \mathbb{N}$, as well as an isomorphism of K^+-modules

$$\operatorname*{colim}_{k \in \mathbb{N}} N_k \xrightarrow{\sim} M.$$

On the other hand, claim 8.7.12 ensures that $N_\bullet$ admits a K^+-flattening sequence. We easily deduce that the same sequence is also flattening for M. □

Corollary 8.7.13. *Let A be a local and locally measurable K^+-algebra, $I \subset K^+$ any ideal, M a finitely generated A/IA-module, and suppose that the structure map $K^+ \to A$ is local. Then the following conditions are equivalent :*

(a) *M is a flat K^+/I-module.*
(b) *For every $c \in K^+$ such that $I \subset c \cdot \mathfrak{m}_K$, the value $\log |c|$ does not break M.*
(c) *M is a K^+/I-flat finitely presented A/IA-module.*

Proof. (a)⇒(b): in light of remark 8.7.7(ii), it suffices to remark that $M \otimes_{K^+} /bK^+$ is a flat K^+/bK^+-module, for every $b \in K^+$ such that $I \subset K^+ b$.

(b)⇒(c): Let us write M as the colimit of a filtered system $(M_\lambda \mid \lambda \in \Lambda)$ of finitely presented A-modules, with surjective transition maps. Then each M_λ is a coherent A-module, and $A/\mathfrak{m}_K A$ is noetherian (remark 8.7.4(iv)), so we may also assume that the induced maps $\varphi_\lambda : M_\lambda/\mathfrak{m}_K M_\lambda \to M/\mathfrak{m}_K M$ are isomorphisms for every $\lambda \in \Lambda$. In view of the commutative diagram

$$\begin{tikzcd} M_\lambda \otimes_{K^+} \kappa \arrow[r, "c_{M_\lambda}"] \arrow[d, "\varphi_\lambda"'] & cM_\lambda \otimes_{K^+} \kappa \arrow[d] \\ M \otimes_{K^+} \kappa \arrow[r, "c_M"] & cM \otimes_{K^+} \kappa \end{tikzcd}$$

it easily follows that $\log |c|$ does not break M_λ, for any $\lambda \in \Lambda$ and any $c \in K^+$ such that $I \subset c \cdot \mathfrak{m}_K$. Thus, if $(b_0, \dots, b_n)$ is the minimal flattening sequence for M_λ, we see that $b_1 \in I$ (lemma 8.7.8(ii)), and therefore M_λ/IM_λ is a flat K^+/I-module for every $\lambda \in \Lambda$. Now, for $\lambda, \mu \in \Lambda$ with $\mu \geq \lambda$, let $\psi_{\lambda\mu} : M_\lambda \to M_\mu$ be the transition map, and set $N_{\lambda\mu} := \mathrm{Ker}(\psi_{\lambda\mu} \otimes_{K^+} K^+/I)$; it follows that $N_{\lambda\mu} \otimes_{K^+} \kappa$ is the kernel of $\psi_{\lambda\mu} \otimes_{K^+} \kappa$. But the latter map is an isomorphism, since the same holds for φ_λ and φ_μ. By Nakayama's lemma, we deduce that $N_{\lambda\mu} = 0$, therefore $M = M_\lambda/IM_\lambda$ for any $\lambda \in \Lambda$, whence (c).

Lastly, (c)⇒(a) is obvious. □

Proposition 8.7.14. *Let A be a locally measurable K^+-algebra. Suppose that*

(a) *$A/\mathfrak{m}_K A$ is a noetherian ring.*
(b) *The Jacobson radical of A contains $\mathfrak{m}_K A$.*

Then we have :

(i) *$A \otimes_{K^+} K$ is a noetherian ring.*
(ii) *Every finitely generated K^+-flat A-module is finitely presented.*
(iii) *If the valuation of K is discrete, A is noetherian.*

Proof. For any K^+-algebra B, and any ideal $I \subset B_K := B \otimes_{K^+} K$, let us set $I^{\mathrm{sat}} := \mathrm{Ker}(B \to B_K/I)$, and notice that $I^{\mathrm{sat}} \otimes_{K^=} K = I$. To begin with, we remark :

Claim 8.7.15. Let B be a measurable K^+-algebra, $I \subset B_K$ an ideal. Then I^{sat} is a finitely generated ideal of B.

Proof of the claim. Clearly B/I^{sat} is a K^+-flat finitely generated B-module, hence it is finitely presented, by proposition 8.7.1; now the claim follows from [36, Lemma 2.3.18(ii)]. $\diamondsuit$

(i): Suppose $(I_k \mid k \in \mathbb{N})$ is an increasing sequence of ideals of $A \otimes_{K^+} K$; assumption (a) implies that there exists $n \in \mathbb{N}$ such that the images of I_n^{sat} and I_m^{sat} agree in $A/\mathfrak{m}_K A$, for every $m \geq n$. This means that

$$(I_m^{\mathrm{sat}}/I_n^{\mathrm{sat}}) \otimes_{K^+} \kappa = 0 \qquad \text{for every } m \geq n. \tag{8.7.16}$$

Next, for any geometric point ξ of $\mathrm{Spec}\, A$, let A_ξ^{sh} be the strict henselization of A at ξ; since the natural map $A \to A_\xi^{\mathrm{sh}}$ is flat, $I_k \otimes_A A_\xi^{\mathrm{sh}}$ is an ideal of $A_\xi^{\mathrm{sh}} \otimes_{K^+} K$, and clearly $I_k^{\mathrm{sat}} \otimes_A A_\xi^{\mathrm{sh}} = (I_k \otimes_A A_\xi^{\mathrm{sh}})^{\mathrm{sat}}$, for every $k \in \mathbb{N}$. Especially $I_k^{\mathrm{sat}} \otimes_A A_\xi^{\mathrm{sh}}$ is a finitely generated ideal of A_ξ^{sh}, by claim 8.7.15. In view of (8.7.16), and assumption (b), Nakayama's lemma then says that $(I_m^{\mathrm{sat}}/I_n^{\mathrm{sat}}) \otimes_A A_\xi^{\mathrm{sh}} = 0$ for every $m \geq n$. Since ξ is arbitrary, we conclude that $I_m^{\mathrm{sat}}/I_n^{\mathrm{sat}} = 0$ for every $m \geq n$, therefore the sequence $(I_k \mid k \in \mathbb{N})$ is stationary.

(ii): Let M be a K^+-flat and finitely generated A-module, pick an A-linear surjection $\varphi : A^{\oplus n} \to M$, and set $N := \mathrm{Ker}\,\varphi$. Since M is K^+-flat, $N/\mathfrak{m}_K N$ is the kernel of $\varphi \otimes_{K^+} \kappa$, and assumption (a) implies that $N/\mathfrak{m}_K N$ is a finitely generated $A/\mathfrak{m}_K A$-module. Hence we may find a finitely generated A-submodule $N' \subset N$ such that $N = N' + \mathfrak{m}_K N$. For any geometric point ξ of $\mathrm{Spec}\, A$, define A_ξ^{sh} as in the foregoing; by proposition 8.7.1 (and by [36, Lemma 2.3.18(ii)]), $N \otimes_A A_\xi^{\mathrm{sh}}$ is a finitely generated A_ξ^{sh}-module; then (b) and Nakayama's lemma imply that $N' \otimes_A A_\xi^{\mathrm{sh}} = N \otimes_A A_\xi^{\mathrm{sh}}$. Since ξ is arbitrary, it follows that $N = N'$; especially, M is finitely presented, as stated.

(iii): It suffices to show that every prime ideal of A is finitely generated ([61, Th.3.4]). However, let $\mathfrak{p} \subset A$ be such a prime ideal, and fix a generator t of $\mathfrak{m}_K$. Suppose first that $t \in \mathfrak{p}$; in that case (a) implies that $\mathfrak{p}/tA$ is a finitely generated ideal of A/tA, so then clearly $\mathfrak{p}$ is finitely generated as well. Next, in case $t \notin \mathfrak{p}$, the quotient $A/\mathfrak{p}$ is a K^+-flat finitely generated A-module, hence it is finitely presented (proposition 8.7.1), so again $\mathfrak{p}$ is finitely generated ([36, Lemma 2.3.18(ii)]). $\square$

Theorem 8.7.17. *Let A be a locally measurable K^+-algebra, M a finitely presented A-module, and suppose that :*

(a) *The valuation of K has finite rank.*
(b) *For every $t \in \mathrm{Spec}\, K^+$, the ring $A \otimes_{K^+} \kappa(t)$ is noetherian.*

Then we have :

(i) *M admits a K^+-flattening sequence.*
(ii) *$\mathrm{Ass}\, M$ is a finite set.*

Proof. (i): Set $X := \mathrm{Spec}\, A$, $S := \mathrm{Spec}\, K^+$, let $f : X \to S$ be the structure morphism, and denote by $\mathscr{M}$ the quasi-coherent $\mathscr{O}_X$-module associated to M. Also, for every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ and every $c \in K^+$, let $c_{\mathscr{F}} : \mathscr{F} \to c\mathscr{F}$ be the unique $\mathscr{O}_X$-linear morphism such that $c_{\mathscr{F}}(U) = c_{\mathscr{F}(U)}$ for every affine open subset $U \subset X$ (notation of definition 8.7.6(ii)). For every $x \in X$, set $K_{f(x)}^+ := \mathscr{O}_{S,f(x)}$; then $K_{f(x)}^+$ is a valuation ring whose valuation we denote $|\cdot|_{f(x)}$, and $\mathscr{O}_{X,x}$ is a locally K_x^+-measurable algebra. By proposition 8.7.11, the $\mathscr{O}_{X,x}$-module $\mathscr{M}_x$ admits a $K_{f(x)}^+$-flattening sequence $(b_{x,0}, \dots, b_{x,n})$; we wish to show that there exists an open neighborhood $U(x)$ of x in X, such that $(b_{x,i}\mathscr{M}/b_{x,i+1}\mathscr{M})_y$ is a $K^+/b_{x,i}^{-1}b_{x,i+1}K^+$-flat module, for every $y \in U(x)$ and every $i = 0, \dots, n-1$. To this aim, we remark :

Claim 8.7.18. Let $\mathscr{N}$ be a quasi-coherent $\mathscr{O}_X$-module of finite type, $x \in X$ a point, and $b \in K^+$ such that $\mathscr{N}_x$ is a flat K^+/bK^+-module. Then there exists an open neighborhood $U \subset X$ of x such that the map

$$c_{\mathscr{N},y} \otimes_{K^+} \kappa(f(y)) : \mathscr{N}_y \otimes_{K^+} \kappa(f(y)) \to c\mathscr{N}_y \otimes_{K^+} \kappa(f(y))$$

is an isomorphism for every $y \in U$ and every $c \in K^+$ with $\log |c|_{f(y)} < \log |b|_{f(y)}$.

Proof of the claim. Let $z \in X(x)$ be any point, and set $t := f(z)$; under our assumptions, $\mathscr{N}_z$ is a flat K^+/bK^+-module. Therefore, the sequence of $\mathscr{O}_{X,z}$-modules

$$0 \to c^{-1}b\mathscr{N}_z \xrightarrow{j} \mathscr{N}_z \xrightarrow{c_{\mathscr{N},z}} c\mathscr{N}_z \to 0$$

is exact, for every $c \in K^+$ with $\log |c| < \log |b|$, and *a fortiori*, whenever $\log |c|_t < \log |b|_t$. If the latter inequality holds, the map $j \otimes_{K^+} \kappa(t)$ vanishes, hence $c_{\mathscr{N},z} \otimes_{K^+} \kappa(t)$ is injective.

Let now $y \in X$ be any point; set $u := f(y)$, pick $c \in K^+$ with $\log |c|_u < \log |b|_u$, set $\mathscr{K}(u) := \mathrm{Ker}(c_{\mathscr{N}} \otimes_{K^+} \kappa(u))$, and suppose that $\mathscr{K}(u)_y \neq 0$. The foregoing shows that in this case $y \notin X(x)$. On the other hand, by assumption $f^{-1}(u)$ is a noetherian scheme, and $\mathscr{N}(u) := \mathscr{N} \otimes_{K^+} \kappa(u)$ is a coherent $\mathscr{O}_{f^{-1}(u)}$-module, therefore $\mathrm{Ass}\, \mathscr{N}(u)$ is a finite set ([61, Th.6.5(i)]). In view of proposition 5.5.4(ii), we conclude that $\mathrm{Ass}\, \mathscr{K}(u)$ is contained in the finite set $\mathrm{Ass}\, \mathscr{N}(u) \setminus X(x)$. Let Z be the topological closure in X of the (finite) set $\bigcup_{u \in S} \mathrm{Ass}\, \mathscr{N}(u) \setminus X(x)$; taking into account lemma 5.5.3(ii,iii), we see that $U := X \setminus Z$ will do. ◊

Fix $x \in X$ and $i \leq n$; taking $\mathscr{N} := b_{x,i}\mathscr{M}/b_{x,i+1}\mathscr{M}$ and $b := b_{x,i}^{-1}b_{x,i+1}$ in claim 8.7.18, and invoking the criterion of corollary 8.7.13, we obtain an open neighborhood U_i of x in X such that $\mathscr{N}_y$ is K^+/bK^+-flat for every $y \in U_i$. Clearly the subset $U(x) := U_1 \cap \cdots \cap U_n$ fulfills the sought condition. Next, pick finitely many points $x_1, \dots, x_k \in X$ and corresponding K^+-flattening sequences $\underline{b}_i$ for $\mathscr{M}_{x_i}$, for each $i = 1, \dots, k$, such that $U(x_1) \cup \cdots \cup U(x_k) = X$; after reordering, the sequence $(\underline{b}_1, \dots, \underline{b}_k)$ becomes K^+-flattening for M (see remark 8.7.7(iii)).

(ii): Let $(b_0, \dots, b_n)$ be a K^+-flattening sequence for M; in view of proposition 5.5.4(ii), it suffices to prove that $\mathrm{Ass}\, b_iM/b_{i+1}M$ is a finite set, for every $i = 0, \dots, n-1$. Taking into account remark 8.7.4(iv), we are then reduced to showing

Claim 8.7.19. Let $\mathscr{N}$ be a quasi-coherent $\mathscr{O}_X$-module, and $b \in K^+$ any element such that $\mathscr{N}_x$ is a K^+/bK^+-flat and finitely presented $\mathscr{O}_{X,x}$-module for every $x \in X$. Then $\mathrm{Ass}\, \mathscr{N}$ is finite.

Proof of the claim. For given $x \in X$ and any geometric point ξ of X localized at x, let A^{sh} be the strict henselization of A at ξ, and B a local and essentially finitely presented K^+-algebra with a local and ind-étale map $B \to A^{\mathrm{sh}}$. We may assume that $\mathscr{N}_x \otimes_A A^{\mathrm{sh}}$ descends to a finitely presented K^+/bK^+-flat B-module N_B ([32, Ch.IV, Cor.11.2.6.1(ii)]). Denote by $\overline{x}$ (resp. by x_B) the closed point of $X(\xi) = \mathrm{Spec}\, A^{\mathrm{sh}}$ (resp. of $\mathrm{Spec}\, B$); in light of corollary 5.4.35 we have

$$x \in \mathrm{Ass}\, \mathscr{N} \Leftrightarrow x \in \mathrm{Ass}_{\mathscr{O}_{X,x}} \mathscr{N}_x \Leftrightarrow \overline{x} \in \mathrm{Ass}_{A^{\mathrm{sh}}} \mathscr{N}_x \otimes_A A^{\mathrm{sh}} \Leftrightarrow x_B \in \mathrm{Ass}_B N_B. \tag{8.7.20}$$

On the other hand, it is easily seen that $\mathrm{Ass}_{K^+} K^+/bK^+$ consists of only one point, namely the maximal point t of $\mathrm{Spec}\, K^+/bK^+$. From corollary 5.5.7, we deduce :

$$x_B \in \mathrm{Ass}_B N_B \Leftrightarrow x_B \in \mathrm{Ass}_B N_B \otimes_{K^+} \kappa(t)$$

and by applying again repeatedly corollary 5.4.35 as in (8.7.20), we see that

$$x_B \in \mathrm{Ass}_B N_B \otimes_{K^+} \kappa(t) \Leftrightarrow x \in \mathrm{Ass}\, \mathscr{N} \otimes_{K^+} \kappa(t).$$

Summing up, we conclude that $\mathrm{Ass} \mathscr{N} = \mathrm{Ass} \mathscr{N} \otimes_{K^+} \kappa(t)$, and the latter is a finite set, by [61, Th.6.5(i)]. □

Corollary 8.7.21. *Let $(K, |\cdot|)$ be a valued field, and A a locally measurable K^+-algebra fulfilling conditions* (a) *and* (b) *of theorem* 8.7.17. *Let also M be a finitely generated A-module, and $\mathfrak{p} \subset A$ any prime ideal such that $M_\mathfrak{p}$ is a finitely presented $A_\mathfrak{p}$-module. We have :*

(i) *There exists $f \in A \backslash \mathfrak{p}$ such that M_f is a finitely presented A_f-module.*
(ii) *A is coherent.*

Proof. (i): We may find a finitely presented A-module M' with an A-linear surjection $\varphi : M' \to M$ such that $\varphi_\mathfrak{p}$ is an isomorphism. Set $M'' := \operatorname{Ker}\varphi$; it follows that $\operatorname{Ass} M'' \subset \operatorname{Ass} M' \setminus \operatorname{Spec} A_\mathfrak{p}$. However, $\operatorname{Ass} M'$ is a finite set (theorem 8.7.17(ii)); therefore the support of $\operatorname{Ker}\varphi$ is contained in a closed subset of $X := \operatorname{Spec} A$ that does not contain $\mathfrak{p}$, and the assertion follows.

(ii): Let M be a finitely presented A-module, $M' \subset M$ a finitely generated submodule, and let $\mathscr{M}'$ be the quasi-coherent $\mathscr{O}_X$-module associated to M'. By remark 8.7.4(iv), we know that $\mathscr{M}'_x$ is a finitely presented $\mathscr{O}_{X,x}$-module for every $x \in X$; from (i), it follows that $\mathscr{M}'$ is a finitely presented $\mathscr{O}_X$-module, and the assertion follows easily. $\square$

In view of theorem 8.7.17 and corollary 8.7.21, it is interesting to have criteria ensuring that the fibres over $\operatorname{Spec} K$ of the spectrum of a locally measurable K^+-algebra are noetherian. We present two results in this direction. To state them, let us make first the following :

Definition 8.7.22. Let $f : X \to Y$ be a morphism of schemes.

(i) We say that f is *absolutely flat* if the following holds. For every geometric point ξ of X, the induced morphism $f_\xi : X(\xi) \to Y(f(\xi))$ is an isomorphism.
(ii) We say that f *has finite fibres* if the set $f^{-1}(y)$ is finite, for every $y \in Y$.
(iii) We say that Y *admits a geometrically unibranch stratification* if every irreducible closed subset Z of Y contains a subset $U \neq \varnothing$, open in Z, and geometrically unibranch.
(iv) We say that a ring homomorphism $\varphi : A \to B$ is *absolutely flat* if $\operatorname{Spec}\varphi$ is absolutely flat.

Remark 8.7.23. In practice, it is often the case that a noetherian scheme admits a geometrically unibranch stratification; for instance, this holds (essentially by definition) for the spectrum of a quasi-excellent noetherian ring (see definition 4.8.3).

Lemma 8.7.24. *Let $f : X \to Y$ be an absolutely flat morphism of schemes. Then $\mathbb{L}_{X/Y} \simeq 0$ in $\mathsf{D}(\mathscr{O}_X\text{-}\mathbf{Mod})$.*

Proof. We easily reduce to the case where both X and Y are local schemes. Let ξ be a geometric point of X localized at the closed point; then $\mathbb{L}_{X(\xi)/X} \simeq 0$ in $\mathsf{D}(\mathscr{O}_{X(\xi)}\text{-}\mathbf{Mod})$ ([36, Th.2.5.37]). Hence, by transitivity ([36, Th.2.5.33]), we are reduced to showing that $\mathbb{L}_{X(\xi)/Y} \simeq 0$ in $\mathsf{D}(\mathscr{O}_{X(\xi)}\text{-}\mathbf{Mod})$. The latter assertion assertion holds, again by virtue of [36, Th.2.5.37], since f_ξ is an isomorphism. $\square$

Proposition 8.7.25. *Let $\varphi : A \to B$ be a local and absolutely flat morphism of local rings. Assume that either one of the following two conditions holds :*

(a) *A is a normal local domain.*
(b) *B is a henselian local ring.*

Then φ is an ind-étale ring homomorphism.

Proof. Pick a geometric point ξ of $\operatorname{Spec} B$ localized at the closed point, and let ξ' denote the image of ξ in $\operatorname{Spec} A$. Let also A^{sh} (resp. B^{sh}) be the strict henselization of A at ξ' (resp. of B at ξ). Notice first that A is a normal local domain if and only if the same holds for B ([33, Ch.IV, Prop.18.8.12(i)]). In case (a) holds, let F_A (resp. F_B) be the field of fractions of A (resp. B). We remark :

Claim 8.7.26. Let R be any normal local domain, and R^{sh} the strict henselization of R at a geometric point localized at the closed point of $\mathrm{Spec}\, R$. We have :

(i) $R = R^{\mathrm{sh}} \cap \mathrm{Frac}\, R$ (where the intersection takes place in $\mathrm{Frac}\, R^{\mathrm{sh}}$).

(ii) For every field extension F of $\mathrm{Frac}\, R$ contained in $\mathrm{Frac}\, R^{\mathrm{sh}}$, the R-algebra $F \cap R^{\mathrm{sh}}$ is ind-étale.

Proof of the claim. (i): More generally, let $C \to D$ be any faithfully flat ring homomorphism, and denote by $\mathrm{Frac}\, C$ (resp. $\mathrm{Frac}\, D$) the total ring of fractions of C (resp. of D); then it is easily seen that $C = D \cap \mathrm{Frac}\, C$, where the intersection takes place in $\mathrm{Frac}\, D$: the proof shall be left as an exercise for the reader.

(ii): We easily reduce to the case where F is a finite extension of $F_R := \mathrm{Frac}\, R$. Then, let E be a finite Galois extension of F_R, contained in a fixed separable closure F^{sep} of $F_R^{\mathrm{sh}} := \mathrm{Frac}\, R^{\mathrm{sh}}$, and containing F; also, denote by R^ν (resp. R_E^ν) the integral closure of R in F^{sep} (resp. in E). Recall that there exists a unique maximal ideal $\mathfrak{p}^{\mathrm{sep}} \subset R^\nu$ lying over the maximal ideal of R, such that $R^{\mathrm{sh}} = (R^\nu)^I_{\mathfrak{p}^{\mathrm{sep}}}$, where $I \subset \mathrm{Gal}(F^{\mathrm{sep}}/F_R)$ is the inertia subgroup associated to $\mathfrak{p}^{\mathrm{sep}}$ ([68, Ch.X, §2, Th.2]); therefore, $F_R^{\mathrm{sh}} = (F^{\mathrm{sep}})^I$. On the other hand, $F = E^H$ for a subgroup $H \subset \mathrm{Gal}(E/F_R)$, and recall that the natural surjection $\mathrm{Gal}(F^{\mathrm{sep}}/F_R) \to \mathrm{Gal}(E/F_R)$ maps I onto the inertia subgroup I_E associated to the maximal ideal $\mathfrak{p}_E := \mathfrak{p}^{\mathrm{sep}} \cap E \subset R_E^\nu$. It follows that H contains I_E. Set $\mathfrak{p}_F := \mathfrak{p}_E \cap F$; then $R_F := (R_E^\nu \cap F)_{\mathfrak{p}_F}$ is a faithfully flat and essentially étale R-algebra ([68, Ch.X, §1, Th.1(1)]). Especially, R^{sh} is also a strict henselization of R_F, and notice that $\mathrm{Frac}\, R_F = F$, so the assertion follows from (i). ◊

By assumption, φ extends to an isomorphism of local domains $A^{\mathrm{sh}} \xrightarrow{\sim} B^{\mathrm{sh}}$; in view of claim 8.7.26(i), we see that $B = F_B \cap A^{\mathrm{sh}}$, and then the proposition follows from claim 8.7.26(ii).

Next, suppose assumption (b) holds. Then, φ extends uniquely to a local ring homomorphism $\varphi^{\mathrm{h}} : A^{\mathrm{h}} \to B$ from the henselization of A ([33, Ch.IV, Th.18.6.6(ii)]); clearly φ^{h} is still absolutely flat, hence we may replace A by A^{h} and assume that both A and B are henselian. Since φ is absolutely flat, the residue field extension $\kappa(A) \to \kappa(B)$ is separable and algebraic; then for every finite extension E of $\kappa(A)$ contained in $\kappa(B)$ there exists a finite étale local A-algebra A_E, unique up to isomorphism, whose residue field is E ([33, Ch.IV, Prop.18.5.15]). The natural map $\kappa(B) \to E \otimes_{\kappa(A)} \kappa(B)$ admits a well defined section, given by the multiplication in $\kappa(B)$, and the latter extends to a section s_E of the induced finite étale ring homomorphism $B \to A_E \otimes_A B$ ([33, Ch.IV, Th.18.5.11]). The composition of s_E with the natural map $A_E \to A_E \otimes_A B$ is a ring homomorphism $A_E \to B$ that extends φ. The construction is clearly compatible with inclusion of subextensions $E \subset E'$, hence let A' be the colimit of the system $(A_E \mid E \subset \kappa(B))$; summing up, φ extends to a local and absolutely flat map $A' \to B$. We may therefore replace A by A' and assume from start that $\kappa(A) = \kappa(B)$. In this case, set $B' := A^{\mathrm{sh}} \otimes_A B$, and notice that $\kappa(A^{\mathrm{sh}}) \otimes_{\kappa(A)} \kappa(B) = \kappa(A^{\mathrm{sh}})$, so that B' is a local ring, ind-étale, faithfully flat over B, and with separably closed residue field. Since B is already henselian, it follows that B' is a strict henselization of B, and therefore the induced map $A^{\mathrm{sh}} \to B'$ is an isomorphism; by faithfully flat descent, we deduce that φ is already an isomorphism, and the proof of the proposition is concluded. □

Lemma 8.7.27. *Let A be a locally measurable K^+-algebra, and set $X := \mathrm{Spec}\, K$. Let also $f : X \to Y$ be a morphism of K^+-schemes with $\Omega_{X/Y} = 0$, and Y a finitely presented K^+-scheme. Then f_ξ is a finitely presented closed immersion, for every geometric point ξ of X.*

Proof. Fix such a geometric point ξ; by definition, we may find a local K^+-scheme Z essentially finitely presented, and an ind-étale morphism $g : X(\xi) \to Z$ of K^+-schemes. We may also assume that the morphism $X(\xi) \to Y$ deduced from f factors through g, in which case the resulting morphism $h : Z \to Y$ is essentially finitely presented. Under the current assumptions, $\Omega_{X(\xi)/Y} = 0$, and then lemma 8.7.24 easily implies that $\Omega_{Z/Y}$ vanishes as well, so

h is essentially unramified, and therefore it factors as the composition of a finitely presented closed immersion $Z \to Z'$ followed by an essentially étale morphism $Z' \to Y$ ([33, Ch.IV, Cor.18.4.7]). The lemma is an immediate consequence. $\square$

Proposition 8.7.28. *Let A be a local, henselian, and locally measurable K^+-algebra, whose structure map $K^+ \to A$ is local. Then A is a measurable K^+-algebra.*

Proof. Pick a local and essentally finitely presented K^+-algebra B and a local and ind-étale map $\varphi : B \to A$ of K^+-algebras. Let ξ be a geometric point of $\mathrm{Spec}\, A$ localized at the closed point, and denote by ξ' the image of ξ in $\mathrm{Spec}\, B$; also, let A^{sh} (resp. B^{sh}) be the strict henselization of A at ξ (resp. of B at ξ'). Then $\Omega_{A/B} = 0$, hence the induced map $\varphi^{\mathrm{sh}} : B^{\mathrm{sh}} \to A^{\mathrm{sh}}$ is a finitely presented surjection (lemma 8.7.27), especially $I := \mathrm{Ker}\, \varphi^{\mathrm{sh}}$ is a finitely generated ideal, and it follows as well that the residue field extension $\kappa(B) \to \kappa(A)$ is separable and algebraic. Let B^{h} be the henselization of B, and for every field extension E of $\kappa(B)$ contained in $\kappa(A)^{\mathrm{sep}} = \kappa(B)^{\mathrm{sep}}$, let B^{h}_E denote the local ind-étale B^{h}-algebra – determined up to isomorphism – whose residue field is E ([33, Ch.IV, Prop.18.5.15]). We may find a finite Galois extension E of $\kappa(B)$ such that I descends to a finitely generated ideal $I_E \subset B^{\mathrm{h}}_E$. For any automorphism $\sigma \in G := \mathrm{Gal}(\kappa(B)^{\mathrm{sep}}/\kappa(B))$, let $\overline{\sigma} \in G_E := \mathrm{Gal}(E/\kappa(B))$ be the image of σ. Recall that G (resp. G_E) is the group of automorphisms of the B^{h}-algebra B^{sh} (resp. B^{h}_E). With this notation, we have the identity

$$\overline{\sigma}(I_E)B^{\mathrm{sh}} = \sigma(I) = I \qquad \text{for every } \sigma \in G$$

and since B^{sh} is a faithfully flat B^{h}-algebra, it follows that I_E is invariant under the action of G_E; by Galois descent, we conclude that I descends to a finitely generated ideal $I_0 \subset B^{\mathrm{h}}$. Set $C := B^{\mathrm{h}}/I_0$, and let C^{sh} denote the strict henselization of C at (the unique lifting of) the geometric point ξ'. Since A is henselian, φ extends uniquely to a local homomorphism $\varphi^{\mathrm{h}} : B^{\mathrm{h}} \to A$ ([33, Ch.IV, Th.18.6.6(ii)]), and it is easily seen that φ^{h} factors through C. By construction, the resulting map $C \to A$ is absolutely flat, so the assertion follows from proposition 8.7.25. $\square$

Lemma 8.7.29. *Let X be a noetherian scheme that admits a geometrically unibranch stratification, and $\mathscr{F}$ a coherent $\mathscr{O}_{X_{\text{ét}}}$-module. Then there exists a partition*

$$X = X_1 \cup \cdots \cup X_k$$

of X into finitely many disjoint irreducible locally closed subsets, such that the following holds. For every $i = 1, \dots, k$, every geometric point $\overline{x}$ of X_i, and every generization $\overline{u}$ of $\overline{x}$ in X_i, every strict specialization map

$$s_{\overline{x},\overline{u}} : \mathscr{F}_{\overline{x}} \to \mathscr{F}_{\overline{u}}$$

is injective (see (2.4.22)*).*

Proof. Arguing by noetherian induction, it suffices to show that every reduced and irreducible closed subscheme W of X contains a subset $U \neq \varnothing$ that is open in W, and such that every strict specialization map $s_{\overline{x},\overline{u}}$ with $\overline{x}$, $\overline{u}$ localized in U, is injective. However, W contains an open subset $U \neq \varnothing$ such that

(a) U is geometrically unibranch
(b) U is affine and irreducible
(c) $\mathscr{F}$ is normally flat along W at every point of U (see [31, Ch.IV,§6.10.1]).

Indeed, (a) holds by assumption; (b) can be easily arranged by shrinking U, since X is noetherian. Lastly, (c) follows from [31, Ch.IV, Prop.6.10.2]. We claim that such U will do. Indeed, let $i : W \to W$ be the closed immersion; set

$$\mathscr{I} := \mathrm{Ker}(\mathscr{O}_X \to i_*\mathscr{O}_W) \qquad \mathscr{R}^\bullet := \bigoplus_{n\in\mathbb{N}} \mathscr{I}^n\mathscr{F}/\mathscr{I}^{n+1}\mathscr{F}$$

and recall that condition (c) means that $\mathscr{R}^\bullet_u$ is a flat $\mathscr{O}_{W,u}$-module, for every $u \in U$. Now, for a given $s_{\overline{x},\overline{u}}$ as in the foregoing, denote by x (resp. u) the support of $\overline{x}$ (resp. of $\overline{u}$) and define descending filtrations by the rule :

$$\mathrm{Fil}^k\mathscr{F}_{\overline{x}} := \mathscr{I}^k_x \cdot \mathscr{F}_{\overline{x}} \qquad \mathrm{Fil}^k\mathscr{F}_{\overline{u}} := \mathscr{I}^k_u \cdot \mathscr{F}_{\overline{u}} \qquad \text{for every } k \in \mathbb{N}.$$

By [61, Th.8.9], both these filtrations are separated, and obviously $s_{\overline{x},\overline{u}}$ is a map of filtered modules; thus, it suffices to show that the induced maps $\mathrm{gr}^k\mathscr{F}_{\overline{x}} \to \mathrm{gr}^k\mathscr{F}_{\overline{u}}$ of associated graded modules are injective, for every $k \in \mathbb{N}$. However, notice the natural identifications :

$$\mathrm{gr}^\bullet\mathscr{F}_{\overline{x}} = \mathscr{R}^\bullet_x \otimes_{\mathscr{O}_{W,x}} \mathscr{O}_{W_{\text{ét}},\overline{x}} \qquad \mathrm{gr}^\bullet\mathscr{F}_{\overline{u}} = \mathscr{R}^\bullet_x \otimes_{\mathscr{O}_{W,x}} \mathscr{O}_{W_{\text{ét}},\overline{u}}.$$

Then, the normal flatness condition reduces to checking that the induced strict specialization map $\mathscr{O}_{W_{\text{ét}},\overline{x}} \to \mathscr{O}_{W_{\text{ét}},\overline{u}}$ is injective; the latter assertion holds by the following :

Claim 8.7.30. Let W be a reduced, irreducible scheme, $\overline{w}$ a geometric point of W, and suppose that W is unibranch at the support w of $\overline{w}$. Then, for every generization $\overline{u}$ of $\overline{w}$ in W, every strict specialization map $\mathscr{O}_{W_{\text{ét}},\overline{w}} \to \mathscr{O}_{W_{\text{ét}},\overline{u}}$ is injective.

Proof of the claim. Let W^ν be the normalization of W, and $\overline{w}^\nu$ a geometric point of W^ν whose image in X is isomorphic to $\overline{w}$, and denote by w^ν the support of $\overline{w}^\nu$. The assumption on w means that the induced morphism $W^\nu(w^\nu) \to W(w)$ is integral, and the residue field extension $\kappa(w) \to \kappa(w^\nu)$ is radicial, hence the natural morphism of $W^\nu(\overline{w}^\nu)$-schemes

$$W(\overline{w}) \times_{W(w)} W^\nu(w^\nu) \to W^\nu(\overline{w}^\nu)$$

is an isomorphism ([33, Ch.IV, Prop.18.8.10]). Since $W^\nu(\overline{w}^\nu)$ is a normal local scheme ([33, Prop.18.8.12(i)]), it follows easily that $W(\overline{w})$ is reduced and irreducible. However, any specialization map is the composition of a localization map, followed by a local ind-étale map of local rings, whence the claim. □

8.7.31. Now, consider – quite generally – a ring homomorphism $A \to B$, with A noetherian. Set $Y := \mathrm{Spec}\, A$, $X := \mathrm{Spec}\, B$, and denote by $f : X \to Y$ the associated morphism of affine schemes.

Lemma 8.7.32. *In the situation of* (8.7.31)*, suppose moreover that f is absolutely flat. Then the following conditions are equivalent :*

(a) *B is noetherian.*
(b) *f has finite fibres.*

Proof. Let ξ be any geometric point of X, and x (resp. y) the support of ξ (resp. of $f(\xi)$).

(a)⇒(b): $X(\xi) \times_{\mathscr{O}_{Y,y}} \kappa(y)$ is the strict henselization of $X_y := f^{-1}(y)$ (with its reduced subscheme structure) at ξ. Under our assumptions, $X_y(\xi)$ is isomorphic to $\mathrm{Spec}\,\kappa(y)$, and (a) implies that $X_y(x)$ is a noetherian scheme of dimension zero. Since x is arbitrary, (b) follows.

(b)⇒(a): Let $I \subset B$ be any ideal, and denote by $\mathscr{I}$ the associated quasi-coherent $\mathscr{O}_X$-module. We need to show that $\mathscr{I}$ is an $\mathscr{O}_X$-module of finite type. Since f_ξ is an isomorphism, we may find a commutative diagram of affine schemes

$$\begin{tikzcd} X' \arrow[r, "g"] \arrow[d, "h"'] & Y' \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

whose left (resp. right) vertical arrow is an étale neighborhood of ξ (resp. of $f(\xi)$), and a quasi-coherent ideal $\mathscr{J} \subset \mathscr{O}_{Y'}$ such that $g^*\mathscr{J} \subset h^*\mathscr{I}$ and such that $(g^*\mathscr{J})_{x'} = (h^*\mathscr{I})_{x'}$ for some

$x' \in h^{-1}(x)$ (notice that g is a flat morphism). Set $\mathscr{M} := \mathscr{O}_{Y'}/\mathscr{J}$, let $z' \in X'$ an arbitrary point, and set $y' := g(z')$; in light of corollary 5.4.35, we have

$$z' \in \mathrm{Ass}\, g^*\mathscr{M} \Leftrightarrow z' \in \mathrm{Ass}\, g^*\mathscr{M}_{|X'(x')} \Leftrightarrow y' \in \mathrm{Ass}\, \mathscr{M}_{|Y'(y')} \Leftrightarrow y' \in \mathrm{Ass}\, \mathscr{M}.$$

In other words, $\mathrm{Ass}\, g^*\mathscr{M} = g^{-1}\mathrm{Ass}\, \mathscr{M}$. However, $\mathrm{Ass}\, \mathscr{M}$ is finite ([61, Th.6.5(i)]), hence the same holds for $\mathrm{Ass}\, g^*\mathscr{M}$, in view of (b). Now, notice that $\mathrm{Ass}\, h^*\mathscr{I}/g^*\mathscr{J} \subset \mathrm{Ass}\, g^*\mathscr{M} \setminus X'(x')$ (proposition 5.5.4(ii)). It follows that there exists an open neighborhood U of x' in X' such that $(h^*\mathscr{I}/g^*\mathscr{J})_{|U} = 0$ (lemma 5.5.3(iii)), *i.e.* $(g^*\mathscr{J})_{|U} = h^*\mathscr{I}_{|U}$; especially, $h^*\mathscr{I}_{|U}$ is an $\mathscr{O}_U$-module of finite type. Since h is an open map, we deduce that $h(U)$ is an open neighborhood of x in X, and $\mathscr{I}_{|h(U)}$ is an $\mathscr{O}_{h(U)}$-module of finite type. Since x is arbitrary, the lemma follows. □

8.7.33. For our second criterion, keep the situation of (8.7.31), and suppose additionaly that, for every geometric point ξ of X, the induced morphism $f_\xi : X(\xi) \to Y(\xi)$ is a closed immersion. Under this weaker assumption, it is not necessarily true that conditions (a) and (b) of lemma 8.7.32 are equivalent. For instance, we have :

Example 8.7.34. Take $A := k[X]$, the free polynomial algebra over a given infinite field k; also, let $(a_i \mid i \in \mathbb{N})$ be a sequence of distinct elements of k. We construct an A-algebra B, as the colimit of the inductive system $(B_i \mid i \in \mathbb{N})$ of A-algebras, such that

- $B_i := k[X] \times k^{i+1}$ (the product of $k[x]$ and $i+1$ copies of k, in the category of rings)
- the structure map $A \to B_i$ is the unique map of k-algebras given by the rule : $X \mapsto (X, a_0, \dots, a_i)$
- the transition maps $B_i \to B_{i+1}$ are given by the rule : $X \mapsto (X, 0, \dots, 0, a_{i+1})$ and $(0, e_i) \mapsto (0, e_i, 0)$ for $i = 0, \dots, i$. (Here $e_0, \dots, e_i$ is the standard basis of the k-vector space k^{i+1}.)

Then one can check that $X := \mathrm{Spec}\, B = \mathrm{Spec}\, k[X] \cup \mathbb{N}$, and $\mathbb{N}$ is an open subset of X with the discrete topology. Moreover, the induced map $\mathrm{Spec}\, B \to Y := \mathrm{Spec}\, A$ restricts to the continuous map $\mathbb{N} \to Y$ given by the rule $i \mapsto \mathfrak{p}_i$, for every $i \in \mathbb{N}$, where $\mathfrak{p}_i$ is the prime ideal generated by $X - a_i$. It follows easily that the condition of (8.7.33) is fulfilled; nevertheless, clearly X has infinitely many maximal points, hence its underlying topological space is not noetherian, and *a fortiori*, B cannot be noetherian.

However, we have the following positive result :

Proposition 8.7.35. *In the situation of* (8.7.33), *suppose additionally that Y admits a geometrically unibranch stratification. Then the following conditions are equivalent :*

(a) *B is noetherian.*
(b) *The topological space underlying X is noetherian.*
(c) *For every geometric point ξ of X there exists a neighborhood X' of ξ in $X_{\text{ét}}$, an unramified Y-scheme Y', and an absolutely flat morphism $X' \to Y'$ of Y-schemes, with finite fibres.*

Proof. Obviously (a)⇒(b).

(c)⇒(a): Indeed, under assumption (c), we may find finitely many geometric points $\xi_1, \dots, \xi_k$ of X, and for every $i = 1, \dots, k$, a neighborhood X'_i of ξ_i in $X_{\text{ét}}$, and a Y-morphism $X'_i \to Y'_i$ with the stated properties, such that moreover, the family $(X'_i \mid i = 1, \dots, k)$ is an étale covering of X. Furthermore, we may assume that X'_i and Y'_i are affine for every $i \leq k$. In this case, lemma 8.7.32 shows that every X'_i is noetherian, and then the same holds for X.

(b)⇒(c): Fix a geometric point ξ of X, and let B^{sh}_ξ (resp. A^{sh}_ξ) denote the strict henselization of B at ξ (resp. of A at $f(\xi)$); by assumption, we may find a (finitely generated) ideal $I \subset A^{\mathrm{sh}}_\xi$ such that f_ξ induces an isomorphism $A^{\mathrm{sh}}_\xi/I \xrightarrow{\sim} B^{\mathrm{sh}}_\xi$. Then we may find an affine étale neighborhood Y' of $f(\xi)$, say $Y' := \mathrm{Spec}\, A'$ for some étale A-algebra A', and an ideal $I' \subset A'$

such that $I'A^{\rm sh}_\xi = I$. Next, we may find an affine étale neighborhood $X' := \mathrm{Spec}\, B'$ of ξ such that f_ξ extends to a morphism $g : X' \to Y'$, and we may further suppose that the corresponding ring homomorphism $A' \to B'$ factors through A'/I', so g factors through a morphism

$$h : X' \to Z := \mathrm{Spec}\, A'/I'$$

and the closed immersion $Z \to Y'$. By construction, ξ lifts to a geometric point ξ' of X', and $h_{\xi'} : X'(\xi') \to Z(h(\xi'))$ is an isomorphism ([33, Ch.IV, Prop.18.8.10]). To conclude the proof, it then suffices to exhibit an open subset $U \subset X'$ containing the support of ξ', and such that the restriction $U \to Z$ of h is absolutely flat with finite fibres.

Claim 8.7.36. Let $\varphi : W \to W'$ be a quasi-finite, separated, dominant and finitely presented morphism of reduced, irreducible schemes, and suppose that W' contains a non-empty geometrically unibranch open subset. Then the same holds for W.

Proof of the claim. After replacing W' by some open subset $U' \subset W'$, and W by $\varphi^{-1}U'$, may assume that W' is affine and unibranch; then we may also suppose that φ is finite ([32, Ch.IV, Th.8.12.6]), in which case W is affine as well, and φ is surjective.

Let $\eta \in W$ and $\eta' \in W'$ be the respective generic points, and denote by $E \subset \kappa(\eta)$ the maximal subfield that is separable over $\kappa(\eta')$. We may then find a reduced and irreducible scheme W'', with generic point η'', such that φ factors as the composition of finite surjective morphisms $\varphi' : W \to W''$, $\varphi'' : W'' \to W'$, and such that $\kappa(\eta'') = E$. By virtue of [32, Ch.IV, Th.8.10.5] and [33, Ch.IV, Prop.17.7.8(ii)], we may then replace W' by a non-empty open subset, and assume that φ' is radicial, and φ'' is étale. Then W'' is geometrically unibranch ([31, Ch.IV, Prop.6.15.10]); hence, we may replace W' by W'', and reduce to the case where φ is radicial. Let p be the characteristic of $\kappa(\eta')$; if $p = 0$, φ is birational, in which case the assertion follows from [31, Prop.6.15.5(ii)]. In case $p > 0$, write $W = \mathrm{Spec}\, C$, $W' = \mathrm{Spec}\, C'$; the induced ring homomorphism $C' \to C$ is finite and injective, and we have $C^{p^n} \subset C'$ for $n \in \mathbb{N}$ large enough. Denote by C^ν (resp. C'^ν) the normalization of the domain C (resp. of C'); it follows easily that $(C^\nu)^{p^n} \subset C'^\nu$, so the morphism $\mathrm{Spec}\, C^\nu \to \mathrm{Spec}\, C'^\nu$ is radicial. On the other hand, since W' is geometrically unibranch, the normalization morphism $\mathrm{Spec}\, C'^\nu \to W'$ is radicial ([30, Ch.0, Lemme 23.2.2]) and therefore the normalization map $\mathrm{Spec}\, C^\nu \to W$ is radicial as well ([31, Ch.IV, Lemme 6.15.3.1(i)]). Then W is geometrically unibranch, again by [30, Ch.0, Lemme 23.2.2]. ◊

From claim 8.7.36 and our assumption on Y, it follows easily that Z admits a geometrically unibranch stratification. The morphism h induces as usual a morphism of étale topoi

$$(X')^\sim_{\mathrm{\acute{e}t}} \underset{h^*}{\overset{h_*}{\rightleftarrows}} Z^\sim_{\mathrm{\acute{e}t}} \tag{8.7.37}$$

as well as a morphism $h^\natural : h^*\mathscr{O}_{Z_{\mathrm{\acute{e}t}}} \to \mathscr{O}_{X'_{\mathrm{\acute{e}t}}}$ of $(X')^\sim_{\mathrm{\acute{e}t}}$-rings. By construction, for every geometric point τ of X', the induced map on stalks $h^\natural_\tau : \mathscr{O}_{Z_{\mathrm{\acute{e}t}},h(\tau)} \to \mathscr{O}_{X'_{\mathrm{\acute{e}t}},\tau}$ is surjective, and $h^\natural_{\xi'}$ is a bijection. It follows easily that $h^\natural_\tau$ is also bijective for every generization τ of ξ'. Now, choose a partition $Z = Z_1 \cup \cdots \cup Z_k$ as in lemma 8.7.29 (with $\mathscr{F} := \mathscr{O}_{Z_{\mathrm{\acute{e}t}}}$), and for given $i \leq k$, suppose that τ and η are two geometric points of $h^{-1}Z_i$, with η a generization of τ. The choice of a strict specialization morphism $X'(\eta) \to X'(\tau)$ yields a commutative diagram

$$\begin{array}{ccc} \mathscr{O}_{Z_{\mathrm{\acute{e}t}},h(\tau)} & \xrightarrow{h^\natural_\tau} & \mathscr{O}_{X'_{\mathrm{\acute{e}t}},\tau} \\ \downarrow & & \downarrow \\ \mathscr{O}_{Z_{\mathrm{\acute{e}t}},h(\eta)} & \xrightarrow{h^\natural_\eta} & \mathscr{O}_{X'_{\mathrm{\acute{e}t}},\eta} \end{array} \tag{8.7.38}$$

whose vertical arrows are strict specialization maps (see remark 2.4.25(i)); in light of lemma 8.7.29, we deduce that $h^\natural_\tau$ is injective, whenever the same holds for $h^\natural_\eta$. Now, let $x' \in X'$ be the support of ξ', and Σ_i the set of maximal points of $h^{-1}Z_i$. Condition (b) and proposition 4.3.45 imply that $h^{-1}Z_i$ is a noetherian topological space, hence Σ_i is a finite set; it follows that the topological closure W of $\bigcup_{i=1}^k \Sigma_i \setminus X'(x')$ in X' is a closed subset that does not contain x'; by construction, $h^\natural$ restricts to a monomorphism on $U := X' \setminus W$, *i.e.* the restriction $h_U : U \to Z$ of h is absolutely flat, as required. It also follows that the fibres of h_U are noetherian topological spaces of dimension zero (cp. the proof of lemma 8.7.32), hence they are finite, and the proof is complete. $\square$

Lemma 8.7.39. *Suppose that the valuation of K has finite rank, and let X be a finitely presented K^+-scheme, $\mathscr{F}$ a coherent $\mathscr{O}_{X_{\text{ét}}}$-module. Then there exists a partition*

$$X = X_1 \cup \cdots \cup X_k$$

of X into finitely many disjoint irreducible locally closed subsets, such that the following holds. For every $i = 1, \dots, k$, every geometric point $\overline{x}$ of X_i, and every generization $\overline{u}$ of $\overline{x}$ in X_i, every strict specialization map

$$s_{\overline{x},\overline{u}} : \mathscr{F}_{\overline{x}} \to \mathscr{F}_{\overline{u}}$$

is injective (see (2.4.22)*).*

Proof. We easily reduce to the case where X is affine, say $X = \operatorname{Spec} A$, and then $\mathscr{F}$ is the coherent $\mathscr{O}_{X_{\text{ét}}}$-module arising from a finitely presented A-module M. By theorem 8.7.17(i), M admits a K^+-flattening sequence $(b_0, \dots, b_n)$. Then we are further reduced to showing the assertion for the subquotients $b_iM/b_{i+1}M$ (that are finitely presented, by corollary 8.7.21(ii)). So, we may assume from start that $f : X \to S_0 := \operatorname{Spec} K^+/bK^+$ is a finitely presented morphism for some $b \in K^+$, and $\mathscr{F}$ is f-flat. For every $t \in S_0$, let

$$i_t : X_t := f^{-1}(t) \to X$$

be the locally closed immersion; since X_t is an excellent noetherian scheme, we may apply lemma 8.7.29 and remark 8.7.23 to produce a partition $X_t = X_{t,1} \cup \cdots \cup X_{t,k}$ by finitely many disjoint irreducible locally closed subsets such that, for every $i = 1, \dots, k$, every geometric point $\overline{x}$ of $X_{t,i}$ and every generization $\overline{u}$ of $\overline{x}$ in $X_{t,i}$, every specialization map

$$(i_t^*\mathscr{F})_{\overline{x}} \to (i_t^*\mathscr{F})_{\overline{u}}$$

is injective. Since $|S_0|$ is a finite set, the lemma will then follow from :

Claim 8.7.40. Let $g : Y \to T$ be any finitely presented morphism of schemes, and $\mathscr{G}$ a finitely presented, quasi-coherent g-flat $\mathscr{O}_Y$-module. Let also $\overline{y}$, $\overline{u}$ be two geometric points of Y with $t := g(\overline{y}) = g(\overline{u})$, and such that $\overline{u}$ is a generization of $\overline{y}$. Let $s_{\overline{y},\overline{u}} : \mathscr{G}_{\overline{y}} \to \mathscr{G}_{\overline{u}}$ be a strict specialization map, and suppose that $s_{\overline{y},\overline{u}} \otimes_{\mathscr{O}_{T,t}} \kappa(t)$ is injective. Then the same holds for $s_{\overline{y},\overline{u}}$.

Proof of the claim. Set $Y' := Y(\overline{y})$, denote by $j : Y' \to Y$ the natural morphism, and set $\mathscr{G}' := j^*\mathscr{G}$. The map $s_{\overline{y},\overline{u}}$ is deduced from a morphism $Y(\overline{u}) \to Y'$ of Y-schemes; the latter factors through a faithfully flat morphism $Y(\overline{u}) \to Y'(u)$, where $u \in Y'$ is the image of the closed point of $Y(\overline{u})$. Hence, $s_{\overline{y},\overline{u}}$ is the composition of the specialization map $s' : \mathscr{G}'_{\overline{y}} \to \mathscr{G}'_u$, and the injective map $\mathscr{G}'_u \to \mathscr{G}'_{\overline{u}}$. Our assumption implies that $s' \otimes_{\mathscr{O}_{T,t}} \kappa(t)$ is injective, and it suffices to show that the same holds for s'. However, Y' is the limit of a cofiltered system $(j_\lambda : Y_\lambda \to Y \mid \lambda \in \Lambda)$ of local, essentially étale Y-schemes. Write $\mathscr{G}_\lambda := j_\lambda^*\mathscr{G}$ for every $\lambda \in \Lambda$, and notice that the transition morphisms $Y_\lambda \to Y_\mu$ are faithfully flat, for every $\lambda \geq \mu$; it follows that $\mathscr{G}'_{\overline{y}}$ is the filtered union of the system of modules $(G_\lambda := \Gamma(Y_\lambda, \mathscr{G}_\lambda) \mid \lambda \in \Lambda)$ (proposition 5.1.15(i)); likewise, $\mathscr{G}'_{\overline{y}} \otimes_{\mathscr{O}_{T,t}} \kappa(t)$ is the filtered union of the submodules $(G_\lambda \otimes_{\mathscr{O}_{T,t}} \kappa(t) \mid \lambda \in \Lambda)$. We are then reduced to checking that all the restrictions $s_\lambda : G_\lambda \to \mathscr{G}'_u$ of s' are injective, and we know already that $s_\lambda \otimes_{\mathscr{O}_{T,t}} \kappa(t)$ is injective for every $\lambda \in \Lambda$. For every such λ, let $u_\lambda \in Y_\lambda$ be the image of u; then s_λ factors through the injective map $\mathscr{G}_{\lambda,u_\lambda} \to \mathscr{G}'_u$ and the specialization map $s'_\lambda : G_\lambda \to \mathscr{G}_{\lambda,u_\lambda}$. Consequently, it suffices to show that s'_λ is injective, and we know already that the same holds for $s'_\lambda \otimes_{\mathscr{O}_{T,t}} \kappa(t)$. However, $\mathscr{G}_{\lambda,u_\lambda}$ is a localization $Q_\lambda^{-1}G_\lambda$, for a multiplicative set $Q_\lambda \subset \Gamma(Y_\lambda, \mathscr{O}_{Y_\lambda})$, and s'_λ is the localization map. The claim therefore boils down to the assertion that, for every $\lambda \in \Lambda$ and every $q \in Q_\lambda$, the endomorphism $q \cdot \mathbf{1}_{G_\lambda}$ is injective on G_λ, and our assumption already ensures that $(q \cdot \mathbf{1}_{G_\lambda}) \otimes_{\mathscr{O}_{T,t}} \kappa(t)$ is injective on $G_\lambda \otimes_{\mathscr{O}_{T,t}} \kappa(t)$. However, let $g_\lambda : Y_\lambda \to T$ be the morphism induced by g; by construction, $\mathscr{G}_\lambda$ is a g_λ-flat $\mathscr{O}_{Y_\lambda}$-module, hence the contention follows from [32, Prop.11.3.7]. $\square$

Proposition 8.7.41. *Let* $(K, |\cdot|)$ *be a valued field, and* A *a locally measurable* K^+*-algebra fulfilling conditions* (a) *and* (b) *of theorem* 8.7.17*. Then, the following conditions are equivalent:*

- (c) *For every geometric point* ξ *of* $X := \operatorname{Spec} A$ *there exists a neighborhood* U *of* ξ *in* $X_{\text{ét}}$*, and a finitely presented* K^+*-scheme* Z *with an absolutely flat morphism of* K^+*-schemes* $X' \to Z$*.*
- (d) Ω_{A/K^+} *is an* A*-module of finite type.*

Proof. (c)⇒(d) follows easily from lemma 8.7.24.

(d)⇒(c): Let $a_1, \dots, a_k \in A$ be a finite system of elements such that $da_1, \dots, da_k$ generate the A-module Ω_{A/K^+}. We define a map of K^+-algebras $A_0 := K^+[T_1, \dots, T_k] \to A$ by the rule: $T_i \mapsto a_i$ for $i = 1, \dots, k$. Clearly $\Omega_{A/A_0} = 0$. Let $Z_0 := \operatorname{Spec} A_0$, and denote by $f : X \to Z_0$ the induced morphism of schemes. Let A^{sh}_ξ (resp. $A^{\text{sh}}_{0,\xi}$) be the strict henselization of A at ξ (resp. of A_0 at $f(\xi)$). According to lemma 8.7.27, the induced map $A^{\text{sh}}_{0,\xi} \to A^{\text{sh}}_\xi$ is surjective, and its kernel is a finitely generated ideal $I \subset A^{\text{sh}}_{0,\xi}$. In this situation, we may argue as in the proof of proposition 8.7.35, to produce an affine étale neighborhood X' of ξ, a finitely presented affine unramified Z_0-scheme Z, and a morphism of Z_0-schemes $h : X' \to Z$ such that $h_\tau : X'(\tau) \to Z(h(\tau))$ is a closed immersion for every geometric point τ of X', and $h_{\xi'}$ is an isomorphism for some lifting ξ' of ξ.

Then we consider the associated morphism of étale topoi as in (8.7.37) and the morphism $h^\natural : h^*\mathscr{O}_{Z_{\text{ét}}} \to \mathscr{O}_{X'_{\text{ét}}}$ of $(X')^\sim_{\text{ét}}$-rings. Again, the induced map on stalks $h^\natural_\tau$ is surjective for every geometric point τ of X', and is bijective if τ is a generization of ξ'. We pick a finite partition $Z = Z_1 \cup \dots \cup Z_k$ as in lemma 8.7.39 (for $\mathscr{F} := \mathscr{O}_Z$). For any $i \leq k$, let τ, η be two geometric points of $h^{-1}Z_i$, such that η is a generization of τ; by considering the commutative diagram (8.7.38), we see again that $h^\natural_\tau$ is injective whenever the same holds for $h^\natural_\eta$.

Now, condition (b) of theorem 8.7.17 easily implies that $h^{-1}Z_i$ is a noetherian topological space, hence its set Σ_i of maximal points is finite. Again we let $x' \in X'$ be the support of ξ', and W the topological closure of $\bigcup_{i=1}^k \Sigma_i \setminus X'(x')$ in X', and it is easily seen that the restriction $U \to Z$ of h is absolutely flat, so (c) holds. $\square$

8.7.42. Henceforth we restrict to the case where the value group Γ of K is not discrete and of rank one. As usual, we consider the almost structure attached to the standard setup attached to $(K, |\cdot|)$.

Definition 8.7.43. In the situation of (8.7.42), let A be any K^{+a}-algebra, and M any A-module.

- (i) We say M is an *almost noetherian* A-module, if every A-submodule of M is almost finitely generated.
- (ii) We say that A is an *almost noetherian* K^{+a}-algebra, if A is an almost noetherian A-module.

Remark 8.7.44. In the situation of (8.7.42), suppose that A is an almost noetherian K^{+a}-algebra. Then the same argument as in the "classical limit" case shows that every almost finitely generated A-module M is almost noetherian. The details shall be left to the reader.

Theorem 8.7.45. *Let A be a locally measurable K^+-algebra. Suppose that both $A \otimes_{K^+} \kappa$ and $A \otimes_{K^+} K$ are noetherian rings. Then A^a is an almost noetherian K^{+a}-algebra.*

Proof. If the valuation of K is discrete, the assertion is proposition 8.7.14(iii). Hence, we may assume that the valuation of K is not discrete. Moreover, let $\mathscr{Q}$ be the set of all locally measurable K^+-algebras B that are quotients of A, and such that B^a is not almost noetherian. We have to show that $\mathscr{Q} = \varnothing$. However, for every $B \in \mathscr{Q}$ the κ-algebra $\overline{B} := B \otimes_{K^+} \kappa$ is a quotient of the noetherian κ-algebra $\overline{A} := A \otimes_{K^+} \kappa$; it follows that the set $\overline{\mathscr{Q}} := \{\overline{B} \mid B \in \mathscr{Q}\}$ admits minimal elements, if it is not empty. In the latter case, we may then replace A by any quotient $B \in \mathscr{Q}$ such that $\overline{B}$ is minimal in $\overline{\mathscr{Q}}$, and therefore assume that for every locally measurable quotient B of A, either $\overline{B} = \overline{A}$, or else B^a is almost noetherian.

Let $I \subset A$ be any ideal; we have to show that I^a is almost finitely generated. By assumption, the image $\overline{I}$ of I in $\overline{A}$ is finitely generated, and the same holds for the ideal $I_K := I \otimes_{K^+} K$ of $A_K := A \otimes_{K^+} K$. Thus, we may find a finitely generated subideal $I_0 \subset I$ whose image in $\overline{A}$ agrees with $\overline{I}$, and such that $I_0 \otimes_{K^+} K = I_K$. After replacing A by A/I_0 and I by I/I_0, we are then reduced to the case where both $\overline{I}$ and I_K vanish. Let $J \subset A$ denote the kernel of the localization map $A \to A_K$, set $S := 1 + \mathfrak{m}_K A$, and let $B := S^{-1}A \times A_K$. Clearly B is a faithfully flat A-algebra, and A/J is a K^+-flat A-module; therefore $(A/J) \otimes_A B$ is a K^+-flat B-module of finite type, and since A_K is noetherian, proposition 8.7.14(ii) implies that $(A/J) \otimes_A B$ is finitely presented. Then A/J is finitely presented as well, and therefore J is a finitely generated ideal. We conclude that there exists $c \in \mathfrak{m}_K$ such that $cJ = 0$. Then notice that $J \cap cA = 0$: indeed, if $a \in J \cap cA$, we have $a = cx$ for some $x \in A$, and $ca = 0$, therefore $c^2 x = 0$, so $x \in J$, and consequently $a = cx = 0$. Now, fix $b \in \mathfrak{m}_K$; since the valuation of K has rank one, and since $I \subset J$, it follows that there exists $n \in \mathbb{N}$ large enough, so that

$$I \cap b^n A = 0.$$

Let $i_0 := \max\{i \in \mathbb{N} \mid I \subset b^i A\}$, and set

$$N := b^{i_0} A/(I + b^{i_0+1}A) \qquad N' := b^{i_0} A/b^{i_0+1}A.$$

Let $\varphi : N' \to N$ be the natural surjection, and set $c^*_N := c_N \circ \varphi : N' \to cN$ for every $c \in K^+$ (notation of definition 8.7.6(ii)).

Claim 8.7.46. There exists $c \in K^+$ with $\log|c| < \log|b|$ such that $c^*_N \otimes_{K^+} \kappa$ is not an isomorphism.

Proof of the claim. Suppose that the claim fails; then it is easily seen that no $\gamma \in \log \Gamma^+$ with $\gamma < \log|b|$ breaks N. For every geometric point ξ of $\mathrm{Spec}\, A$, let A^{sh}_ξ denote the strict henselization of A at ξ; since A^{sh}_ξ is a flat A-algebra, we have a natural identification

$$cN \otimes_A A^{\mathrm{sh}}_\xi = c(N \otimes_A A^{\mathrm{sh}})$$

of A^{sh}-modules; therefore, no $\gamma < \log|b|$ breaks $N \otimes_A A^{\mathrm{sh}}_\xi$. By corollary 8.7.13, we deduce that $N \otimes_A A^{\mathrm{sh}}_\xi$ is a K^+/bK^+-flat and finitely presented A^{sh}_ξ-module, for every geometric point ξ. Hence, $C_\xi := \mathrm{Ker}\, \varphi \otimes_A A^{\mathrm{sh}}_\xi$ is a finitely generated A^{sh}_ξ-module, and $C_\xi \otimes_{K^+} \kappa = 0$, for every geometric point ξ. By Nakayama's lemma, it follows that $C_\xi = 0$ for every such ξ, so finally $\mathrm{Ker}\, \varphi = 0$, which means that $I \subset b^{i_0+1}A$, contradicting the choice of i_0. $\Diamond$

Let c be as in claim 8.7.46, and set $d := cb^{i_0}$; notice the natural isomorphism of $\overline{A}$-modules

$$cN \otimes_{K^+} \kappa \xrightarrow{\sim} \frac{dA}{I \cap dA} \otimes_{K^+} \kappa.$$

Let $\varphi : A \to dA/(I \cap dA)$ be the composition of $d_A : A \to dA$ and the projection $dA \to dA/(I \cap dA)$ (notation of definition 8.7.6(ii)); by construction, $\varphi \otimes_{K^+} \kappa$ is not an isomorphism, hence there exists $x \in I \cap dA$ such that the composition $\varphi_x : A \to dA/xA$ of d_A and the projection $dA \to dA/xA$ induces a map $\varphi_x \otimes_{K^+} \kappa$ with non-trivial kernel. In other words, dA/xA is a cyclic module over a locally measurable quotient B of A such that the projection $\overline{A} \to \overline{B}$ is not an isomorphism, so B^a is almost noetherian. Set $I_0 := (I \cap dA)/xA$; then I_0 is a submodule of dA/xA, and consequently I_0^a is an almost finitely generated B^a-module (remark 8.7.44). Then clearly $(I \cap dA)^a$ is an almost finitely generated ideal of A^a. But by construction, $cI \subset I \cap dA$, and b annihilates $(I \cap dA)/cI$. Since b is arbitrary, this easily implies that I^a is almost finitely generated, as required. □

Corollary 8.7.47. *In the situation of theorem* 8.7.45*, the following holds :*

(i) *Every almost finitely generated A^a-module is almost finitely presented.*
(ii) *Every flat almost finitely generated A^a-module is almost projective of finite rank.*

Proof. Assertion (i) is an easy consequence of theorem 8.7.45 and remark 8.7.44 : the details shall be left to the reader.

(ii): Let M be a flat and almost finitely generated A^a-module; by (i) and [36, Prop.2.4.18(ii)], M is almost projective. It remains to show that there exists $n \in \mathbb{N}$ such that $\Lambda^n_{A^a} M = 0$, or equivalently, that $(\Lambda^n_A M_!)^a = 0$. However, $M_!$ is a flat A-module, so $\mathrm{Ass}\, M_! \subset \mathrm{Ass}\, A$; in view of theorem 8.7.17(ii), we may argue by induction on the cardinality c of $\mathrm{Ass}\, A$, and it suffices to check that, for every $\mathfrak{p} \in \mathrm{Ass}\, A$ there exists $n \in \mathbb{N}$ such that $\Lambda^n_{A_\mathfrak{p}}(M_!)_\mathfrak{p} = 0$. If $c = 0$, we have $A = 0$, and there is nothing to prove. Suppose that $c > 0$ and the assertion is known for every locally measurable K^+-algebra B such that $B \otimes_{K^+} K$ and $B \otimes_{K^+} \kappa$ are noetherian, and such that $\mathrm{Ass}\, B$ has cardinality $< c$. Especially, for a fixed $\mathfrak{p} \in \mathrm{Ass}\, A$, we can cover $\mathrm{Spec}\, A_\mathfrak{p} \setminus \{\mathfrak{p}\}$ by finitely many affine open subsets $\mathrm{Spec}\, B_1, \dots, \mathrm{Spec}\, B_k$, and then the inductive assumption yields $n \in \mathbb{N}$ such that $\Lambda^n_{B_i}(M_! \otimes_A B_i) = \Lambda^n_{B_i}(M \otimes_{A^a} B_i^a)_! = 0$ for every $i = 1, \dots, k$. In other words, $N := \Lambda^n_{A_\mathfrak{p}}(M_\mathfrak{p})$ is a flat $A^a_\mathfrak{p}$-module with $\mathrm{Supp}\, N_! \subset \{\mathfrak{p}\}$.

Let $A^{\mathrm{sh}}_\mathfrak{p}$ denote the strict henselization of A at some geometric point localized at $\mathfrak{p}$; on the one hand, the $A_\mathfrak{p}$-algebra $A^{\mathrm{sh}}_\mathfrak{p}$ is faithfully flat, and $M_! \otimes_A A^{\mathrm{sh}}_\mathfrak{p} = (M \otimes_{A^a} (A^{\mathrm{sh}}_\mathfrak{p})^a)_!$. On the other hand, exterior powers commute with arbitrary base change; thus, we are reduced to showing :

Claim 8.7.48. Let B be a measurable K^+-algebra, and N a flat almost finitely generated B^a-module whose support is contained in $\{s(B)\}$ (notation of (8.3.23)). Then $\Lambda^n_B N_! = 0$ for every sufficently large $n \in \mathbb{N}$.

Proof of the claim. By assumption, we may find an essentially finitely presented K^+-algebra B_0 and an ind-étale and faithfully flat map $B_0 \to B$ of K^+-algebras. Set $X_0 := \mathrm{Spec}\, B_0$ and $X := \mathrm{Spec}\, B$; since $\{s(B_0)\}$ is a constructible subset of X_0, the natural map

$$B \otimes_{B_0} \Gamma_{\{s(B_0)\}} \mathscr{O}_{X_0} \to \Gamma_{\{s(B)\}} \mathscr{O}_X$$

is an isomorphism (lemma 5.4.16(iii)). On the other hand, there exists a finitely generated $s(B_0)$-primary ideal $J \subset B_0$ such that the natural map $\Gamma_{\{s(B_0)\}} \mathscr{O}_{X_0} \to B_0/J$ is injective (lemma 5.5.10 and theorem 5.7.20(i)). It follows that the natural map $\Gamma_{\{s(B)\}} \mathscr{O}_X \to B/JB$ is injective as well. Now, $N_!$ can be written as the colimit of a filtered system $(L_\lambda \mid \lambda \in \Lambda)$ of free B-modules of finite rank; for each $\lambda \in \Lambda$, let $L^\sim_\lambda$ be the quasi-coherent $\mathscr{O}_X$-module arising from L_λ, and define likewise $N^\sim_!$; taking into account lemma 5.4.4(ii.b) we deduce that the natural map

$$N_! = \Gamma_{\{s(B)\}} N^\sim_! = \operatorname*{colim}_{\lambda \in \Lambda} \Gamma_{\{s(B)\}} L^\sim_\lambda \to \operatorname*{colim}_{\lambda \in \Lambda} L_\lambda / J L_\lambda = N_!/JN_!$$

is injective. In other words, N is a B^a/JB^a-module. We may then replace B by B/JB, and assume from start that B has Krull dimension zero. In this situation, we may find a nilpotent ideal $I \subset B$, a valuation ring V that is a measurable K^+-algebra, and a finitely presented surjection $V \to B/I$ (lemma 8.3.29). It suffices to find $n \in \mathbb{N}$ such that $(\Lambda^n_B N_!) \otimes_B B/I = 0$;

hence, we may replace N by N/IN and B by B/I, and assume as well that $B = V/bV$ for some $b \in V$. In this case, the assertion follows easily from proposition 8.3.20. □

9. The almost purity theorem

In this chapter we prove the almost purity theorem. The theorem states that certain pairs $(X_\infty, \{x_\infty\})$, consisting of a local scheme X_∞ and its closed point x_∞, are *almost pure* (see definition 8.2.25(i)). We have tried to axiomatize the minimal set of assumptions on $(X_\infty, \{x_\infty\})$ that are required for the proof (see (9.2.20)); these may look a little cumbersome, even though in practice they are verified in many interesting cases (see example 9.2.27). Regardless, we make no claim that our assumptions are the weakest *reasonable* ones : at the current state of our understanding, it remains entirely conceivable that a different approach would allow to shed some of them. At the same time, some general features are well entrenched and seem to be inherently unavoidable; especially, X_∞ will certainly be the projective limit of a tower of essentially quasi-finite morphisms :

$$\cdots \to X_n \to X_{n-1} \to \cdots \to X_0$$

of affine normal schemes of essentially finite type over a valuation ring K^+ of rank one. Moreover, one requires that the special fibre of X_∞ is perfect.

The proof proceeds by induction on the relative dimension d of X_∞ over $S := \mathrm{Spec}\, K^+$.

- For $d = 0$, the scheme X_∞ is the spectrum of a valuation ring K^+_∞, and the base change $X_\infty \to X_0$ kills ramification precisely when K^+_∞ is deeply ramified (in the sense of [36, Def.6.6.1]); in this situation, the almost purity theorem is equivalent to [36, Prop.6.6.6].
- In section 9.1, we suppose that $d = 1$. In this case, the bottom X_0 of the tower is a strict henselization of a *semi-stable relative curve* over S; so, X admits an ind-étale morphism $g : X \to \mathbb{T}(\gamma)$, where $\gamma \in \Gamma^+$, the monoid of positive elements in the value group Γ of K^+, and $\mathbb{T}(\gamma)$ is the spectrum of the K^+-algebra $R(\gamma) := K^+[\mathsf{X}, \mathsf{Y}]/(\mathsf{XY} - c)$, with $c \in K^+$ any element such that $|c| = \gamma$. There is a basic tower :

$$\cdots \to \mathbb{T}(\gamma_n) \to \mathbb{T}(\gamma_{n-1}) \to \cdots \to \mathbb{T}(\gamma_0)$$

where $\gamma_n := \gamma^{1/p^n}$ for every $n \in \mathbb{N}$, and the morphisms are defined by the rule : $\mathsf{X} \mapsto \mathsf{X}^p$, $\mathsf{Y} \mapsto \mathsf{Y}^p$. The tower $X_\bullet$ is then a strict henselization of the tower $\mathbb{T}(\gamma_\bullet)$.

Now, the essential point is to show that every locally free module of finite rank $\mathscr{F}_\infty$ on $U_\infty := X \setminus \{x_\infty\}$ extends – by direct image under the inclusion $j : U_\infty \to X_\infty$ – to an almost flat $\mathscr{O}_{X_\infty}$-module $j_*\mathscr{F}$. However, x_∞ projects to some $x \in X$, and we may assume that $\mathscr{F}_\infty$ descends to a locally free module of finite rank $\mathscr{F}$ on $U := X(x) \setminus \{x\}$; in this case we show that the direct image of $\mathscr{F}$ under the inclusion $U \subset X(x)$, is a reflexive $\mathscr{O}_{X(x)}$-module. (In the corresponding part of Faltings' original proof, there is a reduction to the case of a tower of regular local rings, and then this direct image is even free.) It follows that $j_*\mathscr{F}_\infty$ can be written as the colimit of a filtered family of reflexive modules.

We are then naturally led to look for a description – as explicit as possible – of all reflexive $\mathscr{O}_{X(x)}$-modules. This latter problem is solved by theorem 9.1.17, which actually holds more generally for a base valuation ring K^+ of arbitrary rank. Namely, every such module is of the form $g^*\mathscr{F}_x$, where $\mathscr{F}$ is a reflexive $\mathscr{O}_{\mathbb{T}(\gamma)}$-module; moreover, $\mathscr{F}$ is a direct sum of generically invertible reflexive $\mathscr{O}_{\mathbb{T}(\gamma)}$-modules; the latter are determined by their restriction to the smooth locus $\mathbb{T}(\gamma)_{\mathrm{sm}}$, and their isomorphism classes form an abelian group naturally isomorphic to $\mathrm{Pic}\,\mathbb{T}(\gamma)_{\mathrm{sm}}$, which is computed by proposition 9.1.13. Summarizing, if the image $y \in S$ of the point x is the generic point (indeed, whenever $y \notin \mathrm{Spec}\, K^+/cK^+$), every reflexive $\mathscr{O}_{X(x)}$-module is free; and more generally, the group of generically invertible reflexive $\mathscr{O}_{X(x)}$-modules is naturally isomorphic to a subquotient of the value group of the valuation ring $\mathscr{O}_{S,y}$. (So that, predictably, the complexity of the classification grows monotonically, as y approaches the closed point of S.)

The upshot is that, if $\mathscr{F}$ is a reflexive $\mathscr{O}_{X(x)}$-module, then c annihilates the functor :

$$\mathscr{M} \mapsto \mathrm{Tor}_i^{\mathscr{O}_{X(x)}}(\mathscr{F}, \mathscr{M}) \qquad \text{for every } i > 0$$

(see claim 9.1.29(ii)). So one can say that $\mathscr{F}$ is *as close to being a flat $\mathscr{O}_{X(x)}$-module, as the value γ is to* $1 \in \Gamma$. This persists after base change to any X_n, except that γ is replaced by γ_n; and if we climb the tower all the way, we see that the almost scheme $j_*\mathscr{F}^a_\infty$ is a flat $\mathscr{O}^a_{X_\infty}$-module.

As usual, once we have established flatness, we are almost home : to conclude, one applies the general trace arguments that have been conveniently packaged in lemmata 8.2.18 and 8.2.11.

• When $d \geq 2$, we do not have any longer an explicit description of reflexive modules, hence we have to resort to more sophisticated homological machinery. This entails a cost, since – as we shall see – for each step of the proof we shall have to add further assumptions on the tower $X_\bullet$: to begin with, we demand that every X_n is the spectrum of a local, essentially smooth K^+-algebra R_n, so that $X_\infty = \mathrm{Spec}\, R_\infty$, where R_∞ is the increasing union of its subalgebras R_n. Now, let $U := X_\infty \setminus \{x_\infty\}$, and suppose that we are given an étale almost finite $\mathscr{O}^a_U$-algebra $\mathscr{A}$; as usual, it suffices to show that $\Gamma(U, \mathscr{A})$ is a flat R^a_∞-module, or equivalently, a flat R^a_n-module, for every $n \in \mathbb{N}$. To this aim, we may try to apply proposition 5.8.19, which translates flatness in terms of a depth condition for the $\mathscr{O}_U$-module $\mathscr{A}_!$; namely, we are reduced to showing that $H^i(U, \mathscr{A}) = 0$ in the range $1 \leq i \leq d-1$. We succeed in proving directly this vanishing when $d \geq 3$, provided one more hypothesis is fulfilled (proposition 9.2.14). The idea is to exploit the Frobenius morphism :

$$\overline{\Phi}_U : U \times_S \mathrm{Spec}\, K^+/\pi^p K^+ \to U \times_S \mathrm{Spec}\, K^+/\pi K^+$$

(where $\pi \in K^+$ is any element such that $1 > |\pi^p| \geq |p|$: see (8.5.2)). Namely, since $\mathscr{A}$ is étale, the Frobenius map $\overline{\Phi}_{\mathscr{A}}$ on $\mathscr{A}$ yields a natural isomorphism (lemma 8.5.5) :

$$\mathscr{A}/\pi\mathscr{A} \xrightarrow{\sim} \overline{\Phi}_{U*}\mathscr{A}/\pi^p\mathscr{A}.$$

Moreover, $\overline{\Phi}_U$ is a flat morphism, therefore we obtain isomorphisms on cohomology :

$$\overline{\Phi}^*_{R_\infty} H^i(U, \mathscr{A}/\pi\mathscr{A}) \xrightarrow{\sim} H^i(U, \mathscr{A}/\pi^p\mathscr{A})$$

where $\overline{\Phi}_{R_\infty} : R_\infty/\pi R_\infty \to R_\infty/\pi^p R_\infty$ is the Frobenius map (see the proof of claim 9.2.15). On the other hand, with the aid of lemma 5.8.26 we show that $H^i(U, \mathscr{A}/\pi\mathscr{A})$ is a R^a_∞-module of *almost finite length* for $1 \leq i \leq d-2$: this refers to a notion of *normalized length* (for R_∞-modules and R^a_∞-modules) which is introduced and thoroughly studied in section 8.3. We have tried to codify in (8.3.49) the minimal conditions that ensure the existence of the normalized length function (this gives us our additional assumption).

The key observation is that the functor $\overline{\Phi}^*_{R_\infty}$ multiplies the normalized length of a module by a factor p^{d+1} (proposition 9.2.4(ii)). On the other hand, suppose – for the sake of argument – that $H^i(U, \mathscr{A}/\pi\mathscr{A})$ has finite (normalized) length $\lambda \geq 0$; using the π-adic filtration of $\mathscr{A}$, we might then bound the length of $H^i(U, \mathscr{A}/\pi^p\mathscr{A})$ by $p \cdot \lambda$; since $d > 0$, we would then conclude that $\lambda = 0$. This line of thought can still be applied to modules of almost finite length; finally, since these cohomology groups are even *almost coherent*, the fact that they have zero length implies that they actually vanish, and then the usual long exact cohomology sequences yield the vanishing of $H^i(U, \mathscr{A})$ in the desired range (see the proof of proposition 9.2.14(ii)).

• When $d = 2$, the previous method fails to produce any information concerning the cohomology of $\mathscr{A}/\pi\mathscr{A}$, hence the whole argument breaks down. Nevertheless, some key ingredients can still be salvaged for use in the final step of the proof. The main new idea (due to Faltings [34]) is to exploit the technique of Witt vectors, in order to construct :

(i) an imbedding of $(U, \mathscr{O}_U)$ into a larger locally ringed space $(\mathbf{A}(U), \mathscr{O}_{\mathbf{A}(U)})$, which admits an automorphism $\boldsymbol{\sigma}_U$ lifting the Frobenius endomorphism of U_0;

(ii) an extension of the algebra $\mathscr{A}$ to an $\mathscr{O}^a_{\mathbf{A}(U)}$-algebra $\mathbf{A}(\mathscr{A})^+$, and an extension of the Frobenius map $\Phi_{\mathscr{A}}$ to an isomorphism :

$$\boldsymbol{\sigma}_{\mathscr{A}} : \boldsymbol{\sigma}_U^* \mathbf{A}(\mathscr{A})^+ \xrightarrow{\sim} \mathbf{A}(\mathscr{A})^+.$$

Actually, to achieve (i) we must first replace U by the scheme $U^\wedge := X^\wedge_\infty \setminus \{x_\infty\}$, where $X^\wedge_\infty := \operatorname{Spec} R^\wedge_\infty$, the spectrum of the p-adic completion of R_∞. There is a natural morphism $U^\wedge \to U$, and we replace likewise $\mathscr{A}$ by its pullback $\mathscr{A}^\wedge$ to $U^\wedge$. (We also show that if the pair $(X^\wedge_\infty, \{x_\infty\})$ is almost pure, then the same holds for the pair $(X_\infty, \{x_\infty\})$, hence this base change is harmless for our purposes.)

The pair $(\mathbf{A}(U), \mathscr{O}_{\mathbf{A}(U)})$ belongs to a category of locally ringed spaces that contains the formal schemes of [26]. The basic theory of these new *ω-formal schemes* is developed in section 8.4. The ω-formal scheme $\mathbf{A}(U)$ is an open subscheme of the *affine ω-formal scheme* $\mathbf{A}(X)$, defined as the spectrum of a topological algebra $\mathbf{A}(R_\infty)^+$ which is constructed in section 4.6, and $\boldsymbol{\sigma}_U$ is deduced from an automorphism $\boldsymbol{\sigma}_{R_\infty}$ of $\mathbf{A}(R_\infty)^+$.

There exists a countable family of closed immersions $(U_\lambda \to \mathbf{A}(U) \mid \lambda \in \Lambda)$, which identify $\mathbf{A}(U)$ with the coequalizer of the two natural projections :

$$\coprod_{\lambda,\lambda' \in \Lambda} U_\lambda \times_{\mathbf{A}(U)} U_{\lambda'} \rightrightarrows \coprod_{\lambda \in \Lambda} U_\lambda$$

such that each U_λ is isomorphic, as a locally ringed space, to an infinitesimal thickening of U. Moreover, for each $\lambda \in \Lambda$ there is $\mu \in \Lambda$, such that $\boldsymbol{\sigma}_U$ restricts to an isomorphism $U_\lambda \xrightarrow{\sim} U_\mu$, and $U_{\lambda\mu} := U_\lambda \times_{\mathbf{A}(U)} U_\mu$ is isomorphic to an infinitesimal thickening of the special fibre U_0 of U (see lemma 8.5.14).

Using these isomorphisms, $\mathscr{A}$ lifts to an étale $\mathscr{O}^a_{U_\lambda}$-algebra $\mathscr{A}_\lambda$, and then $\mathbf{A}(\mathscr{A})^+$ is obtained by gluing these sheaves $\mathscr{A}_\lambda$ along certain natural isomorphisms :

$$\mathscr{A}_\lambda \otimes_{\mathscr{O}_{U_\lambda}} \mathscr{O}_{U_{\lambda\mu}} \xrightarrow{\sim} \mathscr{A}_\mu \otimes_{\mathscr{O}_{U_\mu}} \mathscr{O}_{U_{\lambda\mu}}$$

that are concocted from $\Phi_{\mathscr{A}}$. This construction is fully detailed in (8.5.24)–(8.5.30) : the idea is simple, but the actual implementation requires a certain effort; the gluing itself is an instance of non-flat descent, whose general framework was established in [36, §3.4].

Next, for every $\lambda \in \Lambda$, the closed subscheme $U_\lambda \subset \mathbf{A}(U)$ is the zero locus of a regular element $\vartheta_\lambda \in \mathbf{A}(R_\infty)^+$, and we have short exact sequences of $\mathscr{O}^a_{\mathbf{A}(U)}$-modules (lemma 8.5.34) :

$$0 \to \mathbf{A}(\mathscr{A})^+ \xrightarrow{\vartheta_\lambda} \mathbf{A}(\mathscr{A})^+ \to \mathscr{A}_\lambda \to 0.$$

For every $i \in \mathbb{N}$, set $\mathbf{H}^i(\mathscr{A}) := H^i(\mathbf{A}(U), \mathbf{A}(\mathscr{A})^+)$; together with lemma 8.1.86, we deduce that $\mathbf{H}^1(\mathscr{A})/\vartheta_\lambda \mathbf{H}^1(\mathscr{A})$ is naturally a submodule of $H^1(U, \mathscr{A})$, especially it has almost finite length. Moreover, the pair $(\boldsymbol{\sigma}_U, \boldsymbol{\sigma}_{\mathscr{A}})$ induces a $\boldsymbol{\sigma}_{R_\infty}$-linear endomorphism of $\mathbf{H}^1(\mathscr{A})$, and if we let $I := (p, \vartheta_\lambda) \subset \mathbf{A}(R_\infty)^+$, there follows an isomorphism of R^a_∞/pR^a_∞-modules :

$$\overline{\Phi}^*_{R_\infty}(\mathbf{H}^1(\mathscr{A})/(\boldsymbol{\sigma}^{-1}_{R_\infty} I)\mathbf{H}^1(\mathscr{A})) \xrightarrow{\sim} \mathbf{H}^1(\mathscr{A})/I\mathbf{H}^1(\mathscr{A}).$$

In this situation, we can repeat the argument that helped us for the case $d > 2$: namely, using the π-adic filtration on $\mathbf{H}^1(\mathscr{A})/I\mathbf{H}^1(\mathscr{A})$ we conclude that the normalized length of $\mathbf{H}^1(\mathscr{A})/\vartheta_\lambda \mathbf{H}^1(\mathscr{A})$ equals zero, and then the latter must vanish (see the proof of claim 9.2.22). Finally, we apply (the almost version of) lemma 5.1.35 to deduce that actually $\mathbf{H}^1(\mathscr{A}) = 0$.

We can now prove directly that $H^0(U^\wedge, \mathscr{A}^\wedge)$ is an étale $R^{\wedge a}_\infty$-algebra : indeed, using [36, Prop.3.1.4] we are reduced to showing that the natural map

$$\mu : H^0(U^\wedge, \mathscr{A}^\wedge) \otimes_{R^a_\infty} H^0(U^\wedge, \mathscr{A}^\wedge) \to H^0(U^\wedge, \mathscr{A}^\wedge \otimes_{\mathscr{O}^a_{U^\wedge}} \mathscr{A}^\wedge)$$

is an epimorphism. However, μ lifts to a morphism :

$$\boldsymbol{\mu} : \mathbf{H}^0(\mathscr{A}) \otimes_{\mathbf{A}(R)^+} \mathbf{H}^0(\mathscr{A}) \to \mathbf{H}^0(\mathscr{A} \otimes_{\mathbf{A}(R)^+} \mathscr{A}).$$

Set $C := \mathrm{Coker}\,\boldsymbol{\mu}$; the vanishing of $\mathbf{H}^1(\mathscr{A})$ and $\mathbf{H}^1(\mathscr{A} \otimes_{\mathscr{O}_U^a} \mathscr{A})$ implies that $\mathrm{Coker}\,\mu \simeq C/\vartheta_\lambda C$. However, C is also naturally endowed with a $\boldsymbol{\sigma}_{R_\infty}$-linear endomorphism, hence $C/\vartheta_\lambda C = 0$, again by the same sort of counting argument using normalized lengths.

9.1. **Semistable relative curves.** Throughout this section we let K be a valued field, whose valuation we denote by $|\cdot| : K \to \Gamma \cup \{0\}$ (see [36, §6.1] for our general notations concerning valuations). We shall also continue to use the general notation of (5.7).

9.1.1. Let $K(\mathsf{T})$ be the fraction field of the free polynomial K-algebra $K[\mathsf{T}]$. For every $\gamma \in \Gamma$ one can define an extension of $|\cdot|$ to a *Gauss valuation* $|\cdot|_{0,\gamma} : K(\mathsf{T}) \to \Gamma$ ([36, Ex.6.1.4(iii)]). If $f(\mathsf{T}) := \sum_{i=0}^d a_i \mathsf{T}^i$ is any polynomial, then $|f(\mathsf{T})|_{0,\gamma} = \max\{|a_i| \cdot \gamma^i \mid i = 0, \dots, d\}$. We let $V(\gamma)$ be the valuation ring of $|\cdot|_{0,\gamma}$, and set

$$R(\gamma) := K[\mathsf{T}, \mathsf{T}^{-1}] \cap V(1) \cap V(\gamma).$$

Since $K[\mathsf{T}, \mathsf{T}^{-1}]$ is a Dedekind domain, $R(\gamma)$ is an intersection of valuation rings of the field $K(\mathsf{T})$, hence it is a normal domain. By inspecting the definition we see that $R(\gamma)$ consists of all the elements of the form $f(\mathsf{T}) := \sum_{i=-n}^n a_i \mathsf{T}^i$, such that $|a_i| \leq 1$ and $|a_i| \cdot \gamma^i \leq 1$ for every $i = -n, \dots, n$. Suppose now that $\gamma \leq 1$, and choose $c \in K^+$ with $|c| = \gamma$; then every such $f(\mathsf{T})$ can be written uniquely in the form $\sum_{i=0}^n a_i \mathsf{T}^i + \sum_{j=1}^n b_j (c\mathsf{T}^{-1})^j$, where $a_i, b_j \in K^+$ for every $i, j \leq n$. Conversely, every such expression yields an element of $R(\gamma)$. In other words, we obtain a surjection of K^+-algebras $K^+[\mathsf{X}, \mathsf{Y}] \to R(\gamma)$ by the rule: $\mathsf{X} \mapsto \mathsf{T}$, $\mathsf{Y} \mapsto c\mathsf{T}^{-1}$. Obviously the kernel of this map contains the ideal $(\mathsf{XY} - c)$, and we leave to the reader the verification that the induced map

$$K^+[\mathsf{X}, \mathsf{Y}]/(\mathsf{XY} - c) \to R(\gamma) \tag{9.1.2}$$

is indeed an isomorphism.

9.1.3. Throughout the rest of this section we shall assume that $\gamma \leq 1$. Every $\delta \in \Gamma$ with $\gamma \leq \delta \leq 1$, determines a prime ideal $\mathfrak{p}(\delta) := \{f \in R(\gamma) \mid |f|_{0,\delta} < 1\} \subset R(\gamma)$, such that $\mathfrak{m}_K R(\gamma) \subset \mathfrak{p}(\delta)$. Then it is easy to see that $R(\gamma)_{\mathfrak{p}(\delta)} \subset V(\delta)$, and moreover :

$$R(\gamma)_{\mathfrak{p}(\gamma)} = V(\gamma) \qquad R(\gamma)_{\mathfrak{p}(1)} = V(1)$$

since $V(1)$ (resp. $V(\gamma)$) is already a localization of $K^+[\mathsf{T}]$ (resp. of $K^+[c\mathsf{T}^{-1}]$). In case $\gamma < 1$, (9.1.2) implies that $R(\gamma) \otimes_{K^+} \kappa \simeq \kappa[\mathsf{X}, \mathsf{Y}]/(\mathsf{XY})$, and it follows easily that $\mathfrak{p}(1)$ and $\mathfrak{p}(\gamma)$ correspond to the two minimal prime ideals of $\kappa[\mathsf{X}, \mathsf{Y}]/(\mathsf{XY})$. In case $\gamma = 1$, we have $R(1) \otimes_{K^+} \kappa \simeq \kappa[\mathsf{X}, \mathsf{X}^{-1}]$, and again $\mathfrak{p}(1)$ corresponds to the generic point of $\mathrm{Spec}\,\kappa[\mathsf{X}, \mathsf{X}^{-1}]$. Notice that the natural morphism

$$f_\gamma : \mathbb{T}_K(\gamma) := \mathrm{Spec}\,R(\gamma) \to S$$

restricts to a smooth morphism $f_\gamma^{-1}(\eta) \to \mathrm{Spec}\,K$; moreover the closed fibre $f_\gamma^{-1}(s)$ is geometrically reduced. Notice also that $\mathbb{T}_K(\gamma) \times_S \mathrm{Spec}\,E^+ \simeq \mathbb{T}_E(\gamma)$ for every extension of valued fields $K \subset E$. In the following, we will write just $\mathbb{T}(\gamma)$ in place of $\mathbb{T}_K(\gamma)$, unless we have to deal with more than one base ring.

Proposition 9.1.4. *Keep the notation of* (9.1.3), *and let $g : X \to \mathbb{T}(\gamma)$ be an étale morphism, $\mathscr{F}$ a coherent $\mathscr{O}_X$-module. Set $h := f_\gamma \circ g : X \to S$ and denote by $i_s : h^{-1}(s) \to X$ the natural morphism. Then $\mathscr{F}$ is reflexive at the point $x \in h^{-1}(s)$ if and only if the following three conditions hold:*

(a) *$\mathscr{F}$ is h-flat at the point x.*
(b) *$\mathscr{F}_x \otimes_{K^+} K$ is a reflexive $\mathscr{O}_{X,x} \otimes_{K^+} K$-module.*

(c) *The* $\mathscr{O}_{h^{-1}(s),x}$*-module* $i_s^*\mathscr{F}_x$ *satisfies condition* S_1 *(see definition* 5.5.1(iii)*).*

Proof. Suppose that $\mathscr{F}$ is reflexive at the point x; then it is easy to check that (a) and (b) hold. We prove (c): by remark 5.6.4 we can find a left exact sequence

$$0 \to \mathscr{F}_x \xrightarrow{\alpha} \mathscr{O}_{X,x}^{\oplus m} \xrightarrow{\beta} \mathscr{O}_{X,x}^{\oplus n}.$$

Then $\operatorname{Im}\beta$ is a flat K^+-module, since it is a submodule of the flat K^+-module $\mathscr{O}_{X,x}^{\oplus n}$; hence $\alpha \otimes_{K^+} \mathbf{1}_\kappa : i_s^*\mathscr{F}_x \to \mathscr{O}_{h^{-1}(s),x}^{\oplus m}$ is still injective, so we are reduced to showing that $h^{-1}(s)$ is a reduced scheme, which follows from [33, Ch.IV, Prop.17.5.7] and the fact that $f_\gamma^{-1}(s)$ is reduced.

Conversely, suppose that conditions (a)–(c) hold.

Claim 9.1.5. Let ξ be the generic point of an irreducible component of $h^{-1}(s)$. Then:

(i) $\mathscr{F}_x$ is a torsion-free $\mathscr{O}_{X,x}$-module.

(ii) $\mathscr{O}_{X,\xi}$ is a valuation ring.

(iii) Suppose that the closure of ξ contains x. Then $\mathscr{F}_\xi$ is a free $\mathscr{O}_{X,\xi}$-module of finite rank.

Proof of the claim. (i): By (a), the natural map $\mathscr{F}_x \to \mathscr{F}_x \otimes_{K^+} K$ is injective; since (b) implies that $\mathscr{F}_x \otimes_{K^+} K$ is a torsion-free $\mathscr{O}_{X,x}$-module, the same must then hold for $\mathscr{F}_x$.

(ii): By the going down theorem ([61, Ch.3, Th.9.5]), $g(\xi)$ is necessarily the generic point of an irreducible component of $f_\gamma^{-1}(s)$. The discussion of (9.1.3) shows that $A := \mathscr{O}_{\mathbb{T}(\gamma),g(\xi)}$ is a valuation ring, hence [33, Ch.IV, Prop.17.5.7] says that the ring $B := \mathscr{O}_{X,\xi}$ is an integrally closed domain, and its field of fractions $\operatorname{Frac}(B)$ is a finite extension of the field of fractions of A ([33, Ch.IV, Th.17.4.1]). Let C be the integral closure of A in $\operatorname{Frac}(B)$; then $C \subset B$ and if $\mathfrak{m}_B$ denotes the maximal ideal of B, then $\mathfrak{n} := \mathfrak{m}_B \cap C$ is a prime ideal lying over the maximal ideal of A, so it is a maximal ideal of C. It then follows from [14, Ch.VI, §1, n.3, Cor.3] that the localization $C_\mathfrak{n}$ is a valuation ring; since $C_\mathfrak{n} \subset B$, we deduce from [14, Ch.VI, §1, n.2, Th.1] that $C = B$, whence (ii).

(iii): Suppose that $x \in \overline{\{\xi\}}$. We derive easily from (i) that $\mathscr{F}_\xi$ is a torsion-free $\mathscr{O}_{X,\xi}$-module, so the assertion follows from (ii) and [14, Ch.VI, §3, n.6, Lemma 1]. ◊

By (b), the morphism $\beta_{\mathscr{F},x} \otimes_{K^+} \mathbf{1}_K : \mathscr{F}_x \otimes_{K^+} K \to \mathscr{F}_x^{\vee\vee} \otimes_{K^+} K$ is an isomorphism (notation of (5.6)); since $\mathscr{F}$ is h-flat at x, we deduce easily that $\beta_{\mathscr{F},x}$ is injective and $C := \operatorname{Coker}\beta_{\mathscr{F},x}$ is a torsion K^+-module. To conclude, it remains only to show:

Claim 9.1.6. C is a flat K^+-module.

Proof of the claim. In view of lemma 4.3.35, it suffices to show that $\operatorname{Tor}_1^{K^+}(C, \kappa(s)) = 0$. However, from the foregoing we derive a left exact sequence

$$0 \longrightarrow \operatorname{Tor}_1^{K^+}(C, \kappa(s)) \longrightarrow \mathscr{F}_x \otimes_{K^+} \kappa(s) \xrightarrow{\beta_{\mathscr{F},x} \otimes_{K^+} \kappa(s)} \mathscr{F}_x^{\vee\vee} \otimes_{K^+} \kappa(s).$$

We are thus reduced to showing that $\beta_{\mathscr{F},x} \otimes_{K^+} \kappa(s)$ is an injective map. In view of condition (c), it then suffices to prove that $\beta_{\mathscr{F},\xi}$ is an isomorphism, whenever ξ is the generic point of an irreducible component of $h^{-1}(s)$ containing x. The latter assertion holds by virtue of claim 9.1.5(iii). □

9.1.7. For a given $\rho \in \Gamma$, let us pick $a \in K \setminus \{0\}$ such that $|a| = \rho$; we define the fractional ideal $I(\rho) \subset K[\mathsf{T}, \mathsf{T}^{-1}]$ as the $R(\gamma)$-submodule generated by T and a. The module $I(\rho)$ determines a quasi-coherent $\mathscr{O}_{\mathbb{T}(\gamma)}$-module $\mathscr{I}(\rho)$.

Lemma 9.1.8. *With the notation of* (9.1.7)*:*

(i) $\mathscr{I}(\rho)$ *is a reflexive* $\mathscr{O}_{\mathbb{T}(\gamma)}$*-module for every* $\rho \in \Gamma$.

(ii) *There exists a short exact sequence of $R(\gamma)$-modules:*

$$0 \to I(\rho^{-1}\gamma) \to R(\gamma)^{\oplus 2} \to I(\rho) \to 0.$$

Proof. To start with, let $a \in K \setminus \{0\}$ with $|a| = \rho$.

Claim 9.1.9. If either $\rho \geq 1$ or $\rho \leq \gamma$, then $I(\rho)$ and $I(\rho^{-1}\gamma)$ are rank one, free $R(\gamma)$-modules.

Proof of the claim. If $\rho \geq 1$ (resp. $\rho \leq \gamma$) then $\rho^{-1}\gamma \leq \gamma$ (resp. $\rho^{-1}\gamma \geq 1$), hence it suffices to show that $I(\rho)$ is free of rank one, in both cases. Suppose first that $\rho \geq 1$; in this case, multiplication by a^{-1} yields an isomorphism of $R(\gamma)$-modules $I(\rho) \xrightarrow{\sim} R(\gamma)$. Next, suppose that $\rho \leq \gamma$. Then $I(\rho)$ is the ideal (T, a), where $a = c \cdot b$ for some $b \in K^+$ and $|c| = \gamma$. Therefore $I(\rho) = (\mathsf{T}, \mathsf{T} \cdot (c\mathsf{T}^{-1}) \cdot b) = \mathsf{T} \cdot (1, c\mathsf{T}^{-1}b) = \mathsf{T}R(\gamma)$, and again $I(\rho)$ is a free $R(\gamma)$-module of rank one. ◊

In view of claim 9.1.9, we may assume that $\gamma < \rho < 1$. Let (e_1, e_2) be the canonical basis of the free $R(\gamma)$-module $R(\gamma)^{\oplus 2}$; we consider the $R(\gamma)$-linear surjection $\pi : R(\gamma)^{\oplus 2} \to I(\rho)$ determined by the rule: $e_1 \mapsto \mathsf{T}$, $e_2 \mapsto a$. Clearly $\mathrm{Ker}\,\pi$ contains the submodule $M(\rho) \subset R(\gamma)^{\oplus 2}$ generated by:

$$f_1 := ae_1 - \mathsf{T}e_2 \quad \text{and} \quad f_2 := c\mathsf{T}^{-1}e_1 - ca^{-1}e_2.$$

Let $\mathscr{N}$ be the quasi-coherent $\mathscr{O}_{\mathbb{T}(\gamma)}$-module associated to $N := R(\gamma)^{\oplus 2}/M(\rho)$.

Claim 9.1.10. With the foregoing notation:

(i) N is a flat K^+-module.
(ii) $K^+[c^{-1}] \otimes_{K^+} N$ is a free $K^+[c^{-1}] \otimes_{K^+} R(\gamma)$-module of rank one.
(iii) $K \otimes_{K^+} M(\rho) = K \otimes_{K^+} \mathrm{Ker}\,\pi$.

Proof of the claim. (i): We let $\mathrm{gr}_\bullet R(\gamma)$ be the T-adic grading on $R(\gamma)$ (*i.e.* $\mathrm{gr}_i R(\gamma) = \mathsf{T}^i \cdot K \cap R(\gamma)$ for every $i \in \mathbb{Z}$), and we define a compatible grading on $R(\gamma)^{\oplus 2}$ by setting: $\mathrm{gr}_i R(\gamma)^{\oplus 2} := (\mathrm{gr}_{i-1} R(\gamma) \cdot e_1) \oplus (\mathrm{gr}_i R(\gamma) \cdot e_2)$ for every $i \in \mathbb{Z}$. Since f_1 and f_2 are homogeneous elements, we deduce by restriction a grading $\mathrm{gr}_\bullet M(\rho)$ on $M(\rho)$, and a quotient grading $\mathrm{gr}_\bullet N$ on N, whence a short exact sequence of graded K^+-modules:

$$0 \to \mathrm{gr}_\bullet M(\rho) \xrightarrow{\mathrm{gr}_\bullet j} \mathrm{gr}_\bullet R(\gamma)^{\oplus 2} \to \mathrm{gr}_\bullet N \to 0.$$

However, by inspecting the definitions, it is easy to see that $\mathrm{gr}_\bullet j$ is a split injective map of free K^+-modules, hence $\mathrm{gr}_\bullet N$ is a free K^+-module, and then the same holds for N.

(ii) is easy and shall be left to the reader.

(iii): Similarly, one checks easily that $K \otimes_{K^+} I(\rho)$ is a free $K \otimes_{K^+} R(\gamma)$-module of rank one; then, by (ii) the quotient map $K \otimes_{K^+} N \to K \otimes_{K^+} I(\rho)$ is necessarily an isomorphism, whence the assertion. ◊

Claim 9.1.11. $\mathscr{N}$ is a reflexive $\mathscr{O}_{\mathbb{T}(\gamma)}$-module.

Proof of the claim. Since $\mathscr{N}$ is coherent, it suffices to show that $\mathscr{N}_x$ is a reflexive $\mathscr{O}_{\mathbb{T}(\gamma),x}$-module, for every $x \in \mathbb{T}(\gamma)$ (lemma 5.6.1). Let $y := f_\gamma(x)$; we may then replace $\mathscr{N}$ by its restriction to $\mathbb{T}(\gamma) \times_S S(y)$, which allows to assume that $y = s$ is the closed point of S. In this case, we can apply the criterion of proposition 9.1.4 to the morphism $f_\gamma : \mathbb{T}(\gamma) \to S$. We already know from claim 9.1.10(i) that $\mathscr{N}$ is f_γ-flat. Moreover, by claim 9.1.10(ii) we see that the restriction of $\mathscr{N}$ to $f_\gamma^{-1}(\eta)$ is reflexive. As $\gamma < \rho < 1$, by inspecting the definition and using the presentation (9.1.2), we deduce an isomorphism :

$$\kappa \otimes_{K^+} N \simeq \overline{R}^{\oplus 2}/(\mathsf{X}e_2, \mathsf{Y}e_1) \simeq (\overline{R}/\mathsf{X}\overline{R}) \oplus (\overline{R}/\mathsf{Y}\overline{R})$$

where $\overline{R} := \kappa[\mathsf{X}, \mathsf{Y}]/(\mathsf{XY}) \simeq \kappa \otimes_{K^+} R(\gamma)$. Thus, the $\overline{R}$-module $\kappa \otimes_{K^+} N$ satisfies condition S_1, whence the claim. ◊

It follows from claim 9.1.10(i,iii) that the quotient map $N \to I(\rho)$ is an isomorphism, so $\mathscr{I}(\rho)$ is reflexive, by claim 9.1.11. Next, let us define an $R(\gamma)$-linear surjection $\pi' : R(\gamma)^{\oplus 2} \to M(\rho)$ by the rule: $e_i \mapsto f_i$ for $i = 1, 2$. One checks easily that $\mathrm{Ker}\,\pi'$ contains the submodule generated by the elements $ca^{-1}e_1 - Te_2$ and $ce_1 - aTe_2$, and the latter is none else than the module $M(\rho^{-1}\gamma)$, according to our notation (notice that $\gamma < \rho^{-1}\gamma < 1$). We deduce a surjection of torsion-free $R(\gamma)$-modules $I(\rho^{-1}\gamma) \xrightarrow{\sim} R(\gamma)^{\oplus 2}/M(\rho^{-1}\gamma) \to M(\rho)$, which induces an isomorphism after tensoring by K, therefore $I(\rho^{-1}\gamma) \xrightarrow{\sim} M(\rho)$, which establishes (ii). $\square$

9.1.12. Let $\mathbb{T}(\gamma)_{\mathrm{sm}} \subset \mathbb{T}(\gamma)$ be the largest open subset which is smooth over S. Set $S_\gamma := \mathrm{Spec}\, K^+/cK^+$; it is easy to see that $f_\gamma^{-1}(S \setminus S_\gamma) \subset \mathbb{T}(\gamma)_{\mathrm{sm}}$, and for every $y \in S_\gamma$, the difference $f_\gamma^{-1}(y) \setminus \mathbb{T}(\gamma)_{\mathrm{sm}}$ consists of a single point.

Proposition 9.1.13. *Let $\Delta(\gamma) \subset \Gamma$ be the smallest convex subgroup containing γ. Then there is a natural isomorphism of groups:*

$$\mathrm{Pic}\,\mathbb{T}(\gamma)_{\mathrm{sm}} \xrightarrow{\sim} \Delta(\gamma)/\gamma^{\mathbb{Z}}.$$

Proof. We consider the affine covering of $\mathbb{T}(\gamma)_{\mathrm{sm}}$ consisting of the two open subsets

$$U := \mathrm{Spec}\, K^+[\mathsf{T}, \mathsf{T}^{-1}] \qquad \text{and} \qquad V := \mathrm{Spec}\, K^+[c\mathsf{T}^{-1}, c^{-1}\mathsf{T}]$$

with intersection $U \cap V = \mathrm{Spec}\, K^+[c^{-1}, \mathsf{T}, \mathsf{T}^{-1}]$. We notice that both U and V are S-isomorphic to $\mathbb{G}_{m,S}$, and therefore

$$\mathrm{Pic}\, U = \mathrm{Pic}\, V = 0 \tag{9.1.14}$$

by corollary 5.7.13. From (9.1.14), a standard computation yields a natural isomorphism:

$$\mathrm{Pic}\,\mathbb{T}(\gamma)_{\mathrm{sm}} \xrightarrow{\sim} \mathscr{O}_{\mathbb{T}(\gamma)}(U)^\times \backslash \mathscr{O}_{\mathbb{T}(\gamma)}(U \cap V)^\times / \mathscr{O}_{\mathbb{T}(\gamma)}(V)^\times.$$

(Here, for a ring A, the notation $A^\times$ means the invertible elements of A.) However, $\mathscr{O}_{\mathbb{T}(\gamma)}(U)^\times = (K^+)^\times \cdot (c\mathsf{T}^{-1})^{\mathbb{Z}}$, $\mathscr{O}_{\mathbb{T}(\gamma)}(U \cap V)^\times = (K^+[c^{-1}])^\times \cdot \mathsf{T}^{\mathbb{Z}}$ and $\mathscr{O}_{\mathbb{T}(\gamma)}(V) = (K^+)^\times \cdot \mathsf{T}^{\mathbb{Z}}$, whence the contention. $\square$

9.1.15. Proposition 9.1.13 establishes a natural bijection between the set of isomorphism classes of invertible $\mathscr{O}_{\mathbb{T}(\gamma)_{\mathrm{sm}}}$-modules and the set :

$$]\gamma, 1] := \{\rho \in \Gamma \mid \gamma < \rho < 1\} \cup \{1\}.$$

On the other hand, lemma 5.7.18 yields a natural bijection between $\mathrm{Pic}\,\mathbb{T}(\gamma)_{\mathrm{sm}}$ and the set of isomorphism classes of generically invertible reflexive $\mathscr{O}_{\mathbb{T}(\gamma)}$-modules. Furthermore, lemma 9.1.8 provides already a collection of such reflexive modules, and by inspection of the proof, we see that the family of sheaves $\mathscr{I}(\rho)$ is really parametrized by the subset $]\gamma, 1]$ (since the other values of ρ correspond to free $\mathscr{O}_{\mathbb{T}(\gamma)}$-modules of rank one). The two parametrizations are essentially equivalent. Indeed, let $a \in K \setminus \{0\}$ be any element such that $\rho := |a| \in]\gamma, 1]$. With the notation of the proof of proposition 9.1.13, we can define isomorphisms

$$\varphi : \mathscr{O}_U \xrightarrow{\sim} \mathscr{I}(\rho)_{|U} \qquad \psi : \mathscr{O}_V \xrightarrow{\sim} \mathscr{I}(\rho)_{|V}$$

by letting: $\varphi(1) := \mathsf{T}$ and $\psi(1) := a$. To verify that φ is an isomorphism, it suffices to remark that T is a unit on U, so $\mathscr{I}(\rho)_{|U} = \mathsf{T}\mathscr{O}_U = \mathscr{O}_U$. Likewise, on V we can write $\mathsf{T} = a \cdot (a^{-1}c) \cdot (c^{-1}\mathsf{T})$, so $\mathscr{I}(\rho)_{|V} = a\mathscr{O}_V$, and ψ is an isomorphism. Hence $\mathscr{I}(\rho)$ is isomorphic to the (unique) $\mathscr{O}_{\mathbb{T}(\gamma)}$-module whose global sections consist of all the pairs $(s_U, s_V) \in \mathscr{O}_U(U) \times \mathscr{O}_V(V)$, such that $\mathsf{T}^{-1}s_{U|U\cap V} = a^{-1}s_{V|U\cap V}$. Clearly, under the bijection of proposition 9.1.13, the invertible sheaf $\mathscr{I}(\rho)_{|\mathbb{T}(\gamma)_{\mathrm{sm}}}$ corresponds to the class of ρ in $\Delta(\gamma)/\gamma^{\mathbb{Z}}$. In particular, this shows that the reflexive $\mathscr{O}_{\mathbb{T}(\gamma)}$-modules $\mathscr{I}(\rho)$ are pairwise non-isomorphic for $\rho \in]\gamma, 1]$, and that every reflexive generically invertible $\mathscr{O}_{\mathbb{T}(\gamma)}$-module is isomorphic to one such $\mathscr{I}(\rho)$.

9.1.16. When $\gamma < 1$ and $t \in \mathbb{T}(\gamma)$ is the singular point of the closed fibre, the discussion of (9.1.15) also applies to describe the set $\overline{\mathrm{coh.Div}}(\mathscr{O}_{\mathbb{T}(\gamma),t})$ of isomorphism classes of coherent reflexive fractional ideals of $\mathscr{O}_{\mathbb{T}(\gamma),t}$ (remark 5.6.10(ii)). Indeed, any such module M extends to a reflexive $\mathscr{O}_{\mathbb{T}(\gamma)}$-module (first one uses lemma 5.6.6(ii.b) to extend M to some quasi-compact open subset $U \subset \mathbb{T}(\gamma)$, and then one may extend to the whole of $\mathbb{T}(\gamma)$, via proposition 5.6.7)(i). Hence $M \simeq \mathscr{I}(\rho)_t$ for some $\rho \in]\gamma, 1]$. It follows already that $\mathrm{coh.Div}(\mathscr{O}_{\mathbb{T}(\gamma),t})$ is naturally an abelian group, with multiplication law given by the rule :

$$(M, N) \mapsto M \odot N := j_*j^*(M \otimes_{\mathscr{O}_{\mathbb{T}(\gamma),t}} N) \qquad \text{for any two classes } M, N \in \mathrm{Div}(\mathscr{O}_{\mathbb{T}(\gamma),t})$$

where $j : \mathbb{T}(\gamma)_{\mathrm{sm}} \cap \mathrm{Spec}\, \mathscr{O}_{\mathbb{T}(\gamma),t} \to \mathrm{Spec}\, \mathscr{O}_{\mathbb{T}(\gamma),t}$ is the natural open immersion. Indeed, $M \odot N$ is reflexive (by proposition 5.6.7(i) and corollary 5.6.8), and the composition law $\odot$ is clearly associative and commutative, with $\mathscr{O}_{\mathbb{T}(\gamma),t}$ as neutral element; moreover, for any $\rho \in \Delta(\gamma)$, the class of $\mathscr{I}(\rho)_t$ admits the inverse $j_*((j^*\mathscr{I}(\rho)_t)^\vee)$. Furthermore, the modules $\mathscr{I}(\rho)_t$ are pairwise non-isomorphic for $\rho \in]\gamma, 1]$. Indeed, using the group law $\odot$, the assertion follows once we know that $\mathscr{I}(\rho)_t$ is not trivial, whenever $\rho \in]\gamma, 1[$. However, from the presentation of lemma 9.1.8(ii) one sees that $\mathscr{I}(\rho)_t \otimes_{\mathscr{O}_{\mathbb{T}(\gamma),t}} \kappa(t)$ is a two-dimensional $\kappa(t)$-vector space, so everything is clear. Moreover, a simple inspection shows that the composition law $\odot$ thus defined, agrees with the composition law of the monoid $\mathrm{coh.Div}(\mathscr{O}_{T(\gamma),t})$ given in remark 5.6.10(ii). Summing, we get a natural group isomorphism :

$$\overline{\mathrm{coh.Div}}(\mathscr{O}_{\mathbb{T}(\gamma),t}) \xrightarrow{\sim} \Delta(\gamma)/\gamma^{\mathbb{Z}}.$$

The following theorem generalizes this classification to reflexive modules of arbitrary generic rank.

Theorem 9.1.17. *Let $g : X \to \mathbb{T}(\gamma)$ be an ind-étale morphism, $x \in X$ any point, M a reflexive $\mathscr{O}_{X,x}$-module. Then there exist $\rho_1, \dots, \rho_n \in]\gamma, 1]$ and an isomorphism of $\mathscr{O}_{X,x}$-modules:*

$$M \xrightarrow{\sim} \bigoplus_{i=1}^{n} g^*\mathscr{I}(\rho_i)_x.$$

Proof. Using lemma 5.6.6(ii.b), we are easily reduced to the case where g is étale. Set $t := g(x)$; first of all, if $t \in \mathbb{T}(\gamma)_{\mathrm{sm}}$, then X is smooth over S at the point t, and consequently M (resp. $\mathscr{I}(\rho)_t$) is a free $\mathscr{O}_{X,x}$-module (resp. $\mathscr{O}_{\mathbb{T}(\gamma),t}$-module) of finite rank (by proposition 5.7.10(iii)), so the assertion is obvious in this case. Hence we may assume that $\gamma < 1$ and t is the unique point in the closed fibre of $\mathbb{T}(\gamma) \setminus \mathbb{T}(\gamma)_{\mathrm{sm}}$. Next, let $K^{\mathrm{sh}+}$ be the strict henselization of K^+; denote by $h : \mathbb{T}_{K^{\mathrm{sh}}}(\gamma) \to \mathbb{T}_K(\gamma)$ the natural map, and set $X' := X \times_{\mathbb{T}_K(\gamma)} \mathbb{T}_{K^{\mathrm{sh}}}(\gamma)$. Choose also a point $x' \in X'$ lying over x, and let $t' \in \mathbb{T}_{K^{\mathrm{sh}}}(\gamma)$ be the image of x'; then t' is the unique point of $\mathbb{T}_{K^{\mathrm{sh}}}(\gamma)$ with $h(t') = t$. We have a commutative diagram of ring homomorphisms:

$$\begin{tikzcd}
\mathscr{O}_{\mathbb{T}(\gamma),t} \arrow[r, "g^\natural_x"] \arrow[d, "h^\natural_t"'] & \mathscr{O}_{X,x} \arrow[d] \\
\mathscr{O}_{\mathbb{T}_{K^{\mathrm{sh}}}(\gamma),t'} \arrow[r] & \mathscr{O}_{X',x'}
\end{tikzcd}$$

whence an essentially commutative diagram of functors:

$$\begin{tikzcd}
\mathscr{O}_{\mathbb{T}(\gamma),t}\text{-}\mathbf{Rflx} \arrow[r, "g^*_x"] \arrow[d, "h^*_t"'] & \mathscr{O}_{X,x}\text{-}\mathbf{Rflx} \arrow[d, "\beta"] \\
\mathscr{O}_{\mathbb{T}_{K^{\mathrm{sh}}}(\gamma),t'}\text{-}\mathbf{Rflx} \arrow[r, "\alpha"] & \mathscr{O}_{X',x'}\text{-}\mathbf{Rflx}.
\end{tikzcd}$$

Since K and K^{sh} have the same value group, the discussion in (9.1.16) shows that h_t^* induces bijections on the isomorphism classes of generically invertible modules. On the other hand, proposition 4.3.12 implies that the functor β induces injections on isomorphism classes. Consequently, in order to prove the theorem, we may replace the pair (X, x) by (X', x'), and assume from start that $K = K^{\mathrm{sh}}$.

In terms of the presentation 9.1.2 we can write $\mathbb{T}(\gamma) \times_S \operatorname{Spec} \kappa(s) = Z_1 \cup Z_2$, where Z_1 (resp. Z_2) is the reduced irreducible component on which X (resp. Y) vanishes. By inspecting the definitions, it is easy to check that $Z_1 \simeq \mathbb{A}^1_\kappa \simeq Z_2$ as κ-schemes. Let ξ_i be the generic point of Z_i, for $i = 1, 2$; clearly $\{t\} = Z_1 \cap Z_2$, hence $W_i := X \times_{\mathbb{T}(\gamma)} Z_i$ is non-empty and étale over Z_i, so $\mathscr{O}_{W_i,x}$ is an integral domain, hence $g^{-1}(\xi_i)$ contains exactly one point ζ_i that specializes to x, for both $i = 1, 2$. To ease notation, let us set $A := \mathscr{O}_{X,x}$. Then $\operatorname{Spec} A \otimes_{K^+} \kappa$ consists of exactly three points, namely x, ζ_1 and ζ_2. Set $M(\zeta_i) := M \otimes_A \kappa(\zeta_i)$, and notice that $n := \dim_{\kappa(\zeta_1)} M(\zeta_1) = \dim_{\kappa(\zeta_2)} M(\zeta_2)$, since M restricts to a locally free module over $(\operatorname{Spec} A)_{\mathrm{sm}}$, the largest essentially smooth open S-subscheme of $\operatorname{Spec} A$, which is connected. We choose a basis $\overline{e}_1, \dots, \overline{e}_n$ (resp $\overline{e}'_1, \dots, \overline{e}'_n$) for $M(\zeta_1)^\vee$ (resp. $M(\zeta_2)^\vee$), which we can then lift to a system of sections $e_1, \dots, e_n \in M^\vee_{\zeta_1} := M^\vee \otimes_A \mathscr{O}_{X,\zeta_1}$ (and likewise we construct a system $e'_1, \dots, e'_n \in M^\vee_{\zeta_2}$). Let $\mathfrak{p}_i \subset A$ be the prime ideal corresponding to ζ_i ($i = 1, 2$); after multiplication by an element of $A \setminus \mathfrak{p}_i$, we may assume that $e_1, \dots, e_n \in M^\vee$ (and likewise for $e'_1, \dots, e'_n$). Finally, we set $e''_i := \mathsf{Y} e_i + \mathsf{X} e'_i$ for every $i = 1, \dots, n$; it is clear that the system $(e''_1, \dots, e''_n)$ induces bases of $M(\zeta_i)^\vee$ for both $i = 1, 2$. We wish to consider the map:

$$j : M \to A^{\oplus n} \qquad m \mapsto (e''_1(m), \dots, e''_n(m)).$$

Set $C := \operatorname{Coker} j$, $I := \operatorname{Ann}_A C$, and $B := A/I$.

Claim 9.1.18. (i) The maps $M_{\zeta_i} \to \mathscr{O}^{\oplus n}_{X,\zeta_i}$ induced by j are isomorphisms.

(ii) $\operatorname{Ker} j = \operatorname{Ker} j \otimes_A \mathbf{1}_\kappa = 0$.

(iii) B is a finitely presented K^+-module, and C is a free K^+-module of finite rank.

Proof of the claim. (i): Using Nakayama's lemma, one deduces easily that these maps are surjective; since M_{ζ_i} is a free $\mathscr{O}_{X,\zeta_i}$-module of rank n, they are also necessarily injective.

(ii): Since $\mathscr{O}_{X,x}$ is normal, $\{0\}$ is its only associated prime; then the injectivity of j (resp. of $j \otimes_A \mathbf{1}_\kappa$) follows from (i), and the fact that M (resp. $M \otimes_A \kappa$) satisfies condition S_1, by remark 5.6.4 (resp. by proposition 9.1.4).

(iii): First of all, since A is coherent, I is a finitely generated ideal of A. Let $f : \operatorname{Spec} B \to \operatorname{Spec} K^+$ be the natural morphism. Since C is a finitely presented A-module, its support Z is a closed subset of $\operatorname{Spec} A$. From (i) we see that $Z \cap \operatorname{Spec} A \otimes_{K^+} \kappa \subset \{x\}$; on the other hand, Z is also the support of the closed subscheme $\operatorname{Spec} B$ of $\operatorname{Spec} A$. Therefore

$$f^{-1}(s) \cap \operatorname{Spec} B \subset \{x\}. \tag{9.1.19}$$

Since K^+ is henselian, it follows easily from (9.1.19) and [33, Ch.IV, Th.18.5.11(c″)] that B is a finite K^+-algebra of finite presentation, hence also a finitely presented K^+-module (claim 5.7.8). Since C is a finitely presented B-module, we conclude that C is finitely presented as K^+-module, as well. From (ii) we deduce a short exact sequence: $0 \to M \to A^{\oplus n} \to C \to 0$, and then the long exact Tor sequence yields: $\operatorname{Tor}_1^A(C, \kappa) = 0$ (cp. the proof of claim 9.1.6). We conclude by [14, Ch.II, §3, n.2, Cor.2 of Prop.5]. ◊

Claim 9.1.20. The composed morphism: $\operatorname{Spec} B \to \operatorname{Spec} A \to \mathbb{T}(\gamma)$ is a closed immersion.

Proof of the claim. Let $\mathfrak{p} \subset R(\gamma)$ be the maximal ideal corresponding to t, so that $\mathscr{O}_{\mathbb{T}(\gamma),t} = R(\gamma)_\mathfrak{p}$. From claim 9.1.18(iii) we see that the natural morphism $\psi : R(\gamma)_\mathfrak{p} \to B$ is finite. Moreover, $A/\mathfrak{p}A \simeq R(\gamma)/\mathfrak{p}$, hence $\psi \otimes_{R(\gamma)} \mathbf{1}_{R(\gamma)/\mathfrak{p}}$ is a surjection. By Nakayama's lemma we deduce that ψ is already a surjection, *i.e.* the induced morphism $\operatorname{Spec} B \to \operatorname{Spec} R(\gamma)_\mathfrak{p}$ is a closed immersion. Let $J_\mathfrak{p} := \mathrm{Ker}\,\psi$, $J := R(\gamma) \cap J_\mathfrak{p}$ and $D := R(\gamma)/J$. We are reduced to showing that the induced map $D \to D_\mathfrak{p}$ is an isomorphism. However, let $e \in R(\gamma) \setminus \mathfrak{p}$; since $D_\mathfrak{p} \simeq B$ is finite over K^+, we can find a monic polynomial $P[T] \in K^+[T]$ such that $P(e^{-1}) = 0$, therefore an identity of the form $1 = e \cdot Q(e)$ holds in $D_\mathfrak{p}$ for some polynomial $Q(T) \in K^+[T]$. But then the same identity holds already in the subring D, *i.e.* the element e is invertible in D, and the claim follows. $\diamondsuit$

Now, C is a finitely generated B-module, hence also a finitely generated $R(\gamma)$-module, due to claim 9.1.20. We construct a presentation of C in the following way. First of all, we have a short exact sequence of $R(\gamma) \otimes_{K^+} R(\gamma)$-modules:

$$\underline{E} \ : \ 0 \to \Delta \to R(\gamma) \otimes_{K^+} R(\gamma) \xrightarrow{\mu} R(\gamma) \to 0$$

where μ is the multiplication map. The homomorphism $R(\gamma) \to R(\gamma) \otimes_{K^+} R(\gamma) : a \mapsto 1 \otimes a$ fixes an $R(\gamma)$-module structure on every $R(\gamma) \otimes_{K^+} R(\gamma)$-module (the *right* $R(\gamma)$-module structure), and clearly $\underline{E}$ is split exact, when regarded as a sequence of $R(\gamma)$-modules via this homomorphism. Moreover, in terms of the presentation (9.1.2), the $R(\gamma) \otimes_{K^+} R(\gamma)$-module Δ is generated by the elements $\mathsf{X} \otimes 1 - 1 \otimes \mathsf{X}$ and $\mathsf{Y} \otimes 1 - 1 \otimes \mathsf{Y}$. Let n be the rank of the free K^+-module C (claim 9.1.18(iii)); there follows an exact sequence

$$\underline{E} \otimes_{R(\gamma)} C \ : \ 0 \to \Delta \otimes_{R(\gamma)} C \to R(\gamma)^{\oplus n} \to C \to 0$$

which we may and do view as a short exact sequence of $R(\gamma)$-modules, via the *left* $R(\gamma)$-module structure induced by the restriction of scalars $R(\gamma) \to R(\gamma) \otimes_{K^+} R(\gamma) : a \mapsto a \otimes 1$. The elements $\mathsf{X}, \mathsf{Y} \in R(\gamma)$ act as K^+-linear endomorphisms on C; one can then find bases $(b_i \mid i = 1, \dots, n)$ and $(b'_i \mid i = 1, \dots, n)$ of C, and elements $a_1, \dots, a_n \in K^+ \setminus \{0\}$ such that $\mathsf{X} b_i = a_i b'_i$ for every $i \leq n$. Since $\mathsf{XY} = c$ in $R(\gamma)$, it follows that $\mathsf{Y} b'_i = c a_i^{-1} b_i$ for every $i \leq n$. With this notation, it is clear that $\Delta \otimes_{R(\gamma)} C$, with its left $R(\gamma)$-module structure, is generated by the elements:

$$\mathsf{X} \otimes b_i - 1 \otimes a_i b'_i \quad \text{and} \quad \mathsf{Y} \otimes b'_i - 1 \otimes c a_i^{-1} b_i \quad (i = 1, \dots, n).$$

For every $i \leq n$, let F_i be the $R(\gamma)$-module generated freely by elements $(\varepsilon_i, \varepsilon'_i)$, and $\Delta_i \subset F_i$ the submodule generated by $\mathsf{X}\varepsilon_i - a_i \varepsilon'_i$ and $\mathsf{Y}\varepsilon'_i - c a_i^{-1} \varepsilon_i$. Moreover, let us write $b'_i = \sum_{j=1}^n u_{ij} b_j$ with unique $u_{ij} \in K^+$, let F be the free $R(\gamma)$-module with basis $(e_i \mid i = 1, \dots, n)$, and define $\varphi : F \to \oplus_{i=1}^n F_i$ by the rule: $e_i \mapsto \varepsilon'_i - \sum_{j=1}^n u_{ij} \varepsilon_j$ for every $i \leq n$. We deduce a right exact sequence of $R(\gamma)$-modules:

$$F \oplus \bigoplus_{i=1}^n \Delta_i \xrightarrow{\psi_1} \bigoplus_{i=1}^n F_i \xrightarrow{\psi_2} C \to 0$$

where:

$$\begin{aligned} &\psi_1(f, d_1, \dots, d_n) = \varphi(f) + (d_1, \dots, d_n) && \text{for every } f \in F \text{ and } d_i \in \Delta_i \\ &\psi_2(\varepsilon_i) = b_i \quad \text{and} \quad \psi_2(\varepsilon'_i) = b'_i && \text{for every } i = 1, \dots, n. \end{aligned}$$

Claim 9.1.21. ψ_1 is injective.

Proof of the claim. Let L be the field of fractions of $R(\gamma)$; since the domain of ψ_1 is a torsion-free K^+-module, it suffices to verify that $\psi \otimes_{R(\gamma)} \mathbf{1}_L$ is injective. However, on the one hand $C \otimes_{R(\gamma)} L = 0$, and on the other hand, each $\Delta_i \otimes_{R(\gamma)} L$ is an L-vector space of dimension one, so the claim follows by comparing dimensions. $\diamondsuit$

By inspecting the definitions and the proof of lemma 9.1.8, one sees easily that

$$\Delta_i \simeq I(|a_i^{-1} c|) \quad \text{for every } i \leq n. \tag{9.1.22}$$

Moreover, by remark (4.1.35)(iii), there exists $p \in \mathbb{N}$ and an $R(\gamma)$-linear isomorphism:

$$\mathrm{Ker}\,\psi_2 \xrightarrow{\sim} R(\gamma)^{\oplus p} \oplus \mathrm{Syz}^1_{R(\gamma)}C. \tag{9.1.23}$$

On the other hand, remark (4.1.35)(iii) and claim 9.1.18(ii) also shows that there exists $q \in \mathbb{N}$ and an A-linear isomorphism:

$$M \xrightarrow{\sim} A^{\oplus q} \oplus \mathrm{Syz}^1_A C. \tag{9.1.24}$$

Combining (9.1.23) and (9.1.24) and using lemma 4.1.36, we deduce an $R(\gamma)^{\mathrm{h}}$-linear isomorphism:

$$\begin{aligned}(R(\gamma)^{\mathrm{h}})^{\oplus q} \oplus (R(\gamma)^{\mathrm{h}} \otimes_{R(\gamma)} \mathrm{Ker}\,\psi_2) &\xrightarrow{\sim} (R(\gamma)^{\mathrm{h}})^{\oplus p+q} \oplus (R(\gamma)^{\mathrm{h}} \otimes_{R(\gamma)} \mathrm{Syz}^1_{R(\gamma)}C) \\ &\xrightarrow{\sim} (R(\gamma)^{\mathrm{h}})^{\oplus p+q} \oplus \mathrm{Syz}^1_{R(\gamma)^{\mathrm{h}}}(R(\gamma)^{\mathrm{h}} \otimes_{R(\gamma)} C) \\ &\xrightarrow{\sim} (R(\gamma)^{\mathrm{h}})^{\oplus p+q} \oplus (R(\gamma)^{\mathrm{h}} \otimes_A \mathrm{Syz}^1_A C) \\ &\xrightarrow{\sim} (R(\gamma)^{\mathrm{h}})^{\oplus p} \oplus (R(\gamma)^{\mathrm{h}} \otimes_A M).\end{aligned}$$

By claim 9.1.21 and (9.1.22) it follows that $(R(\gamma)^{\mathrm{h}})^{\oplus p} \oplus (R(\gamma)^{\mathrm{h}} \otimes_A M)$ is a direct sum of modules of the form $R(\gamma)^{\mathrm{h}} \otimes_{R(\gamma)} I(\rho_i)$, for various $\rho_i \in \Gamma$ (recall that $I(1) = R(\gamma)$). Notice that every $I(\rho_i)$ is generically of rank one, hence indecomposable. Then it follows from corollary 4.3.11 that $R(\gamma)^{\mathrm{h}} \otimes_A M$ is a direct sum of various indecomposable $R(\gamma)^{\mathrm{h}}$-modules of the form $R(\gamma)^{\mathrm{h}} \otimes_A g^*\mathscr{I}(\rho_i)_x$. Finally, we apply proposition 4.3.12 to conclude the proof of the theorem. □

9.1.25. Henceforth, and until the end of this section, we suppose that $(K, |\cdot|)$ is an algebraically closed valued field of rank one, with residue characteristic $p > 0$. Following [36, §5.5.1, §6.1.15], one attaches to the valued field K its *standard setup* $(K^+, \mathfrak{m})$, and to every S-scheme X, the categories $\mathscr{O}^a_X$-$\mathbf{Mod}$, $\mathscr{O}^a_X$-$\mathbf{Alg}$ of almost $\mathscr{O}_X$-modules, resp. almost $\mathscr{O}_X$-algebras (relative to the standard setup).

We keep the notation of (9.1.1), so in particular $c \in K^+$ is a fixed element whose valuation we denote $\gamma \in \Gamma^+$. We choose a sequence $(c_n \mid n \in \mathbb{N})$ of elements of K^+, such that $c_0 = c$ and $c^p_{n+1} = c_n$ for every $n \in \mathbb{N}$, and let $R_n := R_0[\mathsf{T}^{1/p^n}, c_n\mathsf{T}^{-1/p^n}]$. In terms of the presentation (9.1.2) we can write

$$R_n = R_0[\mathsf{X}_n, \mathsf{Y}_n] \simeq K^+[\mathsf{X}_n, \mathsf{Y}_n]/(\mathsf{X}_n\mathsf{Y}_n - c_n) \tag{9.1.26}$$

where $\mathsf{X}_n := \mathsf{X}^{1/p^n}$ and $\mathsf{Y}_n := \mathsf{Y}^{1/p^n}$; especially R_n is isomorphic to the K^+-algebra $R(\gamma^{1/p^n})$. Finally, we set $\mathbb{T}_n := \mathrm{Spec}\,R_n$ for every $n \in \mathbb{N}$.

9.1.27. Set $R := \bigcup_{n\in\mathbb{N}} R_n$, and $\mathbb{T} := \mathrm{Spec}\,R$; let $\overline{x}$ be any geometric point of $\mathbb{T}$ whose support is a closed point, and set $X := \mathbb{T}(\overline{x})$. Also, let $x \in X$ be the closed point, set $R^{\mathrm{sh}} := \mathscr{O}_{X,x}$, $U := X \setminus \{x\}$, and denote by $j : U \to X$ the open immersion. Furthermore, for every $n \in \mathbb{N}$, let $\overline{x}_n$ be the image of $\overline{x}$ in $\mathbb{T}_n$; set $X_n := \mathbb{T}_n(\overline{x}_n)$, denote by x_n the closed point of X_n, and let $R^{\mathrm{sh}}_n := \mathscr{O}_{X_n,x_n}$. Also, let $U_n := X_n \setminus \{x_n\}$ and denote by $j_n : U_n \to X_n$ the open immersion.

Proposition 9.1.28. *For every flat quasi-coherent $\mathscr{O}_U$-module $\mathscr{F}$, the $\mathscr{O}^a_X$-module $j_*\mathscr{F}^a$ is flat.*

Proof. We begin with the following :

Claim 9.1.29. Let $n \in \mathbb{N}$ be any integer, and $\mathscr{G}$ a locally free $\mathscr{O}_{U_n}$-module of finite type. Then

(i) $j_{n*}\mathscr{G}$ is a reflexive $\mathscr{O}_{X_n}$-module.
(ii) $c_n \cdot \mathrm{Tor}_i^{\mathscr{O}_{X_n}}(j_{n*}\mathscr{G}, \mathscr{M}) = 0$ for every $i > 0$ and every $\mathscr{O}_{X_n}$-module $\mathscr{M}$.

Proof of the claim. (i): By lemma 5.6.6(ii.a) and proposition 5.6.7, $\mathscr{G}$ extends to a reflexive $\mathscr{O}_{X_n}$-module $\mathscr{G}'$, and we may then descend $\mathscr{G}'$ to a coherent $\mathscr{O}_Y$-module $\mathscr{H}$ on some étale neighborhood $Y \to \mathbb{T}_n$ of $\overline{x}_n$. Denote by $\overline{y}$ the image of $\overline{x}_n$ in Y, let $y \in Y$ be the support of $\overline{y}$, set $U_Y := Y(y) \setminus \{y\}$, and let $j_Y : U_Y \to Y(y)$ be the open immersion; by lemma 5.6.6(iii), the restriction $\mathscr{H}' := \mathscr{H}_{|Y(y)}$ is a reflexive $\mathscr{O}_{Y(y)}$-module, so the natural map $\mathscr{H}' \to j_{Y*}j_Y^*\mathscr{H}'$ is an isomorphism (corollary 5.6.8). By corollary 5.1.19, we deduce that the natural map $\mathscr{G}' \to j_{n*}j_n^*\mathscr{G}' = j_{n*}\mathscr{G}$ is an isomorphism, whence the claim.

(ii): Since $\mathscr{G}$ is locally free, it suffices to show that c_n annihilates $\mathrm{Tor}_i^{R_n^{\mathrm{sh}}}(\mathscr{G}'_{x_n}, M) = 0$ for every $i > 0$ and every R_n^{sh}-module M. Then, by theorem 9.1.17 we may further assume that $\mathscr{G}' = g^*\mathscr{I}(\rho)$ for some $\rho \in]|c_n|, 1]$, where $g : X_n \to \mathbb{T}_n$ is the natural morphism. By inspecting the definition of $\mathscr{I}(\rho)$, we derive a short exact sequence $0 \to \mathscr{I}(\rho) \to \mathscr{O}_{\mathbb{T}_n} \to Q \to 0$ of $\mathscr{O}_{\mathbb{T}_n}$-modules, with $c_n \cdot Q = 0$; then the claim follows easily, using the long exact Tor sequences. $\diamondsuit$

Claim 9.1.30. Let $y \in \mathscr{O}_U$ be any point, and $\mathscr{F}$ any quasi-coherent $\mathscr{O}_U$-module.

(i) the local ring $\mathscr{O}_{U,y}$ is either a field or a valuation ring of rank one.

(ii) $\mathscr{F}$ is a flat $\mathscr{O}_U$-module if and only if it is torsion-free.

Proof of the claim. (i): It suffices to notice that, if $y \in \mathbb{T}_n$ is any non-closed point, the local ring $\mathscr{O}_{\mathbb{T}_n,y}$ is either a field or a valuation rings of rank one : indeed, this is clear in case $y \in \mathbb{T}_n \times_S \mathrm{Spec}\, K$, since $R_n \otimes_{K^+} K$ is a Dedekind domain; otherwise, y is a maximal point of $\mathbb{T}_n \times_S \mathrm{Spec}\, \kappa$, and then the assertion was already remarked in (9.1.3).

(ii): The condition means that $\mathscr{F}_y$ is a torsion-free $\mathscr{O}_{U,y}$-module, for every $y \in U$. The assertion is then an immediate consequence of (i). $\diamondsuit$

Now, let $\mathscr{F}$ be any flat $\mathscr{O}_U$-module, and write $\mathscr{F}$ as the colimit of a filtered family $(\mathscr{F}_\lambda \mid \lambda \in \Lambda)$ of finitely presented $\mathscr{O}_U$-modules (proposition 5.2.19). For every $\lambda \in \Lambda$, we may extend $\mathscr{F}_\lambda$ to a finitely presented $\mathscr{O}_X$-module $\mathscr{F}'_\lambda$ (lemma 5.2.16(ii)). We may then find $n \in \mathbb{N}$ and an affine étale neighborhood $Y \to \mathbb{T}_n$ of $\overline{x}_n$, such that $\mathscr{F}'_\lambda$ descends to a finitely presented $\mathscr{O}_Y$-module $\mathscr{G}'$. Let $\overline{y}$ be the image of $\overline{x}$ in Y, $y \in Y$ the support of $\overline{y}$, and set $U_Y := Y(y) \setminus \{y\}$; notice that $R_Y := \mathscr{O}_{Y,y}$ is also a normal domain. Set $M'_\lambda := \mathscr{G}'_y$, denote by M_λ the maximal R_Y-torsion-free quotient of M'_λ, and let $\mathscr{G}$ be the quasi-coherent $\mathscr{O}_{U_Y}$-module associated to M_λ. Notice that the induced map $\psi_Y : U \to U_Y$ is surjective, and claim 9.1.30(i) easily implies that ψ_Y is flat; it follows that $\psi_Y^*\mathscr{G}$ is the maximal torsion-free quotient of $\mathscr{F}_\lambda$. Then claim 9.1.30(ii) implies that the natural map $\mathscr{F}_\lambda \to \mathscr{F}$ factors through $\psi_Y^*\mathscr{G}$. On the other hand, corollary 5.7.24 says that $\mathscr{G}$ is a finitely presented $\mathscr{O}_{Y(y)}$-module, so $\psi_Y^*\mathscr{G}$ is a finitely presented $\mathscr{O}_U$-module.

Summing up, this shows that $\mathscr{F}$ is the colimit of a filtered system of flat finitely presented $\mathscr{O}_U$-modules; in view of lemma 5.1.10(i), we may therefore assume that $\mathscr{F}$ is finitely presented, and that $\mathscr{F}$ descends to a locally free $\mathscr{O}_{U_n}$-module $\mathscr{G}$ for some $n \in \mathbb{N}$. For every $k \geq n$, let $\varphi_k : U_k \to U_n$ and $\psi_k : U \to U_k$ be the natural morphisms; claim 9.1.29 implies that $\mathscr{F}_k := j_{k*}\varphi_k^*\mathscr{G}$ is a reflexive $\mathscr{O}_{X_k}$-module, for every $k \in \mathbb{N}$, and proposition 5.1.15(ii) says that

$$j_*\mathscr{F} = \operatorname*{colim}_{k \geq n} \psi_k^*\mathscr{F}_k.$$

Now, let $\mathscr{M}$ be any $\mathscr{O}_X$-module. We can compute:

$$c_m \cdot \mathscr{T}or_i^{\mathscr{O}_X}(j_*\mathscr{F}, \mathscr{M}) = \operatorname*{colim}_{k \geq m} c_m \cdot \psi_k^*\mathscr{T}or_i^{\mathscr{O}_{X_k}}(\mathscr{F}_k, \psi_{k*}\mathscr{M}) = 0 \qquad \text{for every } m \in \mathbb{N} \text{ and } i > 0$$

by claim 9.1.29(ii) (cp. the proof of claim 4.3.37), whence the contention. $\square$

Theorem 9.1.31. *With the notation of* (9.1.27)*, the pair* $(X, \{x\})$ *is almost pure.*

Proof. To start out, we remark :

Claim 9.1.32. The natural map $\mathscr{O}_X \to j_*\mathscr{O}_U$ is an isomorphism.

Proof of the claim. In view of proposition 5.1.15(ii), it suffices to show that, for every $n \in \mathbb{N}$, the natural map $\mathscr{O}_{X_n} \to j_{n*}\mathscr{O}_{U_n}$ is an isomorphism. Next, let $t_n \in \mathbb{T}_n$ be the image of x_n, set $U'_n := \mathbb{T}_n \setminus \{t_n\}$ and denote by $j'_n : U'_n \to \mathbb{T}_n$ the open immersion; since the natural morphism $X_n \to \mathbb{T}_n$ is flat, corollary 5.1.19 further reduces to showing that the natural map $\mathscr{O}_{\mathbb{T}_n} \to j'_{n*}\mathscr{O}_{U'_n}$ is an isomorphism. The latter assertion follows immediately from corollary 5.6.8. ◊

Now, let $\mathscr{A}$ be any étale and almost finitely presented $\mathscr{O}^a_U$-algebra. Claim 9.1.32 and lemma 8.2.18(v) show that $j_*\mathscr{A}$ is the normalization $\mathscr{A}^\nu$ of $\mathscr{A}$ over X. Then lemma 8.2.11 and proposition 9.1.28 imply that $\mathscr{A}^\nu$ is an étale $\mathscr{O}^a_X$-algebra. To conclude, it suffices to invoke proposition 8.2.30. □

Remark 9.1.33. (i) The assumption that K is algebraically closed is made only to simplify some notation; as we shall see later, it suffices to assume that $(K, |\cdot|)$ is deeply ramified.

(ii) Theorem 9.1.31 admits the following variant. For every $n \in \mathbb{N}$, let $R_n := K^+[\mathsf{T}^{1/p^n}]$; this defines an inductive system $(R'_n \mid n \in \mathbb{N})$ of K^+-algebras, and again we set $R' := \bigcup_{n\in\mathbb{N}} R'_n$. We let X' be the strict henselization of $\operatorname{Spec} R'$ at a geometric point $\overline{x}'$ whose support is a closed point, and again we denote by $x' \in X'$ the closed point; then we claim that also the pair $(X', \{x'\})$ is almost pure. Indeed, notice that the strict henselization X'_n of $\operatorname{Spec} R'_n$ at the image of $\overline{x}'$ is isomorphic to the K^+-scheme $\mathbb{T}_n(\overline{x}_n)$ appearing in 9.1.27, provided we take $c_n := 1$. It follows that every reflexive $\mathscr{O}_{X'}$-module is free, and therefore proposition 9.1.28 holds in a stronger form (and with easier proof) : if we let $U' := X' \setminus \{x'\}$ and $j' : U' \to X'$ the open immersion, then for every flat quasi-coherent $\mathscr{O}_{U'}$-module $\mathscr{F}$, the $\mathscr{O}_{X'}$-module $j'_*\mathscr{F}$ is flat. After this, the proof of theorem 9.1.31 can be repeated *verbatim*.

9.2. **Almost purity : the smooth case.** In this section, we prove the almost purity theorem for the case of a pair $(X, \{x\})$, such that X is the spectrum of an ind-measurable K^+-algebra, which is the limit of a deeply ramified tower of ind-smooth K^+-algebras. In view of theorem 9.1.31, we may assume that X has dimension at least three, and actually we shall prove the theorem first in the case of dimension > 3 (see the introduction of chapter 9). Then we shall use the $\boldsymbol{\sigma}$-equivariant algebras introduced in (8.5.30), to deal with the remaining case of dimension three.

9.2.1. Let us resume the notation of (8.5), and suppose that R is a measurable K^+-algebra, and M is an $R/\pi R$-module supported at the maximal ideal. We are interested in comparing the normalized lengths of M and of the $R_{(\Phi)}$-module $\overline{\Phi}^*_R M$. This issue must be properly understood; indeed, notice that the invariant $\lambda_{R_{(\Phi)}}(\overline{\Phi}^*_R M)$ depends not only on the ring $R_{(\Phi)}$ and the $R_{(\Phi)}$-module $\overline{\Phi}^*_R M$, but also on the choice of a structure morphism $K^+ \to R_{(\Phi)}$. Two such K^+-algebra structures are manifest on $K^+_{(\Phi)}$; namely, on the one hand we have the natural map:

$$K^+ \to K^+/\pi K^+ \xrightarrow{\overline{\Phi}_{K^+}} K^+_{(\Phi)} \tag{9.2.2}$$

and on the other hand we have the natural surjection :

$$K^+ \to K^+/\pi^p K^+ = K^+_{(\Phi)}. \tag{9.2.3}$$

It follows that $R_{(\Phi)}$ inherits two K^+-algebra structures, via the natural map $K^+_{(\Phi)} \to R_{(\Phi)}$. Notice that, if we view $K^+_{(\Phi)}$ as a K^+-module via (9.2.2), the normalized length $\lambda_{K^+}(K^+_{(\Phi)})$ will not always be finite, since Φ_{K^+} is not always surjective. On the other hand, if we view $K^+_{(\Phi)}$ as a K^+-module via (9.2.3), the invariant $\lambda_{K^+}(K^+_{(\Phi)})$ is always finite, equal to $|\pi^p| \in \Gamma$. For this reason, henceforth, for any measurable K^+-algebra R, we shall always endow $R_{(\Phi)}$ with the K^+-algebra structure deduced from (9.2.3).

Proposition 9.2.4. *Suppose that R is an essentially smooth and measurable K^+-algebra of fibre dimension d. Then :*

(i) *$\overline{\Phi}_R$ is a flat ring homomorphism.*
(ii) $\lambda_{R_{(\Phi)}}(\overline{\Phi}{}^*_R M) = p^{1+d} \cdot \lambda_R(M)$ *for every object M of $R/\pi R$-$\mathbf{Mod}_{\{s\}}$.*

Proof. The assumption means that R is a localization of a smooth K^+-algebra, and the Krull dimension of $\overline{R} := R \otimes_{K^+} \kappa$ equals d. We begin by recalling the following well known :

Claim 9.2.5. Let $\varphi : A \to B$ be an essentially smooth morphism of $\mathbb{F}_p$-algebras, and denote by $\Phi_A : A \to A$ (resp. $\Phi_B : B \to B$) the Frobenius endomorphism. Let also $A_{(\Phi)}$ and $B_{(\Phi)}$ be defined as in (8.5), as well as the relative Frobenius : $\Phi_{B/A} : B \otimes_A A_{(\Phi)} \to B_{(\Phi)}$. Then $\Phi_{B/A}$ is a flat map.

Proof of the claim. Since the formation of (relative and absolute) Frobenius endomorphisms commutes with localization, we may assume that A and B are local rings, in which case φ factors through an essentially étale map $\psi : C := A[T_1, \dots, T_k] \to B$ ([33, Ch.IV, Cor.17.11.4]). We deduce a commutative diagram

$$\begin{tikzcd} C \arrow[r, "\mathbf{1}_C \otimes \Phi_A"] \arrow[d, "\psi"'] & C \otimes_A A_{(\Phi)} \arrow[r, "\Phi_{C/A}"] \arrow[d, "\psi \otimes \mathbf{1}_{A_{(\Phi)}}"'] & C_{(\Phi)} \arrow[d, "\psi \otimes \mathbf{1}_{C_{(\Phi)}}"'] \arrow[dr, "\psi_{(\Phi)}"] & \\ B \arrow[r, "\mathbf{1}_B \otimes \Phi_A"'] & B \otimes_A A_{(\Phi)} \arrow[r, "\mathbf{1}_B \otimes \Phi_{C/A}"'] & B \otimes_C C_{(\Phi)} \arrow[r, "\Phi_{B/C}"'] & B_{(\Phi)} \end{tikzcd}$$

and the identity : $\Phi_{B/C} \circ (\mathbf{1}_B \otimes \Phi_{C/A}) = \Phi_{B/A}$. However, $\Phi_{B/C}$ is an isomorphism [36, Th.3.5.13(ii)], hence we are reduced to showing the claim for the case $B = C$, but it is clear that $C_{(\Phi)}$ is a free $C \otimes_A A_{(\Phi)}$-module of rank p^k. ◊

Claim 9.2.6. Assertion (i) holds for $R = K^+$.

Proof of the claim. We apply the criterion of [61, Th.7.7], which amounts to showing that the natural map

$$(aK^+/\pi K^+) \otimes_{K^+/\pi K^+} K^+_{(\Phi)} \to aK^+_{(\Phi)} = a^p K^+/\pi^p K^+$$

is bijective for every $a \in \mathfrak{m}$ with $|a| \geq |\pi|$. The verification shall be left to the reader. ◊

(i) follows easily from (8.5.1) and claims 9.2.5, 9.2.6.

Next, in order to check the identity (ii), we may assume that M is a finitely presented $R/\pi R$-module, and even that M is flat over K^+/aK^+, for some $a \in \mathfrak{m}$ with $|a| \geq |\pi|$ (claim 8.3.34). By claim 9.2.6 we deduce that $\overline{\Phi}{}^*_{K^+} M = M \otimes_R (R \otimes_{K^+} K^+_{(\Phi)})$ is flat, when regarded as a $K^+/a^p K^+$-module, via the K^+-algebra structure of $K^+_{(\Phi)}$ fixed in (9.2.3). Consequently, $\overline{\Phi}{}^*_R M$ is a flat $K^+/a^p K^+$-module as well, by claim 9.2.5. Let $\overline{M} := M \otimes_{K^+} \kappa$; from theorem 8.3.30(iii.b) we obtain the identities :

$$\boldsymbol{\lambda}_R(M) = |a| \cdot \mathrm{length}_{\overline{R}}(\overline{M}) \qquad \boldsymbol{\lambda}_{R_{(\Phi)}}(\overline{\Phi}{}^*_R M) = |a^p| \cdot \mathrm{length}_{\overline{R}_{(\Phi)}}(\overline{\Phi}{}^*_{\overline{R}} \overline{M}).$$

Thus, we come down to the following :

Claim 9.2.7. $\mathrm{length}_{\overline{R}_{(\Phi)}}(\overline{\Phi}{}^*_{\overline{R}} \overline{M}) = p^d \cdot \mathrm{length}_{\overline{R}}(\overline{M})$.

Proof of the claim. By additivity we reduce easily to the case where $\overline{M} = \kappa(R)$, the residue field of $\overline{R}$. By [61, Th.28.7], $\overline{R}$ is a regular ring of Krull dimension d, hence we may find a regular sequence $t_1, \dots, t_d$ in the maximal ideal of $\overline{R}$, such that $\kappa(R) = \overline{R}/(t_1, \dots, t_d)$. Hence $\overline{\Phi}{}^*_{\overline{R}} \kappa(R) \simeq \overline{R}/(t_1^p, \dots, t_d^p)$, whose length equals p^d, as required. □

9.2.8. Suppose now that $(R_n \mid n \in \mathbb{N})$ is an inductive system of morphisms of measurable K^+-algebras inducing integral ring homomorphisms $R_n \to R_{n+1}/\mathfrak{m}_{R_n} R_{n+1}$ for every $n \in \mathbb{N}$, and $(d_n > 0 \mid n \in \mathbb{N})$ is a sequence of reals fulfilling axioms (a) and (b) of (8.3.49). By lemma 8.3.53(ii), the transition maps $\varphi_{nm} : R_n \to R_m$ (for $m \geq n$) are injective, hence we let

$$R := \bigcup_{n\in\mathbb{N}} R_n \qquad X := \operatorname{Spec} R \qquad \text{and} \qquad X_n := \operatorname{Spec} R_n \qquad \text{for every } n \in \mathbb{N}.$$

Clearly, for every $m \geq n$ we have a commutative diagram :

$$\begin{array}{ccc} R_n/\pi R_n & \xrightarrow{\overline{\Phi}_{R_n}} & R_{n,(\Phi)} \\ \big\downarrow & & \big\downarrow {\scriptstyle \varphi_{nm,(\Phi)}} \\ R_m/\pi R_m & \xrightarrow{\overline{\Phi}_{R_m}} & R_{m,(\Phi)} \end{array}$$

where $\varphi_{nm,(\Phi)} := \varphi_{nm} \otimes_{K^+} \mathbf{1}_{K^+/\pi^p K^+}$, so that :

$$\overline{\Phi}_R = \operatorname*{colim}_{n\in\mathbb{N}} \overline{\Phi}_{R_n}. \tag{9.2.9}$$

Notice that the inductive systems $(R_n/\pi R_n \mid n \in \mathbb{N})$ and $(R_{n,(\Phi)} \mid n \in \mathbb{N})$ still satisfy axioms (a) and (b) : namely we can choose the same sequence $(d_n \mid n \in \mathbb{N})$ of normalizing factors. Especially, every $R_{(\Phi)}$-module N supported over $\{s(R)\}$ admits a normalized length $\lambda(N)$.

Proposition 9.2.10. *In the situation of* (9.2.8)*, suppose that R_n is an essentially smooth K^+-algebra of fibre dimension d, for every $n \in \mathbb{N}$. Then :*

(i) *$\overline{\Phi}_R$ is a flat ring homomorphism.*
(ii) *$\lambda(\overline{\Phi}_R^* M) = p^{1+d} \cdot \lambda(M)$ for every $R/\pi R$-module M supported at $\{s(R)\}$.*

Proof. (i) follows from proposition 9.2.4(i) and (9.2.9). To show (ii), suppose first that M is finitely presented, and let $n \in \mathbb{N}$ such that $M \simeq R \otimes_{R_n} M_n$ for some R_n-module M_n. Clearly

$$\overline{\Phi}_R^*(M) \simeq R_{(\Phi)} \otimes_{R_{n,(\Phi)}} \overline{\Phi}_{R_n}^*(M_n).$$

Taking into account proposition 9.2.4(ii), we may compute :

$$\begin{aligned} \lambda(\overline{\Phi}_R^* M) &= \lim_{k\to\infty} d_{n+k}^{-1} \cdot \lambda_{R_{n+k,(\Phi)}}(\varphi_{n,n+k,(\Phi)}^* \overline{\Phi}_{R_n}^*(M_n)) \\ &= \lim_{k\to\infty} d_{n+k}^{-1} \cdot \lambda_{R_{n+k,(\Phi)}}(\overline{\Phi}_{R_{n+k}}^*(\varphi_{n,n+k}^* M_n)) \\ &= \lim_{k\to\infty} d_{n+k}^{-1} \cdot p^{1+d} \cdot \lambda_{R_{n+k}}(\varphi_{n,n+k}^* M_n) \\ &= p^{1+d} \cdot \lambda(M) \end{aligned}$$

as stated. Next, if M is finitely generated, we may find a filtered system $(N_i \mid i \in I)$ of finitely presented objects of R-$\mathbf{Mod}_{\{s\}}$, with surjective transition maps, such that $M \simeq \operatorname*{colim}_{i\in I} N_i$. Then $\overline{\Phi}_R^*(M) \simeq \operatorname*{colim}_{i\in I} \overline{\Phi}_R^*(N_i)$ and, in view of lemma 8.3.57(ii), we are reduced to the previous case. Finally, let M be an arbitrary R-module supported over $\{s(R)\}$; we write M as the colimit of the filtered system $(M_i \mid i \in I)$ of its finitely generated submodules. It follows from (i) that the filtered system $(\overline{\Phi}_R^* M_i \mid i \in I)$ has injective transition maps, hence we may apply theorem 8.3.62(i) to reduce the sought identity to the case where M is finitely generated, which has already been dealt with. $\square$

Lemma 9.2.11. *In the situation of* (9.2.8)*, let M be an $(R/\pi R)^a$-module supported at $s(R)$, and let us set $N := \overline{\Phi}_{R^a}^*(M)$. Suppose that :*

(a) *M is a submodule of an almost finitely presented R^a-module.*

(b) *N admits a filtration* $(\mathrm{Fil}^i N \mid 0 \leq i \leq p)$, *with* $\mathrm{Fil}^0 N = N$ *and* $\mathrm{Fil}^p N = 0$, *and whose graded quotients* $\mathrm{gr}^i N$ *are subquotients of* M *for every* $i < p$.
(c) *R_n is an essentially smooth K^+-algebra of fibre dimension* $d > 0$, *for every* $n \in \mathbb{N}$.

Then $M = 0$.

Proof. Clearly it suffices to show that $bM = 0$ for every $b \in \mathfrak{m}$, and in view of lemma 8.3.67(ii), the latter will follow, once we have proved that $\lambda(bM) = 0$ for every $b \in \mathfrak{m}$. However, an easy induction using lemma 8.3.67(i) yields:

$$\lambda(b^p N) \leq \sum_{i=0}^{p-1} \lambda(b \cdot \mathrm{gr}^i N) \leq p \cdot \lambda(bM) \tag{9.2.12}$$

where the last inequality holds, due to the fact that $b \cdot \mathrm{gr}^i N$ is a subquotient of bM for every $i < p$. On the other hand, since $\overline{\Phi}_{R^a}$ is a flat morphism (by proposition 9.2.10(i)) we have $\overline{\Phi}{}^*_{R^a}(bM) = b^p N$, and therefore :

$$p^{1+d} \cdot \lambda(bM) = \lambda(b^p N). \tag{9.2.13}$$

Clearly (9.2.12) and (9.2.13) are compatible only if $\lambda(bM) = 0$, as required. $\square$

Proposition 9.2.14. *In the situation of* (9.2.8), *set* $U := X \setminus \{s(R)\}$, *and suppose that :*

(a) *R_n is an essentially smooth K^+-algebra of fibre dimension d, for every* $n \in \mathbb{N}$.
(b) *The transition maps $R_m \to R_n$ are flat for every $m, n \in \mathbb{N}$ with $n \geq m$.*

Let $\mathscr{A}$ be an étale almost finitely presented $\mathscr{O}^a_U$-algebra. We have :

(i) *$H^i(U, \mathscr{A})$ is an almost coherent R^a-module, for every* $i = 0, \dots, d-1$.
(ii) *If $d \geq 3$, then $H^i(U, \mathscr{A}) = 0$ whenever* $1 \leq i \leq d-1$.

Proof. (i): Let $U_n := X_n \setminus \{s(R_n)\}$ for every $n \in \mathbb{N}$, and denote by $\psi_n : X \to X_n$ the natural morphism. Fix $b \in \mathfrak{m}$; by corollary 8.2.24, we may find an integer $n \in \mathbb{N}$, a coherent $\mathscr{O}_{X_n}$-algebra $\mathscr{R}$, and a map $f : (\psi_n^* \mathscr{R})_{|U} \to \mathscr{A}$ such that :

- $\mathrm{Ker}\, f$ and $\mathrm{Coker}\, f$ are annihilated by b.
- For every $x \in U_n$, the map $b \cdot \mathbf{1}_{\mathscr{R},x} : \mathscr{R}_x \to \mathscr{R}_x$ factors through a free $\mathscr{O}_{U_n,x}$-module.

Then, (5.4.3) and lemma 5.8.26 imply that $b \cdot H^i(U_n, \mathscr{R})$ is a finitely presented R_n-module whenever $0 \leq i < d$. From [28, Ch.III, Prop.1.4.1] and our assumption (b), we deduce that $b \cdot H^i(U, \psi_n^* \mathscr{R})$ is a finitely presented R_∞-module for $0 \leq i < d$. Since b is arbitrary, we conclude that $H^i(U, \mathscr{A})$ is almost finitely presented for $0 \leq i < d$, and the assertion follows from lemmata 8.2.4(ii), 5.6.6(ii.a), and assumption (b).

(ii): Set $\mathscr{A}_{/\pi} := (\mathscr{A}/\pi\mathscr{A})_{|U_{/\pi}}$ and $\mathscr{A}_{(\Phi)} := (\mathscr{A}/\pi^p\mathscr{A})_{|U_{(\Phi)}}$ (notation of (8.5.2)). The endomorphism $\Phi_{\mathscr{A}}$ (of the object $\mathscr{A}$ of $\mathscr{O}^a$-$\mathbf{\acute{E}t}$) induces a morphism of étale $\mathscr{O}^a$-algebras

$$\overline{\Phi}{}^*_{\mathscr{A}} := ((\overline{\Phi}_{K^+}, \overline{\Phi}_U), \overline{\Phi}{}^*_{\mathscr{A}}) : (U_{(\Phi)}, \mathscr{A}_{(\Phi)}) \to (U_{/\pi}, \mathscr{A}_{/\pi})$$

which is cartesian, by virtue of lemma 8.5.5.

Claim 9.2.15. If $d \geq 3$, then $H^i(U_{/\pi}, \mathscr{A}_{/\pi}) = 0$ whenever $1 \leq i \leq d-2$.

Proof of the claim. One uses the long exact cohomology sequence associated to the short exact sequence : $0 \to \mathscr{A} \to \mathscr{A} \to \mathscr{A}_{/\pi} \to 0$, together with lemma 8.2.4(i) and assertion (i), to deduce that $H^i(U_{/\pi}, \mathscr{A}_{/\pi})$ is an almost coherent R^a-module for $0 \leq i \leq d-2$. (Notice that $\mathrm{Ker}\, \pi \cdot \mathbf{1}_{\mathscr{A}} = 0$, since $\mathscr{A}$ is an étale $\mathscr{O}^a_U$-algebra, and U is flat over S.) Moreover, $H^i(U_{/\pi}, \mathscr{A}_{/\pi})$ is supported on $s(R)$ for every $i > 0$, since $\mathscr{A}_{/\pi}$ is an étale $\mathscr{O}^a_{U_{/\pi}}$-algebra; thus $H^i(U_{/\pi}, \mathscr{A}_{/\pi})$ has almost finite length, whenever $1 \leq i \leq d-2$ (lemma 8.3.70(ii)). Next, proposition 9.2.4(i) implies that $\overline{\Phi}_U$ is a flat morphism; by [28, Ch.III, Prop.1.4.15], it follows that the natural map :

$$\overline{\Phi}{}^*_{R^a} H^i(U_{/\pi}, \mathscr{A}_{/\pi}) \to H^i(U_{(\Phi)}, \mathscr{A}_{(\Phi)})$$

is an isomorphism of $R^a_{(\Phi)}$-modules for every $i \in \mathbb{N}$. However, using the long exact cohomology sequences associated to the short exact sequences : $0 \to \mathscr{A}/\pi^n\mathscr{A} \to \mathscr{A}/\pi^{n+m}\mathscr{A} \to \mathscr{A}/\pi^m\mathscr{A} \to 0$, and an easy induction, one sees that, for every $i \in \mathbb{N}$, the $R^a_{(\Phi)}$-module $N := H^i(U_{(\Phi)}, \mathscr{A}_{(\Phi)})$ admits a filtration $(\mathrm{Fil}^j N \mid 0 \leq j \leq p)$, whose associated graded pieces $\mathrm{gr}^j N$ are subquotients of $H^i(U_{/\pi}, \mathscr{A}_{/\pi})$, for every $j < p$. The claim now follows from lemma 9.2.11. ◊

Now, for $i > 0$, every almost element of $H^i(U, \mathscr{A})$ is annihilated by some power of π; if additionally $i < d$, then $H^i(U, \mathscr{A})$ is an almost finitely generated module (by assertion (i)), hence it is annihilated by π^n, for sufficiently large $n \in \mathbb{N}$. On the other hand, since $d \geq 3$, it follows from claim 9.2.15 that the scalar multiplication by π is a monomorphism on $H^i(U, \mathscr{A})$ for $2 \leq i \leq d-1$, and an epimorphism on $H^1(U, \mathscr{A})$. The assertion follows easily. □

Theorem 9.2.16. *In the situation of proposition* 9.2.14(ii)*, the pair* $(X, \{s(R)\})$ *is almost pure.*

Proof. Let $j : U \to X$ be the open immersion, and $\mathscr{A}$ any almost finitely presented étale $\mathscr{O}^a_U$-algebra.

Claim 9.2.17. $\Gamma(U, \mathscr{A})$ is a flat R^a-module.

Proof of the claim. For every $n \in \mathbb{N}$, define U_n and $\psi_n : U \to U_n$ as in the proof of proposition 9.2.14; let also $j_n : U_n \to X_n$ be the open immersion. According to proposition 9.2.14(ii), we have $\delta(s(R), j_*\mathscr{A}_!) \geq d+1$; from lemma 5.4.16(ii) we deduce that $\delta(s(R_n), j_{n*}\psi_{n*}\mathscr{A}_!) \geq d+1$, hence $\Gamma(U_n, \psi_{n*}\mathscr{A}_!) = \Gamma(U, \mathscr{A}_!)$ is a flat R_n-module for every $n \in \mathbb{N}$, by proposition 5.8.19. The claim follows. ◊

Claim 9.2.18. For every flat quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, the natural map $\mathscr{F} \to j_*j^*\mathscr{F}$ is an isomorphism.

Proof of the claim. In view of corollary 5.1.17, we may assume that $\mathscr{F} = \mathscr{O}_X$; in this case, using proposition 5.1.15(ii) we reduce to showing that $j_{n*}\mathscr{O}_{U_n} = \mathscr{O}_{X_n}$ (with the notation of the proof of claim 9.2.17). The latter assertion follows from lemma 5.6.36(iv) and corollary 5.6.8. ◊

It follows from claim 9.2.17 that $j_*\mathscr{A}$ is a flat $\mathscr{O}^a_X$-algebra; moreover, $j_*\mathscr{A}$ is also integral and étale over $\mathscr{O}^a_X$, in view of lemmata 8.2.18(v) and 8.2.11 and claim 9.2.18. Next, $j_*\mathscr{A}$ is an almost finite $\mathscr{O}^a_X$-algebra, by lemma 8.2.18(iv). Now the theorem follows from proposition 8.2.30. □

Remark 9.2.19. (i) In the situation of example 8.3.76, suppose that $d \geq 3$ and that $\Gamma \not\simeq \mathbb{Z}$; then all the assumptions of theorem 9.2.16 are fulfilled, and we deduce that $(\mathrm{Spec}\, R, \{s(R)\})$ is an almost pure pair.

(ii) One may show that theorem 9.2.16 still holds for fibre dimension $d = 1$, hence it is natural to expect that it holds as well for the remaining case $d = 2$; this shall be verified under some additional assumption (theorem 9.2.23).

(iii) One may show that every almost finitely generated flat $(R/\pi R)^a$-module is almost projective, and every almost finitely generated projective $(R/\pi R)^a$-module has finite rank. One may ask whether the pair $(X_{/\pi}, \{s(R)\})$ is almost pure, in the situation of proposition 9.2.14(ii), for fibre dimension $d \geq 2$. This question shall be addressed in section 9.5. Here we just remark that, by lemma 8.2.11 and an argument as in the proof of lemma 8.2.18(iv), the question has a positive answer if and only if the direct image $j_*\mathscr{A}$ of any almost finitely generated étale $\mathscr{O}^a_{U_{/\pi}}$-algebra $\mathscr{A}$ is a flat $\mathscr{O}^a_{X_{/\pi}}$-algebra.

9.2.20. We wish now to lift the restriction on the dimension of R. Thus, let us suppose that

- K^+ is deeply ramified.

- R is the increasing union of an inductive system $(R_n \mid n \in \mathbb{N})$ of essentially smooth local K^+-algebras of fibre dimension d, whose transition maps are local and induce integral ring homomorphisms $R_n \to R_{n+1}/\mathfrak{m}_{R_n} R_{n+1}$, for every $n \in \mathbb{N}$.
- The residue field of R has characteristic p, and Φ_R is surjective.

Indeed, in this case $\overline{\Phi}_R$ is an isomorphism, since it is flat by proposition 9.2.4(i), and surjective by assumption. It also follows ([31, Prop.6.1.5] and [32, Th.11.3.10]) that the transition maps $R_m \to R_n$ are flat for every $m, n \in \mathbb{N}$ with $n \geq m$; then example 8.3.75 shows that R is ind-measurable. Resume the notation of of (8.5.23), and let also

$$\psi : U^\wedge \to U$$

be the natural morphism. Also, let $\mathscr{A}$ be any étale almost finitely presented $(\mathscr{O}_U, \mathfrak{m}\mathscr{O}_U)^a$-algebra; we may then perform the constructions of (8.5.30) on the étale $\mathscr{O}^a_{U^\wedge}$-algebra $\mathscr{A}^\wedge := \psi^*\mathscr{A}$; the result is an $\mathscr{O}^a_{\mathbf{A}(U)}$-algebra $\mathbf{A}(\mathscr{A})^+$, endowed with its automorphism $\boldsymbol{\sigma}_{\mathscr{A}}$, that lifts the Frobenius automorphism of $\mathscr{A}_{/p} := \mathscr{A}/p\mathscr{A}$ (lemma 8.5.38).

Proposition 9.2.21. *In the situation of* (9.2.20)*, we have :*

$$H^q(\mathbf{A}(U), \mathbf{A}(\mathscr{A})^+) = 0 \qquad \textit{whenever } 1 \leq q \leq d-1.$$

Proof. To ease notation, let us set $\mathbf{H}^q := H^q(\mathbf{A}(U), \mathbf{A}(\mathscr{A})^+)$. In view of lemma 8.5.34, we have exact sequences :

$$\mathbf{H}^q \xrightarrow{\vartheta_k} \mathbf{H}^q \longrightarrow H^q(U(k), \mathscr{A}(k)) \simeq H^q(U^\wedge, \mathscr{A}^\wedge) \qquad \text{for every } k \in \mathbb{Z} \text{ and every } q \in \mathbb{N}$$

(where ϑ_k is as in (8.5.9)) so that $\mathbf{H}^q/\vartheta_k\mathbf{H}^q$ is naturally a submodule of $H^q(U^\wedge, \mathscr{A}^\wedge)$. On the other hand, we may apply lemma 8.1.86 to the ring extension $R \to R^\wedge$ and the element $t := p$ (which is regular in both R and $R^\wedge$, say by [36, Lemma 7.1.6(i)]), to deduce a natural isomorphism :

$$H^q(U, \mathscr{A}) \xrightarrow{\sim} H^q(U^\wedge, \mathscr{A}^\wedge) \qquad \text{whenever } q > 0.$$

Then proposition 9.2.14(i) implies that $H^q(U^\wedge, \mathscr{A}^\wedge)$ is an almost coherent R^a-module whenever $1 \leq q \leq d-1$, and it follows also that $p^n \cdot H^q(U^\wedge, \mathscr{A}^\wedge) = 0$ for a large enough $n \in \mathbb{N}$, since the latter module is supported over $\{x\}$. So finally, whenever $1 \leq q < d$, $\mathbf{H}^q/\vartheta_k\mathbf{H}^q$ is an R^a-module of almost finite length on which scalar multiplication by p is a nilpotent endomorphism (lemma 8.3.70(ii)).

Claim 9.2.22. $\mathbf{H}^q/\vartheta_k\mathbf{H}^q = 0$ whenever $k \in \mathbb{Z}$ and $1 \leq q \leq d-1$.

Proof of the claim. Define $I_{k,k-1}$ as in remark 8.5.18; it follows from remark 8.5.18(ii) that $\boldsymbol{\sigma}_{\mathscr{A}}$ induces an isomorphism of R^a/pR^a-modules:

$$\overline{\boldsymbol{\sigma}}^*_{\mathscr{A}} : \overline{\Phi}^*_R(\mathbf{H}^q/I_{k+1,k}\mathbf{H}^q) \xrightarrow{\sim} \mathbf{H}^q/I_{k,k-1}\mathbf{H}^q.$$

Taking into account remark 8.5.18(i), we deduce that $\overline{\Phi}^*_R(\mathbf{H}^q/I_{k+1,k}\mathbf{H}^q)$ admits a filtration of length p, whose graded subquotients are quotients of $\mathbf{H}^q/I_{k+1,k}\mathbf{H}^q$ (cp. the proof of claim 9.2.15); in this situation, lemma 9.2.11 shows that $\mathbf{H}^q/I_{k,k+1}\mathbf{H}^q = 0$. Furthermore, remark 8.5.18(iii) implies that multiplication by ϑ_{k+1} is nilpotent on $\mathbf{H}^q/\vartheta_k\mathbf{H}^q$, so the claim follows easily. ◊

Next, choose any monotonically increasing cofinal sequence $(\mathbf{n}_k \mid k \in \mathbb{N})$ of set-theoretic maps $\mathbf{n}_k : \mathbb{Z} \to \mathbb{N}$ with finite support (with the ordering of definition 4.6.37(iii)), such that $\mathbf{n}_0$ is the identically zero map, and $\mathbf{n}_{k+1}$ is a *successor* of $\mathbf{n}_k$ for every $k \in \mathbb{N}$, by which we mean that there are no $\mathbf{m} : \mathbb{Z} \to \mathbb{N}$ such that $\mathbf{n}_k < \mathbf{m} < \mathbf{n}_{k+1}$. We define a filtration $\mathrm{Fil}^\bullet\mathbf{A}(\mathscr{A})^+$ on $\mathbf{A}(\mathscr{A})^+$, by the rule:

$$\mathrm{Fil}^k\mathbf{A}(\mathscr{A})^+ := \vartheta_{\mathbf{n}_k} \cdot \mathbf{A}(\mathscr{A})^+ \qquad \text{for every } k \in \mathbb{N}.$$

Clearly the graded subquotients $\mathrm{gr}^\bullet\mathbf{A}(\mathscr{A})^+$ are isomorphic to sheaves of the form $\mathscr{A}(k)$ for some $k \in \mathbb{Z}$. By combining lemma 8.5.34 and (the almost version of) corollary 8.4.43 we deduce that the natural morphism

$$\mathbf{A}(\mathscr{A})^+ \to R\lim_{\mathbf{n}} \mathscr{A}(\mathbf{n})$$

is an isomorphism in $\mathsf{D}^+(\mathscr{O}^a_{\mathbf{A}(U)}\text{-}\mathbf{Mod})$. With this notation, claim 9.2.22 shows that the natural map $H^q(\mathbf{A}(U), \mathrm{Fil}^k\mathbf{A}(\mathscr{A})^+) \to H^q(\mathbf{A}(U), \mathrm{gr}^k\mathbf{A}(\mathscr{A})^+)$ is the zero morphism in the range $1 \leq q \leq d-1$, and for every $k \in \mathbb{N}$. To conclude the proof of the proposition, it suffices now to remark that The analogue of lemma 5.1.35 holds also for almost modules, and especially, it applies with $\mathscr{F} := \mathbf{A}(\mathscr{A})^+$: indeed, one may take over the proof *verbatim*; one has only to remark that [75, Cor. 3.5.4] still holds for almost modules, and the reference to [75, Lemma 3.5.3] can be replaced by [36, Lemma 2.4.2(iii)]. □

Theorem 9.2.23. *In the situation of* (9.2.20)*, suppose that* $d \geq 2$*. Then the pair* $(X, \{x\})$ *is almost pure.*

Proof. For any given étale almost finitely presented $\mathscr{O}^a_U$-algebra $\mathscr{A}$, define $\mathscr{A}^\wedge$ as in (9.2.20), and set $\mathbf{H}^0(\mathscr{A}) := H^0(\mathbf{A}(U), \mathbf{A}(\mathscr{A})^+)$.

Claim 9.2.24. The natural morphisms :

$$\mathbf{H}^0(\mathscr{A})/\vartheta \cdot \mathbf{H}^0(\mathscr{A}) \to H^0(U^\wedge, \mathscr{A}^\wedge) \leftarrow H^0(U, \mathscr{A}) \otimes_{R^a} R^{\wedge a}$$

are isomorphisms.

Proof of the claim. For the first map one uses lemma 8.5.34 and proposition 9.2.21 with $q := 1$. For the second map, one applies lemma 8.1.86 to the ring homomorphism $R \to R^\wedge$ and the $\mathscr{A}$-regular element $t := p$. ◊

Let $\underline{1}_\mathscr{A} : \mathscr{O}^a_U \to \mathscr{A}$ be the structure morphism of $\mathscr{A}$; then $j_1 := \underline{1}_\mathscr{A} \otimes_{\mathscr{O}^a_U} \mathbf{1}_\mathscr{A}$ and $j_2 := \mathbf{1}_\mathscr{A} \otimes_{\mathscr{O}^a_U} \underline{1}_\mathscr{A}$ are two morphisms of étale $\mathscr{O}^a_U$-algebras $\mathscr{A} \to \mathscr{A} \otimes_{\mathscr{O}^a_U} \mathscr{A}$, and we may consider the $\boldsymbol{\sigma}$-equivariant morphism of $\mathscr{O}^a_{\mathbf{A}(U)}$-algebras :

$$\boldsymbol{\mu} := \mathbf{A}(\mu)^+ \circ (\mathbf{A}(j_1)^+ \otimes_{\mathscr{O}^a_{\mathbf{A}(U)}} \mathbf{A}(j_2)^+) : \mathbf{A}(\mathscr{A})^+ \otimes_{\mathscr{O}^a_{\mathbf{A}(U)}} \mathbf{A}(\mathscr{A})^+ \to \mathbf{A}(\mathscr{A} \otimes_{\mathscr{O}^a_U} \mathscr{A})^+$$

where μ denotes the multiplication morphism of $\mathscr{A} \otimes_{\mathscr{O}^a_U} \mathscr{A}$. After taking global sections, we obtain an equivariant map :

$$H^0(\boldsymbol{\mu}) : \mathbf{H}^0(\mathscr{A}) \otimes_{\mathbf{A}(R_\infty)^{+a}} \mathbf{H}^0(\mathscr{A}) \to \mathbf{H}^0(\mathscr{A} \otimes_{\mathscr{O}^a_U} \mathscr{A}).$$

Consequently, the module $C := \mathrm{Coker}\, H^0(\boldsymbol{\mu})$ is endowed with an isomorphism $\boldsymbol{\sigma}^*_R C \xrightarrow{\sim} C$, whence – by remark 8.5.18(ii) – an isomorphism of R^a/pR^a-modules :

$$\overline{\Phi}^*_R(C/(\vartheta, \vartheta_1)C) \xrightarrow{\sim} C/(\vartheta, p)C. \qquad (9.2.25)$$

Furthermore, claim 9.2.24 (applied to both $\mathscr{A}$ and $\mathscr{A} \otimes_{\mathscr{O}^a_U} \mathscr{A}$), yields a natural identification :

$$C/\vartheta C \xrightarrow{\sim} R^{\wedge a} \otimes_{R^a} \mathrm{Coker}(H^0(U, \mathscr{A}) \otimes_{R^a} H^0(U, \mathscr{A}) \to H^0(U, \mathscr{A} \otimes_{\mathscr{O}^a_U} \mathscr{A}))$$

which shows that $C/\vartheta C$ is an almost coherent R^a-module (proposition 9.2.14(i)) supported over $\{x\}$, especially, it has almost finite length. On the other hand, (9.2.25) and remark 8.5.18(i) imply that $\overline{\Phi}^*_R(C/(\vartheta, \vartheta_1)C)$ admits a filtration of length p, whose graded subquotients are quotients of $C/(\vartheta, \vartheta_1)C$, so that $C/(\vartheta, \vartheta_1)C = 0$ by lemma 9.2.11. Since ϑ_1 acts as a nilpotent endomorphism on $C/\vartheta C$ (again, by remark 8.5.18(iii)), we further deduce that $C/\vartheta C = 0$, in other words, the natural map

$$H^0(U, \mathscr{A}) \otimes_{R^a} H^0(U, \mathscr{A}) \to H^0(U, \mathscr{A} \otimes_{\mathscr{O}^a_U} \mathscr{A}) \qquad (9.2.26)$$

is an epimorphism. Next, let $j : U \to X$ be the open immersion; a direct inspection shows that claim 9.2.18 holds in the present situation (with proof unchanged), whence $j_*\mathscr{O}_U = \mathscr{O}_X$. Then corollary 8.2.16 says that $j_*\mathscr{A}$ is an étale and almost finite $\mathscr{O}_X^a$-algebra. Now the theorem follows from proposition 8.2.30. $\square$

Example 9.2.27. It follows from theorem 9.2.23 that the pair $(\mathrm{Spec}\, R, \{s(R)\})$ of example 9.2.19(i) is still almost pure even if we assume only that $d \geq 2$.

9.3. **Model algebras.** We let $(K, |\cdot|)$ be a valued field of characteristic 0 with value group Γ_K of rank one, such that the residue field κ of K^+ has characteristic $p > 0$.

Definition 9.3.1. The category MA_K of *model K^+-algebras* consists of all the pairs (A, Γ), where $(\Gamma, +)$ is an integral monoid, and A is a Γ-graded K^+-algebra A, fulfilling the following conditions :

(MA1) $\mathrm{gr}_\gamma A$ is a torsion-free K^+-module, with $\dim_K \mathrm{gr}_\gamma A \otimes_{K^+} K = 1$, for every $\gamma \in \Gamma$.
(MA2) $\mathrm{gr}_\alpha A \cdot \mathrm{gr}_\beta A \neq 0$ for every $\alpha, \beta \in \Gamma$, and $(\mathrm{gr}_\gamma A)^n = \mathrm{gr}_{n\gamma} A$ for every $n \in \mathbb{N}$ and $\gamma \in \Gamma$.
(MA3) $\mathrm{gr}_0 A = K^+$.
(MA4) Γ is saturated and Γ^{gp} is a $\mathbb{Z}[1/p]$-module.

The morphisms $(A, \Gamma) \to (A', \Gamma')$ in MA_K are the pairs (f, φ), where $\varphi : \Gamma \to \Gamma'$ is a morphism of monoids, and $f : A \to \Gamma \times_{\Gamma'} A'$ is a morphism of Γ-graded K^+-algebras (see (4.4.1)).

Example 9.3.2. Let M be an integral monoid, $N \to M$ an exact and injective morphism of monoids, $N \to K^+ \setminus \{0\}$ a morphism of monoids, and suppose that :

- M is *divisible*, *i.e.* the k-Frobenius endomorphism of M is surjective for every $k > 0$
- $M^{\mathrm{gp}}/N^{\mathrm{gp}}$ is a $\mathbb{Q}$-vector space.

Let $(\Gamma, +)$ be the image of M in $M^{\mathrm{gp}}/N^{\mathrm{gp}}$, and denote by I the nilradical of the K^+-algebra $A := M \otimes_N K^+$; then A is a Γ-graded K^+-algebra, and I is a Γ-graded ideal (proposition 4.4.12(ii)). We claim that $I \otimes_{K^+} K = 0$ and $(A/I, \Gamma)$ is a model K^+-algebra. Indeed, (MA4) is immediate, and it is easily seen that $\mathrm{gr}_{n\gamma} A = (\mathrm{gr}_\gamma A)^n$ for every $\gamma \in \Gamma$ and $n \in \mathbb{N}$. Moreover, since the map $N \to M$ is exact, the kernel of the map $M \to M^{\mathrm{gp}}/N^{\mathrm{gp}}$ equals N, so $(A/I)_0 = K^+$, *i.e.* (MA3) holds as well. The remaining assertions can be checked after tensoring with K : namely, we have to show that the Γ-graded K-algebra $A_K := M \otimes_N K$ is reduced, with $\dim_K \mathrm{gr}_\gamma A_K = 1$ for every $\gamma \in \Gamma$, and $\mathrm{gr}_\alpha A_K \cdot \mathrm{gr}_\beta A_K \neq 0$ for every $\alpha, \beta \in \Gamma$. However, the morphism $N \to K^+$ extends uniquely to a group homomorphism $N^{\mathrm{gp}} \to K^\times$, and $A_K = (N^{-1}M) \otimes_{N^{\mathrm{gp}}} K$; hence, we may assume that N is a group, in which case Γ is the set-theoretic quotient of M by the translation action of N (lemma 2.3.31(iii)). In this case, choose a representative $\gamma^* \in M$ for every $\gamma \in \Gamma$; it follows easily that $(\gamma^* \otimes 1 \mid \gamma \in \Gamma)$ is a basis of the K-vector space A_K, whence (MA1), and $(\alpha^* \otimes 1) \cdot (\beta^* \otimes 1)$ is a non-zero multiple of $(\alpha+\beta)^* \otimes 1$, which yields (MA2). Lastly, since the nilradical I_K of A_K is Γ-graded (proposition 4.4.12(ii)), we also deduce that $I_K = 0$, as sought.

Remark 9.3.3. (i) The category MA_K admits a tensor product, defined by the rule :

$$(A, \Gamma) \otimes (A', \Gamma') := (A \otimes_{K^+} A', \Gamma \oplus \Gamma').$$

(ii) Let (A, Γ) be any model K^+-algebra, and suppose that $\Gamma = \Gamma_1 \oplus \Gamma_2$ is a given decomposition of Γ as direct sum of monoids. Then it is easily seen that Γ_1 and Γ_2 are integral and saturated, and they fulfill axiom (MA4). There follows a morphism of model K^+-algebras :

$$(\Gamma_1 \times_\Gamma A, \Gamma_1) \otimes (\Gamma_2 \times_\Gamma A, \Gamma_2) \to (A, \Gamma) \tag{9.3.4}$$

(iii) In the situation of (ii), set $A_K := A \otimes_{K^+} K$. Then it is easily seen that (9.3.4) induces an isomorphism of K-algebras

$$(\Gamma_1 \times_\Gamma A_K) \otimes_K (\Gamma_2 \times_\Gamma A_K) \to A_K.$$

In general, (9.3.4) need not be an isomorphism. However, (9.3.4) is an isomorphism, if either Γ_1 or Γ_2 is a torsion abelian group. Indeed, in any case the induced maps

$$\mathrm{gr}_\alpha A \otimes_{K^+} \mathrm{gr}_\beta A \to \mathrm{gr}_{\alpha+\beta} A \qquad a_1 \otimes a_2 \mapsto a_1 a_2$$

will be injective for every $\alpha \in \Gamma_1$ and $\beta \in \Gamma_2$. To check surjectivity, let $a \in \mathrm{gr}_{\alpha+\beta} A$ be any element, and say that $n\beta = 0$; then $a^n \in \mathrm{gr}_{n\alpha} A$, and by (MA2) we know that there exists $a_1 \in \mathrm{gr}_\alpha A$ and $x \in K^+$ such that $a^n = a_1^n x$. It follows that $a = a_1 a_2$ for some $a_2 \in \mathrm{gr}_\beta A_K$ such that $a_2^n \in K^+$. Then $a_2 \in \mathrm{gr}_\beta A$, whence the claim.

(iv) Let $(E, |\cdot|_E)$ be any valued field extension of $(K, |\cdot|)$, such that $|\cdot|_E$ is also a valuation of rank one. Then we have an obvious base change functor :

$$\mathsf{MA}_K \to \mathsf{MA}_E \quad : \quad (A, \Gamma) \mapsto (A \otimes_{K^+} E^+, \Gamma).$$

9.3.5. Let (A, Γ) be any model K^+-algebra. For any subset $\Delta \subset \Gamma$, set

$$A_\Delta := \Delta \times_\Gamma A \qquad A_{\Delta,K} := A_\Delta \otimes_{K^+} K \qquad \Delta[1/p] := \bigcup_{n \in \mathbb{N}} \{\gamma \in \Gamma \mid p^n \gamma \in \Delta\}.$$

Clearly, if Δ is a submonoid of Γ, then A_Δ is a K^+-subalgebra of A, and (A_Δ, Δ) is a subobject of (A, Γ), provided Δ satisfies (MA4). On the other hand, if Δ is saturated, it is easily seen that the same holds for $\Delta[1/p]$, and then the latter does satisfy (MA4).

Also, if Δ is an ideal of Γ, then clearly A_Δ is an ideal of A.

Set $K^* := K^+ \setminus \{0\}$ and let $A^*_\gamma := \mathrm{gr}_\gamma A \setminus \{0\}$ for every $\gamma \in \Gamma$; for any submonoid $\Delta \subset \Gamma$, we deduce a sequence of morphisms of integral monoids :

$$(9.3.6) \qquad 1 \to K^* \xrightarrow{\varphi_\Delta} A^*_\Delta \to \Delta \to 1 \qquad \text{where } A^*_\Delta := \bigoplus_{\gamma \in \Delta} A^*_\gamma$$

(and where the direct sum is formed in the category of K^*-modules), such that $(9.3.6)^{\mathrm{gp}}$ is a short exact complex of abelian groups; then (MA2) implies that φ_Δ is saturated (proposition 3.2.31). Also, (MA1) implies that each A^*_γ is a filtered union of free cyclic K^*-modules, hence φ_Δ is also integral. Furthermore, (MA1) and (MA2) imply that $A^{*\mathrm{gp}}_\Delta$ is Δ^{gp}-graded, and

$$(9.3.7) \qquad \mathrm{gr}_\gamma A^{*\mathrm{gp}}_\Delta = A^*_\gamma \otimes_{K^*} K^\times \qquad \text{for every } \gamma \in \Delta.$$

(details left to the reader). We shall just write A^* and A_K instead of A^*_Γ, and respectively $A_{\Gamma,K}$.

9.3.8. The inclusion map $A^* \to A$ is a morphism of (multiplicative) monoids, hence induces a log structure on $X := \mathrm{Spec}\, A$ (see (6.1.13)). We shall denote

$$\mathbb{S}(A, \Gamma) := (X, (A^*_X)^{\log})$$

the resulting log scheme. Clearly, every morphism $(f, \varphi) : (A, \Gamma) \to (A', \Gamma')$ of model algebras induces a morphism of log schemes

$$\mathbb{S}(f, \varphi) : \mathbb{S}(A', \Gamma') \to \mathbb{S}(A, \Gamma).$$

Especially, by lemma 6.1.11(iv), the map φ_Γ yields a saturated morphism

$$(9.3.9) \qquad \mathbb{S}(A, \Gamma) \to \mathbb{S}(K^+) := \mathbb{S}(K^+, \{1\}).$$

Also, if $(K, |\cdot|) \to (E, |\cdot|_E)$ is an extension of rank one valued fields, the inclusion $A^* \subset (A \otimes_{K^+} E^+)^*$ induces a morphism of log schemes

$$\mathbb{S}(A \otimes_{K^+} E^+, \Gamma) \to \mathbb{S}(A, \Gamma)$$

for any model K^+-algebra (A, Γ), and we remark that the resulting diagram of log schemes

$$\begin{tikzcd} \mathbb{S}(A \otimes_{K^+} E^+, \Gamma) \arrow[r] \arrow[d] & \mathbb{S}(A, \Gamma) \arrow[d] \\ \mathbb{S}(E^+) \arrow[r] & \mathbb{S}(K^+) \end{tikzcd}$$

is cartesian. Indeed, since (6.1.14) is right exact, it suffices to check that the natural map

$$A^* \otimes_{K^*} E^* \to (A \otimes_{K^+} E^+)^*$$

is an isomorphism, which is clear.

Remark 9.3.10. (i) Let (A, Γ) be any model K^+-algebra, and suppose that $\Delta_0 \subset \Gamma$ is a fine and saturated submonoid, such that Δ_0^{gp} is torsion-free. In this case, Δ_0^{gp} is a free abelian group of finite rank, hence (9.3.6)$^{\mathrm{gp}}$ admits a splitting $\sigma : \Delta_0^{\mathrm{gp}} \to A^{*\mathrm{gp}}$. Then, using (9.3.7) and (MA1), it is easily seen that the rule $\gamma \mapsto \sigma(\gamma)$ extends to an isomorphism of Δ_0-graded K-algebras

$$K[\Delta_0] \xrightarrow{\sim} A_{\Delta_0,K}.$$

(ii) In the situation of (i), suppose furthermore, that K is algebraically closed. Pick any $x \in K^\times$ such that $\gamma := |x| \neq 1$, and let $\langle\gamma\rangle \subset \Gamma_K$ be the subgroup generated by γ; we define a group homomorphism $\langle\gamma\rangle \to K^\times$ by the rule : $\gamma^k \mapsto x^k$ for every $k \in \mathbb{Z}$. Since $K^\times$ is divisible, the latter map extends to a group homomorphism

$$\Gamma_K \to K^\times \tag{9.3.11}$$

and since Γ_K is a group of rank one, it is easily seen that (9.3.11) is a right inverse for the valuation map $|\cdot| : K^\times \to \Gamma_K$, whence a decomposition :

$$K^\times \xrightarrow{\sim} (K^+)^\times \oplus \Gamma_K.$$

On the other hand, set $A^*_{\Delta_0,K} := A^*_{\Delta_0} \otimes_{K^*} K^\times$; from (i) we deduce an isomorphism of Δ_0-graded monoids :

$$A^*_{\Delta_0,K} \xrightarrow{\sim} \Delta_0 \oplus K^\times.$$

Combining these two isomorphisms, we deduce a surjection $\tau : A^*_{\Delta_0,K} \to (K^+)^\times$ which is a left inverse to the inclusion $(K^+)^\times \to A^*_0$. Then, for every $\gamma \in \Delta_0$, let us set $C_\gamma := A^*_\gamma \cap \mathrm{Ker}\,\tau$; there follows a (non-canonical) isomorphism of Δ-graded monoids :

$$A^*_{\Delta_0} \xrightarrow{\sim} (K^+)^\times \oplus C \qquad \text{where} \quad C := \bigoplus_{\gamma \in \Delta_0} C_\gamma \subset A^*_{\Delta_0} \quad \text{and} \quad C^{\mathrm{gp}} \simeq \Delta_0^{\mathrm{gp}} \oplus \Gamma_K$$

(details left to the reader).

(iii) Suppose that Γ^{gp} is torsion-free, and K is still algebraically closed. Let us set :

$$\Delta_n := \{\gamma \in \Gamma \mid p^n\gamma \in \Delta_0\} \qquad \text{for every } n \in \mathbb{N}.$$

It is easily seen that Δ_n is still fine and saturated, and since $K^\times$ is divisible, we may extend inductively the splitting σ of (i) to a system of homomorphisms

$$\sigma_n : \Delta_n^{\mathrm{gp}} \to A^{*\mathrm{gp}} \qquad \text{such that} \quad \sigma_{n+1|\Delta_n^{\mathrm{gp}}} = \sigma_n \quad \text{for every } n \in \mathbb{N}$$

whence a compatible system of isomorphisms

$$K[\Delta_n] \xrightarrow{\sim} A_{\Delta_n,K} \qquad \text{for every } n \in \mathbb{N}. \tag{9.3.12}$$

Proceeding as in (ii), we deduce a compatible system of isomorphisms of Δ_n-graded monoids

$$A^*_{\Delta_n} \xrightarrow{\sim} (K^+)^\times \oplus C^{(n)} \qquad \text{such that } C^{(n)} \subset C^{(n+1)} \text{ for every } n \in \mathbb{N}.$$

In this situation, notice that the p-Frobenius automorphism of Γ^{gp} restricts to an isomorphism

$$\Delta_{n+1} \xrightarrow{\sim} \Delta_n \qquad \text{for every } n \in \mathbb{N}.$$

Likewise, since Γ_K is p-divisible, taking p-th powers induces isomorphisms

$$C^{(n+1)} \xrightarrow{\sim} C^{(n)} \qquad \text{for every } n \in \mathbb{N}.$$

Definition 9.3.13. Let (B, Δ) be model K^+-algebra. We say that (B, Δ) is *small*, if the following conditions hold :

(a) $\mathrm{gr}_\gamma B$ is a finitely generated K^+-module for every $\gamma \in \Delta$.
(b) $\Delta = \Delta_0[1/p]$ for some fine and saturated submonoid Δ_0.
(c) $\Delta_0 \times_\Delta B$ is a finitely generated K^+-algebra.

Remark 9.3.14. Notice that condition (a) of definition 9.3.13 and axiom (MA1) imply that $\mathrm{gr}_\gamma B$ is a free K^+-module of rank one, for every small model agebra (B, Δ) and every $\gamma \in \Delta$. Moreover, actually conditions (b) and (c) (together with axioms (MA1) and (MA2)) imply condition (a). Indeed, (c), (MA1) and proposition 4.4.16(ii) imply that $\mathrm{gr}_\gamma B$ is a free K^+-module of rank one, for every $\gamma \in \Delta_0$. Now, in case $n\gamma \in \Delta_0$, (MA2) implies that $\mathrm{gr}_{n\gamma} B$ is generated by an element of the form $z = x_1 \cdots x_n$, for certain $x_1, \dots, x_n \in \mathrm{gr}_\gamma B$. Suppose that $x_1 = ay$ for some $a \in K^+$ and $y \in \mathrm{gr}_\gamma B$; then z is divisible by a in $\mathrm{gr}_{n\gamma} B$, so $a \in (K^+)^\times$, *i.e.* x_1 generates $\mathrm{gr}_\gamma B$. In view of (b), for every $\gamma \in \Delta$ we may find $k \geq 0$ such that $p^k\gamma \in \Delta_0$, so (a) follows.

Remark 9.3.15. Let (B, Δ) be any small model algebra, so that conditions (b) and (c) of definition 9.3.13 are satisfied for some submonoid $\Delta_0 \subset \Delta$.

(i) Choose a decomposition $\Delta_0^\times = G \oplus H$, where G is a free abelian group, and H is the torsion subgroup of Δ_0; we have an isomorphism

$$\Delta_0 \simeq \Lambda \oplus H \qquad \text{with } \Lambda := \Delta_0^\sharp \oplus G$$

(lemma 3.2.10), which induces a decomposition :

$$\Delta \simeq \Lambda[1/p] \oplus H$$

inducing, in turn, an isomorphism of model K^+-algebras :

$$(B, \Delta) \xrightarrow{\sim} (B_{\Lambda[1/p]}, \Lambda[1/p]) \otimes (B_H, H)$$

(remark 9.3.3(iii)). Notice that both B_Λ and B_H are finitely generated K^+-algebras (proposition 4.4.16(i)). In other words, every small model algebra can be written as the tensor product of two small model algebras (B', Δ') and (B'', Δ''), such that Δ'^{gp} is torsion-free, and Δ'' is a finite abelian group whose order is not divisible by p.

(ii) For every $n \in \mathbb{N}$, set $\Delta_n := \{\gamma \in \Delta \mid p^n\gamma \in \Delta_0\}$. Then $B_n := \Delta_n \times_\Delta B$ is a finitely generated K^+-algebra for every $n \in \mathbb{N}$. Indeed, in view of (i), it suffices to check the assertion in case Δ^{gp} is a torsion-free abelian group. Now, pick a system $x_1, \dots, x_k$ of homogeneous generators of the K^+-algebra B_0; by (MA2), for every $i = 1, \dots, k$ there exists a homogeneous element $y_i \in B_n$ and $u_i \in (K^+)^\times$ such that $u_i y_i^{p^n} = x_i$. Let $z \in B_n$ be any homogeneous element; then $z^{p^n} = v x_1^{t_1} \cdots x_k^{t_k}$ for some $v \in K$ and $t_1, \dots, t_k \in \mathbb{N}$. Set $y := y_1^{t_1} \cdots y_k^{t_k}$ and $u := u_1^{t_1} \cdots u_k^{t_k} \in K^\times$; then $y^{p^n} uv = z^{p^n}$, and since Δ^{gp} is torsion-free, we deduce that $yw = z$ for some $w \in K^+$, *i.e.* the system $y_1, \dots, y_k$ generates the K^+-algebra B_n.

(iii) Suppose that Δ is a finite group whose order is not divisible by p. Then we claim that B is a finite étale K^+-algebra. Indeed, in view of remark 9.3.3(iii), it suffices to verify the assertion for Δ a cyclic finite group, say of order n, with $(n, p) = 1$; in the latter case, (MA2) implies that $B \simeq K^+[X]/(X^n - u)$ for some $u \in (K^+)^\times$, whence the contention.

(iv) Let $F \subset \Delta$ be any face. Then (B_F, F) is a small model algebra as well. Indeed, by (i), it suffices to consider the case where Δ^{gp} is a torsion-free abelian group. In this case, set $F_0 := F \cap \Delta_0$; it is easily seen that $F = F_0[1/p]$, and F_0 is a fine and saturated monoid, by lemma 3.1.20(ii) and corollary 3.2.33(ii). Moreover, $B_{F_0} = F_0 \times_F B_{\Delta_0}$ is a finitely generated K^+-algebra, by proposition 4.4.16(i), whence the contention.

(v) Suppose moreover, that K is algebraically closed, and let (B, Δ) be any small model K^+-algebra. Then we have a (non-canonical) isomorphism :

$$K[\Delta] \xrightarrow{\sim} B_K.$$

To exhibit such an isomorphism, we may – in light of (i) – assume that Δ^{gp} is either torsion-free, or a finite group of order not divisible by p. In the latter case, the assertion follows easily from (iii). In case Δ^{gp} is torsion-free, the sought isomorphism is the colimit of the system of isomorphisms (9.3.12).

Lemma 9.3.16. *Let (A, Γ) be a model K^+-algebra, and denote by $\mathscr{F}(\Gamma)$ the filtered family of all fine and saturated submonoids of Γ. We have :*

(i) $A = \operatorname*{colim}_{\Delta \in \mathscr{F}(\Gamma)} A_\Delta$.

(ii) *Suppose that Γ^{gp} is a torsion-free abelian group. Then A is a normal domain.*

Proof. (i) is an immediate consequence of corollary 3.4.1(ii).

(ii): We show first the following :

Claim 9.3.17. Suppose that Γ^{gp} is a torsion-free abelian group. Then A_K is a normal domain.

Proof of the claim. In view of (i), it suffices to show that $A_{\Delta,K}$ is a normal domain, when $\Delta \subset \Gamma$ is fine and saturated. The latter assertion follows from remark 9.3.10(i) and theorem 3.4.16(iii). ♢

Let A^ν be the integral closure of A in A_K; in view of claim 9.3.17 we are reduced to showing that $A = A^\nu$. By proposition 4.4.13(ii), A^ν is Γ-graded. Suppose now that $x \in A^\nu$; we need to show that $x \in A$, and we may assume that $x \in \mathrm{gr}_\gamma A^\nu$ for some $\gamma \in \Gamma$. Hence, let

$$x^n + a_1 x^{n-1} + \cdots + a_n = 0 \qquad \text{with } a_1, \dots, a_n \in A$$

be an integral equation for x over A. If we replace each a_i by its homogeneous component in degree $i\gamma$, we still obtain an integral equation for x, so we may assume that $a_i \in \mathrm{gr}_{i\gamma} A$ for every $i = 1, \dots, n$. Then, (MA2) implies that, for every $i = 1, \dots, n$ there exists $u_i \in K^+$ and $b_i \in \mathrm{gr}_\gamma A$, such that $|u_i| = 1$ and $a_i = u_i b_i^i$. Also, from (MA1) we deduce that there exists $b \in \mathrm{gr}_\gamma A$ such that $b_1, \dots, b_n \in K^+ b$. Set $y := b^{-1} x \in \mathrm{gr}_0 A \otimes_{K^+} K$; clearly y is integral over the subring $\mathrm{gr} A_0$. Lastly, (MA3) shows that $y \in K^+$, whence $x \in \mathrm{gr}_\gamma A$, which proves the contention. □

Proposition 9.3.18. *Let (A, Γ) be a model K^+-algebra, and suppose that Γ_K is divisible. Then (A, Γ) is the filtered union of its small model K^+-subalgebras.*

Proof. In view of lemma 9.3.16(i), we see that (A, Γ) is the colimit of the filtered system of its subobjects $(A_{\Delta[1/p]}, \Delta[1/p])$, for Δ ranging over the fine and saturated submonoids of Γ. We may then assume from start that $\Gamma = \Gamma_0[1/p]$ for some fine and saturated submonoid Γ_0.

Next, in view of remark 9.3.15(i), we may consider separately the cases where Γ_0^{gp} is a torsion-free abelian group, and where $\Gamma = \Gamma_0$ is a finite abelian group.

Suppose first that Γ_0^{gp} is torsion-free, and let $\underline{\gamma} := (\gamma_1, \dots, \gamma_n)$ be a finite system of generators for Γ_0. For every $i = 1, \dots, n$, choose $a_i \in A^*_{\gamma_i}$, and let $B(\underline{\gamma}, \underline{a}) \subset A$ be the K^+-subalgebra generated by $\underline{a} := (a_1, \dots, a_n)$. Clearly the grading of A induces a Γ_0-grading on $B(\underline{\gamma}, \underline{a})$, and $\mathrm{gr}_\beta B(\underline{\gamma}, \underline{a})$ is a finitely generated K^+-module for every $\beta \in \Gamma_0$ (proposition 4.4.16(ii)); by virtue of (MA1), we know that $\mathrm{gr}_\beta B(\underline{\gamma}, \underline{a})$ is then even a free rank one K^+-module, for every

$\beta \in \Gamma_0$. Furthermore, $\underline{a}$ generates a fine submonoid of A^*, so – by proposition 3.6.35(ii) – there exists an integer $k > 0$ such that

$$(\mathrm{gr}_{k\beta} B(\underline{\gamma}, \underline{a}))^n = \mathrm{gr}_{nk\beta} B(\underline{\gamma}, \underline{a}) \qquad \text{for every } \beta \in \Gamma_0 \text{ and every integer } n > 0.$$

Now, for any $\beta \in \Gamma$, let $t > 0$ be an integer such that $p^t\beta \in \Gamma_0$, and pick a generator b of the K^+-module $\mathrm{gr}_{kp^t\beta} B(\underline{\gamma}, \underline{a})$; in light of (MA2) we may find $x \in K^+$ and $c \in \mathrm{gr}_\beta A$, such that $b = c^{kp^t} x$. Since Γ_K is divisible, we may write $x = y^{kp^t} u$ for some $y, u \in K^+$ with $|u| = 1$.

It is easily seen that the K^+-submodule of $\mathrm{gr}_\beta A$ generated by cy does not depend on the choices of c, y, t and k; hence we denote $\mathrm{gr}_\beta C(\underline{\gamma}, \underline{a})$ this submodule; a simple inspection shows that the resulting Γ-graded K^+-module

$$C(\underline{\gamma}, \underline{a}) := \bigoplus_{\beta \in \Gamma} \mathrm{gr}_\beta C(\underline{\gamma}, \underline{a})$$

is actually a Γ-graded K^+-subalgebra of A, for which (MA2) holds, and therefore $(C(\underline{\gamma}, \underline{a}), \Gamma)$ is a small model algebra. Lastly, it is clear that the family of all such $(C(\underline{\gamma}, \underline{a}), \Gamma)$, for $\underline{\gamma}$ ranging over all finite sets of generators of Γ, and $\underline{a}$ ranging over all the finite sequences of elements of A as above, form a cofiltered system of subobjects, whose colimit is (A, Γ). This concludes the proof of the proposition in this case.

Next, suppose that $\Gamma = \Gamma_0$ is a finite abelian group. In view of remark 9.3.3(iii), we are reduced to the case where Γ is cyclic, say $\Gamma = \mathbb{Z}/n\mathbb{Z}$. Fix any generator γ of Γ, and for every $a \in \mathrm{gr}_\gamma A$, let $C(a)$ be the K^+-subalgebra of A generated by a; it is easily seen that the colimit of the filtered family $((C(a), \Gamma) \mid a \in \mathrm{gr}_\gamma A)$ equals (A, Γ). (Details left to the reader.) $\square$

9.3.19. Suppose now that K is algebraically closed, let (B, Δ) be a small model K^+-algebra, with Δ^{gp} torsion-free, and write $\Delta = \Delta_0[1/p]$ for some fine and saturated submonoid Δ_0 such that the K^+-algebra $B_0 := B_{\Delta_0}$ is finitely generated. Let us set

$$\Delta_n := \{\gamma \in \Delta \mid p^n\gamma \in \Delta_0\} \qquad B_n := B_{\Delta_n} \qquad Y_n := \mathrm{Spec}\, B_n \qquad \text{for every } n \in \mathbb{N}.$$

We wish to construct a ladder of log schemes

$$\begin{tikzcd} \cdots \arrow[r, "g_{n+1}"] & (Y_{n+1}, \underline{M}_{n+1}) \arrow[r, "g_n"] \arrow[d, "\varphi_{n+1}"'] & (Y_n, \underline{M}_n) \arrow[r, "g_{n-1}"] \arrow[d, "\varphi_n"'] & \cdots \arrow[r, "g_0"] & (Y_0, \underline{M}_0) \arrow[d, "\varphi_0"] \\ \cdots \arrow[r, "h_{n+1}"] & (S, \underline{N}_{n+1}) \arrow[r, "h_n"] & (S, \underline{N}_n) \arrow[r, "h_{n-1}"] & \cdots \arrow[r, "h_0"] & (S, \underline{N}_0) \end{tikzcd} \tag{9.3.20}$$

such that, for every $n \in \mathbb{N}$:

- φ_n is smooth and saturated, and $\underline{M}_n$ admits a chart, given by a Δ_n-graded fine monoid $P^{(n)}$, such that $P_0^{(n)}$ is sharp, and the inclusion map $P_0^{(n)} \to P^{(n)}$ is flat and saturated, and gives a chart for φ_n
- the morphism of schemes underlying h_n (resp. g_n) is the identity of S (resp. is induced by the inclusion $B_n \subset B_{n+1}$)
- the morphism g_n admits a chart, given by an injective map $P^{(n)} \to \Delta_n \times_{\Delta_{n+1}} P^{(n+1)}$ of Δ_n-graded monoids, whose restriction $P_0^{(n)} \to P_0^{(n+1)}$ gives a chart for h_n.

This will be achieved in several steps, as follows :

- First, let $b_1, \dots, b_k$ be a finite system of generators for B_0. By remark 9.3.10(ii), we have a decomposition $B_0^* = (K^+)^\times \oplus C$ for some submonoid $C \subset B_0^*$, and we may suppose that each b_i lies in C. Let $P^{(0)} \subset B_0^*$ be the submonoid generated by $b_1, \dots, b_k$. The restriction of the surjection $B_0^* \to \Delta_0$ is then still a surjection $\pi : P^{(0)} \to \Delta_0$. Notice that, for every $x \in \mathrm{Ker}\, \pi^{\mathrm{gp}}$, we have either $x \in K^+$ or $x^{-1} \in K^+$; especially, we may find a finite system $x_1, \dots, x_n$ of generators for $\mathrm{Ker}\, \pi^{\mathrm{gp}}$, such that $x_i \in K^+$ for every $i = 1, \dots, n$. Now, let $\Sigma \subset P^{(0)\mathrm{gp}}$ be the submonoid generated by $x_1, \dots, x_n$; after replacing $P^{(0)}$ by $P^{(0)} \cdot \Sigma$, we may assume that $\mathrm{Ker}\,\pi^{\mathrm{gp}} = P_0^{(0)\mathrm{gp}}$, where $P_0^{(0)} := \mathrm{Ker}\,\pi^{\mathrm{gp}} \cap P^{(0)} \subset K^+$ is a fine submonoid.

• Next, by theorem 3.6.45, we may find a finitely generated submonoid $\Sigma' \subset P_0^{(0)\mathrm{gp}} \cap K^+$ such that the induced morphism $P_0^{(0)} \cdot \Sigma' \to P^{(0)} \cdot \Sigma'$ is flat. Clearly

$$P_0^{(0)} \cdot \Sigma' = \mathrm{Ker}\,\pi^{\mathrm{gp}} \cap (P^{(0)} \cdot \Sigma')$$

hence, we may replace $P^{(0)}$ by $P^{(0)} \cdot \Sigma'$, and assume that the morphism $P_0^{(0)} \to P^{(0)}$ is also flat.

• We claim that the map $P_0^{(0)} \to P^{(0)}$ is also saturated. We shall apply the criterion of proposition 3.2.31 : indeed, for given $\gamma \in \Delta_0$ and integer $n > 0$, let x (resp. y) be a generator of the $P_0^{(0)}$-module $P_\gamma^{(0)}$ (resp. $P_{n\gamma}^{(0)}$). Then x (resp. y) is also a generator of the K^+-module $\mathrm{gr}_\gamma B$ (resp. $\mathrm{gr}_{n\gamma} B$), and (MA2) implies that there exists $u \in (K^+)^\times$ such that $uy = x^n$; but since $x, y \in C$, we must have $u = 1$ therefore $(P_\gamma^{(0)})^n = P_{n\gamma}^{(0)}$, as required.

• Next, we claim that the induced map of Δ_0-graded K^+-algebras

$$P^{(0)} \otimes_{P_0^{(0)}} K^+ \to B_0 \tag{9.3.21}$$

is an isomorphism. Indeed, since $P^{(0)}$ contains a set of generators for the K^+-algebra B, the map (9.3.21) is obviously surjective. However, for every $\gamma \in \Delta_0$, the $P_0^{(0)}$-module $P_\gamma^{(0)}$ is free of rank one (remark 3.2.5(iv)), therefore $(P^{(0)} \otimes_{P_0^{(0)}} K^+)_\gamma$ is a free K^+-module of rank one. It follows easily that (9.3.21) is also injective.

• Now, denote by $\underline{N}_0$ (resp. $\underline{M}_0$) the fine log structure on S (resp. on X_0) deduced from the inclusion map $P_0^{(0)} \to K^+$ (resp. $P^{(0)} \to B_0$). By lemma 6.1.11(iv), the inclusion $P_0^{(0)} \to P^{(0)}$ yields a chart for a morphism $\varphi_0 : (Y_0, \underline{M}_0) \to (S, \underline{N}_0)$ that is saturated, as sought. Lastly, since (9.3.21) is an isomorphism, theorem 6.3.37 shows that φ_0 is also smooth.

• Next, according to remark 9.3.10(iii), we have a compatible system of decompositions

$$B_n^* = (K^+)^\times \oplus C^{(n)} \qquad \text{for every } n \in \mathbb{N}$$

with $C^{(0)} = C$, and such that the p-Frobenius induces an isomorphism $\tau_n : C^{(n+1)} \xrightarrow{\sim} C^{(n)}$ for every $n \in \mathbb{N}$. Hence, define inductively an increasing sequence of submonoids

$$P^{(0)} \subset P^{(1)} \subset P^{(2)} \subset \cdots \subset B^* \quad \text{by letting} \quad P^{(n+1)} := \tau_n^{-1} P^{(n)} \text{ for every } n \in \mathbb{N}.$$

Clearly, the grading of B^* restricts to a Δ_n-grading on $P^{(n)}$, and induces a Δ-grading on P. We deduce isomorphisms of Δ_{n+1}-graded monoids :

$$P^{(n+1)} \xrightarrow{\sim} \Delta_{n+1} \times_{\Delta_n} P^{(n)} \qquad \text{for every } n \in \mathbb{N}. \tag{9.3.22}$$

Especially, the induced maps of monoids $P_0^{(n)} \to P^{(n)}$ are still flat and saturated, and $P_0^{(n)}$ is still sharp (remark 9.3.24). The inclusion maps $P^{(n)} \to B_n$ and $P_0^{(n)} \to K^+$ determine an isomorphism of Δ_n-graded K^+-algebras

$$P^{(n)} \otimes_{P_0^{(n)}} K^+ \xrightarrow{\sim} B_n \qquad \text{for every } n \in \mathbb{N} \tag{9.3.23}$$

as well as fine log structures $\underline{M}_n$ on Y_n, and $\underline{N}_n$ on S, whence a morphism of log schemes $\varphi_n : (Y_n, \underline{M}_n) \to (S, \underline{N}_n)$ which is again smooth and saturated. This completes the construction of (9.3.20).

Remark 9.3.24. (i) With the notation of (9.3.19), notice that, by construction, $P^{(n)\mathrm{gp}} \subset C^{(n)\mathrm{gp}}$; especially, since Δ^{gp} is torsion-free, the same holds for $P^{(n)\mathrm{gp}}$. Likewise, the construction shows that the induced map $P_0^{(n)\mathrm{gp}} \to \Gamma_K$ is always injective; especially, $P_0^{(n)\mathrm{gp}}$ is a free abelian group of finite rank, and $P_0^{(n)}$ is sharp. It follows that the $P_0^{(n)}$-module $P_\gamma^{(n)}$ admits a unique generator g_γ, for every $\gamma \in \Delta_n$, and the saturation condition for the inclusion $P_0^{(n)} \to P^{(n)}$ translates as the system of identities :

$$g_\gamma^k = g_{k\gamma} \qquad \text{for every } k \in \mathbb{N} \text{ and every } \gamma \in \Delta_n.$$

(ii) Moreover, set $P := \bigcup_{n\in\mathbb{N}} P^{(n)}$. Notice that the colimit of the maps (9.3.23) is an isomorphism of Δ-graded K^+-algebras

$$P \otimes_{P_0} K^+ \xrightarrow{\sim} B. \tag{9.3.25}$$

Furthermore, since the natural map $P^{(n)}/P_0^{(n)} \to \Delta$ is an isomorphism for every $n \in \mathbb{N}$, we deduce that the grading of P induces an isomorphism :

$$P/P_0 \xrightarrow{\sim} \Delta. \tag{9.3.26}$$

(iii) We also obtain a commutative diagram of log schemes :

$$\begin{tikzcd} \mathbb{S}(B_n, \Delta_n) \arrow[r] \arrow[d] & (Y_n, \underline{M}_n) \arrow[d, "\varphi_n"] \\ \mathbb{S}(K^+) \arrow[r] & (S, \underline{N}_n) \end{tikzcd} \qquad \text{for every } n \in \mathbb{N} \tag{9.3.27}$$

whose left vertical arrow is the saturated morphism (9.3.9), and whose bottom (resp. top) arrow is the identity of S (resp. of Y_n) on the underlying schemes, and is induced by the inclusion map $P_n^{(0)} \to K^*$ (resp. $P^{(n)} \to B_n^*$). It is easily seen that (9.3.27) is cartesian : indeed, since the functor (6.1.14) is right exact, it suffices to check that the natural map

$$P^{(n)} \otimes_{P_0^{(n)}} K^* \to B_n^*$$

is an isomorphism, which is clear from (9.3.23).

(iv) Notice that the inclusion map $i_0 : P_0^{(0)} \to P^{(0)}$ is also local; then corollary 3.4.5 yields a decomposition

$$\vartheta_0 : P^{(0)} \xrightarrow{\sim} P^{(0)\times} \oplus P^{(0)\sharp}$$

such that $\vartheta_0 \circ i_0 = \iota_0 \circ \lambda_0^\sharp$, where $\iota_0 : P^{(0)\sharp} \to P^{(0)\times} \oplus P^{(0)\sharp}$ is the natural inclusion map. By means of the isomorphisms (9.3.22), we may then inductively construct decompositions

$$\vartheta_n : P^{(n)} \xrightarrow{\sim} P^{(n)\times} \oplus P^{(n)\sharp} \qquad \text{for every } n \in \mathbb{N}$$

fitting into a commutative diagram

$$\begin{tikzcd} P_0^{(n)} \arrow[r, "i_n"] \arrow[d, "i_n^\sharp"'] & P^{(n)} \arrow[r, "j_n"] \arrow[d, "\vartheta_n"'] & P^{(n+1)} \arrow[d, "\vartheta_{n+1}"] \\ P^{(n)\sharp} \arrow[r, "\iota_n"] & P^{(n)\times} \oplus P^{(n)\sharp} \arrow[r, "j_n^\times \oplus j_n^\sharp"] & P^{(n+1)\times} \oplus P^{(n+1)\sharp} \end{tikzcd}$$

where i_n, j_n and ι_n are the natural inclusion maps. Now, set

$$B'_n := K^+[P^{(n)\times}] \qquad B''_n := P^{(n)\sharp} \otimes_{P_0^{(n)}} K^+ \qquad \text{for every } n \in \mathbb{N}$$

and let Δ'_n (resp. Δ''_n) be the image of $P^{(n)\times}$ (resp. of $\vartheta_n^{-1}P^{(n)\sharp}$) in Δ_n; since the grading of $P^{(n)}$ induces an isomorphism $P^{(n)}/P_0^{(n)} \xrightarrow{\sim} \Delta_n$, we see that $\Delta_n = \Delta'_n \oplus \Delta''_n$, and by construction, for every $n \in \mathbb{N}$ we have isomorphisms

$$(B_n, \Delta_n) \xrightarrow{\sim} (B'_n, \Delta'_n) \otimes (B''_n, \Delta''_n)$$

of model K^+-algebras, which identify the inclusions $B_n \to B_{n+1}$ with the tensor product of the induced inclusion maps $B'_n \to B'_{n+1}$ and $B''_n \to B''_{n+1}$. Furthermore, set $Y'_n := \operatorname{Spec} B'_n$ and $Y''_n := \operatorname{Spec} B''_n$ for every $n \in \mathbb{N}$. The induced map of monoids $P^{(n)\sharp} \to B''_n$ determines a fine log structure $\underline{M}''_n$ on Y''_n, the map $i^\sharp_n$ gives a chart for a smooth and saturated morphism of log schemes

$$\varphi''_n : (Y''_n, \underline{M}''_n) \to (S, \underline{N}_n)$$

and there follows an isomorphism of $(S, \underline{N}_n)$-schemes (lemma 6.1.4)

$$(Y_n, \underline{M}_n) \stackrel{\sim}{\to} Y'_n \times_S (Y''_n, \underline{M}''_n) \qquad \text{for every } n \in \mathbb{N}. \tag{9.3.28}$$

(v) Starting with (9.3.35), we shall consider the strict henselization B^{sh} of B at a given geometric point $\overline{x}$ of $\mathrm{Spec}\, B \otimes_{K^+} \kappa$. In this situation, let $x \in Y := \mathrm{Spec}\, B$ be the support of $\overline{x}$, and $\mathfrak{p}_x \subset B$ the corresponding prime ideal. Let also P be as in (ii), denote by $\beta : P \to B$ the natural map deduced from (9.3.25), and set $\mathfrak{p} := \beta^{-1}\mathfrak{p}_x$. Moreover, set $\mathfrak{p}_n := \mathfrak{p} \cap P^{(n)}$ and $Q^{(n)} := P^{(n)}_{\mathfrak{p}_n}$ for every $n \in \mathbb{N}$; clearly $\mathfrak{p}_n \subset \mathfrak{p}_{n+1}$ (so the isomorphism (9.3.22) maps $\mathfrak{p}_{n+1}$ onto $\mathfrak{p}_n$), and therefore

$$Q^{(n)} \subset Q^{(n+1)} \qquad \text{for every } n \in \mathbb{N}.$$

Furthermore, the image of $Q^{(n)}$ in Δ^{gp}_n is a localization Γ_n of Δ_n, especially it is still saturated, and the maps (9.3.22) extend to isomorphisms of Γ_{n+1}-graded monoids

$$Q^{(n+1)} \stackrel{\sim}{\to} \Gamma_{n+1} \times_{\Gamma_n} Q^{(n)} \qquad \text{for every } n \in \mathbb{N}.$$

It is also clear that the p-Frobenius of Γ_{n+1} factors through an isomorphism $\Gamma_{n+1} \stackrel{\sim}{\to} \Gamma_n$, for every $n \in \mathbb{N}$. Set $F_n := P^{(n)} \backslash \mathfrak{p}_n$ for every $n \in \mathbb{N}$; notice that, by construction, $F_n \cap P^{(n)}_0 = \{1\}$, hence $F_n \cap P^{(n)}_\gamma$ is either empty or else it contains exactly one element, namely the generator g_γ of $P^{(n)}_\gamma$, by virtue of (i). It follows that $Q^{(n)}_0 = P^{(n)}_0$ for every $n \in \mathbb{N}$, and therefore the inclusion map $Q^{(n)}_0 \to Q^{(n)}$ is still flat and saturated (lemma 3.2.12(ii)). Let $\Gamma := \bigcup_{n\in\mathbb{N}} \Gamma_n$; we conclude that $Q := P_{\mathfrak{p}}$ is a Γ-graded monoid with $Q_0 = P_0$, and we may define

$$B_{\mathfrak{p}} := Q \otimes_{Q_0} K^+.$$

The foregoing shows that $(B_{\mathfrak{p}}, \Gamma)$ is still a small model algebra, and we also obtain a ladder of log schemes with the properties listed in (9.3.19) : namely, set $B_{\mathfrak{p},n} := B_{\mathfrak{p},\Gamma_n} = Q^{(n)} \otimes_{Q^{(n)}_0} K^+$ for every $n \in \mathbb{N}$, and endow $Y_{\mathfrak{p},n} := \mathrm{Spec}\, B_{\mathfrak{p},n}$ with the log structure $\underline{M}_{\mathfrak{p},n}$ determined by the induced map $Q^{(n)} \to B_{\mathfrak{p},n}$. Clearly the geometric point $\overline{x}$ lifts uniquely to a geometric point $\overline{x}_{\mathfrak{p}}$ of $Y_{\mathfrak{p}} := \mathrm{Spec}\, B_{\mathfrak{p}}$, and the localization map $B \to B_{\mathfrak{p}}$ induces an isomorphism

$$Y_{\mathfrak{p}}(\overline{x}_{\mathfrak{p}}) \stackrel{\sim}{\to} Y(\overline{x}).$$

Hence, for the study of the scheme $Y(\overline{x})$, it shall be usually possible to replace the original small algebra B by its localization $B_{\mathfrak{p}}$ thus constructed. In so doing, we gain one more property: indeed, notice that the new chart $Q^{(n)} \to B_{\mathfrak{p},n}$ is local at the geometric point $\overline{x}$, for every $n \in \mathbb{N}$. Now, choose a compatible system of decompositions

$$(Y_{\mathfrak{p},n}, \underline{M}_{\mathfrak{p},n}) \stackrel{\sim}{\to} Y'_{\mathfrak{p},n} \times_S (Y''_{\mathfrak{p},n}, \underline{M}''_{\mathfrak{p},n})$$

as in (iv), and denote by $\overline{x}_{\mathfrak{p},n}$ (resp. $\overline{x}''_{\mathfrak{p},n}$) the image of $\overline{x}_{\mathfrak{p}}$ in $Y_{\mathfrak{p},n}$ (resp. in $Y''_{\mathfrak{p},n}$), for every $n \in \mathbb{N}$. By inspecting the construction, it is easily seen that the chart $Q^{(n)\sharp} \to B''_{\mathfrak{p},n} := Q^{(n)\sharp} \otimes_{Q^{(n)}_0} K^+$ is also local at $\overline{x}''_{\mathfrak{p},n}$.

9.3.29. Before continuing with the general discussion, let us consider the special case where $\Delta_0 \simeq \mathbb{Z}^{\oplus r} \oplus \mathbb{N}^{\oplus s}$, for some $r, s \geq 0$. This extra assumption is noteworthy because, on the one hand it is often verified in practice, and on the other hand, it affords important simplifications in the proof of the almost purity theorem. These simplifications rest on the following :

Proposition 9.3.30. *Keep the notation of* (9.3.19), *and suppose moreover that* $\Delta_0 \simeq \mathbb{Z}^{\oplus r} \oplus \mathbb{N}^{\oplus s}$, *for some* $r, s \geq 0$. *Then there exists* $c \in P_0^{(1)}$ *such that*

$$c^{\varepsilon(n,m)} \cdot \mathrm{Tor}_i^{B_n}(B_{n+m}, M) = 0 \qquad \textit{where} \qquad \varepsilon(n,m) := \sum_{i=0}^{m-1} p^{-(i+n)}$$

for every $n, m \in \mathbb{N}$, *every* $i > 0$, *and every* B_n*-module* M.

Proof. For every $n, m \in \mathbb{N}$, let us set

$$Q^{(n,m)} := P^{(n)} \otimes_{P_0^{(n)}} P_0^{(n+m)} \qquad \text{and} \qquad Q^{(n)} := Q^{(n,n+1)}.$$

So $Q^{(n)}$ is a fine monoid for every $n \in \mathbb{N}$ (theorem 3.2.3); moreover, since Δ_n^{gp} is torsion-free, $P_0^{(n)\mathrm{gp}}$ is a direct summand of $P^{(n)\mathrm{gp}}$, and we deduce easily that $Q^{(n)\mathrm{gp}}$ is a free abelian group, whose rank equals the rank of $P^{(n)\mathrm{gp}}$. Since (9.3.22) restricts to the p-Frobenius map on the submonoid $P^{(n)}$, it follows that the induced map $Q^{(n)} \to P^{(n+1)}$ is injective, for every $n \in \mathbb{N}$. Notice also that $P^{(n+1)}$ is a finitely generated $Q^{(n)}$-module : indeed, if $b_1, \dots, b_k$ is a finite system of generators for the monoid $P^{(n+1)}$, then the system

$$\left\{ \prod_{i=1}^{k} b_i^{r_i} \mid 0 \leq r_1, \dots, r_k < p \right\}$$

generates $P^{(n+1)}$ as a $P^{(n)}$-module, hence also as a $Q^{(n)}$-module. Furthermore, from (9.3.23) we deduce an isomorphism of Δ_{n+m}-graded K^+-algebras

$$P^{(n+m)} \otimes_{Q^{(n,m)}} B_n \xrightarrow{\sim} B_{n+m} \qquad \text{for every } n, m \in \mathbb{N}. \tag{9.3.31}$$

Claim 9.3.32. There exists $c \in P_0^{(1)}$ such that the following holds. For every $n, m \in \mathbb{N}$ we may find a free $Q^{(n,m)}$-submodule L of $P^{(n+m)}$ with

$$c^{\varepsilon(n,m)} \cdot P^{(n+m)} \subset L \tag{9.3.33}$$

Proof of the claim. Under our assumptions, Δ_{n+1} is a free Δ_n-module of rank $N := p^{r+s}$, for every $n \in \mathbb{N}$ (see theorem 6.5.9). Now, choose a basis $\gamma_1, \dots, \gamma_N$ of the free Δ_0-module Δ_1. Consider the $Q^{(0)}$-linear map

$$\beta : (Q^{(0)})^{\oplus N} \to P^{(1)} \qquad e_i^{(0)} \mapsto g_{\gamma_i} \quad \text{for } i = 1, \dots, N$$

where $(e_1^{(n)}, \dots, e_N^{(n)})$ is the standard basis of $(Q^{(n)})^{\oplus N}$, and $g_{\gamma_1}, \dots, g_{\gamma_N}$ are as in remark 9.3.24. Since $Q^{(0)}$ is Δ_0-graded, it is easily seen that β is injective, therefore $\mathrm{Im}(\beta)$ is a free $Q^{(0)}$-submodule of $P^{(1)}$. Say that $x \in P_\gamma^{(1)}$ is a given element; there exist $i \leq N$ and $\delta \in \Delta_0$ such that $\gamma = \gamma_i + \delta$, therefore we may find $c \in P_0^{(1)}$ such that cx lies in the $P_0^{(1)}$-submodule of $P_\gamma^{(1)}$ generated by $g_{\gamma_i} g_\delta$. Since $P^{(1)}$ is a finitely generated $Q^{(0)}$-module, we may then pick $c \in P_0^{(1)}$ such that $cP^{(1)} \subset \mathrm{Im}(\beta)$. Now, set $\beta_0 := \beta$, and define inductively a $Q^{(n)}$-linear map

$$\beta_n : (Q^{(n)})^{\oplus N} \to P^{(n+1)} \qquad e_i^{(n)} \mapsto \tau_n^{-1} \circ \beta_{n-1}(e_i^{(n-1)}) \qquad \text{for } i = 1, \dots, N$$

for every $n > 0$, where $\tau_n : P^{(n+1)} \xrightarrow{\sim} P^{(n)}$ is induced by the p-Frobenius of $P^{(n+1)}$. Clearly $\mathrm{Im}(\beta_n)$ is a free $Q^{(n)}$-module for every $n \geq 1$, and moreover

$$c^{1/p^n} \cdot P^{(n+1)} \subset \mathrm{Im}(\beta_n) \qquad \text{for every } n \in \mathbb{N}. \tag{9.3.34}$$

Next, we construct by induction on m, an injective $Q^{(n,m)}$-linear map

$$\beta_{(n,m)} : (Q^{(n,m)})^{\oplus N^m} \to P^{(n+m)}.$$

To this aim, we set $\beta_{(n,0)} := \mathbf{1}_{P^{(n)}}$, for every $n \in \mathbb{N}$. Suppose then, that $m \geq 0$ and $\beta_{(n,m)}$ has already been defined; then we let $\beta_{(n,m+1)}$ be the composition

$$(Q^{(n,m+1)})^{\oplus N^{m+1}} \xrightarrow{\vartheta^{\oplus N^m}} (Q^{(n+1,m)})^{\oplus N^m} \xrightarrow{\beta_{(n+1,m)}} P^{(n+m+1)}$$

where

$$\vartheta := \beta_n \otimes_{P_0^{(n+1)}} P_0^{(n+m+1)} : (Q^{(n,m+1)})^{\oplus N} \to Q^{(n+1,m)}.$$

Again, one checks that ϑ is injective, by remarking that $Q^{(n,m)}$ inherits a Δ_n-grading from $P^{(n)}$. A simple inspection and an easy induction on m, starting from (9.3.34), show that (9.3.33) holds with $L := \mathrm{Im}(\beta_{(n,m)})$, for every $n, m \in \mathbb{N}$. $\Diamond$

We see from claim 9.3.32 that multiplication by $c^{\varepsilon(n,m)}$ on B_{n+m} factors through the free B_n-module $L \otimes_{Q^{(n,m)}} B_n$, whence the assertion. $\Box$

9.3.35. Let us return to the situation of (9.3.19). Fix a geometric point $\overline{x}$ of $Y := \mathrm{Spec}\, B$, localized on $Y \times_S \mathrm{Spec}\, \kappa$, and for every $n \in \mathbb{N}$, let $\overline{x}_n$ be the image of $\overline{x}$ in Y_n. Set

$$B^{\mathrm{sh}} := \mathscr{O}_{Y(\overline{x}),\overline{x}} \qquad \text{and} \qquad B_n^{\mathrm{sh}} := \mathscr{O}_{Y_n(\overline{x}_n),\overline{x}_n} \quad \text{for every } n \in \mathbb{N}.$$

For the next step, we shall apply the method of (8.3.47), to construct a normalized length function for B^{sh}-modules. To this aim, we need an auxiliary model K^+-algebra, defined as follows.

First, notice that the grading $P \to \Delta$ extends to a morphism of monoids

$$\pi_{\mathbb{Q}} : P_{\mathbb{Q}} \to \Delta_{\mathbb{Q}}$$

(notation of (3.3.20)); *i.e.* $P_{\mathbb{Q}}$ is a $\Delta_{\mathbb{Q}}$-graded monoid. Since $K^\times$ is a divisible group, the inclusion map $P_0 \to K^*$ extends to a group homomorphism $P_{\mathbb{Q},0}^{\mathrm{gp}} \to K^\times$, and it is easily seen that the latter restricts to a morphism of monoids

$$P_{\mathbb{Q},0} \to K^*.$$

So finally, we may set

$$A := P_{\mathbb{Q}} \otimes_{P_{\mathbb{Q},0}} K^+.$$

Taking into account (9.3.25), we deduce an isomorphism of Δ-graded K^+-algebras

$$B \xrightarrow{\sim} A_\Delta. \tag{9.3.36}$$

Moreover, arguing as in remark 9.3.10(iii), we get an isomorphism

$$K[\Delta_{\mathbb{Q}}] \xrightarrow{\sim} A_K \tag{9.3.37}$$

fitting into a commutative diagram of K-algebras

$$\begin{tikzcd} K[\Delta] \arrow[r, "\sim"] \arrow[d] & B_K \arrow[d] \\ K[\Delta_{\mathbb{Q}}] \arrow[r, "\sim"] & A_K \end{tikzcd}$$

whose top horizontal arrow is the isomorphism of remark 9.3.15(v), and whose right (resp. left) vertical arrow is deduced from (9.3.36) (resp. is induced by the natural inclusion $\Delta \subset \Delta_{\mathbb{Q}}$).

Lemma 9.3.38. *With the notation of* (9.3.35), *we have :*

(i) *$P_{\mathbb{Q},\gamma}$ is a free $P_{\mathbb{Q},0}$-module of rank one, for every $\gamma \in \Delta_{\mathbb{Q}}$.*
(ii) *$P_{\mathbb{Q},0}$ is a sharp monoid, and the inclusion map $P_{\mathbb{Q},0} \to P_{\mathbb{Q}}$ is flat and saturated.*

Proof. (i): We can write $P_\mathbb{Q}$ as the colimit of the system of monoids

$$(P_{[n]};\ \mu_{P,n,m} : P_{[n]} \to P_{[nm]} \mid n, m \in \mathbb{N})$$

such that $P_{[n]} := P^{(0)}$ for every $n \in \mathbb{N}$, and $\mu_{P,n,m}$ is the m-Frobenius map of $P^{(0)}$, for every $n, m \in \mathbb{N}$. Likewise, we may write $\Delta_\mathbb{Q}$ as colimit of a system of Frobenius endomorphisms $(\Delta_{0,[n]};\ \mu_{\Delta,n,m} \mid n, m \in \mathbb{N})$, and $\pi_\mathbb{Q}$ is the colimit of the corresponding system of maps $(\pi_{[n]} : P_{[n]} \to \Delta_{0,[n]} \mid n \in \mathbb{N})$ where $\pi_{[n]} := \pi$ for every $n \in \mathbb{N}$. Hence, for any $\gamma \in \Delta_0$ there exists $n \in \mathbb{N}$ such that γ is the image of some $\gamma_n \in \Delta_{0,[n]}$, and

$$P_{\mathbb{Q},\gamma} = \operatorname*{colim}_{k\in\mathbb{N}} P_{[nk],k\gamma_n} = \operatorname*{colim}_{k\in\mathbb{N}} P_{k\gamma_n}.$$

However, we know that $P_{k\gamma_n}$ is free of rank one, generated by a unique element $g_{k\gamma_n}$; moreover, the transition maps $\mu_{P,n,k}$ send g_{γ_n} onto $g_{k\gamma_n}$, for every $k \in \mathbb{N}$ (remark 9.3.24). This implies that $P_{\mathbb{Q},\gamma}$ is generated by the image of g_{γ_n}, whence (i).

(ii): The flatness follows from (i). Since the inclusion map $P_0^{(0)} \to P^{(0)}$ is saturated, it is easily seen that the same holds for the inclusion $P_{\mathbb{Q},0} \to P_\mathbb{Q}$. Lastly, the sharpness of $P_{\mathbb{Q},0}$ is likewise deduced from the sharpness of $P_0^{(0)}$ (details left to the reader). □

9.3.39. Lemma 9.3.38(ii) implies that the pair $(A, \Delta_\mathbb{Q})$ fulfills axiom (MA2), and we have thus our sought auxiliary model K^+-algebra. Since $P_{\mathbb{Q},0}$ is sharp, the $P_{\mathbb{Q},0}$-module $P_{\mathbb{Q},\gamma}$ admits a unique generator g_γ, for every $\gamma \in \Delta_\mathbb{Q}$. The image $g_\gamma \otimes 1$ of g_γ in A is a generator of the direct summand A_γ, which is a free K^+-module of rank one; hence, there exists a unique $a_\gamma \in K$ such that $\gamma \otimes a_\gamma$ gets mapped to $g_\gamma \otimes 1$, under the isomorphism (9.3.37).

After choosing an order-preserving isomorphism (8.3.50), we may define a function

$$f_A : \Delta_\mathbb{Q} \to \mathbb{R} \quad : \quad \gamma \mapsto \log |a_\gamma|.$$

The inclusion $A_\gamma \cdot A_\delta \subset A_{\gamma+\delta}$ translates as the inequality

$$f_A(\gamma) + f_A(\delta) \geq f_A(\gamma + \delta) \qquad \text{for every } \gamma, \delta \in \Delta_\mathbb{Q}.$$

Likewise, the saturation condition of axiom (MA2) translates as the identity

$$f_A(n\gamma) = n \cdot f_A(\gamma) \qquad \text{for every } \gamma \in \Delta_\mathbb{Q} \text{ and every } n \in \mathbb{N}.$$

We also fix a (Banach) norm

$$\Delta_\mathbb{R}^{\mathrm{gp}} \to \mathbb{R} \qquad \gamma \mapsto \|\gamma\|.$$

We can then state the following

Lemma 9.3.40. *With the notation of* (9.3.39)*, we have :*

(i) *There exists a Δ_0^{gp}-rational subdivision Θ of the convex polyhedral cone $(\Delta_\mathbb{R}^{\mathrm{gp}}, \Delta_\mathbb{R})$, such that*

$$f_A(\gamma + \delta) = f_A(\gamma) + f_A(\delta) \qquad \textit{for every } \sigma \in \Theta \textit{ and every } \gamma, \delta \in \sigma \cap \Delta_\mathbb{Q}.$$

(ii) *Especially, the function f_A is of* Lipschitz type, i.e. *there exists a real constant $C_A > 0$ such that*

$$|f_A(\gamma) - f_A(\gamma')| \leq C_A \cdot \|\gamma - \gamma'\| \qquad \textit{for every } \gamma, \gamma' \in \Delta_\mathbb{Q}^{\mathrm{gp}}.$$

Proof. (Here, $\Delta_\mathbb{R}^{\mathrm{gp}} := \Delta_0^{\mathrm{gp}} \otimes_\mathbb{Z} \mathbb{R}$ and $\Delta_\mathbb{R} \subset \Delta_\mathbb{R}^{\mathrm{gp}}$ is the cone spanned by Δ_0.) Combining proposition 3.3.28 and lemma 3.3.33, we find a Δ_0^{gp}-rational subdivision Θ of $\Delta_\mathbb{R}$ such that

$$P_{\mathbb{Q},\gamma+\delta} = P_{\mathbb{Q},\gamma} + P_{\mathbb{Q},\delta} \qquad \text{for every } \sigma \in \Theta \text{ and every } \gamma, \delta \in \sigma \cap \Delta_\mathbb{Q}.$$

This means that $g_\gamma \cdot g_\delta = g_{\gamma+\delta}$ for every $\sigma \in \Theta$ and every $\gamma, \delta \in \sigma \cap \Delta_\mathbb{Q}$, whence (i).

(ii): Clearly, for every $\sigma \in \Theta$ we may find a constant C_σ such that the stated inequality holds – with C_A replaced by C_σ – for every $\gamma, \gamma' \in \sigma \cap \Delta_\mathbb{Q}$. It is easily seen that $C_A := \max(C_\sigma \mid \sigma \in \Theta)$ will do (details left to the reader). □

9.3.41. Fix $n \in \mathbb{N}$, pick a subdivision Θ of $(\Delta_{\mathbb{R}}^{\mathrm{gp}}, \Delta_{\mathbb{R}})$ as in lemma 9.3.40, and let $\Theta^s \subset \Theta$ be the subset of all $\sigma \in \Theta$ that span $\Delta_{\mathbb{R}}^{\mathrm{gp}}$. Also, for every $\gamma \in \Delta_{\mathbb{Q}}^{\mathrm{gp}}$, denote by $[\gamma]$ the class of γ in $\Delta_{\mathbb{Q}}^{\mathrm{gp}}/\Delta_n^{\mathrm{gp}}$. For any subset $\Sigma \subset \Delta_{\mathbb{R}}^{\mathrm{gp}}$ we let

$$A_\Sigma := A_{\Sigma \cap \Delta_{\mathbb{Q}}}$$

(where the right-hand side is defined as in (9.3.5)). In light of lemma 3.3.26 we obtain a decomposition of $B_n = A_{\Delta_n} = A_{[0]}$ as sum of K^+-subalgebras :

$$B_n = \sum_{\sigma \in \Theta^s} A_{[0] \cap \sigma}$$

and a corresponding decomposition of the B_n-module $A_{[\gamma]}$ as sum of $A_{[0]\cap\sigma}$-modules

$$A_{[\gamma]} = \sum_{\sigma \in \Theta^s} A_{[\gamma]\cap\sigma} \qquad \text{for every } \gamma \in \Delta_{\mathbb{Q}}^{\mathrm{gp}}.$$

Lemma 9.3.42. *With the notation of* (9.3.41) *and* (9.3.19)*, the following holds :*

(i) *There exists an integer $N \in \mathbb{N}$ independent of n such that, for every $\gamma \in \Delta_{\mathbb{Q}}^{\mathrm{gp}}$, the B_n-module $A_{[\gamma]}$ admits a system of generators of cardinality at most N.*
(ii) *For every $\gamma \in \Delta_{\mathbb{Q}}^{\mathrm{gp}}$, the coherent $\mathscr{O}_{Y_n}$-module $A_{[\gamma]}^{\sim}$ associated to $A_{[\gamma]}$ is φ_n-Cohen-Macaulay at every point of $Y_n \times_S \operatorname{Spec}\kappa$, and* $\operatorname{Supp} A_{[\gamma]}^{\sim} = Y_n$.

Proof. (i): According to proposition 3.3.22(ii), for every $\sigma \in \Theta^s$ and every $\gamma \in \Delta_{\mathbb{Q}}^{\mathrm{gp}}$, the subset $\Sigma_{\sigma,\gamma} := \Delta_n^{\mathrm{gp}} \cap (\sigma - \gamma)$ is a finitely generated $(\Delta_n^{\mathrm{gp}} \cap \sigma)$-module; hence, let us fix a finite system of generators $G_{\sigma,\gamma}$ for $\Sigma_{\sigma,\gamma}$. In light of lemma 9.3.40(i) we see that the finite set

$$G'_{\sigma,\gamma} := \{g_{\gamma+\delta} \otimes 1 \mid \delta \in G_{\sigma,\gamma}\} \subset A$$

generates the $A_{[0]\cap\sigma}$-module $A_{[\gamma]\cap\sigma}$. Consequently, the finite set $\bigcup_{\sigma\in\Theta^s} G'_{\sigma,\gamma}$ generates the B_n-module $A_{[\gamma]}$. On the other hand, since $\mathscr{S}_{\Delta_n^{\mathrm{gp}},\sigma}$ is a finitely generated Δ_n^{gp}-module (proposition 3.3.35(ii)) it is clear that the cardinality of $G_{\sigma,\gamma}$ is bounded by a constant N_σ that is independent of γ, and then $A_{[\gamma]}$ is generated by at most $N_n := \sum_{\sigma\in\Theta^s} N_\sigma$ elements. It remains to show that the estimate for N_n is independent of n. However, notice that, for every $\sigma \in \Theta^s$, the automorphism of $\Delta_{\mathbb{Q}}^{\mathrm{gp}}$ given by multiplication by p^n, induces a natural bijection

$$\mathscr{S}_{\Delta_n^{\mathrm{gp}},\sigma} \xrightarrow{\sim} \mathscr{S}_{\Delta_0^{\mathrm{gp}},\sigma}$$

that sends each Δ_n-module $\Delta_n^{\mathrm{gp}} \cap (\sigma - v)$ onto the Δ_0-module

$$\Delta_0^{\mathrm{gp}} \cap (\sigma - p^n v) = \Delta_0 \otimes_{\Delta_n} (\Delta_n^{\mathrm{gp}} \cap (\sigma - v))$$

(where the extension of scalars $\Delta_n \to \Delta_0$ is the isomorphism given by the rule : $\gamma \mapsto p^n\gamma$ for every $\gamma \in \Delta_n \cap \sigma$). Thus, we see that $N := N_0$ will already do.

(ii): Notice the natural isomorphisms of κ-algebras :

$$B_n \otimes_{K^+} \kappa \xrightarrow{\sim} P^{(n)} \otimes_{P_0^{(n)}} \kappa \qquad \text{for every } n \in \mathbb{N}. \tag{9.3.43}$$

Taking into account lemma 6.7.6, we deduce that φ_n is Cohen-Macaulay at every point of $Y_n \times_S \operatorname{Spec}\kappa$. It follows immediately that $A_{[\gamma]}^{\sim}$ is φ_n-Cohen Macaulay for every $\gamma \in \Delta_{\mathbb{Q}}^{\mathrm{gp}}$. Lastly, since $A_{[\gamma]}$ is a flat K^+-module, the support of $A_{[\gamma]}^{\sim}$ is the topological closure of the support of the $B_{n,K}$-module $A_{[\gamma],K}$. However, the latter is free (of rank one), hence its support is $Y_{n,K} := \operatorname{Spec} B_{n,K}$. Since B_n is a flat K^+-algebra, the topological closure of $Y_{n,K}$ in Y_n is Y_n, whence the contention. $\square$

9.3.44. Let us choose N as in lemma 9.3.42(i), and endow the set $\mathscr{M}_N(B_n^a)$ with the uniform structure defined in (8.3.72) (relative to the standard setup attached to K^+); we consider the mapping

$$\Delta_{\mathbb{Q}}^{\mathrm{gp}} \to \mathscr{M}_N(B_n^a) \qquad \gamma \mapsto A^a_{[\gamma]}.$$

Lemma 9.3.45. *Keep the notation of* (3.3.34) *and* (8.3.72)*, and let also $C_A > 0$ be the constant appearing in lemma* 9.3.40(ii)*. Then :*

$$(A^a_{[\gamma]}, A^a_{[\gamma']}) \in E_{C_A\cdot\|\gamma-\gamma'\|\cdot 2}$$

for every $\Sigma \in \mathscr{S}_{\Delta_n^{\mathrm{gp}},\Delta_{\mathbb{R}}}$ and every $\gamma, \gamma' \in \Omega(\Delta_{\mathbb{R}}, \Sigma) \cap \Delta_{\mathbb{Q}}^{\mathrm{gp}}$.

Proof. By inspecting the definitions, we get an isomorphism of B_n-modules

$$A_{[\gamma]} \xrightarrow{\sim} \bigoplus_{\delta\in\Sigma}\{\delta\otimes a \in K[\Delta_n^{\mathrm{gp}}] \mid \log|a| \geq f_A(\gamma+\delta)\} \tag{9.3.46}$$

for every $\Sigma \in \mathscr{S}_{\Delta_n^{\mathrm{gp}},\Delta_{\mathbb{R}}}$ and every $\gamma \in \Omega(\Delta_{\mathbb{R}}, \Sigma) \cap \Delta_{\mathbb{Q}}^{\mathrm{gp}}$. Hence, suppose more generally that Σ, Σ' are two elements of $\mathscr{S}_{\Delta_n^{\mathrm{gp}},\Delta_{\mathbb{R}}}$ such that $\Sigma \subset \Sigma'$, and let $\gamma \in \Omega(\Delta_{\mathbb{R}}, \Sigma) \cap \Delta_{\mathbb{Q}}^{\mathrm{gp}}$, $\gamma' \in \Omega(\Delta_{\mathbb{R}}, \Sigma') \cap \Delta_{\mathbb{Q}}^{\mathrm{gp}}$ be any two elements; from lemma 9.3.40(ii) we get, for any $b \in K^+$ with $\log|b| \geq C_A\|\gamma'-\gamma\|$, a B_n-linear map

$$\tau_{b,\gamma'-\gamma} : A_{[\gamma]} \to A_{[\gamma']}$$

which, under the identifications (9.3.46), corresponds to the map given by the rule : $\delta\otimes a \mapsto \delta\otimes ba$ for every $\delta \in \Sigma$ and every $a \in K$ such that $\log|a| \geq f(\gamma+\delta)$. In case $\Sigma = \Sigma'$, also $\tau_{b,\gamma-\gamma'}$ is well defined, and clearly

$$\tau_{b,\gamma'-\gamma}\circ\tau_{b,\gamma-\gamma'} = b^2\cdot\mathbf{1}_{A_{[\gamma']}} \qquad \tau_{b,\gamma-\gamma'}\circ\tau_{b,\gamma'-\gamma} = b^2\cdot\mathbf{1}_{A_{[\gamma]}}$$

whence the contention. $\square$

Remark 9.3.47. Let $\gamma_1, \gamma_2, \gamma'_1, \gamma'_2 \in \Delta_{\mathbb{Q}}^{\mathrm{gp}}$ be four elements, and set $\gamma_3 := \gamma_1+\gamma_2$, $\gamma'_3 := \gamma'_1+\gamma'_2$; suppose that

$$\Delta_n^{\mathrm{gp}} \cap (\Delta_{\mathbb{R}} - \gamma_i) \subset \Delta_n^{\mathrm{gp}} \cap (\Delta_{\mathbb{R}} - \gamma'_i) \qquad \text{for } i = 1,2,3.$$

With the notation of the proof of lemma 9.3.45, for every $b_1, b_2 \in K^+$ such that

$$\log|b_i| \geq C_A\|\gamma'_i - \gamma_i\| \qquad (i=1,2)$$

we obtain a commutative diagram of B_n-linear maps

$$\begin{tikzcd} A_{[\gamma_1]}\otimes_{B_n} A_{[\gamma_2]} \arrow[r] \arrow[d, "\tau_{b_1,\gamma'_1-\gamma_1}\otimes\tau_{b_2,\gamma'_2-\gamma_2}"'] & A_{[\gamma_3]} \arrow[d, "\tau_{b_1b_2,\gamma'_3-\gamma_3}"] \\ A_{[\gamma'_1]}\otimes_{B_n} A_{[\gamma'_2]} \arrow[r] & A_{[\gamma'_3]} \end{tikzcd}$$

whose horizontal arrows are the restrictions of the multiplication map of the B_n-algebra B : the detailed verification shall be left as an exercise for the reader.

Theorem 9.3.48. *In the situation of* (9.3.35)*, the K^+-algebra B^{sh} is ind-measurable.*

Proof. First, notice that, for every $k \geq 0$, the inclusion map $B_n \to B_{n+k}$ induces a radicial morphism $B_n\otimes_{K^+}\kappa \to B_{n+k}\otimes_{K^+}\kappa$, and therefore also an isomorphism of B_{n+k}-algebras

$$B_n^{\mathrm{sh}}\otimes_{B_n} B_{n+k} \xrightarrow{\sim} B_{n+k}^{\mathrm{sh}} \tag{9.3.49}$$

([33, Ch.IV, Prop.18.8.10]). Notice as well, that B_n^{sh} is a measurable K^+-algebra (see definition 8.3.3(ii)), hence we have a well defined normalized length function λ_n on isomorphism classes of B_n^{sh}-modules, characterized as in theorem 8.3.30. Thus, our task is to exhibit a sequence $(d_n \mid n \in \mathbb{N})$ of normalizing factors fulfilling conditions (a) and (b) of definition 8.3.51, for the directed system $(B_n^{\mathrm{sh}} \mid n \in \mathbb{N})$, whose colimit is B^{sh}. To this aim, let M be any object of $B_n^{\mathrm{sh}}\text{-}\mathbf{Mod}_{\mathrm{coh},\{s\}}$; in view of (9.3.49), we need to estimate

$$\lambda_{k+n}(B_{k+n} \otimes_{B_n} M) = \lambda_{k+n}\left(\bigoplus_{[\gamma]\in\Delta^{\mathrm{gp}}_{k+n}/\Delta^{\mathrm{gp}}_n} A_{[\gamma]} \otimes_{B_n} M\right).$$

To this aim, let us introduce the function

$$l_M : \Delta^{\mathrm{gp}}_{\mathbb{Q}} \to \mathbb{R} \qquad \gamma \mapsto \lambda_n(A_{[\gamma]} \otimes_{B_n} M).$$

Claim 9.3.50. Let Σ be as in lemma 9.3.45, set $J := \mathrm{Ann}_{B_0^{\mathrm{sh}}}(M/\mathfrak{m}M)$, and suppose that M admits a generating set of cardinality k. Then the restriction of l_M to $\Omega(\Delta_{\mathbb{R}}, \Sigma) \cap \Delta^{\mathrm{gp}}_{\mathbb{Q}}$ is *of Lipschitz type, i.e.* there exists a real constant $C'_A > 0$ independent of n, Σ and M such that

$$|l_M(\gamma) - l_M(\gamma')| \leq C'_A \cdot k \cdot \mathrm{length}_{B_n^{\mathrm{sh}}}(B_n^{\mathrm{sh}}/JB_n^{\mathrm{sh}}) \cdot \|\gamma - \gamma'\|$$

for every $\gamma, \gamma' \in \Omega(\Delta_{\mathbb{R}}, \Sigma) \cap \Delta^{\mathrm{gp}}_{\mathbb{Q}}$.

Proof of the claim. Let $I \subset B_0^{\mathrm{sh}}$ be any finitely generated $\mathfrak{m}_{B_n^{\mathrm{sh}}}$-primary ideal contained in the annihilator of M. Also, let N' be the cardinality of a finite set of generators for the B_n^{sh}-module M, and N the constant provided by lemma 9.3.42(i); clearly $A_{[\gamma]} \otimes_{B_n} M$ admits a system of generators of cardinality $\leq NN'$, for every $\gamma \in \Delta^{\mathrm{gp}}_{\mathbb{Q}}$. Furthermore, an argument as in the proof of [36, Lemma 2.3.7(iv)] shows that the induced mapping

$$\mathscr{M}_N(B_n) \to \mathscr{M}_{NN'}(B_n^{\mathrm{sh}}/IB_n^{\mathrm{sh}}) \qquad L \mapsto L \otimes_{B_0} M$$

is of Lipschitz type; more precisely, it sends the entourage E_r into E_{2r}, for every $r \in \mathbb{R}_{>0}$ (details left to the reader). Then the claim follows by combining lemmata 8.3.73 and 9.3.45. $\diamondsuit$

From proposition 3.3.35(iv), we see that, for every $\Sigma \in \mathscr{S}_{\Delta_n^{\mathrm{gp}},\Delta_{\mathbb{R}}}$, the subset $\Omega(\Delta_{\mathbb{R}}, \Sigma) \cap \Delta^{\mathrm{gp}}_{\mathbb{Q}}$ is dense in $\Omega(\Delta_{\mathbb{R}}, \Sigma)$; taking into account claim 9.3.50, it follows that l_M extends (uniquely) to a function $\Delta^{\mathrm{gp}}_{\mathbb{R}} \to \mathbb{R}$, whose restriction to every $\Omega(\Delta_{\mathbb{R}}, \Sigma)$ is continuous and still satisfies the same Lipschitz type estimate. Notice that l_M descends to a function on the torus

$$\bar{l}_M : T_n := \Delta^{\mathrm{gp}}_{\mathbb{R}}/\Delta^{\mathrm{gp}}_n \to \mathbb{R}.$$

Hence, set $r := \mathrm{rk}_{\mathbb{Z}}\Delta^{\mathrm{gp}}_0$, let us fix a basis $e_1, \dots, e_r$ for the free $\mathbb{Z}$-module Δ^{gp}_0, and define

$$\Omega_n := \left\{\sum_{i=1}^r t_i e_i \mid -\frac{1}{2p^n} \leq t_i < \frac{1}{2p^n} \;\text{ for } i = 1, \dots, r\right\}$$

so $\Omega_n \subset \Delta^{\mathrm{gp}}_{\mathbb{R}}$ is a *fundamental domain* for the lattice Δ^{gp}_n; in view of theorem 8.3.30(ii.b), we reach the following identity :

$$\lambda_{k+n}(B_{k+n} \otimes_{B_n} M) = \frac{1}{[\kappa(B^{\mathrm{sh}}_{k+n}) : \kappa(B^{\mathrm{sh}}_n)]} \cdot \sum_{\gamma\in\Delta^{\mathrm{gp}}_{k+n}\cap\Omega_n} l_M(\gamma).$$

Now, set

$$d_n := p^{nr} \cdot [\kappa(B^{\mathrm{sh}}_n) : \kappa(B^{\mathrm{sh}}_0)]^{-1} \qquad \text{for every } n \in \mathbb{N}. \tag{9.3.51}$$

We claim that $(d_n \mid n \in \mathbb{N})$ is a suitable sequence of normalizing factors for B^{sh}. Indeed, recall that $\Omega(\Delta_{\mathbb{R}}, \Sigma)$ is linearly constructible for every $\Sigma \in \mathscr{S}_{\Delta_0^{\mathrm{gp}},\Delta_{\mathbb{R}}}$ (proposition 3.3.35(iii)), especially, the bounded function l_M is continuous outside a subset of $\Delta^{\mathrm{gp}}_{\mathbb{R}}$ of (Riemann) measure zero, hence it is integrable on every bounded measurable subset of $\Delta_{\mathbb{R}}$. It follows that $\bar{l}_M$ is integrable relative to the invariant measure $d\mu_n$ on T_n of total volume equal to 1; lastly, a simple inspection yields

$$\lambda(B \otimes_{B_n} M) := \lim_{k \to +\infty} d_{k+n}^{-1} \cdot \lambda_{k+n}(B_{k+n} \otimes_{B_n} M) = d_n^{-1} \int_{T_n} \bar{l}_M d\mu_n$$

which shows that condition (a) of definition 8.3.51 holds for this choice of normalizing factors.

In order to check condition (b), set $\Delta_{\mathbb{R}}(\rho) := \{v \in \Delta_{\mathbb{R}} \mid \|v\| \leq \rho\}$ for every $\rho > 0$. Notice that the automorphism of $\Delta_{\mathbb{R}}^{\mathrm{gp}}$ given by multiplication by p^n restricts to a bijection

$$\Omega(\Delta_{\mathbb{R}}, \Delta_n) \xrightarrow{\sim} \Omega(\Delta_{\mathbb{R}}, \Delta_0) \qquad \text{for every } n \in \mathbb{N}.$$

Then, by claim 3.3.40, we see that there exists ρ_0 such that

$$\Delta_{\mathbb{R}}(p^{-n}\rho_0) \subset \Omega_n \cap \Omega(\Delta_{\mathbb{R}}, \Delta_n) \qquad \text{for all } n \in \mathbb{N}. \tag{9.3.52}$$

Especially, for every $\rho \leq \rho_0$ we may regard $\Delta_{\mathbb{R}}(p^{-n}\rho)$ as a measurable subset of T_n, whose measure $\mathrm{Vol}(\rho)$ is strictly positive and independent of n.

Claim 9.3.53. Let $I \subset B_0^{\mathrm{sh}}$ be any finitely generated $\mathfrak{m}_{B_0^{\mathrm{sh}}}$-primary ideal. Then there exists a real constant $C_I > 0$ such that

$$d_n^{-1} \cdot \mathrm{length}_{B_n^{\mathrm{sh}}}(B_n^{\mathrm{sh}}/(I + \mathfrak{m})B_n^{\mathrm{sh}}) \leq C_I \qquad \text{for every } n \in \mathbb{N}.$$

Proof of the claim. On the one hand, we may write B_n^{sh} as the direct sum of the B_0^{sh}-modules $A_{[\gamma] \cap \Delta_{\mathbb{Q}}} \otimes_{B_0} B_0^{\mathrm{sh}}$, for γ ranging over the elements of $\Delta_n^{\mathrm{gp}}/\Delta_0^{\mathrm{gp}}$. There are p^{rn} such direct summands, and each of them admits a generating system of cardinality $\leq N$, where N is the constant provided by lemma 9.3.42(i). It follows that

$$\mathrm{length}_{B_0^{\mathrm{sh}}}(B_n^{\mathrm{sh}}/(I + \mathfrak{m})B_n^{\mathrm{sh}}) \leq Np^{rn} \cdot \mathrm{length}_{B_0^{\mathrm{sh}}}(B_0^{\mathrm{sh}}/(I + \mathfrak{m}B_0^{\mathrm{sh}})).$$

On the other hand, say that $l := \mathrm{length}_{B_n^{\mathrm{sh}}}(B_n^{\mathrm{sh}}/(I+\mathfrak{m})B_n^{\mathrm{sh}})$; this means that may find a filtration of $B_n^{\mathrm{sh}}/(I + \mathfrak{m})B_n^{\mathrm{sh}}$ of length l, consisting of B_n^{sh}-submodules, whose graded subquotients are all isomorphic to $\kappa(B_n^{\mathrm{sh}})$. Given such a filtration, we easily obtain a filtration of $B_n^{\mathrm{sh}}/(I+\mathfrak{m})B_n^{\mathrm{sh}}$ of length $l \cdot [\kappa(B_n^{\mathrm{sh}}) : \kappa(B_0^{\mathrm{sh}})]$, consisting of B_0^{sh}-submodules, whose graded subquotients are all isomorphic to $\kappa(B_0^{\mathrm{sh}})$. Therefore

$$l \cdot [\kappa(B_n^{\mathrm{sh}}) : \kappa(B_0^{\mathrm{sh}})] = \mathrm{length}_{B_0^{\mathrm{sh}}}(B_n^{\mathrm{sh}}/(I + \mathfrak{m})B_n^{\mathrm{sh}})$$

whence the claim. $\Diamond$

Now, fix $\varepsilon > 0$, and suppose there is given a finitely generated $\mathfrak{m}_{B_0^{\mathrm{sh}}}$-primary ideal $I \subset B_0^{\mathrm{sh}}$, and a surjection of finitely presented $B_n^{\mathrm{sh}}/IB_n^{\mathrm{sh}}$-modules $M \to M'$, generated by k elements, such that

$$d_n^{-1} \cdot (\lambda_n(M) - \lambda_n(M')) > \varepsilon.$$

Set $C(I, k) := 2kC'_A C_I$, where C'_A and C_I are as in claims 9.3.50 and 9.3.53; since

$$l_M(0) - l_{M'}(0) = \lambda_n(M) - \lambda_n(M')$$

we may estimate :

$$\begin{aligned} |\lambda(B^{\mathrm{sh}} \otimes_{B_n^{\mathrm{sh}}} M) - \lambda(B^{\mathrm{sh}} \otimes_{B_n^{\mathrm{sh}}} M')| &\geq d_n^{-1} \cdot \int_{\Delta_{\mathbb{R}}(p^{-n}\rho)} (\bar{l}_M - \bar{l}_{M'}) d\mu_n \\ &\geq \mathrm{Vol}(\rho) \cdot (d_n^{-1}(l_M(0) - l_{M'}(0)) - C(I, k) \cdot p^{-n}\rho) \\ &\geq \mathrm{Vol}(\rho) \cdot (\varepsilon - C(I, k) \cdot \rho) \end{aligned}$$

for every $\rho \leq \rho_0$. Therefore, if we set

$$\rho(k, \varepsilon, I) := \min\{\rho_0, 2^{-1}C(I, k)^{-1}\varepsilon\}$$

it is easily seen that the sought condition (b) holds with $\delta(k, \varepsilon, I) := 2^{-1} \cdot \mathrm{Vol}(\rho(k, \varepsilon, I)) \cdot \varepsilon$. $\square$

9.4. **Almost purity : the case of model algebras.** In this section we prove the almost purity theorem for (henselizations of) model algebras. The general outline of the proof is the same as that for smooth K^+-algebras; however, all of the various ingredients have to be tweaked a little to make them work with model algebras. We keep the notation of (9.3.35) throughout this section.

9.4.1. Let M be an almost finitely presented $(B^{\mathrm{sh}})^a$-module; recall (lemma 8.3.67(ii)) that a submodule $M' \subset M$ supported at $s(B^{\mathrm{sh}})$ vanishes if and only if its normalized length vanishes. In the proof of almost purity for model algebras, we shall encounter certain modules that are not necessarily almost finitely presented; hence, our first task is to extend this vanishing criterion to a suitable class of B^{sh}-modules. Since the same idea shall be applied again in section 9.6, we shall introduce a convenient framework – more general than what is required for our present needs – wherein the method can be applied.

Thus, suppose $R_\bullet := (R_n \mid n \in \mathbb{N})$ is a system of rings, whose transition maps are injective ring homomorphisms, and such that R_m is a finitely presented R_n-module, for every $m \geq n$; denote by R the colimit of the system $R_\bullet$. Fix an integer $n \in \mathbb{N}$, and let $\mathscr{C}_n$ be a full subcategory of the category $R_n\text{-}\mathbf{Mod}_{\mathrm{fp}}$ of finitely presented R_n-modules, such that

- $\mathrm{Ob}(\mathscr{C}_n)$ contains B_m, for every $m \geq n$
- $\mathscr{C}_n$ is closed under finite direct sums.

Let also

$$T_n : \mathscr{C}_n \to R_n\text{-}\mathbf{Mod}_{\mathrm{fp}}$$

be an R_n*-linear functor*. The latter condition means that the induced map

$$\mathrm{Hom}_{\mathscr{C}_n}(M, N) \to \mathrm{Hom}_{R_n}(T_n M, T_n N)$$

is R_n-linear, for all objects M, N of $\mathscr{C}_n$. For every $b \in R_n$, and every R_n-module M, let $b_M := b \cdot \mathbf{1}_M$; since $T_n \mathbf{1}_M = \mathbf{1}_{T_n M}$, we deduce that

(9.4.2) $$T_n(b_M) = b_{T_n M} \qquad \text{for every } b \in R_n \text{ and } M \in \mathrm{Ob}(\mathscr{C}_n).$$

It also follows that T_n commutes with finite direct sums; recall the standard argument : if $M = M_1 \oplus \cdots \oplus M_k$, the natural injections $e_i : M_i \to M$ and surjections $\pi_j : M \to M_j$ satisfy the identities

$$\sum_{i=1}^{k} e_i \circ \pi_i = \mathbf{1}_M \qquad \pi_i \circ e_j = \begin{cases} \mathbf{1}_{M_i} & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

Then the maps $T_n(e_i)$, $T_n(\pi_j)$ satisfy the corresponding identites, and the latter yield a decomposition $T_n M = T_n M_1 \oplus \cdots \oplus T_n M_k$. Moreover, for every object M of $\mathscr{C}_n$ we have a natural R_n-linear map

$$\psi_M : M \otimes_{R_n} T_n R_n \to T_n M$$

obtained as follows. Any $m \in M$ induces a R_n-linear map $\mu_m : R_n \to M$ by the rule : $b \mapsto bm$ for every $b \in R_n$; whence a pairing

$$M \times T_n R_n \to T_n M \quad : \quad (m, x) \mapsto T_n \mu_m(x) \qquad \text{for every } m \in M \text{ and } x \in T_n R_n.$$

The R_n-linearity of T_n easily implies that the latter pairing is R_n-bilinear, so it factors through a unique map ψ_M as sought. Furthermore, a simple inspection shows that any morphism $f : M \to N$ in $\mathscr{C}_n$ induces a commutative diagram :

(9.4.3) $$\begin{tikzcd} M \otimes_{R_n} T_n R_n \arrow[r, "\psi_M"] \arrow[d, "f \otimes_{R_n} T_n R_n"'] & T_n M \arrow[d, "T_n f"] \\ N \otimes_{R_n} T_n R_n \arrow[r, "\psi_N"] & T_n N. \end{tikzcd}$$

9.4.4. For any integer $m \geq n$, denote by $\varphi_{n,m} : R_n \to R_m$ the transition map; let now M be a finitely presented R_m-module, such that $\varphi^*_{n,m}M$ is an object of $\mathscr{C}_n$. Then we may endow $T_n(\varphi^*_{n,m}M)$ with a natural R_m-module structure; namely, we have an induced R_n-linear map

$$R_m \to \mathrm{End}_{R_n}(T_n(\varphi^*_{n,m}M)) \quad : \quad b \mapsto T_n(b \cdot \mathbf{1}_M).$$

Denote by $T_m M$ the resulting R_m-module; taking (9.4.2) into account, it is easily seen that

$$\varphi^*_{n,m}T_m M = T_n(\varphi^*_{n,m}M).$$

Since both $T_n(\varphi^*_{n,m}M)$ and R_m are finitely presented R_n-modules, we conclude that $T_m M$ is a finitely presented R_m-module (details left to the reader). Let $\mathscr{C}_m$ be the full subcategory of R_m-$\mathbf{Mod}_{\mathrm{fp}}$ consisting of those R_m-modules M such that $\varphi^*_{n,m}M$ lies in $\mathscr{C}_n$. Summing up, for every $m \geq n$ we have obtained an R_m-linear functor

$$T_m : \mathscr{C}_m \to R_m\text{-}\mathbf{Mod}_{\mathrm{fp}}$$

and if $p \geq m \geq n$, we have an essentially commutative diagram of functors :

$$\begin{tikzcd} \mathscr{C}_p \arrow[r, "T_p"] \arrow[d] & R_p\text{-}\mathbf{Mod}_{\mathrm{fp}} \arrow[d] \\ \mathscr{C}_m \arrow[r, "T_m"] & R_m\text{-}\mathbf{Mod}_{\mathrm{fp}} \end{tikzcd}$$

whose vertical arrows are the restriction of scalars. Furthermore, a simple inspection shows that

(9.4.5) $\qquad \psi_{R_m}(b \otimes b') = b \cdot T_n\varphi_{n,m}(b') \qquad$ for every $m \geq n$, every $b \in B_m$ and $b' \in B_n$.

Definition 9.4.6. From the discussion of (9.4.1), it follows easily that

$$TR := \operatorname*{colim}_{m \geq n} T_n R_m = \operatorname*{colim}_{m \geq n} T_m R_m$$

carries a natural R-module structure. With this notation :

(i) We say that TR is the *R-module presented by T_n*.
(ii) Let M be an R-module; we say that M is *presentable* if there exists $n \in \mathbb{N}$ and a functor T_n as in (9.4.1), such that M is isomorphic to the R-module presented by T_n.
(iii) Let $(V, \mathfrak{m})$ be any basic setup, and suppose that R is a V-algebra. Let M be an R^a-module (for the almost structure given by $(V, \mathfrak{m})$). We say that M is *presentable*, if there exists a presentable R-module N with an isomorphism $M \xrightarrow{\sim} N^a$ of R^a-modules.
(iv) In the situation of (iii), denote by $\mathscr{P}(R^a) \subset \mathscr{M}(R^a)$ the subset of isomorphism classes of all presentable R^a-modules (notation of [36, Def.2.3.1(ii)]), and $\overline{\mathscr{P}}(R^a)$ the topological closure of $\mathscr{P}(R^a)$ in $\mathscr{M}(R^a)$. Let M be an R^a-module. We say that M is *almost presentable* if the isomorphism class of M lies in $\overline{\mathscr{P}}(R^a)$.

Example 9.4.7. In this section, we shall be concerned exclusively with the following special case of the general situation contemplated in (9.4.1). For any fixed integer $n \in \mathbb{N}$, we shall take $\mathscr{C}_n$ to be the smallest full subcategory of B^{sh}_n-$\mathbf{Mod}_{\mathrm{coh}}$ (notation of definition 4.3.26) such that :

- $\mathscr{C}_n$ contains the B^{sh}_n-module $A^{\mathrm{sh}}_{[\gamma]} := A_{[\gamma]} \otimes_{B_n} B^{\mathrm{sh}}_n$, for every $\gamma \in \Delta^{\mathrm{gp}}$ (notation of (9.3.41))
- $\mathscr{C}_n$ is closed under finite direct sums.

Especially, $\mathscr{C}_n$ contains B^{sh}_m, for every $m \geq n$, as required, and the notion of presentable B^{sh}-module is well defined. Moreover, we shall consider exclusively the almost structure arising from the standard setup attached to K^+ (see [36, §6.1.15]), so the presentable and almost presentable $(B^{\mathrm{sh}})^a$-modules are defined for this structure.

Remark 9.4.8. With the notation of definition 9.4.6, we have :

(i) Clearly, TR is also the R-module presented by T_m, for every $m \geq n$.

(ii) Every finitely presented R-module M is presentable. Indeed, we may find $n \in \mathbb{N}$ and a finitely presented R_n-module M_n with an isomorphism of R-modules $M \xrightarrow{\sim} R \otimes_{R_n} M_n$. Then pick any full subcategory $\mathscr{C}_n$ of $R_n\text{-}\mathbf{Mod}_{\mathrm{fp}}$ as in (9.4.1), and let T_n be the functor given by the rule : $N \mapsto M_n \otimes_{R_n} N$ for every $N \in \mathrm{Ob}(\mathscr{C}_n)$. Clearly $TR = M$, whence the contention.

(iii) It follows immediately from (ii) that, in the situation of definition 9.4.6(iii,iv) every almost finitely presented R^a-module is almost presentable.

(iv) In the special case of example 9.4.7, suppose furthermore that $\Delta_0 \simeq \mathbb{Z}^{\oplus r} \oplus \mathbb{N}^{\oplus s}$ for some $r, s \geq 0$. In this case, we claim that the converse of (iii) holds : every almost presentable $(B^{\mathrm{sh}})^a$-module is almost finitely presented. Indeed, suppose that M is presented by the B^{sh}_n-linear functor T_n; it suffices to show that M^a is an almost finitely presented $(B^{\mathrm{sh}})^a$-module. However, from claim 9.3.32, we see that, for every $m \geq 0$, there exists a free B^{sh}_n-module L of finite rank, and an injective B^{sh}_n-linear map $L \to B^{\mathrm{sh}}_{n+m}$ whose cokernel is annihilated by the constant $c^{\varepsilon(n,m)}$ of (9.3.33). Since ψ_L is clearly an isomorphism, it follows easily that both kernel and cokernel of $\psi_{B^{\mathrm{sh}}_{n+m}}$ are annihilated by $c^{\varepsilon(n,m)}$. Let

$$\psi_{B^{\mathrm{sh}}} : B^{\mathrm{sh}} \otimes_{B^{\mathrm{sh}}_n} T_n B^{\mathrm{sh}}_n \to TB^{\mathrm{sh}} = M$$

be the colimit of the system of maps $(\psi_{B^{\mathrm{sh}}_{n+m}} \mid m \geq 0)$; it follows that both kernel and cokernel of $\psi_{B^{\mathrm{sh}}}$ are annihilated by c^{2/p^n}. Since n is arbitrary, the claim follows.

(v) Whereas the tensor product of two almost finitely presented R^a-modules is still almost finitely presented, it is not clear whether the tensor product of two almost presentable R^a-modules is again almost presentable (not even in the special case of example 9.4.7). This deficiency will force us to take a rather long detour in the final step of the proof of almost purity. On the other hand, the problem obviously disappears if we are in the situation contemplated in (iv) : see remark 9.4.23.

(vi) Likewise, suppose that $f : M_1 \to M_2$ is a R^a-linear morphism, with M_1 and M_2 almost presentable; then it is not clear whether $\mathrm{Coker}\, f$ and $\mathrm{Ker}\, f$ are almost presentable.

Lemma 9.4.9. *Let M be an almost presentable $(B^{\mathrm{sh}})^a$-module. Then there exists $b \in K^+ \setminus \{0\}$ such that*

$$\mathrm{Ann}_M(b) = \mathrm{Ann}_M(c) \qquad \textit{for every } c \in K^+ b \setminus \{0\}.$$

Proof. For any $n \in \mathbb{N}$, let $\Omega_n \subset \Delta^{\mathrm{gp}}_{\mathbb{R}}$ be a fundamental domain for Δ^{gp}_n, as in the proof of theorem 9.3.48. By assumption, for every $b \in \mathfrak{m}$ there exists a presentable B^{sh}-module N, and a morphism $M \to N^a$ whose kernel is annihilated by b. We are then easily reduced to showing the corresponding assertion for N^a, so we may assume that M is presentable, say

$$M = (TB^{\mathrm{sh}})^a = \bigoplus_{[\gamma] \in \Delta^{\mathrm{gp}}/\Delta^{\mathrm{gp}}_n} (T_n A^{\mathrm{sh}}_{[\gamma]})^a = \bigoplus_{\gamma \in \Omega_n \cap \Delta^{\mathrm{gp}}} (T_n A^{\mathrm{sh}}_{[\gamma]})^a$$

for some $n \in \mathbb{N}$ and some functor T_n as in (9.4.1). For every $\Sigma \in \mathscr{S}_{\Delta^{\mathrm{gp}}_n, \Delta_{\mathbb{R}}}(\Omega_n)$ such that $\Omega_n \cap \Omega(\Delta_{\mathbb{R}}, \Sigma) \cap \Delta^{\mathrm{gp}} \neq \varnothing$, fix an element γ_Σ in the latter subset (notation of (3.3.34)). Notice that the norm $\|\cdot\|$ is bounded on Ω_n; by lemma 9.3.45, it follows that we may find $b \in K^+ \setminus \{0\}$ such that the following holds. For every $\gamma \in \Omega_n \cap \Delta^{\mathrm{gp}}$ there exists $\Sigma \in \mathscr{S}_{\Delta^{\mathrm{gp}}_n, \Delta_{\mathbb{R}}}(\Omega_n)$ with

$$(A^a_{[\gamma]}, A^a_{[\gamma_\Sigma]}) \in E_{\log|b|}.$$

Especially, there exists a B^{sh}_n-linear map $A^{\mathrm{sh}}_{[\gamma]} \to A^{\mathrm{sh}}_{[\gamma_\Sigma]}$ whose kernel is annihilated by b. Then, since $\mathscr{S}_{\Delta^{\mathrm{gp}}_n, \Delta_{\mathbb{R}}}(\Omega_n)$ is a finite set, we are reduced to showing the corresponding vanishing assertion for the torsion in the B^{sh}_n-modules $T_n A^{\mathrm{sh}}_{[\gamma_\Sigma]}$. However, the latter are finitely presented B^{sh}_n-modules, so the contention is easily deduced from corollary 5.7.24(ii). $\square$

Proposition 9.4.10. *Let M be an almost presentable $(B^{\mathrm{sh}})^a$-module, $N \subset M$ a submodule supported at $s(B^{\mathrm{sh}})$, and suppose that $\lambda(N) = 0$. Then $N = 0$.*

Proof. We have to show that $ax = 0$ for every $a \in \mathfrak{m}$ and every $x \in N_!$. However, let $b \in \mathfrak{m}$ such that $a \in b^2K^+$; by assumption, there exists $n \in \mathbb{N}$ and a B_n^{sh}-linear functor T_n as in (9.4.1), with a $(B^{\mathrm{sh}})^a$-linear morphism $M \to (TB^{\mathrm{sh}})^a$ whose kernel is annihilated by b. Let $h : M_! \to TB^{\mathrm{sh}}$ be the induced B^{sh}-linear map; it then suffices to show that $h(x)$ is annihilated by b. We may then assume from start that $M = (TB^{\mathrm{sh}})^a$, $N = (B^{\mathrm{sh}}x)^a$ for some $x \in TB^{\mathrm{sh}}$, and $\lambda(B^{\mathrm{sh}}x) = 0$. After replacing n by a possibly larger integer, we may also assume that $x \in T_nB_n^{\mathrm{sh}}$ (see remark 9.4.8(i)). According to lemma 8.3.57(i), we have

$$\lim_{k\to\infty} d_{n+k}^{-1} \cdot \lambda_{n+k}(B_{n+k}^{\mathrm{sh}}x) = 0. \tag{9.4.11}$$

For every $[\gamma] \in \Delta_{n+k}^{\mathrm{gp}}/\Delta_n^{\mathrm{gp}}$, set $A_{[\gamma]}^{\mathrm{sh}}x := T_nA_{[\gamma]}^{\mathrm{sh}} \cap B_{n+k}^{\mathrm{sh}}x$. Clearly

$$B_{n+k}^{\mathrm{sh}}x = \bigoplus_{[\gamma]\in\Delta_{n+k}^{\mathrm{gp}}/\Delta_n^{\mathrm{gp}}} A_{[\gamma]}^{\mathrm{sh}}x.$$

For every $\rho \in \mathbb{R}_{>0}$, define $\Delta_{\mathbb{R}}(\rho)$ as in the proof of theorem 9.3.48, and pick $\rho_0 > 0$ such that (9.3.52) holds. By lemma 9.3.45, there exists $b \in K^+$ such that $(A_{[\gamma]}, B_n) \in E_{|b|}$ for every $\gamma \in \Delta_{\mathbb{R}}(p^{-n}\rho_0) \cap \Delta^{\mathrm{gp}}$. It follows easily that, for every $\gamma \in \Delta_{\mathbb{R}}(p^{-n}\rho_0) \cap \Delta^{\mathrm{gp}}$, we may find B_n^{sh}-linear maps

$$A_{[\gamma]}^{\mathrm{sh}} \underset{g}{\overset{f}{\rightleftarrows}} B_n^{\mathrm{sh}} \qquad \text{such that} \quad g \circ f = b \cdot \mathbf{1}_{A_{[\gamma]}^{\mathrm{sh}}} \quad \text{and} \quad f \circ g = b \cdot \mathbf{1}_{B_n^{\mathrm{sh}}}.$$

Say that $\gamma \in \Delta_m^{\mathrm{gp}}$ for some $m \geq n$; in view of (9.4.3), we get a commutative diagram :

$$\begin{tikzcd}
B_m^{\mathrm{sh}} \otimes_{B_n^{\mathrm{sh}}} T_nB_m^{\mathrm{sh}} \arrow[d, "\psi_{B_m^{\mathrm{sh}}}"'] & A_{[\gamma]}^{\mathrm{sh}} \otimes_{B_n^{\mathrm{sh}}} T_nB_n^{\mathrm{sh}} \arrow[l, "j_{[\gamma]}"'] \arrow[d, "\psi_{A_{[\gamma]}^{\mathrm{sh}}}"'] \arrow[r, shift left, "f\otimes T_nB_n^{\mathrm{sh}}"] & T_nB_n^{\mathrm{sh}} \arrow[l, shift left, "g\otimes T_nB_n^{\mathrm{sh}}"] \arrow[d, equal] \\
T_nB_m^{\mathrm{sh}} & T_nA_{[\gamma]}^{\mathrm{sh}} \arrow[l, "T_nj_{[\gamma]}"'] \arrow[r, shift left, "T_nf"] & T_nB_n^{\mathrm{sh}} \arrow[l, shift left, "T_ng"]
\end{tikzcd}$$

where $j_{[\gamma]} : A_{[\gamma]}^{\mathrm{sh}} \to B_m^{\mathrm{sh}}$ is the inclusion map. From this and from (9.4.5), a simple inspection shows that T_nf and T_ng restrict to B_n^{sh}-linear maps

$$A_{[\gamma]}^{\mathrm{sh}}x \underset{g'}{\overset{f'}{\rightleftarrows}} B_n^{\mathrm{sh}}x$$

whose compositions both ways are again $b \cdot \mathbf{1}$ (details left to the reader). Thus,

$$\lambda_n(bB_n^{\mathrm{sh}}x) \leq \lambda_n(f'A_{[\gamma]}^{\mathrm{sh}}x) \leq \lambda_n(A_{[\gamma]}^{\mathrm{sh}}x) \qquad \text{for every } \gamma \in \Delta_{\mathbb{R}}(p^{-n}\rho_0) \cap \Delta^{\mathrm{gp}}.$$

If $r := \mathrm{rk}_{\mathbb{Z}}\Delta_n^{\mathrm{gp}}$, we deduce the lower bound

$$d_{n+k}^{-1} \cdot \lambda_{n+k}(B_{n+k}^{\mathrm{sh}}x) \geq d_n^{-1}p^{-kr} \cdot \lambda_n(bB_n^{\mathrm{sh}}x) \cdot c_{n+k}$$

where c_{n+k} is the cardinality of $\Delta_{\mathbb{R}}(p^{-n}\rho_0) \cap \Delta_{n+k}^{\mathrm{gp}}$. Taking the limit for $k \to \infty$, we obtain

$$\lim_{k\to\infty} d_{n+k} \cdot \lambda_{n+k}(B_{n+k}^{\mathrm{sh}}) \geq d_n^{-1} \cdot \lambda_n(B_n^{\mathrm{sh}}bx) \cdot \mathrm{Vol}(\rho_0)$$

where $\mathrm{Vol}(\rho_0)$ is the measure of $\Delta(\rho_0)$ (see the proof of theorem 9.3.48). Comparing with (9.4.11), we conclude that $\lambda_n(B_n^{\mathrm{sh}}bx) = 0$. From theorem 8.3.62(iii) it follows that $bx = 0$, as required. $\square$

Proposition 9.4.12. *Every almost presentable $(B^{\mathrm{sh}})^a$-module supported at $s(B^{\mathrm{sh}})$ has almost finite length.*

Proof. Let M be such a module, and $b \in \mathfrak{m}$ any element; we have to show that $\lambda(bM) \in \mathbb{R}$. Pick $a \in \mathfrak{m}$ such that $b \in K^+a^2$; by definition, we may find a presentable $(B^{\mathrm{sh}})^a$-module M' with $(B^{\mathrm{sh}})^a$-linear maps

$$M' \underset{g}{\overset{f}{\rightleftarrows}} M \qquad \text{such that} \quad g \circ f = a \cdot \mathbf{1}_{M'} \quad \text{and} \quad f \circ g = a \cdot \mathbf{1}_M.$$

Say that $M' = (TB^{\mathrm{sh}})^a$, for some $n \in \mathbb{N}$ and some functor T_n on a category $\mathscr{C}_n$ as in (9.4.1); set $X_n := \operatorname{Spec} B_n^{\mathrm{sh}}$, $Z := \{s(B_n^{\mathrm{sh}})\} \subset X_n$, and for every B_n^{sh}-module N, let $N^\sim$ be the quasi-coherent $\mathscr{O}_X$-module arising from N.

Claim 9.4.13. For every finitely presented B_n^{sh}-module N, the B_n^{sh}-submodule $\Gamma_Z N^\sim$ is also finitely presented.

Proof of the claim. We may find a local, essentially étale B_n-algebra $(C, \mathfrak{m}_C)$, and a finitely presented C-module N_0 with an isomorphism $N_0 \otimes_C B_n^{\mathrm{sh}} \xrightarrow{\sim} N$. Set $W := \operatorname{Spec} C$, $Z_W := \{\mathfrak{m}_C\}$, and let $N_0^\sim$ be the quasi-coherent $\mathscr{O}_W$-module arising from N_0. From corollary 5.7.24, we see that $\Gamma_{Z_W} N_0^\sim$ is a finitely generated C-module. On the other hand, let $\varphi : X_n \to W$ be the natural morphism; since φ is flat, Z_W is constructible and $Z = \varphi^{-1} Z_W$, the induced map

$$B_n^{\mathrm{sh}} \otimes_C \Gamma_{Z_W} N_0^\sim \to \Gamma_Z N^\sim$$

is an isomorphism (lemma 5.4.16(iii)). Hence $\Gamma_Z N^\sim$ is a finitely generated B_n^{sh}-module. To conclude, it suffices to recall that B_n^{sh} is a coherent ring (see definition 8.3.3(i)). $\Diamond$

In view of claim 9.4.13, we may consider the functor

$$T'_n : \mathscr{C}_n \to B_n^{\mathrm{sh}}\text{-}\mathbf{Mod}_{\mathrm{coh}} \qquad M \mapsto \Gamma_Z(T_n M)^\sim$$

which is again of the type considered in (9.4.1), so we may define $M'' := (T'B^{\mathrm{sh}})^a$. Clearly g factors through a map $g' : M \to M''$, and if $f' : M'' \to M$ is the restriction of f, we still have $f' \circ g' = a\mathbf{1}_M$ and $g' \circ f' = a\mathbf{1}_{M''}$. We are then easily reduced to showing that $\lambda(aM'') < +\infty$. Hence, we may replace T_n by T'_n and assume from start that $M = P^a$, where P is a B^{sh}-module supported at $s(B^{\mathrm{sh}})$ and presented by T_n. In this case, theorem 8.3.62(i) implies that

$$\lambda(bM) = \lim_{k \to \infty} \lambda(B^{\mathrm{sh}} \cdot bT_{n+k} B_{n+k}^{\mathrm{sh}})$$

and lemma 8.3.57(i) says that

$$\lambda(B^{\mathrm{sh}} \cdot bT_{n+k} B_{n+k}^{\mathrm{sh}}) = \lim_{j \to \infty} d_{n+k+j}^{-1} \cdot \lambda_{n+k+j}(B_{n+k+j}^{\mathrm{sh}} \cdot bT_{n+k} B_{n+k}^{\mathrm{sh}}).$$

We conclude that

$$\lambda(bM) \leq \liminf_{k \to \infty} d_{n+k}^{-1} \cdot \lambda_{n+k}(bT_{n+k} B_{n+k}^{\mathrm{sh}}). \tag{9.4.14}$$

Now, for any $N \in \mathbb{N}$, let Ω_N be the fundamental domain for Δ_N^{gp} defined as in the proof of theorem 9.3.48; we may choose N large enough, so that $C_A \cdot \|\gamma\| < \log |b|$ for every $\gamma \in \Omega_N$, where C_A is the constant appearing in lemma 9.3.40(ii). Then, for every $\delta \in \Omega_n \cap \Delta_N^{\mathrm{gp}}$ and every $\Sigma \in \mathscr{S}_{\Delta_n^{\mathrm{gp}}, \Delta_{\mathbb{R}}}(\delta + \Omega_N)$ such that

$$\Omega_{\delta,\Sigma} := (\delta + \Omega_N) \cap \Omega(\Delta_{\mathbb{R}}, \Sigma) \cap \Delta^{\mathrm{gp}} \neq \varnothing$$

let us pick some $\gamma_{\delta,\Sigma} \in \Omega_{\delta,\Sigma}$. By lemma 9.3.45, for every $\gamma \in \Omega_{\delta,\Sigma}$ we may find B_n^{sh}-linear maps

$$A_{[\gamma]}^{\mathrm{sh}} \underset{g}{\overset{f}{\rightleftarrows}} A_{[\gamma_{\delta,\Sigma}]}^{\mathrm{sh}} \qquad \text{such that} \quad g \circ f = b \cdot \mathbf{1}_{A_{[\gamma]}^{\mathrm{sh}}} \quad \text{and} \quad f \circ g = b \cdot \mathbf{1}_{A_{[\gamma_{\delta,\Sigma}]}}$$

whence :

$$\lambda_n(b \cdot T_n A_{[\gamma]}^{\mathrm{sh}}) \leq \lambda_n(\operatorname{Im} T_n f) \leq \lambda_n(T_n A_{[\gamma_{\delta,\Sigma}]}) \qquad \text{for every } \gamma \in \Omega_{\delta,\Sigma}.$$

Since both $\Omega_n \cap \Delta_N^{\mathrm{gp}}$ and $\mathscr{S}_{\Delta_n^{\mathrm{gp}},\Delta_{\mathbb{R}}}(\delta + \Omega_N)$ are finite sets (proposition 3.3.35(i)), we conclude that there exists a real constant $C > 0$ such that

$$\lambda_n(b \cdot T_n A^{\mathrm{sh}}_{[\gamma]}) \leq C \qquad \text{for every } \gamma \in \Delta^{\mathrm{gp}}.$$

Set $r := \mathrm{rk}_{\mathbb{Z}} \Delta_0^{\mathrm{gp}}$; we deduce that

$$\lambda_{n+k}(bT_{n+k}B^{\mathrm{sh}}_{n+k}) = d_n^{-1} \cdot p^{-rk} \cdot \sum_{\gamma \in \Omega_n \cap \Delta^{\mathrm{gp}}_{n+k}} \lambda_n(b \cdot T_n A^{\mathrm{sh}}_{[\gamma]}) \leq d_n^{-1} \cdot C.$$

Comparing with (9.4.14), the proposition follows. □

9.4.15. Now, let $\pi \in \mathfrak{m}$ be any element with $|\pi|^p \geq |p|$; our next task is to examine the behaviour of normalized lengths for $B^{\mathrm{sh}}/\pi B^{\mathrm{sh}}$-modules under Frobenius. To ease notation, for any $n \in \mathbb{N}$ we let $\overline{\Phi}_n := \overline{\Phi}_{B_n^{\mathrm{sh}}}$ and $\overline{\Phi} := \overline{\Phi}_{B^{\mathrm{sh}}}$ (see (8.5)). To begin with, [36, Th.3.5.13(ii)] implies that

$$\overline{\Phi}_n = \mathbf{1}_{B_n^{\mathrm{sh}}} \otimes_{B_n} \overline{\Phi}_{B_n}.$$

On the other hand, notice that

$$\overline{\Phi}_{B_n} = \boldsymbol{p}_{P^{(n)}} \otimes_{P_0^{(n)}} \overline{\Phi}_{K^+} : P^{(n)} \otimes_{P_0^{(n)}} K^+/\pi K^+ \to P^{(n)} \otimes_{P_0^{(n)}} K^+_{(\Phi)}$$

where $\boldsymbol{p}_{P^{(n)}}$ is the p-Frobenius map of $P^{(n)}$; the latter factors through an isomorphism $P^{(n)} \xrightarrow{\sim} P^{(n-1)}$, and $\overline{\Phi}_{K^+}$ is an isomorphism as well (recall that K is algebraically closed). Hence $\overline{\Phi}_{B_n}$ factors through an isomorphism $B_n/\pi B_n \xrightarrow{\sim} B_{n-1,(\Phi)}$, and taking into account (9.3.49), we deduce that $\overline{\Phi}_n$ factors as the composition of an isomorphism

$$\Psi_n : B_n^{\mathrm{sh}}/\pi B_n^{\mathrm{sh}} \xrightarrow{\sim} B^{\mathrm{sh}}_{n-1,(\Phi)}$$

and the natural inclusion map $B^{\mathrm{sh}}_{n-1,(\Phi)} \to B^{\mathrm{sh}}_{n,(\Phi)}$. Therefore

$$\overline{\Phi} = \operatorname*{colim}_{n \in \mathbb{N}} \overline{\Phi}_n$$

is an isomorphism. The map Ψ_n can be further analyzed as a composition :

$$B_n^{\mathrm{sh}}/\pi B_n^{\mathrm{sh}} \xrightarrow{\mathbf{1}_{B_n^{\mathrm{sh}}} \otimes_{K^+} \overline{\Phi}_{K^+}} R_n := \overline{\Phi}^*_{K^+}(B_n^{\mathrm{sh}}/\pi B_n^{\mathrm{sh}}) \xrightarrow{\overline{\Psi}_n} B^{\mathrm{sh}}_{n-1,(\Phi)}$$

where $\overline{\Psi}_n$ is an isomorphism of K^+-algebras (cp. (8.5.1)). Clearly, both R_n and $B^{\mathrm{sh}}_{n-1,(\Phi)}$ are measurable K^+-algebras (with the $K^+/\pi^p K^+$-algebra structure fixed in (9.2.3)). The first observation is the following :

Lemma 9.4.16. *Let M be any $B_n^{\mathrm{sh}}/\pi B_n^{\mathrm{sh}}$-module supported at $s(B_n^{\mathrm{sh}})$. Then*

$$\lambda_{n-1}(\Psi_n^* M) = p \cdot \lambda_n(M).$$

Proof. Since $\overline{\Psi}_n$ is an isomorphism of K^+-algebras, it suffices to show that

$$\lambda_{R_n}(\overline{\Phi}^*_{K^+} M) = p \cdot \lambda_n(M)$$

for every $B_n^{\mathrm{sh}}/\pi B_n^{\mathrm{sh}}$-module M supported at $s(B_n)$. Also, since $\overline{\Phi}_{K^+}$ is an isomorphism, we may invoke theorem 8.3.62(i) to reduce to the case where M is a finitely presented module supported at $s(B_n^{\mathrm{sh}})$. Next, by claim 8.3.34 we may further assume that M is a flat K^+/aK^+-module, for some $a \in \mathfrak{m}$ such that $|a| \geq |\pi|$. In this case, $\overline{\Phi}^*_{K^+} M$ is a flat $K^+_{(\Phi)}/aK^+_{(\Phi)}$-module, *i.e.* a flat $K^+/a^p K^+$-module. Set $\overline{M} := M \otimes_{K^+} \kappa$; it is easily seen that $(\overline{\Phi}^*_{K^+} M) \otimes_{K^+} \kappa = \overline{\Phi}^*_{K^+} \overline{M}$, so we may compute using theorem 8.3.30(iii.b)

$$\lambda_n(M) = \log |a| \cdot \mathrm{length}_{B_n^{\mathrm{sh}}}(\overline{M}) \qquad \lambda_{R_n}(\overline{\Phi}^*_{K^+} M) = \log |a^p| \cdot \mathrm{length}_{R_n}(\overline{\Phi}^*_{K^+} \overline{M}).$$

The assertion is then reduced to the identity

$$\mathrm{length}_{B^{\mathrm{sh}}_n}(\overline{M}) = \mathrm{length}_{R_n}(\overline{\Phi}^*_{K^+}\overline{M})$$

which is obvious. $\square$

Proposition 9.4.17. *Let M be any $B^{\mathrm{sh}}/\pi B^{\mathrm{sh}}$-module supported at $s(B^{\mathrm{sh}})$. Then :*

$$\lambda(\overline{\Phi}^*M) = p^d \cdot \lambda(M) \qquad \textit{where } d := \dim B^{\mathrm{sh}}.$$

Proof. Notice first that we have a commutative diagram of ring homomorphisms

$$\begin{tikzcd} B^{\mathrm{sh}}_n/\pi B^{\mathrm{sh}}_n \arrow[r, "\Psi_n"] \arrow[d] \arrow[dr, "\overline{\Phi}_n"] & B^{\mathrm{sh}}_{n-1,(\Phi)} \arrow[d] \\ B^{\mathrm{sh}}_{n+1}/\pi B^{\mathrm{sh}}_{n+1} \arrow[r, "\Psi_{n+1}"'] & B^{\mathrm{sh}}_{n,(\Phi)} \end{tikzcd} \qquad \text{for every } n \in \mathbb{N}$$

whose vertical arrows are the natural inclusion maps. Set $r := \mathrm{rk}_{\mathbb{Z}}\Delta_0^{\mathrm{gp}}$.

Claim 9.4.18. $[\kappa(B^{\mathrm{sh}}_{n+1}) : \kappa(B^{\mathrm{sh}}_n)] = p^{r-d+1}$ for every $n \in \mathbb{N}$.

Proof of the claim. Let $x_n \in Y_n$ be the support of $\overline{x}_n$ (notation of (9.3.35)), and denote by Z_n the topological closure of $\{x_n\}$ in Y_n. It has already been remarked that the inclusion map $B_n \otimes_{K^+} \kappa \to B_{n+1} \otimes_{K^+} \kappa$ is radicial; especially, the residue field extension $\kappa(x_n) \to \kappa(x_{n+1})$ is purely inseparable, and indeed – arguing as in (9.4.15) – we easily see that the image of $\kappa(x_n)$ in $\kappa(x_{n+1})$ is the subfield $\kappa(x_{n+1})^p$. On the other hand, from (9.3.49) we deduce that

$$\kappa(B^{\mathrm{sh}}_{n+1}) = \kappa(B^{\mathrm{sh}}_n) \cdot \kappa(x_{n+1})$$

and since $\kappa(B^{\mathrm{sh}}_n)$ is a separable extension of $\kappa(x_n)$, it follows that

$$[\kappa(B^{\mathrm{sh}}_{n+1}) : \kappa(B^{\mathrm{sh}}_n)] = [\kappa(x_{n+1}) : \kappa(x_n)] = [\kappa(x_{n+1}) : \kappa(x_{n+1})^p].$$

From (9.3.43) we deduce that

$$\mathrm{tr.\,deg}(\kappa(x_n)/\kappa) = \dim \kappa[\Delta_n] - \dim Z_n = r - d + 1$$

([31, Ch.IV, Prop.5.2.1] and lemma 5.8.8(v)). Now, since κ is algebraically closed, we may find a subextension $E \subset \kappa(x_n)$, purely transcendental over κ, such that $\kappa(x_n)$ is a finite separable extension of E; in this case, $\kappa(x) = E \otimes_{E^p} \kappa(x)^p$ (*e.g.* by [36, Th.3.5.13(ii)]), so we are reduced to showing that $[E : E^p] = p^{\mathrm{tr.\,deg}\,(E/\kappa)}$, which is clear. $\Diamond$

Now, let first M be a finitely presented $B^{\mathrm{sh}}/\pi B^{\mathrm{sh}}$-module supported at $s(B^{\mathrm{sh}})$; we may find $n \in \mathbb{N}$ and a $B^{\mathrm{sh}}_n/\pi B^{\mathrm{sh}}_n$-module M_n supported at $s(B^{\mathrm{sh}}_n)$, with an isomorphism $M \stackrel{\sim}{\to} B^{\mathrm{sh}} \otimes_{B^{\mathrm{sh}}_n} M_n$ of B^{sh}-modules. There follows an isomorphism :

$$\overline{\Phi}^*M \stackrel{\sim}{\to} B^{\mathrm{sh}}_{(\Phi)} \otimes_{B^{\mathrm{sh}}_{n,(\Phi)}} \overline{\Phi}^*_n M_n$$

so we may compute :

$$\begin{aligned}
\lambda(\overline{\Phi}^*M) &= \lim_{k\to\infty} d^{-1}_{n+k} \cdot \lambda_{n+k}(B^{\mathrm{sh}}_{n+k,(\Phi)} \otimes_{B^{\mathrm{sh}}_{n,(\Phi)}} \overline{\Phi}^*_n M_n) \\
&= \lim_{k\to\infty} d^{-1}_{n+k} \cdot \lambda_{n+k}(\overline{\Phi}^*_{n+k}(B^{\mathrm{sh}}_{n+k} \otimes_{B^{\mathrm{sh}}_n} M_n)) \\
&= \lim_{k\to\infty} d^{-1}_{n+k} \cdot \lambda_{n+k}(\Psi^*_{n+k+1}(B^{\mathrm{sh}}_{n+k+1} \otimes_{B^{\mathrm{sh}}_n} M_n)) \\
&= \lim_{k\to\infty} p \cdot d^{-1}_{n+k} \cdot \lambda_{n+k+1}(B^{\mathrm{sh}}_{n+k+1} \otimes_{B^{\mathrm{sh}}_n} M_n) \qquad \text{(by lemma 9.4.16)} \\
&= \lim_{k\to\infty} p^d \cdot d^{-1}_{n+k+1} \cdot \lambda_{n+k+1}(B^{\mathrm{sh}}_{n+k+1} \otimes_{B^{\mathrm{sh}}_n} M_n) \qquad \text{(by (9.3.51) and claim 9.4.18)} \\
&= p^d \cdot \lambda(M)
\end{aligned}$$

as stated. Next, if M is a finitely generated $B^{\mathrm{sh}}/\pi B^{\mathrm{sh}}$-module, we may write M as the colimit of a filtered family of finitely presented $B^{\mathrm{sh}}/\pi B^{\mathrm{sh}}$-modules, with surjective transition maps; then theorem 8.3.62(i) reduces the assertion to the case of a finitely presented module, which has just been dealt with. Lastly, if M is a general $B^{\mathrm{sh}}/\pi B^{\mathrm{sh}}$-module, we may write M as the colimit of the filtered system $(M_i \mid i \in I)$ of its finitely generated B^{sh}-submodules; since $\overline{\Phi}$ is an isomorphism, the induced filtered system $(\overline{\Phi}^* M_i \mid i \in I)$ has still injective transition maps, so theorem 8.3.62(i) reduces the assertion for M to the corresponding assertion for the M_i, which is already known. $\square$

Lemma 9.4.16 and proposition 9.4.17 extend as usual to almost modules. As a corollary, we can generalize to $(B^{\mathrm{sh}})^a$-modules the vanishing criterion of lemma 9.2.11; indeed, we have :

Corollary 9.4.19. *Let M be a $(B^{\mathrm{sh}}/\pi B^{\mathrm{sh}})^a$-module supported at $s(B^{\mathrm{sh}})$, and let us set $N := \overline{\Phi}^*(M)$. Suppose that :*

(a) *M is a submodule of an almost presentable $(B^{\mathrm{sh}})^a$-module.*
(b) *N admits a filtration $(\mathrm{Fil}^i N \mid 0 \leq i \leq p)$, with $\mathrm{Fil}^0 N = N$ and $\mathrm{Fil}^p N = 0$, and whose graded quotients $\mathrm{gr}^i N$ are subquotients of M for every $i < p$.*
(c) $\dim B^{\mathrm{sh}} > 1$.

Then $M = 0$.

Proof. We proceed as in the proof of lemma 9.2.11 : by proposition 9.4.10, it suffices to show that $\lambda(bM) = 0$ for every $b \in \mathfrak{m}$. However, from (b) and lemma 8.3.67(i) we deduce that $\lambda(b^p N) \leq p \cdot \lambda(bM)$ for every $b \in \mathfrak{m}$. On the other hand, $\overline{\Phi}^* bM = b^p N$, since $\overline{\Phi}$ is an isomorphism. Set $d := \dim B^{\mathrm{sh}}$; then $p^d \cdot \lambda(bM) = \lambda(b^p N)$, by proposition 9.4.17; since $\lambda(bM) < +\infty$ (proposition 9.4.12) and $d > 1$, the contention follows. $\square$

9.4.20. After this preparation, we are finally ready to generalize to model algebras the vanishing result of proposition 9.2.21. Namely, in the situation of (8.5.23) we take $R := B^{\mathrm{sh}}$, and set

$$X := \mathrm{Spec}\, B^{\mathrm{sh}} \qquad X_n := Y_n(\overline{x}_n) \qquad U_n := X_n \setminus \{s(B^{\mathrm{sh}}_n)\} \qquad \text{for every } n \in \mathbb{N}$$

(notation of (9.3.35)). Let also $\psi : U^\wedge \to U$ and $\psi_n : U \to U_n$ be the natural morphism, for every $n \in \mathbb{N}$. Then, to any étale almost finitely presented $\mathscr{O}^a_U$-algebra $\mathscr{A}$, we attach the étale $\mathscr{O}^a_{U^\wedge}$-algebra $\mathscr{A}^\wedge := \psi^* \mathscr{A}$, and the constructions of (8.5.27)–(8.5.30) yield a $\boldsymbol{\sigma}$-equivariant topological $\mathscr{O}^a_{\mathbf{A}(U)}$-algebra $\mathbf{A}(\mathscr{A})^+$. Denote $d := \dim B^{\mathrm{sh}}$; we begin with the following more general :

Lemma 9.4.21. *Fix $b \in \mathfrak{m}_K$, and let $\mathscr{A}$ be any étale almost finitely presented $\mathscr{O}^a_{U_{/b}}$-algebra. We have :*

(i) *$H^q(U_{/b}, \mathscr{A})$ is an almost presentable $(B^{\mathrm{sh}})^a$-module, for every $q = 0, \dots, \dim X_{/b} - 2$.*
(ii) *If $b \neq 0$, then $H^q(U_{/b}, \mathscr{A}) = 0$ for every $q = 1, \dots, \dim X_{/b} - 2$.*

Proof. (i): We may write $X_{/b}$ as the limit of a filtered system $(W_i \mid i \in I)$ of local S-schemes, such that the following holds. For each $i \in I$, there exists $n(i) \in \mathbb{N}$ and an affine étale neighborhood $Z_i \to (Y_{n(i)})_{/b}$ of $\overline{x}_{n(i)}$ such that $W_i = Z_i(w_i)$ for some point $w_i \in Z_i$ (and then, necessarily, the image of w_i in $Y_{n(i)}$ is the support of $\overline{x}_{n(i)}$). For every $i \in I$, let $g_i : U_{/b} \to W_i$ be the induced morphism. By corollary 8.2.24, for every $c \in \mathfrak{m}_K$ we may find $i \in I$, a finitely presented $\mathscr{O}_{W_i}$-algebra $\mathscr{R}$ and a morphism $f : g_i^* \mathscr{R} \to \mathscr{A}$ such that :

- $\mathrm{Ker}\, f$ and $\mathrm{Coker}\, f$ are annihilated by c.
- For every $x \in U_w := W_i \setminus \{w_i\}$, the map $c \cdot \mathbf{1}_{\mathscr{R},x}$ factors through a free $\mathscr{O}_{W_i,x}$-module.

To ease notation, set

$$W := W_i \qquad w := w_i \qquad g := g_i \qquad Z := Z_i \qquad n := n(i).$$

Since c is arbitrary, it suffices to show that the B^{sh}-module $c \cdot H^q(U_{/b}, g^*\mathscr{R})$ is presentable, for every $q = 0, \dots, \dim X_{/b} - 2$ (see example 9.4.7). Indeed, for every B^{sh}_n-module (resp. $\mathscr{O}_{W,w}$-module) M, let $M^\sim$ denote the quasi-coherent $\mathscr{O}_{X_n}$-module (resp. $\mathscr{O}_W$-module) associated to M, and consider the functor

$$T_n : \mathscr{C}_n \to B^{\mathrm{sh}}_n\text{-}\mathbf{Mod} \qquad M \mapsto c \cdot H^q((U_n)_{/b}, (\mathscr{R}_w \otimes_{\mathscr{O}_{W,w}} M)^\sim).$$

It is easily seen that the natural map

$$\operatorname*{colim}_{m \geq n} T_n B^{\mathrm{sh}}_m \to c \cdot H^q(U_{/b}, g^*\mathscr{R})$$

is an isomorphism, hence we are reduced to showing that $T_n M$ is a finitely presented B^{sh}_n-module, for every object M of $\mathscr{C}_n$, and notice the natural B^{sh}_n-linear isomorphism

$$H^q(U_w, (\mathscr{R}_w \otimes_{B_n} A_{[\gamma]})^\sim) \otimes_{\mathscr{O}_{W,w}} B^{\mathrm{sh}}_n \xrightarrow{\sim} T_n A^{\mathrm{sh}}_{[\gamma]} \qquad \text{for every } \gamma \in \Delta^{\mathrm{gp}}$$

(corollary 5.1.19). For any $z \in U_w$, multiplication by c on $(\mathscr{R}_{W,w} \otimes_{B_n} A_{[\gamma]})^\sim_z$ factors through $\mathscr{O}_{W,z} \otimes_{B_n} A^{\oplus t}_{[\gamma]}$, for some $t \in \mathbb{N}$. Let $\varphi : Z \to S$ be the structure map, and set $C := H^0(Z, \mathscr{O}_Z)$; it follows easily from lemmata 9.3.42(ii) and 5.6.36(iii) that $(C \otimes_{B_n} A_{[\gamma]})^\sim$ is φ-Cohen-Macaulay at w, in which case (5.4.3) and lemmata 5.8.26, 5.6.36(i) imply that the $\mathscr{O}_{W,w}$-module $c \cdot H^q(U_w, (\mathscr{R}_{W,w} \otimes_{B_n} A_{[\gamma]})^\sim)$ is finitely presented for $q = 0, \dots, \dim X_{/b} - 2$. The assertion follows.

(ii): Pick $c \in \mathfrak{m}_K$ such that $|c^p| \geq \max(|b|, |p|)$, and let $N \in \mathbb{N}$ be the smallest integer such that $|c^N| \leq |b|$. Define a descending filtration on $\mathscr{A}$, by the rule : $\mathrm{Fil}_i \mathscr{A} := c^i \mathscr{A}$ for $i = 0, \dots, N$. Notice that the associated graded $\mathscr{O}^a_{U_{/b}}$-module $\mathrm{gr}_i \mathscr{A}$ is isomorphic to $\mathscr{A}/c\mathscr{A}$ for every $i = 0, \dots, N-1$, and $\mathrm{gr}_N \mathscr{A}$ is isomorphic to $\mathscr{A}/bc^{1-N}\mathscr{A}$. We easily reduce to showing the vanishing of $H^j(U_{/c}, \mathrm{gr}_i \mathscr{A})$ for $j = 1, \dots, \dim X_{/c} - 2$ and $i = 1, \dots, N$. Hence we may replace b by c, and assume from start that $|b^p| \geq |p|$ and that there exists an étale $\mathscr{O}^a_{U_{/b^p}}$-algebra $\mathscr{A}'$ such that $(\mathscr{A}'/b\mathscr{A}')_{|U_{/b}} = \mathscr{A}$. We now argue as in the proof of claim 9.2.15 : since $\overline{\Phi}$ is an isomorphism, lemma 8.5.5 implies that the natural map

$$\overline{\Phi}^* H^j(U_{/b}, \mathscr{A}) \to H^j(U_{/b^p}, \mathscr{A}')$$

is an isomorphism of $(B^{\mathrm{sh}}_{(\Phi)})^a$-modules, for every $j \in \mathbb{N}$. On the other hand, the filtration $\mathrm{Fil}_\bullet \mathscr{A}'$ defined as in the foregoing induces a filtration on $H^i(U_{/b^p}, \mathscr{A}')$ of length p, whose associated graded pieces are subquotients of $H^j(U_{/b}, \mathscr{A})$. The latter is supported at $s(B^{\mathrm{sh}})$ for $j > 0$, since $\mathscr{A}$ is a quasi-coherent $\mathscr{O}^a_{U_{/b}}$-algebra. Then the assertion follows from (i) and corollary 9.4.19. □

Proposition 9.4.22. *In the situation of* (9.4.20)*, suppose that* $d \geq 3$*. Then :*

$$H^q(\mathbf{A}(U), \mathbf{A}(\mathscr{A})^+) = 0 \qquad \textit{whenever } 1 \leq q \leq d-2.$$

Proof. Set $\mathbf{H}^q := H^q(\mathbf{A}(U), \mathbf{A}(\mathscr{A})^+)$ for every $q \in \mathbb{N}$. Arguing as in the proof of proposition 9.2.21, we reduce to showing that $\mathbf{H}^q/\vartheta_k \mathbf{H}^q = 0$ for every $k \in \mathbb{Z}$, in the range $1 \leq q \leq d-2$, and we also see that the latter quotient is a $(B^{\mathrm{sh}})^a$-submodule of $H^q(U, \mathscr{A})$.

In view of proposition 9.4.12 and lemmata 9.4.9 and 9.4.21(i), we see that $\mathbf{H}^q/\vartheta \mathbf{H}^q$ is a B^{sh}-module of almost finite length, on which scalar multiplication by p is nilpotent, for every $q = 1, \dots, d-2$. Now, in order to show that this quotient vanishes, we may repeat *verbatim* the proof of claim 9.2.22; the only change is that we appeal to corollary 9.4.19, instead of lemma 9.2.11 : the details shall be left to the reader. □

Remark 9.4.23. Suppose now that $d \geq 3$ and $\Delta_0 = \mathbb{Z}^{\oplus r} \oplus \mathbb{N}^{\oplus s}$ for some $r, s \in \mathbb{N}$. As already noted (remark 9.4.8(iv)), in this case every almost presentable $(B^{\mathrm{sh}})^a$-module is almost coherent. It becomes then possible to repeat *verbatim* the proof of theorem 9.2.23 : the details shall be left to the reader. The resulting almost purity theorem is already more general than the one found in [34]. However, we will not spell out here this statement, and instead we shall move on to the proof of almost purity for a general model algebra.

9.4.24. Keep the situation of (9.4.20). For any $b \in \mathfrak{m}_K$, and any $\mathscr{O}^a_{U_{/b}}$-algebra $\mathscr{B}$, let

$$\mu_{\mathscr{B}} : \Gamma(U_{/b}, \mathscr{B}) \otimes_{(B^{\mathrm{sh}})^a} \Gamma(U_{/b}, \mathscr{B}) \to \Gamma(U_{/b}, \mathscr{B} \otimes_{\mathscr{O}^a_U} \mathscr{B})$$

be the natural map, and set

$$C_{\mathscr{B}} := \mathrm{Coker}\, \mu_{\mathscr{B}}.$$

Now, let $\mathscr{A}$ be any étale almost finitely presented $\mathscr{O}^a_{U_{/b}}$-algebra; clearly $C_{\mathscr{A}}$ is a $(B^{\mathrm{sh}})^a$-module supported at $s(B^{\mathrm{sh}})$, and from lemma 9.4.21(i) we see that, if $\dim X_{/b} \geq 2$, then $C_{\mathscr{A}}$ is a quotient of an almost presentable $(B^{\mathrm{sh}})^a$-module.

Lemma 9.4.25. *With the notation of* (9.4.24)*, suppose that* $\dim X_{/b} \geq 2$*. Then the* $(B^{\mathrm{sh}})^a$*-module* $C_{\mathscr{A}}$ *has almost finite length, for every étale almost finitely presented* $\mathscr{O}^a_{U_{/b}}$*-algebra* $\mathscr{A}$*.*

Proof. Notice that we cannot appeal to proposition 9.4.12, since it is not known at this point whether $C_{\mathscr{A}}$ is almost presentable. Instead we use a direct argument. Namely, fix $a \in \mathfrak{m}$, and pick $n \in \mathbb{N}$ and a coherent $\mathscr{O}_{X_{n/b}}$-algebra $\mathscr{R}_n$ with a morphism $\mathscr{R} := \psi_n^*(\mathscr{R}^a_{n|U_{n/b}}) \to \mathscr{A}$ fulfilling the conditions of corollary 8.2.24. It is easily seen that the image of the induced morphism $C_{\mathscr{R}} \to C_{\mathscr{A}}$ contains $a^2 C_{\mathscr{A}}$, hence it suffices to show that $a^2 C_{\mathscr{R}}$ has almost finite length. To ease notation, denote by $\mathscr{B}$ the quasi-coherent $\mathscr{O}_{X_{n/b}}$-algebra associated to $B^{\mathrm{sh}}/bB^{\mathrm{sh}}$, and set $\mathscr{R}^{(2)}_n := \mathscr{R}_n \otimes_{\mathscr{O}_{X_n}} \mathscr{R}_n$; by virtue of (5.4.2), we have an exact sequence

$$\Gamma(X_{n/b}, \mathscr{R}^{(2)}_n \otimes_{\mathscr{O}_{X_n}} \mathscr{B}) \to \Gamma(U_{n/b}, \mathscr{R}^{(2)}_n \otimes_{\mathscr{O}_{X_n}} \mathscr{B}) \to H := H^1_{\{s(B^{\mathrm{sh}}_n)\}}(\mathscr{R}^{(2)}_n \otimes_{\mathscr{O}_{X_n}} \mathscr{B}) \to 0$$

from which we see that $C_{\mathscr{R}}$ is a quotient of H^a, and taking into account proposition 9.4.12, we are therefore reduced to showing that $a^2 H^a$ is almost presentable. To show the latter assertion, one remarks that multiplication by a^2 on $\mathscr{R}^{(2)}_{n,y}$ factors through a free $\mathscr{O}_{X_{n/b},y}$-module of finite rank, for every $y \in U_{n/b}$, and then one argues as in the proof of lemma 9.4.21(i) : the details shall be left to the reader. $\square$

9.4.26. Let A be the auxiliary model K^+-algebra constructed in (9.3.35). Set

$$\Delta_{(m)} := \{\gamma \in \Delta_{\mathbb{Q}} \mid m\gamma \in \Delta\} \qquad D_{(m)} := A_{\Delta_{(m)}} \qquad \text{for every integer } m > 0.$$

Thus, $D^{\mathrm{sh}}_{(m)} := D_{(m)} \otimes_B B^{\mathrm{sh}}$ is a $(\Delta^{\mathrm{gp}}_{(m)}/\Delta^{\mathrm{gp}})$-graded B^{sh}-algebra, and it determines a quasi-coherent $\mathscr{O}_X$-algebra $\mathscr{D}_{(m)}$. For any $b \in \mathfrak{m}_K$, denote $j_{/b} : U_{/b} \to X_{/b}$ (resp. $i_{/b} : X_{/b} \to X$) the open (resp. closed) immersion, and for any $\mathscr{O}^a_{U_{/b}}$-algebra $\mathscr{A}$, set as well

$$\mathscr{D}_{(m)/b} := i^*_{/b}\mathscr{D}_{(m)} \qquad \mathscr{A}_{(m)} := \mathscr{A} \otimes_{\mathscr{O}^a_U} j^*_{/b}\mathscr{D}^a_{(m)/b}.$$

Proposition 9.4.27. *With the notation of* (9.4.26)*, let* $\mathscr{A}$ *be an étale almost finitely presented* $\mathscr{O}_{U_{/b}}$*-algebra, and suppose that :*

(a) $\lambda(C_{\mathscr{A}}) = 0$ *(notation of* (9.4.24)*).*
(b) $\dim X_{/b} \geq 2$.

Then there exists an integer $m > 0$ *such that* $(p, m) = 1$ *and such that* $j_{/b*}\mathscr{A}_{(m)}$ *is an étale and almost finitely presented* $\mathscr{D}^a_{(m)/b}$*-algebra.*

Proof. Notice that we cannot any longer appeal to proposition 9.4.10, to deduce from (a) that $C_{\mathscr{A}}$ vanishes, since at this point is not known whether $C_{\mathscr{A}}$ is almost presentable. We use instead an ad hoc argument, exploiting the combinatorial properties of the algebras B_n.

Indeed, fix $a \in \mathfrak{m}$ and apply corollary 8.2.24 to find $n \in \mathbb{N}$ and a coherent $\mathscr{O}_{U_{n/b}}$-algebra $\mathscr{R}_n$ with a map $\mathscr{R} := \psi_n^* \mathscr{R}_n \to \mathscr{A}_*$ of $\mathscr{O}_{U_{/b}}$-algebras, with kernel and cokernel annihilated by a. Let $e \in \Gamma(U_{/b}, \mathscr{A} \otimes_{\mathscr{O}_U^a} \mathscr{A})_*$ be the diagonal idempotent of $\mathscr{A}$ (see remark 8.2.6(i)). In view of (9.3.49), it is easily seen that the cokernel of the induced morphism

$$\Gamma(U_{n/b}, \mathscr{R}_n \otimes_{\mathscr{O}_{U_n}} \mathscr{R}_n) \otimes_{B_n} B^a \to \Gamma(U_{/b}, \mathscr{A} \otimes_{\mathscr{O}_U^a} \mathscr{A})$$

is annihilated by a^2. We deduce that there exists $e_n \in \Gamma(U_{n/b}, \mathscr{R}_n \otimes_{\mathscr{O}_{U_n}} \mathscr{R}_n)$ whose image in $\Gamma(U_{/b}, \mathscr{A} \otimes_{\mathscr{O}_U^a} \mathscr{A})_*$ equals $a^3 e$ (details left to the reader). Furthermore, a diagram chase as usual shows that the natural map $C_{\mathscr{R}^a} \to C_{\mathscr{A}}$ induces an isomorphism of the $(B^{\mathrm{sh}})^a$-submodule $a^3 C_{\mathscr{R}}$ onto a quotient of a submodule of $C_{\mathscr{A}}$; in view of assumption (a), we conclude that

$$\lambda(a^3 C_{\mathscr{R}^a}) = 0.$$

Denote by $\overline{e}_n \in (C_{\mathscr{R}^a})_*$ the image of $a^3 e_n$; by virtue of lemma 8.3.57 and theorem 8.3.30(ii.b), we derive :

$$\lim_{k\to\infty} p^{-(n+k)r} \cdot \sum_{[\gamma]\in\Delta^{\mathrm{gp}}_{n+k}/\Delta^{\mathrm{gp}}_n} \lambda_n(A^{\mathrm{sh}}_{[\gamma]}\overline{e}_n) = 0 \qquad (\text{where } r := \mathrm{rk}_{\mathbb{Z}}\Delta^{\mathrm{gp}}_0). \tag{9.4.28}$$

Claim 9.4.29. There exists a sequence $(\gamma_i \mid i \in \mathbb{N})$ of elements of Δ such that

$$\lim_{i\to\infty} \gamma_i = 0 \qquad \text{and} \qquad \lim_{i\to\infty} \lambda_n(A^{\mathrm{sh}}_{[\gamma_i]}\overline{e}_n) = 0.$$

Proof of the claim. Let $\Omega_n \subset \Delta^{\mathrm{gp}}_{\mathbb{R}}$ be the fundamental domain for Δ^{gp}_n defined as in the proof of theorem 9.3.48; also, fix a Banach norm $\|\cdot\|$ on $\Delta^{\mathrm{gp}}_{\mathbb{R}}$, and set

$$\Delta^{\mathrm{gp}}_{\mathbb{R}}(\rho) := \{\gamma \in \Delta^{\mathrm{gp}}_{\mathbb{R}} \mid \|\gamma\| \leq \rho\} \quad \Delta(\rho) := \Delta \cap \Delta^{\mathrm{gp}}_{\mathbb{R}}(\rho) \qquad \text{for every } \rho > 0.$$

Then $\Delta(\rho) \subset \Omega_n$ for every sufficiently small $\rho > 0$, and it is easily seen that there exists a real constant $C > 0$ such that

$$\sharp(\Delta(\rho) \cap \Delta^{\mathrm{gp}}_{n+k}) \geq C \cdot p^{(n+k)r} \cdot \rho^r \qquad \text{for every } k \in \mathbb{N} \text{ and every } \rho > 0$$

(where $\sharp(\Sigma)$ denotes the cardinality of a set Σ : details left to the reader). Suppose now that the claim fails; then there exist $\varepsilon, \rho > 0$ such that $\lambda_n(A^{\mathrm{sh}}_{[\gamma]}\overline{e}_n) > \varepsilon$ for every $\gamma \in \Delta(\rho)$. It follows that :

$$\lim_{k\to\infty} p^{-(n+k)r} \cdot \sum_{\gamma\in\Omega_n\cap\Delta^{\mathrm{gp}}_{n+k}} \lambda_n(A^{\mathrm{sh}}_{[\gamma]}\overline{e}_n) > C \cdot \rho^r \cdot \varepsilon$$

which contradicts (9.4.28). ◊

Let $(\gamma_i \mid i \in \mathbb{N})$ be a sequence as in claim 9.4.29, and for every $i \in \mathbb{N}$ set $l_i := \lambda_n(A^{\mathrm{sh}}_{[\gamma_i]}\overline{e}_n)$ and pick some $a_i \in K^+$ such that $\log|a_i| \in [l_i, 2l_i]$; by lemma 8.3.46 we have $a_i \cdot A^{\mathrm{sh}}_{[\gamma_i]}\overline{e}_n = 0$ for every $i \in \mathbb{N}$. To ease notation, for every $\gamma \in \Delta^{\mathrm{gp}}_{\mathbb{Q}}$ denote by $(A^{\mathrm{sh}}_{[\gamma]})^\sim$ the quasi-coherent $\mathscr{O}_{U_n}$-module associated to $A^{\mathrm{sh}}_{[\gamma]}$, and set

$$M^{(1)}_{[\gamma]} := \Gamma(U_{n/b}, \mathscr{R}_n \otimes_{\mathscr{O}_{U_n}} (A^{\mathrm{sh}}_{[\gamma]})^\sim) \qquad M^{(2)}_{[\gamma]} := \Gamma(U_{n/b}, \mathscr{R}_n \otimes_{\mathscr{O}_{U_n}} \mathscr{R}_n \otimes_{\mathscr{O}_{U_n}} (A^{\mathrm{sh}}_{[\gamma]})^\sim).$$

Notice that

$$\Gamma(U_{/b}, \mathscr{R}) = \bigoplus_{[\gamma]\in\Delta^{\mathrm{gp}}/\Delta^{\mathrm{gp}}_n} M^{(1)}_{[\gamma]} \qquad \text{and} \qquad \Gamma(U_{/b}, \mathscr{R} \otimes_{\mathscr{O}_U} \mathscr{R}) = \bigoplus_{[\gamma]\in\Delta^{\mathrm{gp}}/\Delta^{\mathrm{gp}}_n} M^{(2)}_{[\gamma]}$$

and the natural map $\Gamma(U_{/b}, \mathscr{R}) \otimes_{B^{\mathrm{sh}}} \Gamma(U_{/b}, \mathscr{R}) \to \Gamma(U_{/b}, \mathscr{R} \otimes_{\mathscr{O}_U} \mathscr{R})$ is the direct sum of its $(\Delta^{\mathrm{gp}}/\Delta^{\mathrm{gp}}_n)$-graded summands :

$$\mu_{[\gamma]} : \bigoplus_{[\beta]\in\Delta^{\mathrm{gp}}/\Delta^{\mathrm{gp}}_n} M^{(1)}_{[\beta]} \otimes_{B^{\mathrm{sh}}_n} M^{(1)}_{[\gamma-\beta]} \to M^{(2)}_{[\gamma]}.$$

Especially, $C_{\mathscr{R}^a}$ is a $(\Delta^{\mathrm{gp}}/\Delta^{\mathrm{gp}}_n)$-graded $(B^{\mathrm{sh}}_n)^a$-module, and it follows that $a_i a^3 A^{\mathrm{sh}}_{[\gamma_i]} e_n$ lies in the image of $\mu_{[\gamma_i]}$, for every $i \in \mathbb{N}$.

Let $C_A > 0$ be the constant provided by lemma 9.3.40(ii), and for every $\gamma, \gamma' \in \Delta^{\mathrm{gp}}$ such that $\Delta^{\mathrm{gp}} \cap (\Delta_{\mathbb{R}} - \gamma) \subset \Delta^{\mathrm{gp}} \cap (\Delta_{\mathbb{R}} - \gamma')$, and every $c \in K^+$ with $\log|c| \geq C_A\|\gamma' - \gamma\|$, denote by $\tau_{c,\gamma'-\gamma} : A_{[\gamma]} \to A_{[\gamma']}$ the B_n-linear maps defined as in the proof of lemma 9.3.45; after tensoring with $\mathscr{R}_n$ (resp. with $\mathscr{R}_n \otimes_{\mathscr{O}_{U_n}} \mathscr{R}_n$) we obtain B^{sh}_n-linear maps

$$\tau^{(1)}_{c,\gamma'-\gamma} : M^{(1)}_{[\gamma]} \to M^{(1)}_{[\gamma']} \qquad (\text{resp. } \tau^{(2)}_{c,\gamma'-\gamma} : M^{(2)}_{[\gamma]} \to M^{(2)}_{[\gamma']}).$$

Claim 9.4.30. For every $\rho > 0$ there exists an integer $m > 0$ with $(p, m) = 1$, and a real number $\varepsilon > 0$ such that the following holds. For every $\beta_1, \beta_2 \in \Delta^{\mathrm{gp}}_{\mathbb{R}}(\rho)$ with $\|\beta_1 + \beta_2\| < \varepsilon$ we may find $\beta'_1, \beta'_2 \in \Delta^{\mathrm{gp}}_{(m)}$ such that

(i) $\beta'_1 + \beta'_2 = 0$
(ii) $\|\beta'_i - \beta_i\| < \log|a|$ for $i = 1, 2$
(iii) $\Delta^{\mathrm{gp}}_n \cap (\Delta_{\mathbb{R}} - \beta_i) \subset \Delta^{\mathrm{gp}}_n \cap (\Delta_{\mathbb{R}} - \beta'_i)$ for $i = 1, 2$.

Proof of the claim. Recall that Δ^{gp} is p-divisible, hence $\Delta^{\mathrm{gp}}_{(m)} = \Delta^{\mathrm{gp}}_{(pm)}$ for every integer $m > 0$, so the condition that $(p, m) = 1$ can always be arranged, if all the other conditions are already fulfilled. Now, for every $\gamma \in \Delta^{\mathrm{gp}}_{\mathbb{R}}$, let $\overline{\Omega}(\gamma)$ be the topological closure of the subset $\Omega(\Delta_{\mathbb{R}}, \Delta^{\mathrm{gp}}_n \cap (\Delta_{\mathbb{R}} - \gamma))$ in $\Delta^{\mathrm{gp}}_{\mathbb{R}}$ (notation of (3.3.34)); in view of condition (i) and of proposition 3.3.35(v), condition (iii) is implied by :

$$\beta'_1 \in \overline{\Omega}(\beta_1) \cap (-\overline{\Omega}(\beta_2)). \tag{9.4.31}$$

Moreover, suppose that

$$\|\beta'_1 - \beta_1\| \leq 2^{-1} \cdot \log|a|. \tag{9.4.32}$$

Then

$$\|\beta'_2 - \beta_2\| = \|\beta'_2 + \beta_1 - \beta_1 - \beta_2\| \leq \| - \beta'_1 + \beta_1\| + \|\beta_1 + \beta_2\| < 2^{-1}\log|a| + \varepsilon.$$

Hence condition (ii) will hold, provided $\varepsilon \leq 2^{-1} \cdot \log|a|$. Thus, we have to exhibit $m > 0$ and $\varepsilon > 0$ such that, for every $\beta_1, \beta_2 \in \Delta^{\mathrm{gp}}_{\mathbb{R}}(\rho)$ with $\|\beta_1 + \beta_2\| < \varepsilon$, there exists $\beta'_1 \in \Delta^{\mathrm{gp}}_{(m)}$ fulfilling (9.4.31) and (9.4.32). Then, by proposition 3.3.35(i) and lemma 3.3.42, it suffices to find $\beta'_1 \in \Delta^{\mathrm{gp}}_{\mathbb{R}}$ fulfilling these two latter conditions. By way of contradiction, suppose that such β'_1 cannot always be found : this means that there exists a sequence $\underline{\beta} := ((\beta_{1,k}, \beta_{2,k}) \mid k \in \mathbb{N})$ of pairs of elements in $\Delta^{\mathrm{gp}}_{\mathbb{R}}(\rho)$, such that

(a) $\|\beta_{1,k} + \beta_{2,k}\| < 2^{-k}$ for every $k \in \mathbb{N}$
(b) $\overline{\Omega}(\beta_{1,k}) \cap (-\overline{\Omega}(\beta_{2,k})) \cap (\Delta^{\mathrm{gp}}_{\mathbb{R}}(2^{-1}\log|a|) + \beta_{1,k}) = \varnothing$ for every $k \in \mathbb{N}$.

However, by proposition 3.3.35(i), after replacing $\underline{\beta}$ by a subsequence we may assume that both $\overline{\Omega}(\beta_{1,k})$ and $\overline{\Omega}(\beta_{2,k})$ are independent of k. Since $\Delta^{\mathrm{gp}}_{\mathbb{R}}(\rho)$ is a compact subset, we may also assume that the sequence $\underline{\beta}$ converges to a pair (β'_1, β'_2) of elements of $\Delta^{\mathrm{gp}}_{\mathbb{R}}(\rho)$. Clearly, $\beta'_1 + \beta'_2 = 0$; since $\beta_{i,k} \in \overline{\Omega}(\beta_{i,k})$ for $i = 1, 2$ and every $k \in \mathbb{N}$, we also have $\beta'_i \in \overline{\Omega}(\beta_{i,k})$ for $i = 1, 2$. Lastly, we have $\|\beta'_1 - \beta_{1,k}\| < 2^{-1}\log|b|$ for every sufficiently large $k \in \mathbb{N}$; this contradicts (b), and the claim follows. ◊

Fix $\rho > 0$ such that $\Omega_n \subset \Delta^{\mathrm{gp}}_{\mathbb{R}}(\rho/2)$, and pick $\varepsilon > 0$ and $m > 0$ as in claim 9.4.30; set

$$\mathscr{R}_{(m)} := \mathscr{R} \otimes_{\mathscr{O}_{U_{/b}}} j^*_{/b}\mathscr{D}_{(m)/b}.$$

Also, fix $i \in \mathbb{N}$ such that
$$\|\gamma_i\| < \min(\varepsilon, \rho/2).$$
Then, if $\beta_1 \in \Omega_n$, both β_1 and $\beta_2 := \gamma_i - \beta_1$ lie in $\Delta^{\mathrm{gp}}(\rho)$, and therefore there exist $\beta'_1, \beta'_2 \in \Delta^{\mathrm{gp}}_{(m)}$ fulfilling conditions (i)–(iii) of claim 9.4.30. Now, for each $\beta_1 \in \Omega_n \cap \Delta^{\mathrm{gp}}$, choose such a pair (β'_1, β'_2); furthermore, let $c \in K^+$ be any element such that
$$C_A \log|a| \leq \log|c| \leq 2C_A \log|a|.$$
In view of remark 9.3.47, we obtain a commutative diagram of B_n^{sh}-linear maps :
$$\begin{tikzcd}
\bigoplus\limits_{\beta_1 \in \Omega_n \cap \Delta^{\mathrm{gp}}} M^{(1)}_{[\beta_1]} \otimes_{B_n^{\mathrm{sh}}} M^{(1)}_{[\gamma_i - \beta_1]} \arrow[r, "\mu_{[\gamma_i]}"] \arrow[d, "\bigoplus\limits_{\beta_1 \in \Omega_n \cap \Delta^{\mathrm{gp}}} \left(\tau^{(1)}_{c, \beta'_1 - \beta_1} \otimes \tau^{(1)}_{c, \beta_1 - \beta'_1 - \gamma_i}\right)"'] & M^{(2)}_{[\gamma_i]} \arrow[d, "\tau^{(2)}_{c^2, -\gamma_i}"] \\
\bigoplus\limits_{\beta_1 \in \Omega_n \cap \Delta^{\mathrm{gp}}} M^{(1)}_{[\beta'_1]} \otimes_{B_n^{\mathrm{sh}}} M^{(1)}_{[-\beta'_1]} \arrow[r] & M^{(2)}_{[0]}
\end{tikzcd}$$
whose bottom horizontal arrow is a restriction of the $[0]$-graded summand of the natural map
$$\mu_{\mathscr{R}_{(m)}} : \Gamma(U_{/b}, \mathscr{R}_{(m)}) \otimes_{D_{(m)}} \Gamma(U_{/b}, \mathscr{R}_{(m)}) \to \Gamma(U_{/b}, \mathscr{R}_{(m)} \otimes_{\mathscr{D}_{(m)}} \mathscr{R}_{(m)}).$$
The B_n-module $A_{[\gamma_i]}$ contains an element which gets identified with $\gamma_i \otimes c$ under the isomorphism (9.3.46), hence
$$c^3 e_n \in \tau^{(2)}_{c^2, -\gamma_i}(A^{\mathrm{sh}}_{[\gamma_i]} e_n).$$
We deduce that $a_i a^3 c^3 e_n$ lies in the image of $\mu_{\mathscr{R}_D}$, and since i is arbitrary, it follows that $a^3 c^3 e_n$ lies in the image of $(\mu^a_{\mathscr{R}_D})_*$, so $a^6 c^3 e$ is an almost element of the image of the corresponding morphism
$$\mu_{\mathscr{A}_{(m)}} : \Gamma(U_{/b}, \mathscr{A}_{(m)}) \otimes_{D_{(m)}} \Gamma(U_{/b}, \mathscr{A}_{(m)}) \to \Gamma(U_{/b}, \mathscr{A}_{(m)} \otimes_{\mathscr{D}_{(m)}} \mathscr{A}_{(m)}).$$
Lastly, since a can be taken arbitrarily small, we conclude that e lies in the image of $\mu_{\mathscr{A}_{(m)}*}$.

Claim 9.4.33. The natural map $\mathscr{D}_{(m)/b} \to j_{/b*} j^*_{/b} \mathscr{D}_{(m)/b}$ is an isomorphism.

Proof of the claim. Indeed, for every $k \in \mathbb{N}$, let $j_{k/b} : U_{k/b} \to X_{k/b}$ be the open immersion, set
$$\Delta_{k,(m)} := \{\gamma \in \Delta_{\mathbb{Q}} \mid m\gamma \in \Delta_k\} \qquad D_k := \bigoplus_{\gamma \in \Delta_{k,(m)}} A_\gamma$$
and denote by $\mathscr{D}_k$ the coherent $\mathscr{O}_{X_{k/b}}$-algebra determined by $D_k \otimes_{B_k} B_k^{\mathrm{sh}}/bB^{\mathrm{sh}}$. In view of proposition 5.1.15(ii), we reduce to showing that the natural map $\mathscr{D}_k \to j_{k/b*} j^*_{k/b} \mathscr{D}_k$ is an isomorphism for every $k \in \mathbb{N}$. Then, we may write $X_{k/b}$ as the limit of a filtered system $(g_i : W_i \to Y_{k/b} \mid i \in I)$ of étale $Y_{k/b}$-schemes; set $U_i := W_i \setminus g_i^{-1}(x_n)$ (where $x_n \in Y_n$ is the support of $\overline{x}_n$), let $j_i : U_i \to W_i$ the open immersion, $\varphi_i : W_i \to S$ the composition of g_i with the structure morphism $Y_k \to S$, and denote by $\mathscr{D}_i$ the coherent $\mathscr{O}_{W_i}$-algebra determined by $D_k \otimes_{B_k} \mathscr{O}_{W_i}$; invoking again proposition 5.1.15(ii), we further reduce to checking that the natural map $\mathscr{D}_i \to j_{i*} j_i^* \mathscr{D}_i$ is an isomorphism. However, lemmata 5.6.36(iii) and 9.3.42(ii) imply that $\mathscr{D}_i$ is φ_i-Cohen-Macaulay. Then, from lemma 5.6.36(iv) and assumption (b) we see that $\delta'(w, \mathscr{O}_{W_i}) \geq 2$ for every $i \in I$, and every point $w \in W_i \setminus U_i$. The assertion follows from the exact sequence (5.4.2). ◊

The proposition follows from claim 9.4.33 and corollary 8.2.16. □

Theorem 9.4.34. *Keep the situation of* (9.4.20), *and suppose as well that* $d \geq 3$. *Then the pair* $(X, \{s(B^{\mathrm{sh}})\})$ *is almost pure.*

Proof. Since X is a normal scheme, the pair $(X, \{s(B^{\mathrm{sh}})\})$ is normal, so it suffices to check that every étale almost finitely presented $\mathscr{O}_U^a$-algebra $\mathscr{A}$ extends to an étale almost finite $\mathscr{O}_X^a$-algebra (proposition 8.2.30). On the one hand, lemmata 9.4.25 and 8.3.46 imply that some power of π annihilates $C_{\mathscr{A}}$; on the other hand, arguing as in the proof of theorem 9.2.23, we see that $\overline{\Phi}^*(C_{\mathscr{A}}/\pi C_{\mathscr{A}})$ admits a filtration of length p, whose subquotients are quotients of $C/\pi C_{\mathscr{A}}$. In this situation, the proof of corollary 9.4.19 shows that $\lambda(C_{\mathscr{A}}/\pi C_{\mathscr{A}}) = 0$, and then, by an easy induction we get $\lambda(C_{\mathscr{A}}) = 0$.

Let $j : U \to X$ be the open immersion; by proposition 9.4.27, we deduce that there exists an integer $m > 0$ such that $j_*\mathscr{A}_{(m)}$ is an étale almost finitely presented $\mathscr{D}^a_{(m)}$-algebra.

The last step consists in descending $j_*\mathscr{A}_{(m)}$ to an étale almost finitely presented $\mathscr{O}_X^a$-algebra, and to this aim we shall apply the technique of section 8.6. Indeed, notice first that the field of fractions of $D_{(m)}$ is a finite Galois extension of the field of fractions of B, whose Galois group G admits a natural isomorphism

$$G \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}}(\Delta^{\mathrm{gp}}, \boldsymbol{\mu}_m) \qquad \sigma \mapsto \rho_\sigma \tag{9.4.35}$$

(where $\boldsymbol{\mu}_m \subset K^\times$ is the subgroup of m-th roots of one : see (7.3.31)). Since $D_{(m)}$ is a normal domain (lemma 9.3.16(ii)), it follows that G is the automorphism group of the B-algebra $D_{(m)}$, and $D^G_{(m)} = B$. Then G is also the automorphism group of the B^{sh}-algebra $D^{\mathrm{sh}}_{(m)}$ (resp. of the $\mathscr{O}_X$-algebra $\mathscr{D}_{(m)}$), and $(D^{\mathrm{sh}}_{(m)})^G = B^{\mathrm{sh}}$ (resp. and $\mathscr{D}^G_{(m)} = \mathscr{O}_X$). Consequently, G acts on $\mathscr{A}_{(m)}$ and $\mathscr{A}^G_{(m)} = \mathscr{A}$. Set $E := \Gamma(U, \mathscr{A}_{(m)})$; then E is an étale $(D^{\mathrm{sh}}_{(m)})^a$-algebra, and corollary 8.6.28(ii) reduces to showing that the induced G-action on E is horizontal. To this aim, fix $\sigma \in G$, and denote by $I_\sigma \subset D_{(m)}$ the ideal generated by the elements of the form $x - \sigma(x)$, for x ranging over all the elements of $D_{(m)}$; we have to show that σ acts trivially on $E^\sigma := E/I_\sigma E$.

Claim 9.4.36. $D^a_{(m)}/I^a_\sigma$ is a flat K^{+a}-algebra.

Proof of the claim. Recall that each graded summand A_γ of $D_{(m)}$ is a free K^+-module generated by an element $g_\gamma \otimes 1$ (notation of (9.3.39)), and by inspecting the constructions, it is easily seen that – under the identification (9.4.35) – the action of σ is determined by the rule :

$$g_\gamma \otimes 1 \mapsto g_\gamma \otimes \rho_\sigma(m\gamma) \qquad \text{for every } \gamma \in \Delta_{(m)}.$$

Notice that $1 - \zeta$ is invertible in K^+, whenever $\zeta \in \boldsymbol{\mu}_m \setminus \{1\}$; thus, I_σ is generated by the direct sum of the A_γ, for those $\gamma \in \Delta_{(m)}$ such that $\rho_\sigma(m\gamma) \neq 1$; especially, I_σ is a graded ideal. The claim will follow, once we have shown that, for every $\gamma \in \Delta_{(m)}$, the graded summand $(I_\sigma)^a_\gamma$ is either 0 or the whole of A^a_γ. Now, $(I_\sigma)_\gamma$ is generated by all products $A_{\gamma_1}A_{\gamma_2}$ where $\gamma_1 + \gamma_2 = \gamma$, $\gamma_1, \gamma_2 \in \Delta_{(m)}$ and $\rho_\sigma(m\gamma_1) \neq 1$; for every such pair (γ_1, γ_2), let $c(\gamma_1, \gamma_2) \in K^+$ be the unique element such that

$$(g_{\gamma_1} \otimes 1) \cdot (g_{\gamma_2} \otimes 1) = g_\gamma \otimes c(\gamma_1, \gamma_2).$$

By inspecting the constructions in (9.3.35), we see that

$$c(p^k\gamma_1, p^k\gamma_2) = c(\gamma_1, \gamma_2)^{p^k} \qquad \text{for every } k \in \mathbb{Z}.$$

Notice that $\rho_\sigma(p^{-k}m\gamma_1) \neq 1$ for every $k \in \mathbb{N}$; therefore

$$g_\gamma \otimes c(\gamma_1, \gamma_2)^{p^{-k}} = (g_{p^{-k}\gamma_1} \otimes 1) \cdot (g_{p^{-k}\gamma_2} \otimes 1) \cdot (g_{p^{-k}\gamma} \otimes 1)^{p^k - 1} \in I_\sigma \qquad \text{for every } k \in \mathbb{N}.$$

Since k is arbitrary, the contention follows. $\Diamond$

Since E is a flat $D_{(m)}$-algebra, we have $E^\sigma = E \otimes_{D^a_{(m)}} D^a_{(m)}/I^a_\sigma$, and it follows – by claim 9.4.36 – that $E/I_\sigma E$ is a flat K^{+a}-algebra, hence we are reduced to showing that σ acts trivially on $E^\sigma_K := E^\sigma \otimes_{K^+} K$. However, set $U_K := U \times_S \mathrm{Spec}\, K$ and $E_K := E \otimes_{K^+} K$; clearly

$E_K = \Gamma(U_K, \mathscr{A}_{(m)})$ descends to the étale and finitely presented $B^{\mathrm{sh}} \otimes_{K^+} K$-algebra $\Gamma(U_K, \mathscr{A})$, therefore the G-action on E_K is horizontal (corollary 8.6.28(ii)). To conclude, it suffices to remark that $E_K^\sigma = E_K / I_\sigma E_K$. $\square$

9.4.37. Let now (A, Γ) be any model K^+-algebra; set $S := \mathrm{Spec}\, K^+$, $X := \mathrm{Spec}\, A$, and let $\overline{x}$ be any geometric point of X, localized on the closed subset $Z := X \times_S \mathrm{Spec}\, \kappa$. Suppose furthermore, that K^+ is deeply ramified (see [36, Def.6.6.1]), and let $(K^+, \mathfrak{m}_K)$ be the standard setup associated to K^+ (see [36, §6.1.15]); then we have the corresponding sheaf $\mathscr{O}^a_{X(\overline{x})}$ of K^{+a}-algebras on $X(\overline{x})$, and we may state the following *almost purity* theorem :

Theorem 9.4.38. *With the notation of* (9.4.37), *the pair* $(X(\overline{x}), Z(\overline{x}))$ *is almost pure.*

Proof. Set $A^{\mathrm{sh}} := \mathscr{O}_{X(\overline{x}),\overline{x}}$, $A^{\mathrm{sh}}_K := A^{\mathrm{sh}} \otimes_{K^+} K$, and suppose that $A^{\mathrm{sh}}_K \to C_K$ is a finite étale ring homomorphism. Let C be the integral closure of A^{sh} in C_K. As in the proof of theorem 9.4.34, it is easily seen that the pair $(X(\overline{x}), Z(\overline{x}))$ is normal; by proposition 8.2.30, it then suffices to prove that the induced morphism $(A^{\mathrm{sh}})^a \to C^a$ of K^{+a}-algebras, is étale and almost finite (and indeed, weakly unramified would already be enough).

Claim 9.4.39. We may assume that K is algebraically closed.

Proof of the claim. Indeed, let E be an algebraic closure of K, and $|\cdot|_E$ a valuation on E which extends $|\cdot|$. Set $B := A \otimes_{K^+} E^+$, choose a geometric point $\overline{y}$ of $Y := \mathrm{Spec}\, B$ whose image in X is isomorphic to $\overline{x}$, and let $B^{\mathrm{sh}} := \mathscr{O}_{Y(\overline{y}),\overline{y}}$. Then (B, Γ) is a model E^+-algebra (remark 9.3.3(iv)). Let $D_K := C_K \otimes_{A^{\mathrm{sh}}} B^{\mathrm{sh}}$, and denote by D the integral closure of B^{sh} in D_K. Let also W_E be the integral closure of K^+ in E, and notice that B^{sh} is a localization of $A^{\mathrm{sh}} \otimes_{K^+} W_E$ ([33, Ch.IV, Prop.18.8.10]). Moreover, W_E^a is a filtered colimit of étale almost finite projective K^{+a}-algebras of finite rank ([36, Prop.6.3.8, Rem.4.3.10(i) and Prop.6.6.2]), therefore $(C \otimes_{K^+} W_E)^a$ is integrally closed in $(C_K \otimes_{K^+} W_E)^a$ ([36, Prop.8.2.31(i)]), and it follows easily that $D^a = (C \otimes_{A^{\mathrm{sh}}} B^{\mathrm{sh}})^a$. Suppose now, that D^a is an almost finite and étale $(B^{\mathrm{sh}})^a$-algebra; then [36, §3.4.1] implies that C is an almost finite étale $(A^{\mathrm{sh}})^a$-algebra, whence the claim. $\Diamond$

Henceforth, we assume that K is algebraically closed. In this case, let $\mathscr{B}$ be the family of all small model K^+-subalgebras of (A, Γ); for every $(B, \Delta) \in \mathscr{B}$ let $\overline{x}_B$ be the image of $\overline{x}$ in $X_B := \mathrm{Spec}\, B$, and let $B^{\mathrm{sh}} := \mathscr{O}_{X_B(\overline{x}_B),\overline{x}_B}$. In view of proposition 9.3.18, we have

$$A^{\mathrm{sh}} = \operatorname*{colim}_{(B,\Delta)\in\mathscr{B}} B^{\mathrm{sh}}$$

([33, Ch.IV, Prop.18.8.18(ii)]). We may then find $(B, \Delta) \in \mathscr{B}$, and a finite étale ring homomorphism $B^{\mathrm{sh}}_K := B^{\mathrm{sh}} \otimes_{K^+} K \to C'_K$, with an isomorphism $C_K \xrightarrow{\sim} C'_K \otimes_{B^{\mathrm{sh}}} A^{\mathrm{sh}}$ of A^{sh}_K-algebras ([33, Ch.IV, Prop.17.7.8(ii)] and [32, Ch.IV, Th.8.10.5]). Let C' be the integral closure of B^{sh} in C'_K, and suppose that C'^a is an almost finite and étale $(B^{\mathrm{sh}})^a$-algebra; then proposition 8.2.17 shows that $(C' \otimes_{B^{\mathrm{sh}}} A^{\mathrm{sh}})^a$ is integrally closed in $(C_K)^a$, *i.e.* it equals C^a, so the latter shall be an almost finite and étale $(A^{\mathrm{sh}})^a$-algebra, as required. This shows that we may replace throughout (A, Γ) by (B, Δ), and assume from start that (A, Γ) is a small model K^+-algebra. By remark 9.3.15(i) we have $(A, \Gamma) = (A', \Gamma') \otimes (A'', \Gamma'')$, where Γ'^{gp} is torsion-free, and Γ'' is a finite abelian group of order prime to p, so that A'' is an étale K^+-algebra (remark 9.3.15(iii)); therefore, if $\overline{x}'$ is the image of $\overline{x}$ in $X' := \mathrm{Spec}\, A'$, we have a natural isomorphism $A^{\mathrm{sh}} \xrightarrow{\sim} \mathscr{O}_{X'(\overline{x}'),\overline{x}'}$ of K^+-algebras. In other words, we may assume that $A = A'$, and that Γ^{gp} is torsion-free.

In this case, notice that X has finite Krull dimension; let $x \in Z(\overline{x})$ be the support of $\overline{x}$; corollary 8.2.31 easily reduces to checking that the pair $(X(\overline{x}), \{x\})$ is almost pure.

To this aim, we shall apply the discussion of (9.3.19), that yields

- a Γ-graded monoid P such that P_0 is sharp, and a local map $P_0 \to K^+$ such that $A = P \otimes_{P_0} K^+$

- increasing exhaustive filtrations $(\Gamma_n \mid n \in \mathbb{N})$ of Γ, and $(P^{(n)} \mid n \in \mathbb{N})$ of P, such that $P^{(n)}$ is Γ_n-graded submonoid of P, for every $n \in \mathbb{N}$
- a ladder (9.3.20), whose vertical arrows are smooth morphisms of log schemes

$$\varphi_n : (Y_n, \underline{M}_n) \to (S, \underline{N}_n)$$

where $Y_n := \operatorname{Spec} A_n$, with $A_n := P^{(n)} \otimes_{P_0^n} K^+$, and the induced map $P^{(n)} \to A_n$ (resp. $P_0^{(n)} \to K^+$) provides a chart for $\underline{M}_n$ (resp. for $\underline{N}_n$). Also, the underlying morphisms of schemes $Y_{n+1} \to Y_n$ are finite and surjective, induced by the inclusion map $P^{(n)} \to P^{(n+1)}$.

Moreover, following remark 9.3.24(iv), we also choose a compatible system of decompositions

$$(9.4.40) \qquad (Y_n, \underline{M}_n) \xrightarrow{\sim} Y'_n \times_S (Y''_n, \underline{M}''_n) \qquad \text{for every } n \in \mathbb{N}$$

such that $Y'_n = \operatorname{Spec} K^+[P^{(n)\times}]$, $Y''_n := \operatorname{Spec} A''_n$, with $A''_n := P^{(n)\sharp} \otimes_{P_0^{(n)}} K^+$, the induced map $P^{(n)\sharp} \to A''_n$ provides a chart for $\underline{M}''_n$, and φ_n factors as the composition of a smooth morphism of log schemes

$$\varphi''_n : (Y''_n, \underline{M}''_n) \to (S, \underline{N}_n)$$

and the projection $(Y_n, \underline{M}_n) \to (Y''_n, \underline{M}''_n)$ deduced from (9.4.40). Furthermore, for every $n \in \mathbb{N}$, denote by $\overline{x}_n$ the image of $\overline{x}$ in the closed fibre $Z_n \subset Y_n$ of the structure morphism $\varphi_n : Y_n \to S$; according to remark 9.3.24(v), we may also assume – after replacing A by a suitable localization – that the induced morphism $P^{(n)} \to \mathscr{O}_{Y_n, \overline{x}_n}$ is local, for every $n \in \mathbb{N}$.

We argue now by induction on $d := \dim Z_0(\overline{x}_0)$. If $d = 0$, then the support of $\overline{x}_0$ is a maximal point of Z_0; hence, for every $n \in \mathbb{N}$, the support of $\overline{x}_n$ is a maximal point of Z_n. In this case, we know that $(\log \varphi_n)^\sharp_{\overline{x}_n}$ is an isomorphism (theorem 6.7.8(iii.a)), and since $P_0^{(n)}$ is sharp, and both maps $P_0^{(n)} \to K^+$ and $P^{(n)} \to \mathscr{O}_{Y_n, \overline{x}_n}$ are local, we see that the inclusion $P_0^{(n)} \to P^{(n)}$ induces an isomorphism

$$P_0^{(n)} \xrightarrow{\sim} P^{(n)\sharp} \qquad \text{for every } n \in \mathbb{N}$$

whence $B''_n = K^+$, and we get an isomorphism

$$K^+[P^{(n)\times}] \xrightarrow{\sim} B_n \qquad \text{for every } n \in \mathbb{N}.$$

By construction, the transition maps $Y_{n+1} \to Y_n$ are induced by the inclusions of abelian groups $P^{(n)\times} \subset P^{(n+1)\times}$, for every $n \in \mathbb{N}$. It is easily seen that $\mathscr{O}_{Y_n, \overline{x}_n}$ is the valuation ring of a Gauss valuation of rank one, extending the valuation of K ([36, Ex.6.1.4(iv)]); then A^{sh} is a valuation ring as well. From (9.3.22) we see that the p-Frobenius maps induce isomorphisms $P^{(n+1)} \to P^{(n)}$ for every $n \in \mathbb{N}$, and then [36, Prop.6.6.6] tells us that A^{sh} is deeply ramified. In this situation the theorem is a rephrasing of [36, Prop.6.6.2] : we leave to the reader the task of spelling out the details.

Next, suppose that $d = 1$; in this case, we know that $\operatorname{Coker}(\log \varphi_n^\sharp)^{\mathrm{gp}}_{\overline{x}_n}$ is a free abelian group of rank $r \leq 1$ (theorem 6.7.8(ii,iii.c)). If $r = 0$, then $(\log \varphi_n^\sharp)^{\mathrm{gp}}_{\overline{x}_n}$ is an isomorphism, and then the same holds for $(\log \varphi_n^\sharp)_{\overline{x}_n}$ (corollary 3.2.32(i)); we may thus repeat the foregoing considerations for the case $d = 0$, so we still have $Y_n = \operatorname{Spec} K^+[P^{(n)\times}]$ for every $n \in \mathbb{N}$. In this case A_n is a K^+-algebra of the type $R(1)$ as in (9.1.1), and the tower $(Y_n \mid n \in \mathbb{N})$ is of the type described in (9.1.25), so the contention is none else than theorem 9.1.31.

If $r = 1$, denote by $\overline{y}'_n$ (resp. $\overline{y}''_n$) the image of $\overline{x}_n$ in the closed fibre Z'_n (resp. Z''_n) of the projection $\varphi'_n : Y'_n \to S$ (resp. $\varphi''_n : Y''_n \to S$); then $\operatorname{Coker}(\log \varphi''_0)^{\sharp\mathrm{gp}}_{\overline{y}''_0}$ is again a free abelian group of rank 1, therefore

$$(9.4.41) \qquad \dim Z''_0(\overline{y}''_0) \geq 1$$

(theorem 6.7.8(ii)). On the other hand, we point out the following general :

Claim 9.4.42. Let κ be a field, T_1, T_2 two κ-schemes of finite type, and $t \in T := T_1 \times_\kappa T_2$ any point. Denote by $t_1 \in T_1$ and $t_2 \in T_2$ the images of t. Then we have

$$\dim T(t) \geq \dim T_1(t_1) + \dim T_2(t_2).$$

Proof of the claim. To ease notation, set $R_i := \mathscr{O}_{T_i,t_i}$ for $i := 1, 2$; according to [31, Ch.IV, Prop.6.1.1] we have

$$\dim \mathscr{O}_{T,t} = \dim R_1 + \dim \mathscr{O}_{T,t} \otimes_{R_1} \kappa(t_1).$$

However, $\mathscr{O}_{T,t}$ is a localization of $R_1 \otimes_\kappa R_2$, hence $\mathscr{O}_{T,t} \otimes_{R_1} \kappa(t_1)$ is a localization of $R_2 \otimes_\kappa \kappa(t_1)$. It follows that the induced map $R_2 \to \mathscr{O}_{T,t} \otimes_{R_1} \kappa(t_1)$ is flat and local; therefore $\dim \mathscr{O}_{T,t} \otimes_{R_1} \kappa(t_1) \geq \dim R_2$, by the going down theorem. $\diamondsuit$

Combining (9.4.41) and claim 9.4.42 we see that $\dim Z'_0(\overline{y}'_0) = 0$ and $\dim Z''_0(\overline{y}''_0) = 1$. Let X' be the limit of the projective system of schemes $(Y'_n \mid n \in \mathbb{N})$, and denote by $\overline{x}'$ the image of $\overline{x}$ in X'. The foregoing case $d = 0$ shows that $\mathscr{O}_{X'(\overline{x}'),\overline{x}'}$ is a deeply ramified valuation ring of rank one. Set $X''_n := Y''_n \times_S X'(\overline{x}')$, and let $\overline{x}''_n$ be the image of $\overline{x}$ in X''_n, for every $n \in \mathbb{N}$; clearly $X(\overline{x})$ is the limit of the projective system of $X'(\overline{x}')$-schemes $(X''_n(\overline{x}''_n) \mid n \in \mathbb{N})$. Thus, we may replace S by $X'(\overline{x}')$, $P^{(n)}$ by $P^{(n)\sharp}$, and Γ_n by $\Gamma_n/P^{(n)\times}$, after which we may assume that $P^{(n)}$ is sharp, $\Gamma_n^{\mathrm{gp}} \simeq \mathbb{Z}$, the inclusion map $P_0^{(n)} \to P^{(n)}$ is still flat and saturated (corollary 3.1.49(i) and lemma 3.2.12(iii)), and Γ_n is still saturated (lemma 3.2.9(ii)) for every $n \in \mathbb{N}$. Then Γ_n is isomorphic to either $\mathbb{N}$ or $\mathbb{Z}$.

Suppose that $\Gamma_n = \mathbb{Z}$ for every $n \in \mathbb{N}$. In this case, the $P_0^{(n)}$-modules $P_1^{(n)}$ and $P_{-1}^{(n)}$ are both free of rank one, with unique generators $e_1^{(n)}$ and respectively $e_{-1}^{(n)}$; moreover, the monoid $P^{(n)}$ is generated by $P_0^{(n)} \cup \{e_1^{(n)}, e_{-1}^{(n)}\}$, and the inclusions $P^{(n)} \to P^{(n+1)}$ map $e_1^{(n)} \mapsto (e_1^{(n+1)})^p$ and $e_{-1}^{(n)} \mapsto (e_{-1}^{(n+1)})^p$, for every $n \in \mathbb{N}$. Furthermore, notice that $e_1^{(n)} \cdot e_{-1}^{(n)} \in P_0^{(n)} \setminus \{1\}$, since $P^{(n)}$ is sharp. Summing up, we conclude that in this case A_n is a K^+-algebra of the type $R(\gamma)$ as in (9.1.1), for some $\gamma \in \mathfrak{m}$, and the tower $(Y_n \mid n \in \mathbb{N})$ is again of the type described in (9.1.25). Then, again the contention is theorem 9.1.31.

If $\Gamma_n = \mathbb{N}$ for every $n \in \mathbb{N}$, a similar analysis shows that $A_n = K^+[\mathbb{N}]$, and the inclusion maps $A_n \subset A_{n+1}$ are induced by the p-Frobenius endomorphism of $\mathbb{N}$, for every $n \in \mathbb{N}$. This is precisely the situation contemplated in remark 9.1.33(ii), so also this case is taken care of.

Lastly, in case $d \geq 2$, the assertion is none else than theorem 9.4.34. The proof is concluded. $\square$

9.5. **Purity of the special fibre.** This section studies pairs $(X_{/p}, \{x\})$, where x is the closed point of $X_{/p}$ (notation of (5.7)), with X as in (9.4.20). We shall completely characterize the cases where such a pair is almost pure, and describe precisely the obstruction to almost purity in the other cases. To begin with, $(K, |\cdot|)$ is a valued field with non-discrete value group Γ of rank one. We shall resume the notation of (8.3.1).

9.5.1. Let A be any (commutative, unitary) ring; we denote by $A^\iota \subset A$ the subset of idempotent elements of A. We define a structure of commutative unitary ring on A^ι as follows. The multiplication of A^ι is just the restriction of the multiplication law of A. The addition on A^ι is given by the rule : $a +^\iota b := a + b - 2ab$ (where $a + b$ denotes the addition law of A, and likewise the subtraction is taken in A). Notice that the zero element of A is the neutral element for this addition on A^ι. As an exercise, the reader may check that this addition law satisfies associativity with respect to the multiplication, and that every element admits an opposite (indeed, $a +^\iota a = 0$ for every $a \in A^\iota$). Moreover, every ring homomorphism $f : A \to B$ induces by restriction a ring homomorphism $f^\iota : A^\iota \to B^\iota$.

Lemma 9.5.2. *Let A be a K^+-algebra, and $b_0 \in \mathfrak{m}_K$ such that the following holds :*

(TF/b_0) *For every $b \in \mathfrak{m}_K$ that divides b_0, the quotient A/bA has no non-zero $\mathfrak{m}_K$-torsion elements.*

Then we have :

(i) *There is a natural isomorphism of K^+-algebras :*

$$A^a_* \xrightarrow{\sim} \lim_{b\in\mathfrak{m}_K} A/\mathrm{Ann}_A(b).$$

(ii) *If furthermore,* $\mathrm{Supp}\, b_0 A = \mathrm{Spec}\, A$, *then the induced map $A^\iota \to (A^a_*)^\iota$ is bijective.*

Proof. (i): In view of [36, (2.2.4)], we can write A^a_* as the limit of the cofiltered system

$$(\mathrm{Hom}_{K^+}(bK^+, A) \mid b \in \mathfrak{m}_K)$$

and under the natural identification $\mathrm{Hom}_{K^+}(bK^+, A) \xrightarrow{\sim} A$, this is the same as the limit of the cofiltered system $(A_{(b)} \mid b \in \mathfrak{m}_K)$, where $A_{(b)} := A$ for every $b \in \mathfrak{m}_K$, and for b dividing b', the transition map $A_{(b)} \to A_{(b')}$ is scalar multiplication by $b^{-1}b'$. For every $b \in \mathfrak{m}_K$, denote by $A'_{(b)}$ the image of the induced map $A^a_* \to A_{(b)}$; clearly, A^a_* is also the limit of the cofiltered system $(A'_{(b)} \mid b \in \mathfrak{m}_K)$. On the other hand, let $f : \mathfrak{m}_K \to A$ be any almost element; the image of f in $A'_{(b)}/bA'_{(b)}$ is clearly a $\mathfrak{m}_K$-torsion element, so it vanishes by assumption, whenever b divides b_0, *i.e.* $A'_{(b)} = bA = A/\mathrm{Ann}_A(b)$ for every such b. Since the subset of all b that divide b_0 is cofinal in $\mathfrak{m}_K$, the assertion follows.

(ii): For any ring R, an idempotent of R is the same as the datum of a partition of $\mathrm{Spec}\, R$ as a disjoint union of two open subsets. Assertion (i) implies that an idempotent $e \in A^a_*$ is the same as a compatible system of idempotents of $R_{(b)} := A/\mathrm{Ann}_A(b)$ for all $b \in \mathfrak{m}_K$. Set $X_b := \mathrm{Spec}\, R_{(b)}$; we conclude that e is the same as a compatible system of partitions $X_b = U_b \cup U'_b$ by disjoint open subsets. However, if $\mathrm{Supp}\, b_0 A = X := \mathrm{Spec}\, A$, the schemes X and X_b have the same underlying topological space, whenever b divides b_0. The contention is an immediate consequence. □

Remark 9.5.3. Let A be a K^+-algebra and $b_0 \in \mathfrak{m}_K$. The following assertions are immediate :

(i) If A satisfies condition (TF/b_0), then the same holds for A/bA, for every $b \in K^+$.

(ii) Suppose that for, every maximal ideal $\mathfrak{m} \subset A$, the localization $A_\mathfrak{m}$ satisfies condition (TF/b_0). Then the same holds for A (details left to the reader).

(iii) Suppose that A is the colimit of a filtered system $(A_\lambda \mid \lambda \in \Lambda)$ of K^+-algebras fulfilling condition (TF/b_0), and such that all transition morphisms $A_\lambda \to A_\mu$ are pure, when regarded as maps of K^+-modules (see (8.6.20)). Then A satisfies condition (TF/b_0) as well.

Lemma 9.5.4. *Let A be a K^+-algebra whose Jacobson radical contains $\mathfrak{m}_K A$, and suppose that either one of the following conditions holds :*

(a) *A is a locally measurable K^+-algebra.*

(b) *K is a field of characteristic zero, with residue field κ of characterisitic $p > 0$, and A is a small model K^+-algebra.*

Then A fulfills condition (TF/b_0) *of lemma* 9.5.2*, for every $b_0 \in \mathfrak{m}_K$.*

Proof. Suppose first that (a) holds. For given $b_0 \in \mathfrak{m}_K$, let x be a $\mathfrak{m}_K$-torsion element of $A' := A/b_0A$, and set $M := A'x$. We wish to show that $M = 0$. To this aim, it suffices to show that $M_\mathfrak{m} = 0$ for every maximal ideal $\mathfrak{m} \subset A$. Thus, we may replace A by $A_\mathfrak{m}$, and assume that A is local and locally measurable. In this case, M is a finitely presented A-module (remark 8.7.4(iv)), and therefore it admits a minimal K^+-flattening sequence $(c_0, \dots, c_n)$ (proposition 8.7.11). Suppose now that $M \neq 0$; then $n > 1$, and clearly every b in $\mathfrak{m}_K$ that divides b_0 breaks M (see definition 8.7.6(ii)); taking $\gamma := \log|b| < \log|c_1|$ in lemma 8.7.8(ii), we then get a contradiction.

Next, notice that every finitely generated flat K^+-algebra fulfills condition (a), since such a K^+-algebra is finitely presented (proposition 5.7.1). On the other hand, if A fulfills condition (b), we may write A as the increasing union of a system of finitely generated subalgebras $(A_n \mid n \in \mathbb{N})$ as in remark 9.3.15(ii). More precisely, each A_n is a direct summand of the K^+-module A, hence A_n/b_0A_n is a direct summand of A/b_0A, for every $n \in \mathbb{N}$ and $b_0 \in K^+$. It follows easily that the claim holds as well in case (b). $\square$

Remark 9.5.5. (i) Suppose that K and A fulfill condition (b) of lemma 9.5.4, and define the system $(A_n \mid n \in \mathbb{N})$ as in the proof of lemma 9.5.4, so that each A_n satisfies condition (TF$/b_0$) for every $b_0 \in \mathfrak{m}_K$. Fix any prime ideal $\mathfrak{p} \subset A$ with $\mathfrak{m}_K \subset \mathfrak{p}$, and let $\mathfrak{p}_n := \mathfrak{p} \cap A_n$, for every $n \in \mathbb{N}$. Then the henselization A^{h} of $A_{\mathfrak{p}}$ is the colimit of the system of henselizations $(A^{\mathrm{h}}_n \mid n \in \mathbb{N})$ of the localizations $A_{n,\mathfrak{p}_n}$. However, a simple inspection of the definition shows that the transition maps $A_n \to A_{n+1}$ are pure for every $n \in \mathbb{N}$, and induce radicial maps $A_n \otimes_{K^+} \kappa \to A_{n+1} \otimes_{K^+} \kappa$, hence $A^{\mathrm{h}}_{n+1} = A^{\mathrm{h}}_n \otimes_{A_n} A_{n+1}$, and we deduce that the transition maps $A^{\mathrm{h}}_n \to A^{\mathrm{h}}_{n+1}$ are pure as well. From remark 9.5.3(iii), we conclude that A^{h} fulfills condition (TF$/b_0$) for every $b_0 \in \mathfrak{m}_K$.

(ii) Likewise, if K and A are as in (i), then for every geometric point ξ of $\operatorname{Spec} A \otimes_{K^+} \kappa$, the strict henselization of A at ξ fulfills condition (TF$/b_0$) for every $b_0 \in \mathfrak{m}_K$: the details shall be left to the reader.

Lemma 9.5.6. *Let A be a K^+-algebra whose Jacobson radical contains $\mathfrak{m}_K A$, and suppose that either one of the following conditions holds :*

(a) *There exists $b \in \mathfrak{m}_K$ such that A is a flat K^+/bK^+-algebra.*

(b) *A is locally measurable and $A/\mathfrak{m}_K A$ is noetherian.*

Then there exists $b_0 \in \mathfrak{m}_K$ such that $\operatorname{Supp} b_0 A = \operatorname{Spec} A$.

Proof. In case (a) holds, we claim that the lemma holds for any b_0 such that $|b_0| > |b|$. Indeed, notice that, for such b_0, the natural maps

$$b_0 A \leftarrow \frac{b_0 K^+}{bK^+} \otimes_{K^+} A \to B := \frac{K^+}{b_0^{-1} bK^+} \otimes_{K^+} A$$

are all isomorphisms of A-modules. Moreover, the closed immersion $\operatorname{Spec} K^+/b_0^{-1}bK^+ \to \operatorname{Spec} K^+/bK^+$ is radicial, so the natural morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ is a homeomorphism ([31, Ch.IV, Prop.2.4.5(i)]). The assertion follows straightforwardly.

Next, under assumption (b), proposition 8.7.14(i) and theorem 8.7.17(i) imply that A admits a minimal K^+-flattening sequence $(c_0, \dots, c_n)$. Now, if $A = 0$, there is nothing to prove. If $A \neq 0$, we have $n > 1$ and we shall show that the lemma holds with any $b_0 \in \mathfrak{m}_K$ such that $|b_0| > |c_1|$. To this aim, notice that

$$\operatorname{Spec} A = (\operatorname{Spec} A/c_1 A) \cup (\operatorname{Supp} c_1 A) \quad \text{and} \quad \operatorname{Supp} b_0 A = (\operatorname{Supp} b_0 A/c_1 A) \cup (\operatorname{Supp} c_1 A)$$

for every such b_0; hence we are reduced to showing that $\operatorname{Supp} b_0 A/c_1 A = \operatorname{Spec} A/c_1 A$. We may thus replace A by $A/c_1 A$, and assume from start that A is a flat K^+/c_1K^+-algebra, which is the case already covered. $\square$

9.5.7. We consider the site $\mathscr{S} := (K^{+a}\text{-}\mathbf{Alg})^o_{\mathrm{fpqc}}$ whose objects are all the K^{+a}-algebras A such that A_* is U-small (where U is a fixed universe). This site is not a U-site, hence we choose a second universe U' such that $\mathsf{U} \in \mathsf{U}'$, and we let $T := \mathscr{S}^\sim_{\mathsf{U}'}$ (notation of definition 2.1.15(ii)), which is therefore a U'-topos. Denote also by $\mathscr{O}_*$ the presheaf on $\mathscr{S}$ given by the rule :

$$\operatorname{Spec} A \mapsto A_* \qquad \text{for every } K^{+a}\text{-algebra } A.$$

Lemma 9.5.8. *$\mathscr{O}_*$ is a T-ring.*

Proof. Let $\mathrm{Spec}\, A$ be any object of $\mathscr{S}$, and denote by $J(A)$ the set of covering sieves of $\mathrm{Spec}\, A$, partially ordered by inclusion. Then $J(A)$ admits a cofinal subset, consisting of all the sieves $\mathscr{J}_f$ generated by a single morphism $f^o : \mathrm{Spec}\, B \to \mathrm{Spec}\, A$, corresponding to a faithfully flat morphism of K^{+a}-algebras $f : A \to B$. Since the functor $B \mapsto B_*$ on K^{+a}-algebras is left exact, we easily deduce that $\mathscr{O}_*$ is a separated presheaf. By claim 2.1.24(iii), it then suffices to check that the natural morphism $\mathscr{O}_* \to \mathscr{O}_*^+$ is an isomorphism of presheaves. The latter comes down to the following assertion. Let $f : A \to B$ be any faithfully flat morphism; then the natural map

$$A_* \to \mathrm{Equal}(B_* \rightrightarrows (B \otimes_A B)_*)$$

is an isomorphism. Since B_* is left exact, we are then reduced to showing that the natural morphism

$$A \to \mathrm{Equal}(B \rightrightarrows B \otimes_A B)$$

is an isomorphism, which is clear. □

Remark 9.5.9. (i) Let X be an S-scheme; following [36, §5.7], we attach to X its *almost scheme* X^a, which is a sheaf (and indeed, a U-sheaf) on the site $\mathscr{S}$. Recall the explicit description of X^a from *loc.cit.* One picks any Zariski hypercovering $Z_\bullet \to X$ where each Z_i is a disjoint union of affine open subsets of X; then X is the colimit of the system $Z_\bullet$, and therefore we have an induced isomorphism in T :

$$X^a \xrightarrow{\sim} \operatorname*{colim}_{\Delta^o} Z_\bullet^a.$$

(ii) In the situation of (i), notice that the natural map

$$\mathscr{O}_X(X) \to \mathrm{Equal}(\mathscr{O}_{Z_0}(Z_0) \rightrightarrows \mathscr{O}_{Z_1}(Z_1))$$

is an isomorphism. Arguing as in the proof of lemma 9.5.8, we deduce that the induced map $\mathscr{O}_X(X)_*^a \to \mathscr{O}_*(X^a)$ is an isomorphism.

(iii) Let $(U_i \mid i \in I)$ be any covering of X consisting of affine open subsets. We deduce easily from (i) that the family $(U_i^a \mid i \in I)$ generates a covering sieve of X^a (for the canonical topology of T : details left to the reader).

(iv) Recall that X^a is connected and non-empty if and only if the ring of X^a-sections $\Gamma(X^a, \mathbb{Z}_T)$ of the constant sheaf $\mathbb{Z}_T$ has no idempotent elements other than 0 and 1 (see example 2.2.7). On the other hand, it is easily seen that the natural map $\mathbb{Z}_T \to \mathscr{O}_*$ induces a bijection on the rings of idempotents

$$\Gamma(X^a, \mathbb{Z}_T)^\iota \xrightarrow{\sim} \Gamma(X^a, \mathscr{O}_*)^\iota.$$

In view of (i), we conclude that the almost scheme X^a is connected and non-empty if and only if $(\mathscr{O}_X(X)_*^a)^\iota = \{0, 1\}$.

Lemma 9.5.10. *Let X be an S-scheme, and $(U_i \mid i \in I)$ a covering of X consisting of affine open subsets, and suppose that, for every $i \in I$ there exists $b_i \in \mathfrak{m}_K$ such that :*

(a) *The K^+-algebra $\mathscr{O}_X(U_i)$ satisfies* (TF/b_i).

(b) $\mathrm{Supp}\, b_i\mathscr{O}_X(U_i) = U_i$.

Then we have :

(i) *The natural map $\mathscr{O}_X(X)^\iota \to \mathscr{O}_*(X^a)^\iota$ is bijective.*

(ii) *Especially, X is connected if and only if the same holds for X^a.*

Proof. To begin with, notice that the rule : $U \mapsto \mathscr{O}_X(U)^\iota$ (resp. $U \mapsto \mathscr{O}_*(U^a)^\iota$) defines a sheaf of rings on the Zariski site of X which we denote $\mathscr{O}_X^\iota$ (resp. $\mathscr{O}_*^\iota$: details left to the reader). Moreover, there is a natural morphism of sheaves of rings

(9.5.11) $$\mathscr{O}_X^\iota \to \mathscr{O}_*^\iota$$

and (i) just states that this map induces an isomorphism on global sections. To show the latter assertion, let us first remark :

Claim 9.5.12. Let $(U_i \mid i \in I)$ be an affine open covering of the S-scheme X, and suppose that, for every $i \in I$ there exists $b_i \in \mathfrak{m}_K$ such that (b) holds. Then (9.5.11) is a monomorphism.

Proof of the claim. Denote by $\mathscr{I}$ the kernel of (9.5.11), and let $U \subset X$ be any open subset; it suffices to show that $\mathscr{I}(U \cap U_i) = 0$ for every $i \in I$, hence we may replace X by U_i, and assume that X is affine, say $X = \operatorname{Spec} A$, and there exists $b \in \mathfrak{m}_K$ such that

$$\operatorname{Supp} bA = X. \tag{9.5.13}$$

For any $a \in A$, set $U_a := \operatorname{Spec} A_a$; it suffices to show that $\mathscr{I}(U_a) = 0$ for every $a \in A$. However, let $e \in \mathscr{I}(U_a)$; this means that e is an idempotent of A_a, and $\mathfrak{m}_K$ annihilates e. Write $e = a^{-m}f$ for some $f \in A$ and $m \in \mathbb{N}$; we may then find an integer $n \in \mathbb{N}$ such that $ba^n f = 0$ in A. On the other hand, (9.5.13) implies that $\operatorname{Ann}_A(b)$ is included in the nilradical of A, so $a^n f$ is nilpotent in A, hence e is nilpotent in A_a, and finally $e = 0$, since e is an idempotent. ◊

Now, lemma 9.5.2(ii) says that the map $\mathscr{O}^\iota_X(U_i) \to \mathscr{O}^\iota_*(U_i)$ deduced from (9.5.11) is bijective for every $i \in I$. Taking into account claim 9.5.12, assertion (i) follows easily (details left to the reader). Assertion (ii) is an immediate consequence of (i) and remark 9.5.9(iv). □

Definition 9.5.14. Let X be an S-scheme. We call

$$\mathbf{Cov}(X^a) := (\mathscr{O}^a_X\text{-}\mathbf{\acute{E}t}_{\mathrm{fr}})^o$$

the category of *étale coverings* of X^a.

Lemma 9.5.15. *Let X be any quasi-compact S-scheme. We have :*

(i) $\mathbf{Cov}(X^a)$ *is equivalent to the category of locally constant bounded objects of* T/X^a.
(ii) *If X^a is connected and non-empty,* $\mathbf{Cov}(X^a)$ *is a pregalois category (*[36, Def.8.2.14 and (8.2.20)]*).*

Proof. (i): Denote by X^a_{Zar} the subsite of $\mathscr{S}$ whose objects are the almost schemes of the form U^a, where $U \subset X$ is any (Zariski) affine open subset. For every quasi-coherent $\mathscr{O}^a_X$-algebra $\mathscr{A}$, we have a well defined sheaf of K^+-algebras on X^a_{Zar} given by the rule : $U^a \mapsto \mathscr{A}(U)_*$, that we denote by $\mathscr{A}_*$. Moreover, the inclusion functor $X^a_{\mathrm{Zar}} \to \mathscr{S}$ induces a morphism of ringed topoi

$$h : (T/X^a, \mathscr{O}_*) \to ((X^a_{\mathrm{Zar}})^\sim, \mathscr{O}^a_{X*})$$

hence for every object $\mathscr{A}$ of $\mathbf{Cov}(X^a)$ we obtain an $\mathscr{O}_*$-algebra $h^*\mathscr{A}_*$ on T/X^a. Now, consider the presheaf on T/X^a

$$\operatorname{Spec}\mathscr{A} \quad : \quad Y \mapsto \operatorname{Hom}_{\mathscr{O}_{*|Y}\text{-}\mathbf{Alg}}(h^*\mathscr{A}_{*|Y}, \mathscr{O}_{*|Y}) \qquad \text{for every } Y \in \operatorname{Ob}(T/X^a).$$

It follows easily from [36, Prop.8.2.23] that, for every affine open subset $U \subset X$, the restriction of $\operatorname{Spec}\mathscr{A}$ to T/U^a, is a locally constant bounded object of the latter topos. Since X is quasi-compact, remark 9.5.9(iii) then implies that $\operatorname{Spec}\mathscr{A}$ is itself locally constant and bounded, so the rule $\mathscr{A} \mapsto \operatorname{Spec}\mathscr{A}$ yields a functor from $\mathbf{Cov}(X^a)$ to the category of locally constant bounded objects of T/X^a. We need to show that this functor is an equivalence. The latter assertion can be checked locally on X; so we may assume that X is affine, in which case one concludes by invoking again [36, Prop.8.2.23].

(ii) follows from (i) and [36, 8.2.21]. □

9.5.16. Let X be a quasi-compact S-scheme such that X^a is connected and non-empty. It follows from lemma 9.5.15(ii) and [36, Lemma 8.2.15], that the category $\mathbf{Cov}(X^a)$ admits a fibre functor ξ to the category of finite sets, and it is therefore a Galois category. In this situation, the general theory of [42, Exp.V] attaches to $\mathbf{Cov}(X^a)$ a profinite group, which we denote

$$\pi_1(X^a_{\text{ét}}, \xi)$$

and we call the *étale fundamental group of X^a pointed at ξ*. Then $\mathbf{Cov}(X^a)$ is equivalent to the category of finite (discrete) sets with a continuous action of $\pi_1(X^a_{\text{ét}}, \xi)$.

Example 9.5.17. (i) Suppose that K is a henselian valued field, let $b \in \mathfrak{m}_K$ be any non-zero element, and choose any fibre functor ξ for the category $\mathbf{Cov}(S^a)$; according to lemma 8.2.39(i) and [36, §5.1.12], there is a natural equivalence $\mathbf{Cov}(S^a_{/b}) \xrightarrow{\sim} \mathbf{Cov}(S^a)$; composing with ξ, we deduce a fibre functor for $\mathbf{Cov}(S^a_{/b})$, which we denote again ξ, and then the closed immersion $S_{/b} \to S$ induces an isomorphism of topological groups :

$$\pi_1(S^a_{\text{ét}}, \xi) \xrightarrow{\sim} \pi_1(S^a_{/b,\text{ét}}, \xi).$$

(ii) On the other hand, suppose that K is deeply ramified, and fix a fibre functor ξ' for the category of étale coverings of $\mathrm{Spec}\, K$; after composition with the base change functor $\mathbf{Cov}(S^a) \to \mathbf{Cov}(\mathrm{Spec}\, K)$, we deduce another fibre functor (which we denote again ξ') for $\mathbf{Cov}(S^a)$, and the almost purity theorem implies that the open immersion $\mathrm{Spec}\, K \to S$ induces an isomorphism of topological groups

$$\mathrm{Gal}(K^{\mathrm{a}}/K) \xrightarrow{\sim} \pi_1((\mathrm{Spec}\, K)_{\text{ét}}, \xi') \xrightarrow{\sim} \pi_1(S^a_{\text{ét}}, \xi').$$

9.5.18. Suppose now that $(K, |\cdot|)$ is deeply ramified, of characteristic zero, with residue field κ of characteristic $p > 0$, and let (B, Δ) be a small model K^+-algebra. We set $X := \mathrm{Spec}\, B$, and consider :

- a cofiltered system $\underline{Y} := (Y_\lambda \to X \mid \lambda \in \Lambda)$ of absolutely flat morphisms of S-schemes, with affine transition morphisms
- a filtered system $((E_i, |\cdot|_i) \mid i \in I)$ of algebraic valued field extensions of K

and we denote by Y the limit of the cofiltered system

$$(Y_{\lambda,i} := Y_\lambda \times_S \mathrm{Spec}\, E_i^+ \mid (\lambda, i) \in \Lambda \times I).$$

Proposition 9.5.19. *In the situation of* (9.5.18), *suppose that Y_λ is quasi-compact and quasi-separated, for every $\lambda \in \Lambda$. Then, for any $b \in \mathfrak{m}_K$ we have :*

(i) *Every étale almost finitely presented $\mathscr{O}_{Y_{/b}}$-algebra has finite rank.*

(ii) *The induced functor*

$$\mathop{2\text{-colim}}_{(\lambda,i)\in\Lambda\times I} \mathbf{Cov}(Y^a_{\lambda,i/b}) \to \mathbf{Cov}(Y^a_{/b})$$

is an equivalence.

Proof. To begin with, let us notice :

Claim 9.5.20. We may assume that Y_λ is affine for every $\lambda \in \Lambda$.

Proof of the claim. Without loss of generality, we may assume that the ordered indexing set Λ admits a final element λ_0. For every affine open subset $U \subset Y_{\lambda_0}$, the induced system $\underline{U} := (Y_\lambda \times_{Y_{\lambda_0}} U \mid \lambda \in \Lambda)$ consists of affine schemes; set $U_{/b} := Y_{/b} \times_{Y_{\lambda_0}} U$. If assertion (i) is known for the system $\underline{U}$, then every almost finitely presented étale $\mathscr{O}^a_{U_{/b}}$-algebra has finite rank; since U is arbitrary, assertion (i) follows for $\underline{Y}$.

Next, for every open subset $U \subset Y_{\lambda_0}$, and every $(\lambda, i) \in \Lambda \times I$, set $U_{\lambda,i/b} := Y_{\lambda,i/b} \times_{Y_{\lambda_0}} U$. Let $\mathscr{C}$ be the category whose objects are all the pairs $(U, \mathscr{A})$, where $U \subset Y_{\lambda_0}$ is an open subset, and $\mathscr{A}$ is an object of

$$\mathscr{C}_U := \underset{(\lambda,i)\in\Lambda\times I}{2\text{-colim}}\, \mathbf{Cov}(U^a_{\lambda,i/b}).$$

Any inclusion $U \subset U'$ of open subsets of Y_{λ_0} induces a restriction functor

$$\mathscr{C}_{U'} \to \mathscr{C}_U \quad : \quad \mathscr{A} \mapsto \mathscr{A}_{|U}$$

(we leave to the reader the task of spelling out the precise definition). The morphisms $(U, \mathscr{A}) \to (U', \mathscr{A}')$ in $\mathscr{C}$ are the pairs (j, φ), where $j : U \to U'$ is an open immersion of Y_{λ_0}-schemes, and $\varphi : \mathscr{A} \to \mathscr{A}'_{|U}$ is a morphism in $\mathscr{C}_U$. There is a natural functor $\varphi : \mathscr{C} \to (Y_{\lambda_0})_{\mathrm{Zar}}$ from $\mathscr{C}$ to the category of all open subsets of Y_{λ_0}, and it is easily seen that φ is a fibration. On the other hand, consider the category $\mathscr{C}'$ whose objects are the pairs $(U, \mathscr{A})$, where $U \subset Y_{\lambda_0}$ is an open subset, and $\mathscr{A}$ is an object of $\mathbf{Cov}(U^a_{/b})$. The morphisms $(U, \mathscr{A}) \to (U', \mathscr{A}')$ are defined as for the foregoing category $\mathscr{C}$; then also the natural functor $\varphi' : \mathscr{C}' \to (Y_{\lambda_0})_{\mathrm{Zar}}$ is a fibration, and furthermore we have a well defined cartesian functor $F : \mathscr{C} \to \mathscr{C}'$ of $(Y_{\lambda_0})_{\mathrm{Zar}}$-categories, whose restriction $\varphi^{-1}(Y_{\lambda_0}) \to \varphi'^{-1}(Y_{\lambda_0})$ is the functor considered in (ii) (details left to the reader). Suppose now that Y_{λ_0} is separated (and quasi-compact), and let $\mathscr{S}$ be the sieve of $(Y_{\lambda_0})_{\mathrm{Zar}}$ generated by the family $\mathscr{F}$ of all the affine open subsets of Y_{λ_0}; notice that $\mathscr{F}$ is stable under arbitrary finite intersections. For every $U \in \mathscr{F}$, define the cofiltered system $\underline{U}$ as in the foregoing, and suppose that assertion (ii) holds for the system $\underline{U}$; it is easily seen that $\mathscr{S}$ is a sieve of 2-descent, for both fibrations φ and φ', and in light of theorem 1.5.30(i), we deduce that (ii) holds for $\underline{Y}$ as well.

Lastly, let $\underline{Y}$ be an arbitrary cofiltered system, with Y_{λ_0} quasi-separated and quasi-compact; in this case, we denote by $\mathscr{S}'$ the sieve of $(Y_{\lambda_0})_{\mathrm{Zar}}$ generated by the family $\mathscr{F}'$ all quasi-compact and separated open subsets of Y_{λ_0}. Again, $\mathscr{S}'$ is of 2-descent for φ and φ', and $\mathscr{F}'$ is stable under finite intersections. Moreover, the foregoing case shows that, if (ii) holds for all cofiltered systems $\underline{U}$ of affine schemes, then it holds as well for all systems $\underline{U}'$ such that $U'_{\lambda_0} \in \mathscr{F}'$. Invoking again theorem 1.5.30(i), we conclude the proof of the claim. ◊

Suppose now that Y_λ is affine for every $\lambda \in \Lambda$, and for every $i \in I$ denote by $Y^{\mathrm{h}}_{\lambda,i}$ the henselization of $Y_{\lambda,i}$ along $Y_{\lambda,i/b}$; then the limit of the cofiltered system $(Y^{\mathrm{h}}_{\lambda,i} \mid \lambda \in \Lambda,\ i \in I)$ is naturally isomorphic to the henselization Y^{h} of Y along $Y_{/b}$. By lemma 8.2.39(i), the induced functors

$$\mathbf{Cov}((Y^{\mathrm{h}})^a) \to \mathbf{Cov}(Y^a_{/b}) \qquad \mathbf{Cov}((Y^{\mathrm{h}}_{\lambda,i})^a) \to \mathbf{Cov}(Y^a_{\lambda,i/b})$$

are equivalences, for every $i \in I$ and $\lambda \in \Lambda$, and the same holds for the corresponding functors on almost finitely presented étale algebras ([36, Th.5.5.7(iii)]). In order to prove (ii), it suffices therefore to show that the functor

$$\underset{(\lambda,i)\in\Lambda\times I}{2\text{-colim}}\, \mathbf{Cov}((Y^{\mathrm{h}}_{\lambda,i})^a) \to \mathbf{Cov}((Y^{\mathrm{h}})^a)$$

is an equivalence. Likewise, (i) holds, provided every étale almost finitely presented $\mathscr{O}^a_{Y^{\mathrm{h}}}$-algebra has finite rank; the latter assertion is easily established, by remarking that Y^{h} is a flat S-scheme, and that, obviously, every étale finitely presented $\mathscr{O}_{Y^{\mathrm{h}}} \otimes_{K^+} K$-algebra has finite rank (details left to the reader). We remark :

Claim 9.5.21. The pair $(Y^{\mathrm{h}}, Y_{/b})$ (resp. $(Y^{\mathrm{h}}_{\lambda,i}, Y_{\lambda,i/b})$, for every $(\lambda, i) \in \Lambda \times I$) is almost pure.

Proof of the claim. We consider the assertion for $Y_{\lambda,i}$; the same argument will apply to Y as well. Fix $\lambda \in \Lambda$ and $i \in I$, and let $(\xi_j \mid j \in J)$ be a small set of geometric points of $Y_{\lambda,i/b}$ such that $|Y_{\lambda,i/b}| = \bigcup_{j\in J} |\xi_j|$. Set $Y' := \coprod_{j\in J} Y_{\lambda,i}(\xi_j)$, and let $f : Y' \to Y^{\mathrm{h}}_{\lambda,i}$ be the natural morphism of schemes. According to proposition 8.2.34, it suffices to check that the pair $(Y', f^{-1}Y_{\lambda,i/b})$ is almost pure, and that comes down to showing that the pair $(Y_{\lambda,i}(\xi_j), Y_{\lambda,i}(\xi_j) \times_S \operatorname{Spec} K^+/bK^+)$ is almost pure, for every $j \in J$. However, set $B_i := B \otimes_{K^+} E_i^+$; notice that E_i^+ is a deeply ramified valuation ring, and (B_i, Δ) is a model E_i^+-algebra. Furthermore, the induced morphism $Y_{\lambda,i} \to \operatorname{Spec} B_i$ is absolutely flat, so the assertion follows from the almost purity theorem 9.4.38. $\Diamond$

Set $U := Y^{\mathrm{h}} \setminus Y_{/b}$, and $U_{\lambda,i} := Y^{\mathrm{h}}_{\lambda,i} \setminus Y_{\lambda,i/b}$ for every $(\lambda, i) \in \Lambda \times I$. In view of claim 9.5.21, we are reduced to checking that the induced functor

$$\underset{(\lambda,i)\in\Lambda\times I}{2\text{-colim}}\, \mathbf{Cov}(U_{\lambda,i}) \to \mathbf{Cov}(U)$$

is an equivalence. The latter assertion is just a special case of lemma 7.1.6. $\square$

Corollary 9.5.22. *In the situation of proposition* 9.5.19*, if the almost scheme $Y^a_{\lambda,i/b}$ is connected for every $(\lambda, i) \in \Lambda \times I$, then the same holds for the almost scheme $Y^a_{/b}$.*

Proof. From proposition 9.5.19(ii) and lemma 9.5.15(i), we deduce that the natural map

$$\underset{(\lambda,i)\in\Lambda\times I}{\operatorname{colim}}\, \Gamma(Y^a_{\lambda,i/b}, \mathbb{Z}/2\mathbb{Z}_T) \to \Gamma(Y^a_{/b}, \mathbb{Z}/2\mathbb{Z}_T)$$

is a bijection. Then the assertion follows from the criterion of example 2.2.7(ii). $\square$

9.5.23. Let $(K, |\cdot|)$ be a henselian valued field fulfilling the conditions of (9.5.18), and fix an algebraic closure K^{a} of K; then the valuation $|\cdot|$ admits a unique extension $|\cdot|^{\mathrm{a}}$ to K^{a}, and as usual, we denote by $K^{\mathrm{a}+}$ the corresponding valuation ring. Let also $b \in K^+$ be a given element; for every S-scheme X we set :

$$X_\kappa := X \times_S \operatorname{Spec} \kappa \qquad X_{K^{\mathrm{a}+}} := X \times_S \operatorname{Spec} K^{\mathrm{a}+}.$$

We consider a quasi-compact and quasi-separated S-scheme X such that the following holds. There exists a covering $(U_i \mid i \in I)$ of X consisting of affine open subsets, and for every $i \in I$:

- an absolutely flat morphism of S-schemes $V_i \to \operatorname{Spec} B_i$, where (B_i, Δ_i) is a model K^+-algebra
- an element $b_i \in \mathfrak{m}_K$ and an isomorphism $U_i \xrightarrow{\sim} V_i \times_S S_{/b_i}$ of S-schemes.

Lemma 9.5.24. *In the situation of* (9.5.23)*, the scheme X_κ is geometrically connected if and only if the almost scheme $X^a_{K^{\mathrm{a}+}}$ is connected.*

Proof. Let κ^{a} be the residue field of $K^{\mathrm{a}+}$; the natural morphism $X_\kappa \times_S \operatorname{Spec} \kappa^{\mathrm{a}} \to X_{K^{\mathrm{a}+}}$ is a homeomorphism ([31, Ch.IV, Prop.2.4.5(i)]), so we are reduced to showing that $X^a_{K^{\mathrm{a}+}}$ is connected if and only if the same holds for $X_{K^{\mathrm{a}+}}$. To this aim, it suffices to show that the affine open subset U_i fulfills conditions (a) and (b) of lemma 9.5.10, for every $i \in I$. However, since V_i is a flat S-scheme, U_i is a flat $S_{/b_i}$-scheme, so condition (b) for U_i follows from lemma 9.5.6. Lastly, in view of remark 9.5.3(ii) it suffices to check that the strict henselization $\mathscr{O}_{X,\xi}$ fulfills condition (TF$/b$) for every $b \in \mathfrak{m}_K$, and every geometric point ξ of X. The latter assertion follows immediately from remarks 9.5.5(ii), 9.5.3(i) and [33, Ch.IV, Prop.18.8.10]. $\square$

9.5.25. Keep the situation of (9.5.23), and suppose moreover that $(K, |\cdot|)$ is deeply ramified (and still henselian), and $X^a_{K^{\mathrm{a}+}}$ is connected and non-empty. Pick a fibre functor $F_{K^{\mathrm{a}}}$ for the category $\mathbf{Cov}(X^a_{K^{\mathrm{a}+}})$ (see (9.5.16)). The natural morphisms $X^a_{K^{\mathrm{a}+}} \to X^a \to S^a$ induce functors $\mathbf{Cov}(S^a) \to \mathbf{Cov}(X^a) \to \mathbf{Cov}(X^a_{K^{\mathrm{a}+}})$, and composing with $F_{K^{\mathrm{a}}}$ we deduce fibre functors

$$F_S : \mathbf{Cov}(S^a) \to \mathbf{Set} \qquad F : \mathbf{Cov}(X^a) \to \mathbf{Set}.$$

We may then state :

Proposition 9.5.26. *In the situation of* (9.5.25)*, the induced sequence of topological groups :*

$$1 \to \pi_1(X^a_{K^{\mathrm{a}+}}, F_{K^{\mathrm{a}}}) \to \pi_1(X^a, F) \to \mathrm{Gal}(K^{\mathrm{a}}/K) \to 1$$

is exact.

Proof. The Galois group $\mathrm{Gal}(K^{\mathrm{a}}/K)$ is naturally identified with $\pi_1(S^a, F_S)$, as in example 9.5.17(ii). Then, according to [42, Exp.V, Prop.6.9], the surjectivity of the second map amounts to the following :

Claim 9.5.27. For every connected étale covering $S' \to S^a$, the almost scheme $Y^a := X^a \times_{S^a} S'$ is connected.

Proof of the claim. Indeed, S' is isomorphic to $(\mathrm{Spec}\, E^+)^a$ for some finite extension E of K, and therefore Y^a is isomorphic to the almost scheme attached to the S-scheme $Y := X \times_S \mathrm{Spec}\, E^+$. Notice that Y fulfills as well the conditions of (9.5.23), and since X_κ is geometrically connected, Y_κ is connected. Therefore $Y^a_{K^{\mathrm{a}+}}$ is connected (lemma 9.5.24), so the same holds for Y^a. ◊

Next, let $\mathscr{E}$ denote the filtered category of all finite extensions of K contained in K^{a} (with morphisms given by the inclusion maps); pick a cleavage for the fibration $\mathsf{s} : \mathsf{Morph}(S^a\text{-}\mathbf{Sch}) \to S^a\text{-}\mathbf{Sch}$ (notation of (1.1.17)), and for every $E \in \mathscr{E}$, set $X_E := X \times_S \mathrm{Spec}\, E^+$. In light of lemma 9.5.15, the rule $E \mapsto \mathbf{Cov}(X^a_E)$ extends to a pseudo-functor

$$\mathbf{Cov}(X^a_\bullet) : \mathscr{E} \to \mathbf{Galois}$$

which depends on the chosen cleavage (see definition 1.6.6(i)). Next, let $E \in \mathscr{E}$ be any element, and recall that by definition, F is isomorphic to the functor given by the rule :

$$(f : Y \to X^a) \mapsto F_{K^{\mathrm{a}}}(f \times_{X^a} X^a_{K^{\mathrm{a}+}}) \qquad \text{for every étale covering } f.$$

Denote

$$\Delta_E : X^a_{K^{\mathrm{a}+}} \to X^a_E \times_{X^a} X^a_{K^{\mathrm{a}+}}$$

the morphism induced by the projection $X^a_{K^{\mathrm{a}+}} \to X^a_E$ and the identity of $X^a_{K^{\mathrm{a}+}}$, and let $\xi_E \in F(X^a_E) = F_{K^{\mathrm{a}}}(X^a_E \times_{X^a} X^a_{K^{\mathrm{a}+}})$ be the unique element that lies in the image of the induced map $F_{K^{\mathrm{a}}}(\Delta_E)$ of finite sets. Notice that, for every inclusion $E \subset E'$ of elements of $\mathscr{E}$, the image of $\xi_{E'}$ under the induced map $F(X^a_{E'}) \to F(X^a_E)$ agrees with ξ_E. Furthermore, denote by

$$F_E : \mathbf{Cov}(X^a_E) \stackrel{\sim}{\to} \mathbf{Cov}(X^a)/X^a_E \to \mathbf{Set}$$

the subfunctor of $F_{|X^a_E}$ selected by ξ_E (see (1.6.19)). The discussion of (1.6.21) shows that the rule $E \mapsto (\mathbf{Cov}(X^a_E), F_E)$ extends to a pseudo-functor

$$(\mathbf{Cov}(X^a_\bullet), F_\bullet) : \mathscr{E} \to \mathbf{fibre.Fun}$$

(see definition 1.6.6(iii)). From proposition 9.5.19(ii) we then deduce an equivalence :

$$2\text{-}\operatorname*{colim}_{\mathscr{E}} (\mathbf{Cov}(X^a_\bullet), F_\bullet) \stackrel{\sim}{\to} (\mathbf{Cov}(X^a_{K^{\mathrm{a}+}}), F_{K^{\mathrm{a}}})$$

(notation of (1.6)). For every $E \in \mathscr{E}$, let $H_E \subset \pi_1(X^a, F)$ be the stabilizer of ξ_E under the $\pi_1(X^a, F)$-action on $F(X^a_E)$. In view of corollary 1.6.25, the proposition is now reduced to the following

Claim 9.5.28. $\bigcap_{E \in \mathscr{E}} H_E = \mathrm{Ker}(\pi_1(X^a, F) \to \pi_1(S^a, F_S))$.

Proof of the claim. Let $\mathscr{E}' \subset \mathscr{E}$ be the subset of all finite Galois extension of K (contained in K^{a}). For every $E \in \mathscr{E}'$, the action of $\pi_1(X^a, F)$ on $F(X^a_E) = F_S(\mathrm{Spec}\, E^+)$ is obtained by restriction along the map $p_E : \pi_1(X^a, F) \to \pi_1(S^a, F_S) \to \mathrm{Gal}(E/K)$; it is then clear that $H_E = \mathrm{Ker}\, p_E$. Since $\mathscr{E}'$ is cofinal in $\mathscr{E}$, the claim follows. □

Lemma 9.5.29. *Suppose that* K *is algebraically closed, let* (B,Δ) *be a small model* K^+*-algebra,* $b\in\mathfrak{m}_K$ *any non-zero element, and set* $Y:=\mathrm{Spec}\,B$*. Then we have :*

(i) *The underlying topological space* $|Y_{/b}|$ *of* $Y_{/b}$ *is noetherian.*
(ii) *Each irreducible component of* $Y_{/b}$ *has dimension* $\dim_{\mathbb{Q}}\Delta^{\mathrm{gp}}\otimes_{\mathbb{Z}}\mathbb{Q}$*.*
(iii) $Y_{/b}$ *is a Cohen-Macaulay scheme. Especially,* $\mathrm{Ass}\,\mathscr{O}_{Y_{/b}}$ *is the set of maximal points of* $Y_{/b}$*.*

Proof. Define Δ_n and B_n as in remark 9.3.15(ii), and set $Y_n:=\mathrm{Spec}\,B_n$ for every $n\in\mathbb{N}$; clearly $Y_{/b}$ is the colimit of the resulting system $(Y_{n/b}\mid n\in\mathbb{N})$ of $S_{/b}$-schemes, and it is easily seen that the transition morphisms $g_n:Y_{n/b}\to Y_{0/b}$ are surjective and radicial for every $n\in\mathbb{N}$. It follows that the natural morphism $g:Y_{/b}\to Y_{0/b}$ is a homeomorphism on the underlying topological spaces; especially, (i) holds, and to prove (ii) it suffices to show that every irreducible component of $Y_{0/b}$ has dimension $\dim_{\mathbb{Q}}\Delta^{\mathrm{gp}}\otimes_{\mathbb{Z}}\mathbb{Q}=\mathrm{rk}_{\mathbb{Z}}\Delta_0^{\mathrm{gp}}$.

Now, pick a decomposition $(B,\Delta)=(B',\Delta')\otimes(B'',\Delta'')$ as in remark 9.3.15(i), set $Y':=\mathrm{Spec}\,B'$, and let $f:Y_{/b}\to Y'_{/b}$ be the induced morphism of schemes. Then f is étale and surjective (remark 9.3.15(iii)), hence $Y_{/b}$ is Cohen-Macaulay if and only if the same holds for $Y'_{/b}$ (lemma 5.6.36(iii)). It then suffices to prove (ii) and (iii) for the model algebra (B',Δ'); hence we may assume from start that Δ^{gp} is a torsion-free abelian group. Since the continuous map $|g|$ is a homeomorphism, we have

$$\delta(y,\mathscr{O}_{Y_{/b}})=\delta(g(y),g_*\mathscr{O}_{Y_{/b}})\geq\min_{n\in\mathbb{N}}\delta(g(y),g_{n*}\mathscr{O}_{Y_{n/b}})\qquad\text{for every } y\in Y$$

(lemma 5.4.4(ii.b)). However, it has already been remarked that the structure morphism $Y_n\to S$ is Cohen-Macaulay at every point of the closed subset $Y_{n/b}$ (see the proof of lemma 9.3.42(ii)), therefore the scheme $Y_{n/b}$ is Cohen-Macaulay (lemma 5.6.36(iv)) for every $n\in\mathbb{N}$, whence (iii).

Lastly, (ii) follows easily from corollary 6.5.8(ii) and lemma 5.8.8(iv,v). □

9.5.30. Suppose now that $(K,|\cdot|)$ is deeply ramified, fix a non-zero element $b\in\mathfrak{m}_K$, and let (B,Δ) be a small model K^+-algebra. Set $(Y,\mathscr{M}):=\mathbb{S}(B,\Delta)$ (notation of (9.3.8)) and $X:=Y(\overline{x})$, with $\overline{x}$ a given geometric point of $Y_{/b}$; denote by x the closed point of X. As in (9.3.9), there exists a natural morphism of log schemes $f:(Y,\mathscr{M})\to\mathbb{S}(K^+)$, and we denote by $f_{(\overline{x})}:(X,\mathscr{M}(\overline{x}))\to\mathbb{S}(K^+)$ its composition with the natural morphism $(X,\mathscr{M}(\overline{x}))\to(Y,\mathscr{M})$. Denote by $r:|X|\to\mathbb{N}\cup\{\infty\}$ the rank function of $\mathrm{Coker}(\log f_{(\overline{x})})$ (see (6.2.20); in fact, it shall be seen that $r(y)\in\mathbb{N}$ for every $y\in X$). We may now state :

Theorem 9.5.31. *In the situation of* (9.5.30), *suppose that* $\dim X_{/b}\geq 3$*. Then we have :*

(i) *If either* $r^{-1}r(x)\cap X_{/b}\neq\{x\}$*, or* $r^{-1}r(x)=\{x\}$*, the pair* $(X_{/b},\{x\})$ *is almost pure.*
(ii) *In case the two conditions of* (i) *fail, the following holds :*
  (a) *There exists a unique point* $x_+\in X\setminus X_{/b}$ *such that* $r^{-1}r(x)=\{x,x_+\}$*.*
  (b) *Let* $\overline{x}_+$ *be any geometric point of* X *localized at* x_+*, and* $U^+:=X(\overline{x}_+)\setminus\{\overline{x}_+\}$*. Then* $(X_{/b},\{x\})$ *is an almost pure pair if and only if* $U^+_{\text{ét}}$ *is simply connected.*

Proof. Set as usual $U:=X\setminus\{x\}$, and notice that $\delta(x,\mathscr{O}_{X_{/b}})\geq 3$, by virtue of lemma 9.5.29(iii) and corollary 5.4.35. Taking into account corollary 5.4.22, we deduce that $U_{/b}$ is connected.

Claim 9.5.32. We may assume that $(K,|\cdot|)$ is a strictly henselian (deeply ramified) valued field.

Proof of the claim. Indeed, let $\overline{s}$ denote the image of $\overline{x}$ in S, denote $K^{\mathrm{sh}+}$ the strict henselization of K^+ at the geometric point $\overline{s}$, and set $(Y^\dagger,\mathscr{M}^\dagger):=\mathbb{S}(B\otimes_{K^+}K^{\mathrm{sh}+},\Delta)$. Let $\overline{x}^\dagger$ be the image of $\overline{x}$ in $Y^\dagger$; clearly the natural morphism

$$g:(X,\mathscr{M}(\overline{x}))\to(Y^\dagger(\overline{x}^\dagger),\mathscr{M}^\dagger(\overline{x}^\dagger))$$

is an isomorphism. Moreover, let $f^\dagger:(Y^\dagger(\overline{x}^\dagger)\mathscr{M}^\dagger(\overline{x}^\dagger))\to\mathbb{S}(K^{\mathrm{sh}+})$ be the morphism defined as in (9.3.9), and $r^\dagger$ the rank function of $\mathrm{Coker}(\log f^\dagger)$; the discussion of (9.3.8), combined with lemma 6.1.24, implies that $r = r^\dagger \circ g$. Hence, we may replace throughout K^+ by $K^{\mathrm{sh}+}$, whence the claim. ◊

Claim 9.5.33. We may assume that K is algebraically closed.

Proof of the claim. By claim 9.5.32, we may already assume that $(K, |\cdot|)$ is strictly henselian. We proceed next as in the proof of claim 9.5.32 : let K^{a} be an algebraic closure of K, and set $(Y^\dagger, \mathscr{M}^\dagger) := \mathbb{S}(B \otimes_{K^+} K^{\mathrm{a}+}, \Delta)$. Under the current assumptions, the residue field κ of K^+ is separably closed, and $K^{\mathrm{a}+}$ is an integral K^+-algebra; hence the induced map $Y^\dagger_{/b} \to Y_{/b}$ is a homeomorphism, $\overline{x}$ lifts uniquely to a geometric point $\overline{x}^\dagger$ of $Y^\dagger$, and the projection

$$g : Y^\dagger(\overline{x}^\dagger) \to X$$

induces an isomorphism $Y^\dagger(\overline{x}^\dagger) \xrightarrow{\sim} X_{K^{\mathrm{a}+}}$. Moreover, let $f^\dagger : (Y^\dagger(\overline{x}^\dagger), \mathscr{M}^\dagger(\overline{x}^\dagger)) \to \mathbb{S}(K^{\mathrm{a}+})$ be the morphism defined as in (9.3.9), and $r^\dagger$ the rank function of $\mathrm{Coker}(\log f^\dagger)$; arguing as in the proof of claim 9.5.32, we see that $r^\dagger = r \circ g$. Since g is surjective and $g \times_S S_{/b}$ is a homeomorphism, it follows easily that the conditions stated in (i) and (ii) hold for the function r, whenever the corresponding conditions hold for $r^\dagger$. Furthermore, say that $r^{\dagger-1}r(x) = \{x^\dagger, x^\dagger_+\}$ for a point $x^\dagger_+ \in X^\dagger \setminus X^\dagger_{/b}$, let $\overline{x}^\dagger_+$ be a geometric point of $X^\dagger$ localized at $x^\dagger_+$ and lifting $\overline{x}_+$; and set $U^{\dagger+} := X^\dagger(\overline{x}^\dagger_+) \setminus \{\overline{x}^\dagger_+\}$; then it is easily seen that the induced morphism $U^{\dagger+} \to U^+$ is an isomorphism.

On the other hand, pick any geometric point ξ of $(U_{/b})_{K^{\mathrm{a}+}}$ (notation of (9.5.23)). The geometric point ξ induces geometric points of $U_{/b}$ and $X_{/b}$, that we denote again ξ. We have to show that the induced map $\pi_1(U_{/b}, \xi) \to \pi_1(X_{/b}, \xi)$ is an isomorphism. However, under the current assumptions, U_κ is geometrically connected (notation of (9.5.23)): in light of lemma 9.5.24 and proposition 9.5.26, it then suffices to check that the corresponding map $\pi_1((U_{/b})_{K^{\mathrm{a}+}}, \xi) \to \pi_1((X_{/b})_{K^{\mathrm{a}+}}, \xi)$ is an isomorphism. Summing up, this shows that we may replace K by K^{a}, whence the claim. ◊

Henceforth we assume that K is algebraically closed, and in view of remark 9.3.15(i,iii), we may assume as well that Δ^{gp} is a torsion-free abelian group (see the proof of theorem 9.4.38). In this case, the discussion of (9.3.19) and remark 9.3.24(iv) yield a decomposition

$$(B, \Delta) = (B', \Delta') \otimes (B'', \Delta'') \tag{9.5.34}$$

such that Δ' is an abelian group, $B' = K^+[\Delta']$, and $B'' = P \otimes_{P_0} K^+$ for a Δ''-graded sharp monoid P with an injective local map $P_0 \to K^+$. We may then endow Y (resp. S) with the log structure $\underline{M}$ deduced from the natural map $\beta : P \to B$ (resp. $P_0 \to K^+$). Moreover, set $Y'' := \mathrm{Spec}\, B''$, denote by $\overline{x}''$ the image of $\overline{x}$ in Y'', and x'' the support of $\overline{x}''$; by remark 9.3.24(v), we may assume – after replacing B by a suitable localization – that the resulting map $P \to \mathscr{O}_{Y'', \overline{x}''}$ is local as well; especially :

$$\kappa(x'') = \kappa \qquad \text{and} \qquad \underline{M}^\sharp_{\overline{x}} = P. \tag{9.5.35}$$

Furthermore, remark 9.3.24(iii) implies that the resulting commutative diagram

$$\begin{tikzcd} (Y, \mathscr{M}) \arrow[r] \arrow[d, "f"'] & (Y, \underline{M}) \arrow[d, "f^\dagger"] \\ \mathbb{S}(K^+) \arrow[r] & (S, \underline{N}) \end{tikzcd}$$

is cartesian. Hence, let $r^\dagger$ denote the rank function of $\mathrm{Coker}(\log f^\dagger)$, and $g : X \to Y$ the natural morphism; from lemma 6.1.24, we deduce that $r = r^\dagger \circ g$, which allows to reinterpret as follows the conditions for r stated in (i) and (ii). First, due to (9.3.26), we see that $r(x) = \dim_{\mathbb{Q}} \Delta''^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$. By the same token, it is clear that, for every geometric point $\overline{y}$ of Y, the stalk $\mathrm{Coker}(\log f^\dagger_{\overline{y}})$ shall be in any case a quotient of Δ'', hence $r^\dagger(y) \leq r(x)$, if y denotes the support of $\overline{y}$. Lastly, let $\beta_{\overline{y}} : P \to \mathscr{O}_{Y,\overline{y}}$ be the map induced by β; then we have

$$r^\dagger(y) = r(x) \qquad \text{if and only if} \qquad P_+ := P \setminus P_0 \subset \beta_{\overline{y}}^{-1}(\mathfrak{m}_{\overline{y}}).$$

Thus, let us set

$$Z''_+ := \mathrm{Spec}\, B''/P_+B'' \qquad Z_+ := Z''_+ \times_{Y''} X.$$

Summing up, we have

Claim 9.5.36. (i) $Z_+ = \{z \in X \mid r(z) = r(x)\}$.

(ii) $r^{-1}r(x) \cap X_{/b} = \{x\}$ if and only if the image of x in $\mathrm{Spec}\, K^+[\Delta']$ is the maximal point of $\mathrm{Spec}\, \kappa[\Delta']$.

(iii) $r^{-1}r(x) \setminus X_{/b} \neq \varnothing$ if and only if Δ'' is sharp.

(iv) Assertion (ii.a) of the theorem holds for r.

Proof of the claim. Assertion (i) is an immediate consequence of the foregoing discussion : details left to the reader. Next, notice that $C := B''/P_+B''$ is in any case a quotient of K^+. If Δ'' is sharp, then P_+B'' is the direct sum of the graded terms B''_δ with $\delta \neq 0$, so $C = K^+$. If Δ'' is not sharp, then say that $\delta \in (\Delta'')^\times \setminus \{0\}$, and pick arbitrary non-zero elements $a \in B''_\delta$ and $a' \in B''_{-\delta}$; then $a \cdot a' \in P_+B'' \cap B''_0$, so C is a quotient of $K^+/aa'K^+$. This easily implies (iii). By the same token, we see that $Z_{+/b}$ is isomorphic to the strict henselization of $\mathrm{Spec}\, \kappa[\Delta']$ at the image of $\overline{x}$ in $\mathrm{Spec}\, \kappa[\Delta']$, whence (ii). Lastly, if both (ii) and (iii) hold, we see that $Z''_+ = \mathrm{Spec}\, K^+$, and [36, Ex.6.1.4(iv) and Prop.6.6.6] tell us that Z_+ is isomorphic to the spectrum of a deeply ramified strictly henselian valuation ring of rank one, whence (iv). ♢

We shall now continue as in the proof of theorem 9.4.34. Indeed, let $\mathscr{A}$ be any étale almost finitely presented $\mathscr{O}^a_{U_{/b}}$-algebra, and define the $(B^{\mathrm{sh}})^a$-module $C_{\mathscr{A}}$ as in (9.4.24).

Claim 9.5.37. $\lambda(C_{\mathscr{A}}) = 0$.

Proof of the claim. From lemma 9.4.25, we know already that $C_{\mathscr{A}}$ has almost finite length. Now, let $a \in \mathfrak{m}_K$, such that $|a^p| \geq \max(|b|, |p|)$; since $\mathscr{A}$ is a flat $\mathscr{O}_{U_{/b}}$-algebra, we have isomorphisms of $\mathscr{O}_{U_{/b}}$-modules :

$$\mathrm{Im}(a \cdot \mathbf{1}_{\mathscr{A}}) \xrightarrow{\sim} \mathscr{A}/ba^{-1}\mathscr{A} \qquad \text{and} \qquad \mathrm{Ker}(a \cdot \mathbf{1}_{\mathscr{A}}) = ba^{-1}\mathscr{A} \xrightarrow{\sim} \mathscr{A}/a\mathscr{A}.$$

On the other hand, since $\dim X_{/b} \geq 3$, lemma 9.4.21(ii) says that $H^1(U_{/b}, \mathscr{A}/c\mathscr{A})$ for every $c \in \mathfrak{m}_K$; we deduce a natural isomorphism

$$H^0(U_{/b}, \mathscr{A}) \otimes_{K^+} K^+/aK^+ \xrightarrow{\sim} H^0(U_{/b}, \mathscr{A}/a\mathscr{A}).$$

The same can be repeated with $\mathscr{A}$ replaced by $\mathscr{A} \otimes_{\mathscr{O}_{/U_{/b}}} \mathscr{A}$, and it follows that

$$C_{\mathscr{A}/a\mathscr{A}} = C_{\mathscr{A}}/aC_{\mathscr{A}}.$$

By the same token, we have $C_{\mathscr{A}/a^p\mathscr{A}} = C_{\mathscr{A}}/a^pC_{\mathscr{A}}$. On the other hand, since $\overline{\Phi}$ is an isomorphism, lemma 8.5.5 yields an isomorphism of $(B^{\mathrm{sh}})^a$-modules

$$\overline{\Phi}^* C_{\mathscr{A}/a\mathscr{A}} \xrightarrow{\sim} C_{\mathscr{A}/a^p\mathscr{A}}.$$

Arguing as in the proof of corollary 9.4.19, we deduce that $\lambda(C_{\mathscr{A}}/aC_{\mathscr{A}}) = \lambda(C_{\mathscr{A}/a\mathscr{A}}) = 0$. Then, the claim follows by an easy induction. ♢

From claim 9.5.37 and proposition 9.4.27 we deduce that there exists an integer $m > 0$ such that $(p, m) = 1$ and $j_{/b*}\mathscr{A}_{(m)}$ is an étale and almost finitely presented $\mathscr{D}^a_{(m)/b}$-algebra, and we need to descend this algebra to an étale almost finitely presented $\mathscr{O}^a_{X_{/b}}$-algebra. Notice first that the isomorphisms (9.5.34) induce a corresponding decomposition

$$(D_{(m)}, \Delta_{(m)}) = (K^+[\Delta'_{(m)}], \Delta'_{(m)}) \otimes (D''_{(m)}, \Delta''_{(m)})$$

where $D''_{(m)} := A_{\Delta''_{(m)}}$ (notation of (9.3.5)), whence an isomorphism of B-algebras :

$$D^{\mathrm{sh}}_{(m)} := D_{(m)} \otimes_B B^{\mathrm{sh}} = D''_{(m)} \otimes_{B''} (B^{\mathrm{sh}} \otimes_{K^+[\Delta']} K^+[\Delta'_{(m)}])$$

and since $(p,m)=1$, the B-algebra $B^{\mathrm{sh}} \otimes_{K^+[\Delta']} K^+[\Delta'_{(m)}]$ is a finite direct product of copies of B^{sh}. Hence, let $\mathscr{D}_{/b}$ be the quasi-coherent $\mathscr{O}_X$-algebra determined by the finite B^{sh}-algebra $D''_{(m)} \otimes_{B''} B^{\mathrm{sh}}/bB^{\mathrm{sh}}$, and set

$$\mathscr{E} := j_{/b*}(\mathscr{A} \otimes_{\mathscr{O}_U} j^*_{/b}\mathscr{D}_{/b}).$$

It follows that $\mathscr{D}_{(m)/b}$ is a finite direct sum of copies of $\mathscr{D}_{/b}$, and $j_{/b*}\mathscr{A}_{(m)} = \mathscr{E} \otimes_{\mathscr{D}_{/b}} \mathscr{D}_{(m)/b}$. Especially, since the natural map $\mathscr{D}_{/b} \to \mathscr{D}_{(m)/b}$ admits a left inverse, it is easily seen that $\mathscr{E}$ is an étale and almost finitely presented $\mathscr{D}^a_{/b}$-algebra, and we are thus reduced to showing that $\mathscr{E}$ descends to an étale and almost finitely presented $\mathscr{O}^a_{X_{/b}}$-algebra.

To ease notation, set $\Gamma := \Delta' \oplus \Delta''_{(m)}$ and $D := K^+[\Delta'] \otimes_{K^+} D''_{(m)}$. Let G be the automorphism group of the B-algebra D; there is a natural isomorphism

$$\rho : G \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}}(\Delta''^{\mathrm{gp}}, \boldsymbol{\mu}_m) = (\Delta'')^{\mathrm{gp}\vee} \otimes_{\mathbb{Z}} \boldsymbol{\mu}_m$$

and we have both $(D/bD)^G = B/bB$ and $\mathscr{E}^G_{|U_{/b}} = \mathscr{A}$. Set $E := \Gamma(X_{/b}, \mathscr{E})$; then corollary 8.6.28(ii) reduces to showing that the induced G-action on E is horizontal. Thus, pick any $\sigma \in G$, and let $I_\sigma \subset D$ be the ideal generated by the elements of the form $a - \sigma(a)$, for a ranging over all the elements of D; we have to check that σ acts trivially on $E^\sigma := E/I_\sigma E$. However, set

$$X^\sigma := \mathrm{Spec}\, D/I_\sigma D \otimes_B B^{\mathrm{sh}}/bB^{\mathrm{sh}} \qquad U^\sigma := X' \times_X U \qquad Z^\sigma := X^\sigma \setminus U^\sigma$$

and denote by $\mathscr{E}^\sigma$ the quasi-coherent $\mathscr{O}^a_{X^\sigma}$-algebra determined by E^σ. By construction, σ acts trivially on $\mathscr{E}^\sigma_{|U^\sigma}$, therefore

$$s - \sigma(s) \in \Gamma_{Z^\sigma}\mathscr{E}^\sigma_! \qquad \text{for every } s \in E^\sigma_!.$$

However, $\mathscr{E}^\sigma_!$ is a flat $\mathscr{O}_{X^\sigma}$-module, and Z^σ is closed and constructible in X^σ; taking into account theorem 5.4.20(ii) and lemma 5.4.19(iii), we conclude that σ acts trivially on E^σ, provided

$$Z^\sigma \cap \mathrm{Ass}\, \mathscr{O}_{X^\sigma} = \varnothing.$$

By inspecting the proof of claim 9.4.36, we see that $I^a_\sigma = A^a_{J_\sigma}$ (notation of (9.3.41)), where $J_\sigma \subset \Gamma$ is the ideal generated by

$$F_\sigma := \{\delta \in \Delta''_{(m)} \mid \rho_\sigma(m\delta) \neq 1\}.$$

Claim 9.5.38. J_σ is a radical ideal of Γ.

Proof of the claim. Indeed, say that $\delta^n \in J_\sigma$ for some integer $n > 0$; we may assume that $n = p^k$ for some integer $k > 0$, and therefore $\delta^{p^k} = \delta_1 \cdot \delta_2$, for some $\delta_1 \in \Gamma$, and $\delta_2 \in F_\sigma$. Since Γ is uniquely p-divisible, we may write $\delta = \delta_1^{p^{-k}} \cdot \delta_2^{p^{-k}}$; but clearly $\delta_2^{p^{-k}} \in F_\sigma$, whence the claim. ◊

Denote $\mathrm{Min}(\Gamma/J_\sigma)$ the set of prime ideals of Γ that are minimal among those that contain J_σ (these are in natural bijection with the minimal prime ideals of the pointed monoid Γ/J_σ). Set

$$I_{\mathfrak{p}} := A_{\mathfrak{p}} \qquad \text{and} \qquad X_{\mathfrak{p}} := \mathrm{Spec}\, D/I_{\mathfrak{p}} \otimes_B B^{\mathrm{sh}}/bB^{\mathrm{sh}} \qquad \text{for every } \mathfrak{p} \in \mathrm{Spec}\,\Gamma$$

(notation of (9.3.41)); from claim 9.5.38 and lemma 3.1.15 we get an injective map :

$$D/I_\sigma \to \prod_{\mathfrak{p} \in \mathrm{Min}(\Gamma/J_\sigma)} D/I_{\mathfrak{p}} \tag{9.5.39}$$

and a simple inspection shows that the induced map (9.5.39) $\otimes_B B^{\mathrm{sh}}/bB^{\mathrm{sh}}$ is still injective. Notice that $D/I_{\mathfrak{p}} \simeq D_{\Gamma \setminus \mathfrak{p}}$ is a small K^+-algebra for every $\mathfrak{p} \in \mathrm{Spec}\,\Gamma$ (remark 9.3.15(iv));

on the one hand, lemma 9.5.29(iii) and proposition 5.5.4(vi) imply that $\mathrm{Ass}\,\mathscr{O}_{X_\mathfrak{p}}$ is the set of maximal points of $X_\mathfrak{p}$. On the other hand, from (9.5.39) we get a finite and surjective morphism

$$\bigcup_{\mathfrak{p}\in\mathrm{Min}(\Gamma/J_\sigma)} X_\mathfrak{p} \to X^\sigma.$$

Notice as well that $X_\mathfrak{p} \times_{X^\sigma} X_\mathfrak{q} = X_{\mathfrak{p}\cup\mathfrak{q}}$, for every $\mathfrak{p},\mathfrak{q} \in \mathrm{Spec}\,\Gamma$. Especially, if $\mathfrak{p} \neq \mathfrak{q}$, then lemma 9.5.29(ii) implies that the intersection of the closed subsets $X_\mathfrak{p}$ and $X_\mathfrak{q}$ is nowhere dense in both of them. Taking into account proposition 5.5.4(ii,iv), we deduce that $\mathrm{Ass}\,\mathscr{O}_{X^\sigma}$ is the set of maximal points of X^σ.

Claim 9.5.40. Let $\sigma \in G$ be an element such that $Z^\sigma \cap \mathrm{Ass}\,\mathscr{O}_{X^\sigma} \neq \varnothing$. Then we have :

(i) Δ'' is a sharp monoid.
(ii) The image of x in $\mathrm{Spec}\,K^+[\Delta']$ is the maximal point of $\mathrm{Spec}\,\kappa[\Delta']$.
(iii) $J_\sigma = \mathfrak{m}_\Gamma$.

Proof of the claim. Suppose that $z \in Z^\sigma \cap \mathrm{Ass}\,\mathscr{O}_{X^\sigma}$, let d be the dimension of the topological closure of $\{z\}$ in Z^σ, say that $z \in X_\mathfrak{p}$ for some $\mathfrak{p} \in \mathrm{Min}(\Gamma/J_\sigma)$, and set $Y_\mathfrak{p} := \mathrm{Spec}\,D/I_\mathfrak{p}$. By the foregoing, z is a maximal point of $X_\mathfrak{p}$, hence the image of z in $Y_\mathfrak{p}$ is a maximal point of $Y_{\mathfrak{p}/b}$, so $d = \dim Y_{\mathfrak{p}/b}$. Since Δ' is a group, we have $\Delta' \subset \Gamma^\times \subset \Gamma\backslash\mathfrak{p}$, therefore $d \geq \dim \Gamma^\times \otimes_\mathbb{Z} \mathbb{Q} \geq r := \dim_\mathbb{Q} \Delta'^{\mathrm{gp}} \otimes_\mathbb{Z} \mathbb{Q}$, by lemma 9.5.29(ii). On the other hand, Z^σ is a closed subset of

$$\mathrm{Spec}\,\kappa(x'') \times_{Y''} X^\sigma_{/b} \subset \mathrm{Spec}\,(K^+[\Delta'] \otimes_{K^+} D''_{(m)} \otimes_{B''} \kappa(x'')) \times_Y X_{/b}.$$

Since the map $D''_{(m)}$ is a filtered union of finite B''-algebras, (9.5.35) implies that $d \leq r$. We conclude that $d = r = \dim \Gamma^\times \otimes_\mathbb{Z} \mathbb{Q}$. This means that (i) holds, and that the image of z in $\mathrm{Spec}\,K^+[\Delta']$ is the maximal point of the closed subset $\mathrm{Spec}\,\kappa[\Delta']$, so (ii) follows as well. We also deduce that $\mathfrak{p} = \mathfrak{m}_\Gamma$ must be the maximal ideal of Γ, which implies (iii). ◊

Assertions (i) and (ii.a) of the theorem already follow from claims 9.5.40 and 9.5.36. Suppose now that $Z^\sigma \cap \mathrm{Ass}\,\mathscr{O}_{X^\sigma} \neq \varnothing$ for some $\sigma \in G$; the foregoing shows that, in this case, there exists a unique point x_+ of $X \setminus X_{/b}$, such that $|Z_+| = \{x, x_+\}$. Let $s := \dim \Delta''$ and denote by $(\mathrm{Spec}\,\Delta'')_{s-1}$ the set of all prime ideals of Δ'' of height $s-1$; with this notation, we have :

Claim 9.5.41. Let U^+ and $\overline{x}_+$ be as in (ii.b), and fix a geometric point ξ of U^+. Then there exists a natural group isomorphism :

$$\pi_1(U^+_{\mathrm{\acute{e}t}}, \xi) \xrightarrow{\sim} \frac{(\Delta'')^{\mathrm{gp}\vee}}{\sum_{\mathfrak{q}\in(\mathrm{Spec}\,\Delta'')_{s-1}} (\Delta''_\mathfrak{q})^{\sharp\mathrm{gp}\vee}} \otimes_\mathbb{Z} \widehat{\mathbb{Z}}(1) \qquad \text{with} \quad \widehat{\mathbb{Z}}(1) := \lim_{n\in\mathbb{N}} \boldsymbol{\mu}_n(K^+)$$

where $(\Delta'')^{\mathrm{gp}\vee} := \mathrm{Hom}_{\mathbb{Z}[1/p]}((\Delta'')^{\mathrm{gp}}, \mathbb{Z}[1/p])$ and likewise for the sugroups $(\Delta''_\mathfrak{q})^{\sharp\mathrm{gp}\vee}$.

Proof of the claim. Let $\overline{y}$ be the image of $\overline{x}_+$ in Y, and K' the algebraic closure in $\kappa(\overline{x}_+)$ of $\mathscr{O}_{Z_+,x_+}$. Set $Y_+ := \mathrm{Spec}\,K' \times_S Y''$, and pick any lifting $\overline{y}_+$ of $\overline{y}$ to Y_+. On the one hand, the induced morphism $Y_+(\overline{y}_+) \to Y(\overline{y})$ is an isomorphism; on the other hand, recall that $\mathrm{Spec}\,K \times_S Y'' = \mathrm{Spec}\,K[\Delta'']$ (remark 9.3.15(v));. Thus, we may replace K by K', Δ by Δ'', and reach the following situation : Δ is sharp, U^+ is of the form $Y_+(\overline{y}_+) \setminus \{\overline{y}_+\}$, where Y_+ is the scheme underlying $(Y_+, \underline{M}_+) := \mathrm{Spec}\,(K, \Delta)$, and $\overline{y}_+$ is the unique geometric point (up to isomorphism) such that the chart $\Delta \to \underline{M}_{+,\overline{y}_+}$ is local. Now, let us write $\Delta = \Delta_0[1/p]$ for some fine and saturated submonoid Δ_0, and define $\Delta_n \subset \Delta$ as in (9.3.19), for every $n \in \mathbb{N}$. Set $(Y_{n,K}, \underline{M}_n) := \mathrm{Spec}(K, \Delta_n)$, let $\overline{y}_n$ be the image of $\overline{y}_+$ in $Y_{n,K}$, and define $U^+_n := Y_{n,K}(\overline{y}_n) \setminus \{\overline{y}_n\}$ for every $n \in \mathbb{N}$. Denote also by ξ_n the image of ξ under the induced morphism $U^+ \to U^+_n$, for every $n \in \mathbb{N}$. In view of lemma 7.1.6 and proposition 1.6.14, the natural map

$$\pi_1(U^+_{\mathrm{\acute{e}t}}, \xi) \to \lim_{n\in\mathbb{N}} \pi_1(U^+_{n,\mathrm{\acute{e}t}}, \xi_n) \tag{9.5.42}$$

is an isomorphism. By construction, the charts $\Delta_n \to \underline{M}_{n,\overline{y}_n}$ are local, for every $n \in \mathbb{N}$. Notice also that the p-Frobenius endomorphism of Δ_n induces an isomorphism $\Delta_n \xrightarrow{\sim} \Delta_{n-1}$, and moreover $\dim \Delta_n = s$ for every $n \in \mathbb{N}$. Thus, set

$$G := \frac{\Delta_0^\vee}{\sum_{\mathfrak{q}\in(\mathrm{Spec}\,\Delta_0)_{s-1}} \Delta_{0,\mathfrak{q}}^\vee} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}(1)$$

(where $(-)^\vee$ denotes the usual dual $\mathrm{Hom}_{\mathbf{Mnd}}(-,\mathbb{N})$). Taking into account (7.3.30) and theorem 7.3.55, we obtain a natural isomorphism between $\pi_1(U^+_{\text{ét}},\xi)$ and the limit of the system of finite abelian groups $(G_n \mid n \in \mathbb{N})$, such that $G_n := G$, and the transition map $G_{n+1} \to G_n$ is induced by the p-Frobenius endomorphism of Δ_0, for every $n \in \mathbb{N}$. In other words, these transition maps are the p-Frobenius endomorphism of G, so the limit is isomorphic to $G \otimes_{\mathbb{Z}} \mathbb{Z}[1/p]$. The claim follows easily. $\diamondsuit$

Next, let $\tau \in G$ be any element, and suppose that J_τ is strictly contained in $\mathfrak{m}_\Gamma$; then claim 9.5.40(iii). shows that τ acts trivially on E^τ, and *a fortiori*, also on E^σ, again due to claim 9.5.40(iii). Now, let $\mathfrak{p} \subset \Gamma$ be any prime ideal different from $\mathfrak{m}_\Gamma$; so $\mathfrak{p} = \Delta' \oplus \mathfrak{p}''$ for a prime ideal $\mathfrak{p}'' \subset \Delta''_{(m)}$ different from the maximal ideal. Suppose that, for some $\tau \in G$ we have

$$F_\tau \subset \mathfrak{p}'' \tag{9.5.43}$$

Then clearly $J_\tau \subset \mathfrak{p}$, and therefore τ acts trivially on our E^σ. However, (9.5.43) means that $\rho_\tau(m\gamma) = 1$ for every $\gamma \in \Delta''_{(\mathfrak{m})} \setminus \mathfrak{p}''$. Set $\mathfrak{q} := \mathfrak{p}'' \cap \Delta''$; the latter condition is equivalent to

$$\rho_\tau(\gamma) = 1 \qquad \text{for every } \gamma \in \Delta'' \setminus \mathfrak{q}$$

i.e. ρ_τ lies in the subgroup

$$G_\mathfrak{q} := (\Delta''_\mathfrak{q})^{\sharp\mathrm{gp}\vee} \otimes_{\mathbb{Z}} \boldsymbol{\mu}_m \subset (\Delta'')^{\mathrm{gp}\vee} \otimes_{\mathbb{Z}} \boldsymbol{\mu}_m.$$

Finally, suppose that the group $\pi_1(U^+_{\text{ét}},\xi)$ vanishes; by claim 9.5.41, this means that $G = \sum_{\mathfrak{q}\in(\mathrm{Spec}\,\Delta'')_{s-1}} G_\mathfrak{q}$; we conclude that the whole of G acts trivially on E^σ, so $\mathscr{E}$ descends to an étale almost finitely presented $\mathscr{O}^a_X$-algebra, which shows that the pair $(X,\{x\})$ is indeed almost pure, as stated.

Lastly, suppose that $r^{-1}r(x) = \{x,x_+\}$ and $\pi_1(U^+_{\text{ét}},\xi)$ does not vanish; we have to show that in this case the pair $(X_{/b},\{x\})$ is not almost pure, and arguing as in the proof of claim 9.5.33, we may assume again that K is algebraically closed. Now, notice first that the map $\pi_1(U^+_{\text{ét}},\xi) \to G$ deduced from (9.5.42) is injective, and its image is the direct summand $G[1/p]$ of G (notation of the proof of claim 9.5.41); this allows to construct a universal (étale) covering $V \to U^+$, as follows. Let N be the order of $G[1/p]$; the quotient map $G \to G[1/p]$ corresponds to an étale covering $V_0 \to U^+_0$ such that $V = U^+ \times_{U^+_0} V_0$, and the discussion of (7.3.39) shows that, in turn, V_0 is the fibre product in a cartesian diagram

$$\begin{tikzcd} V_0 \arrow[r] \arrow[d] & \mathrm{Spec}\,K'[Q_0] \arrow[d] \\ U^+_0 \arrow[r] & \mathrm{Spec}\,K'[\Delta''_0] \end{tikzcd}$$

where K' is as in the proof of claim 9.5.41, and Q_0 is a fine and saturated monoid with

$$\Delta''_0 \subset Q_0 \subset \frac{1}{N}\Delta''_0 \qquad \text{and such that} \qquad Q_0^{\mathrm{gp}}/\Delta_0''^{\mathrm{gp}} = G[1/p]^\vee.$$

Then the right vertical arrow is induced by the inclusion map $\Delta''_0 \to Q_0$, and the bottom arrow is induced by the chart $\Delta''_0 \to \mathscr{O}_{U^+_0,\xi_0}$ of $\underline{M}_{n|U^+_0}$. Next, we wish to extend V to a tamely ramified covering of the whole of Y. Namely, set

$$Q := (\Delta''^{\mathrm{gp}} + Q_0^{\mathrm{gp}}) \cap \Delta''_{\mathbb{Q}} \qquad \text{and} \qquad Y_Q := \mathrm{Spec}\,A_{\Delta'\oplus Q}$$

where $(A, \Delta_{\mathbb{Q}})$ is the auxiliary model algebra constructed in (9.3.35), such that $B = A_\Delta$. It is easily seen that $Q^{\mathrm{gp}}/\Delta''^{\mathrm{gp}} = G[1/p]^\vee$. More precisely, the natural map

$$\Delta'' \otimes_{\Delta_0} Q_0 \to Q$$

is an isomorphism of monoids (details left to the reader), whence a cartesian diagram of schemes

$$\begin{tikzcd} V \arrow[r] \arrow[d] & Y_Q \arrow[d] \\ U^+ \arrow[r] & Y. \end{tikzcd}$$

Since V is connected, it follows that the same holds for $X_Q := X \times_Y Y_Q$; hence, $\overline{x}$ lifts uniquely to a geometric point $\overline{x}_Q$ of Y_Q, and the natural morphism $Y_Q(\overline{x}_Q) \to X_Q$ is an isomorphism. By the same token, since $V = U^+ \times_X X_Q$, we see that $\overline{x}_+$ lifts uniquely to a geometric point $\overline{x}_{Q+}$ of $X_Q \setminus X_{Q/b}$, and the induced morphism $X_Q(\overline{x}_{Q+}) \setminus \{\overline{x}_{Q+}\} \to V$ is an isomorphism. Let x_Q be the closed point of X_Q; since $V_{\text{ét}}$ is simply connected (proposition 1.6.20(ii)), so assertion (ii) of theorem applies to X_Q, and shows that the pair $(X_{Q/b}, \{x_Q\})$ is almost pure.

Claim 9.5.36(i) also shows that $Z_{Q+} := \{x_Q, |\overline{x}_{Q+}|\}$ is a closed subset of X_Q; set $U_Q := X_Q \setminus \{x_Q\}$, $W := X \setminus Z_+$ and $W_Q := X_Q \setminus Z_Q$, and consider the essentially commutative diagram

$$\begin{tikzcd} \mathbf{Cov}(X^a_{Q/b}) \arrow[d] & \mathbf{Cov}(X^a_Q) \arrow[l] \arrow[r] \arrow[d] & \mathbf{Cov}(X_{Q,K}) \arrow[d] \\ \mathbf{Cov}(U^a_{Q/b}) & \mathbf{Cov}(W^a_Q) \arrow[l, "\alpha_1"'] \arrow[r, "\alpha_2"] & \mathbf{Cov}(W_{Q,K}) \end{tikzcd}$$

(where, as usual, $X_{Q,K} := X_Q \backslash X_{Q/b}$, and likewise for $W_{Q,K}$ and W_K; notice that $W_{Q/b} = U_{Q/b}$). The two top horizontal arrows are both equivalences, by lemma 8.2.39(i) and by the almost purity theorem 9.4.38. We have also just remarked that the left vertical arrow is an equivalence. Since $V_{\text{ét}}$ is simply connected, the right vertical arrow is an equivalence as well, by virtue of lemma 7.1.7(ii.b). Lastly, α_2 is an equivalence, again by theorem 9.4.38 and corollary 8.2.31. Finally, we deduce that α_1 is an equivalence. Let $g : X_Q \to X$ be the natural morphism, and set $\mathscr{B} := g_*\mathscr{O}_{X_Q}$.

Claim 9.5.44. The restriction $W_{Q,K} \to W_K$ of g is a finite étale covering.

Proof of the claim. To ease notation, set $M := \Delta''$ and $N := Q_0$; by inspecting the constructions, we see that

$$M \subset N \subset N' := (\textstyle\sum_{\mathfrak{q} \in (\operatorname{Spec} M)_{s-1}} M^\vee_{\mathfrak{q}})^\vee \subset M_{\mathbb{Q}}.$$

Moreover, consider the morphism of objects of $\mathscr{K}$ (see (6.6.2) and example 6.6.5(i,ii)) :

$$\begin{tikzcd} \operatorname{Spec}(K,N)^\sharp \arrow[r, "\operatorname{Spec}(K,j)"] \arrow[d, "\psi_N"'] & \operatorname{Spec}(K,M)^\sharp \arrow[d, "\psi_M"] \\ (\operatorname{Spec} N)^\sharp \arrow[r, "(\operatorname{Spec} j)^\sharp"] & (\operatorname{Spec} M)^\sharp \end{tikzcd}$$

induced by the inclusion map $j : M \to N$, and set

$$U_M := \operatorname{Spec} M \backslash \{\mathfrak{m}_M\} \qquad \text{and} \qquad U_N := \operatorname{Spec} N \backslash \{\mathfrak{m}_N\}.$$

We easily reduce to showing that the restriction $\psi^{-1}U_N \to \psi^{-1}U_M$ of $\operatorname{Spec} K[j]$ is an étale morphism of schemes. However, we know already that $\operatorname{Spec}(K,j)$ is an étale morphism of log schemes (proposition 6.3.34), hence it suffices to check that the restriction $U_N \to U_M$ of $(\operatorname{Spec} j)^\sharp$ is a strict morphism of fans (corollary 6.3.27(i)). This, in turn, comes down to checking that $j^\sharp_{\mathfrak{p}} : M^\sharp_{\mathfrak{p}} \to N^\sharp_{\mathfrak{p}}$ is an isomorphism, for every prime ideal $\mathfrak{p} \subset M$ of height $s-1$.

To this aim, it suffices to show that the corresponding map $M^\sharp_\mathfrak{p} \to N'^\sharp_\mathfrak{p}$ is an isomorphism for every such $\mathfrak{p}$, or equivalently, that the same holds for the dual map $(N'_\mathfrak{p})^\vee \to (M_\mathfrak{p})^\vee$ (proposition 3.4.12(iv)). However, notice that, quite generally $(T_\mathfrak{t})^\vee = \{\tau \in T^\vee \mid \tau(T\setminus\mathfrak{t}) = 0\}$ for every monoid T and every prime ideal $\mathfrak{t} \subset T$. Therefore :

$$\begin{aligned}(N'_\mathfrak{p})^\vee &= \{\tau \in (\textstyle\sum_{\mathfrak{q}\in(\mathrm{Spec}\,M)_{s-1}} M^\vee_\mathfrak{q})^{\vee\vee} \mid \tau(M\setminus\mathfrak{p}) = 0\} \\ &= \{\tau \in \textstyle\sum_{\mathfrak{q}\in(\mathrm{Spec}\,M)_{s-1}} M^\vee_\mathfrak{q} \mid \tau(M\setminus\mathfrak{p}) = 0\} \\ &= \textstyle\sum_{\mathfrak{q}\in(\mathrm{Spec}\,M)_{s-1}} M^\vee_{\mathfrak{q}\cap\mathfrak{p}} = M^\vee_\mathfrak{p}\end{aligned}$$

as contended. $\Diamond$

From claim 9.5.44 and proposition 8.2.30, we deduce that $\mathscr{B}^a_{|W}$ is an étale $\mathscr{O}^a_W$-algebra of finite rank, and notice that the category $\mathscr{B}^a$-$\mathbf{\acute{E}t}_{\mathrm{fr}}$ is equivalent to the category $\mathscr{O}^a_{W_Q}$-$\mathbf{\acute{E}t}_{\mathrm{fr}}$ (notation of definition 8.2.25). Hence, consider the essentially commutative diagram

$$\begin{array}{ccc}\mathscr{O}^a_W\text{-}\mathbf{\acute{E}t}_{\mathrm{fr}} & \longrightarrow & \mathscr{O}^a_{U/b}\text{-}\mathbf{\acute{E}t}_{\mathrm{fr}} \\ \downarrow & & \downarrow \\ \mathsf{Desc}(\mathscr{O}^a\text{-}\mathbf{\acute{E}t}_{\mathrm{fr}}, \{W_Q \to W\}) & \longrightarrow & \mathsf{Desc}(\mathscr{O}^a\text{-}\mathbf{\acute{E}t}_{\mathrm{fr}}, \{U_{Q/b} \to U_{/b}\})\end{array}$$

(notation of (8.5.4) and (1.5.27)). Since α_1 is an equivalence, it is easily seen that the same holds for the bottom horizontal arrow. Also, the two vertical arrows are equivalences, by faithfully flat descent. We conclude that the top horizontal arrow is also an equivalence. We reach then the essentially commutative diagram

$$\begin{array}{ccccc}\mathbf{Cov}(X^a_{/b}) & \longleftarrow & \mathbf{Cov}(X^a) & \longrightarrow & \mathbf{Cov}(X_K) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{Cov}(U^a_{/b}) & \longleftarrow & \mathbf{Cov}(W^a) & \longrightarrow & \mathbf{Cov}(W_K)\end{array}$$

and summing up the foregoing, we see that all the horizontal arrows in this last diagram are equivalences. However, the restriction $X_{Q,K} \to X_K$ of g is not étale, so the right vertical arrow is not an equivalence, and therefore neither is the left vertical one, as claimed. $\square$

9.6. **Almost purity : the log regular case.** In this section, we prove an almost purity theorem for certain towers of regular log schemes.

9.6.1. Let $\underline{M}_0$ be a log structure on the Zariski site of a local scheme X_0, such that $(X_0, \underline{M}_0)$ is a regular log scheme, and say that $X_0 = \mathrm{Spec}\,B_0$ for a local ring B_0 which is necessarily noetherian, normal and Cohen-Macaulay (corollary 6.5.29). Let $x_0 \in X_0$ be the closed point, and $\beta_0 : P \to B_0$ a chart for $\underline{M}_0$ which is sharp at x_0. Especially, P is a fine and saturated monoid, and $A := B_0/\mathfrak{m}_P B_0$ is a regular local ring; we denote by $\mathfrak{m}_A$ (resp. $\mathfrak{m}_{B_0}$) the maximal ideal of A (resp. of B_0). We assume furthermore that :

(a) The characteristic p of the residue field $\kappa(x_0)$ of A is positive.
(b) The Frobenius endomorphism Φ_{B_0} of B_0/pB_0 is a finite ring homomorphism.

Notice that (b) implies that B_0/pB_0 is excellent (theorem 4.8.42(i)) and also that $\Omega^1_{A/\mathbb{Z}} \otimes_A \kappa(x_0)$ is a finite dimensional $\kappa(x_0)$-vector space. Set

$$P^{(n)} := \{\gamma \in P_\mathbb{Q} \mid \gamma^{p^n} \in P\} \qquad \text{for every } n \in \mathbb{N}.$$

The p^n-Frobenius map of $P^{(n)}$ identifies $P^{(n)}$ with its submonoid P; in other words, the inclusion map $P \to P^{(n)}$ is naturally identified with the p^n-Frobenius endomorphism of P. Fix a sequence $(f_1, \dots, f_r)$ of elements of B_0 whose image in A is *maximal* in the sense of remark 4.7.28(iii). Notice that $\dim_{\kappa(x_0)} \mathfrak{m}_A/\mathfrak{m}_A^2 = \dim A$, since A is regular; it follows that

$$r = \dim A + \dim_{\kappa(x_0)} \Omega^1_{\kappa(x_0)/\mathbb{Z}}$$

by virtue of the short exact sequence (4.7.2), if $p \in \mathfrak{m}_A^2$, and otherwise, by virtue of proposition 4.7.14. Denote also by $\overline{f}_i \in A$ the image of f_i, for every $i = 1, \dots, r$. According to corollary 4.7.27, the ring

$$A_n := A[T_1, \dots, T_r]/(T_1^{p^n} - \overline{f}_1, \dots, T_r^{p^n} - \overline{f}_r)$$

is regular. For every $n \in \mathbb{N}$, we set :

$$B'_n := P^{(n)} \otimes_P B_0 \qquad B''_n := B_0[T_1, \dots, T_r]/(T_1^{p^n} - f_1, \dots, T_r^{p^n} - f_r) \qquad B_n := B'_n \otimes_{B_0} B''_n.$$

Lemma 9.6.2. *The induced maps*

$$B_n \to B_{n+1} \qquad \textit{and} \qquad B_n/pB_n \to B_{n+1}/pB_{n+1}$$

are injective, for every $n \in \mathbb{N}$.

Proof. The natural map $B_n \to B_{n+1}$ factors as the composition

$$B'_n \otimes_{B_0} B''_n \to B'_{n+1} \otimes_{B_0} B''_n \to B'_{n+1} \otimes_{B_0} B''_{n+1} \qquad \text{for every } n \in \mathbb{N}.$$

Notice that $B_0 = B''_0$, and B''_{n+1} is a free B''_n-module (of rank p^r) for every $n \in \mathbb{N}$; we are then reduced to checking that the maps $B'_n \to B'_{n+1}$ and $B'_n/pB'_n \to B'_{n+1}/pB'_{n+1}$ are injective for every $n \in \mathbb{N}$. However, set $G := P^{(n+1)}/P^{(n)}$, and notice that $P^{(n+1)}$ is a G-graded monoid, with $P_0^{(n+1)} = P^{(n)}$, hence B'_{n+1} is a G-graded B'_n-algebra with $(B'_{n+1})_0 = B'_n$ for every $n \in \mathbb{N}$. The assertion follows. $\square$

9.6.3. In view of lemma 9.6.2, we may set

$$B'' := \bigcup_{n \in \mathbb{N}} B''_n \qquad B := \bigcup_{n \in \mathbb{N}} B_n \qquad P^{(\infty)} := \bigcup_{n \in \mathbb{N}} P^{(n)}$$

and clearly

$$B = P^{(\infty)} \otimes_P B'' \qquad \text{for every } n \in \mathbb{N}$$

from which we see that B is naturally a $P^{(\infty)}/P^{(n)}$-graded B_n-algebra, for every $n \in \mathbb{N}$. Also, the induced morphism $\operatorname{Spec} B_{n+1}/pB_{n+1} \to \operatorname{Spec} B_n/pB_n$ is radicial and surjective, so $\operatorname{Spec} B_n/pB_n$ is a local scheme for every $n \in \mathbb{N}$; on the other hand, the map $B_0 \to B_n$ is finite, so every point of $\operatorname{Spec} B_n$ specializes to a point of $\operatorname{Spec} B_n/pB_n$. We conclude that B_n is a local ring, and we denote by $\mathfrak{m}_{B_n}$ its maximal ideal, for every $n \in \mathbb{N}$. Let $\underline{M}_n$ be the log structure on the Zariski site of $X_n := \operatorname{Spec} B_n$ deduced from the natural map $\beta_n : P^{(n)} \to B_n$; notice that $B_n/\mathfrak{m}_{P^{(n)}} B_n = A_n$ is a regular local ring of dimension equal to $\dim A$. Since we have as well $\dim P^{(n)} = \dim P$, it follows that $(X_n, \underline{M}_n)$ is regular at the closed point $x_n \in X_n$. Then theorem 6.5.46 shows that $(X_n, \underline{M}_n)$ is a regular log scheme. Thus, we have obtained a tower of finite morphisms of regular log schemes

$$\cdots \to (X_{n+1}, \underline{M}_{n+1}) \to (X_n, \underline{M}_n) \to \cdots \to (X_0, \underline{M}_0) \tag{9.6.4}$$

which we call the *maximal tower* associated to the chart $\beta_0 : P \to B_0$ and the maximal sequence $(f_1, \dots, f_r)$. The limit of the tower (9.6.4) is a log scheme $(X, \underline{M})$ whose log structure admits a chart $P^{(\infty)} \to B$ which is sharp at the closed point.

Remark 9.6.5. Keep the notation of (9.6.3), and let $s \in \mathbb{N}$ be any integer; from remark 4.7.28(iii) it is easily seen that the sequence $((X_{n+s}, \underline{M}_{n+s}) \mid n \in \mathbb{N})$, obtained by removing from (9.6.4) the first s terms, is the maximal tower associated to the chart β_n and the maximal sequence $(f_1^{1/p^s}, \dots, f_r^{1/p^s})$.

9.6.6. Let now $y \in (X_0, \underline{M}_0)_{\mathrm{tr}}$ be any point, and $n \in \mathbb{N}$ any integer. The chart β_0 extends uniquely to a morphism of monoids $\beta_0^{\mathrm{gp}} : P^{\mathrm{gp}} \to B_{0,y} := \mathscr{O}_{X_0,y}$, and we have a natural isomorphism

$$B_n \otimes_{B_0} B_{0,y} \xrightarrow{\sim} P^{(n)\mathrm{gp}} \otimes_{P^{\mathrm{gp}}} B_{0,y} \otimes_{B_0} B''_n.$$

Especially, $B_n \otimes_{B_0} B_{0,y}$ is a free $B_{0,y}$-module of rank p^{nd}, where

$$d := \dim P + r = \dim P + \dim A + \dim_{\kappa(x_0)} \Omega^1_{\kappa(x_0)/\mathbb{Z}}.$$

However, since $(X_0, \underline{M}_0)$ is regular, we have $\dim A + \dim P = \dim B_0$. Summing up, we find

$$d = \dim B_0 + \dim_{\kappa(x_0)} \Omega^1_{\kappa(x_0)/\mathbb{Z}}.$$

Next, let $z \in \operatorname{Spec} B_0/pB_0 \subset X_0$ be any point, and $\mathfrak{p}_z \subset B_0$ the corresponding prime ideal; as in the foregoing, the chart β_0 extends uniquely to a morphism $\beta_{\mathfrak{p}} : P_{\mathfrak{p}} \to B_{0,z} := \mathscr{O}_{X_0,z}$, where $\mathfrak{p} := \beta_0^{-1}\mathfrak{p}_z$. As already remarked, the point z lifts uniquely to a point $z_n \in X_n$, and on the other hand, there exists a unique prime ideal $\mathfrak{p}^{(n)} \subset P^{(n)}$ containing $\mathfrak{p}$, and the inclusion map $j_{\mathfrak{p}} : P_{\mathfrak{p}} \to P^{(n)}_{\mathfrak{p}^{(n)}}$ is naturally identified with the p^n-Frobenius endomorphism of $P_{\mathfrak{p}}$. By lemma 3.2.10, there exists an isomorphism of monoids $P_{\mathfrak{p}} \xrightarrow{\sim} G \times Q$, with $G := P^{\times}_{\mathfrak{p}}$ and $Q := P^{\sharp}_{\mathfrak{p}}$; we may then find a corresponding decomposition $P^{(n)}_{\mathfrak{p}^{(n)}} = G^{(n)} \times Q^{(n)}$ that identifies $j_{\mathfrak{p}}$ with the product of maps of monoids $G \to G^{(n)}$ and $Q \to Q^{(n)}$. Summing up, there follows an isomorphism of B_0-algebras

$$\mathscr{O}_{X_n,z_n} \xrightarrow{\sim} P^{(n)}_{\mathfrak{p}^{(n)}} \otimes_{P_{\mathfrak{p}}} B_{0,z} \otimes_{B_0} B''_n \xrightarrow{\sim} (Q^{(n)} \otimes_Q B_{0,z}) \otimes_{B_{0,z}} (G^{(n)} \otimes_G B_{0,z}) \otimes_{B_0} B''_n.$$

Fix a basis $g_1, \dots, g_s$ of the free $\mathbb{Z}$-module G, and set $f_{r+i} := \beta_{\mathfrak{p}}(g_i)$ for $i = 1, \cdots, s$; clearly

$$(G^{(n)} \otimes_G B_{0,z}) \otimes_{B_0} B''_n = B_{0,z}[T_1, \dots, T_{r+s}]/(T_1^{p^n} - f_1, \dots, T_{r+s}^{p^n} - f_{r+s}).$$

On the other hand, set $A_z := B_{0,z}/\mathfrak{m}_Q B_{0,z}$; we have

$$\begin{aligned} \dim P &= \dim Q + s && \text{(corollary 3.4.10(i))} \\ \dim B_{0,z} &= \dim Q + \dim A_z && \text{(since } (X_0, \underline{M}_0) \text{ is regular)} \\ \dim B_{0,z} &= d - \dim_{\kappa(z)} \Omega^1_{\kappa(z)/\mathbb{Z}} && \text{(proposition 4.8.36).} \end{aligned}$$

Therefore $r + s = \dim A_z + \dim_{\kappa(z)} \Omega^1_{\kappa(z)/\mathbb{Z}}$. However, $A_{z_n} := \mathscr{O}_{X_n,z_n}/\mathfrak{m}_{Q^{(n)}}\mathscr{O}_{X_n,z_n}$ is a regular local ring, since $(X_n, \underline{M}_n)$ is regular, and by inspecting the construction we see that

$$A_{z_n} = A_z[T_1, \dots, T_{r+s}]/(T_1^{p^n} - f_1, \dots, T_{r+s}^{p^n} - f_{r+s})$$

hence the image of the system $(f_1, \dots, f_{r+s})$ yields a basis of either $\Omega^1_{A_z/\mathbb{Z}} \otimes_{A_z} \kappa(z)$ or $\boldsymbol{\Omega}_{A_z}$, depending on whether or not $p \in \mathfrak{m}^2_{A_z}$ (corollary 4.7.27). In conclusion, we see that the induced sequence of morphisms of log schemes

$$\cdots \to (X_{n+1}(z_n), \underline{M}_{n+1}(z_n)) \to (X_n(z_n), \underline{M}_n(z_n)) \to \cdots \to (X_0(z), \underline{M}_0(z))$$

(notation of (6.7.11)) is the maximal tower associated to the induced chart $Q \to \mathscr{O}_{X_0,z}$ and the maximal sequence $(f_1, \dots, f_{r+s})$.

9.6.7. Let $\Phi_{B_n} : B_n/pB_n \to B_n/pB_n$ be the Frobenius endomorphism of B_n/pB_n; taking into account lemma 9.6.2, we see that $\Phi_{B_{n+1}}$ factors through a unique ring homomorphism

$$\overline{\Phi}_{B_{n+1}} : B_{n+1}/pB_{n+1} \to B_n/pB_n \qquad \text{for every } n \in \mathbb{N}.$$

Lemma 9.6.8. *The map $\overline{\Phi}_{B_{n+1}}$ is surjective for every $n \in \mathbb{N}$.*

Proof. Let us start out with the following general :

Claim 9.6.9. Let $\varphi : R \to S$ be an injective, finite and radicial ring homomorphism. Then φ is an isomorphism if and only if $\Omega^1_{S/R} = 0$.

Proof of the claim. We may assume that $\Omega^1_{S/R} = 0$, and we show that φ is an isomorphism. To this aim, it suffices to show that

$$\varphi \otimes_R R/\mathfrak{m} : R' := R/\mathfrak{m} \to S' := S/\mathfrak{m}S$$

is an isomorphism for every maximal ideal $\mathfrak{m} \subset R$. However, $\Omega^1_{S'/R'} = \Omega^1_{S/R} \otimes_R R' = 0$, so we may replace R by R' and S by S', and assume from start that R is a field. In this case, S is a local unramified R-algebra, so S must be a finite separable field extension of R, by [33, Ch.IV, Cor.17.4.2]. But S is also a radicial extension of R, so $S = R$. $\diamondsuit$

Claim 9.6.10. Let $p > 0$ be a prime integer, R a local $\mathbb{F}_p$-algebra whose Frobenius endomorphism Φ_R is a finite map, k_R the residue field of R, and $I \subset R$ an ideal. Set $R_0 := R/I$, and let $(g_1, \dots, g_n)$ be a finite sequence of elements of R such that $dg_1, \dots, dg_n$ is a system of generators for the k_R-vector space $\Omega^1_{R_0/\mathbb{Z}} \otimes_R k_R$. Then $R = R^p[I, g_1, \dots, g_n]$.

Proof of the claim. Set $S := R^p[I, g_1, \dots, g_n]$. The inclusion map $S \to R$ is clearly radicial, and it is finite, since Φ_R is finite; hence claim 9.6.9 reduces to checking that $\Omega^1_{R/S}$ vanishes. By Nakayama's lemma, it then suffices to show that $\Omega^1_{R/S} \otimes_R k_R = 0$. However, let M (resp. N) be the R-submodule of $\Omega := \Omega^1_{R/\mathbb{Z}}$ generated by $\{da \mid a \in I\}$ (resp. by $\{dg_1, \dots, dg_n\}$); then $\Omega^1_{R/S} = \Omega/(M + N)$ ([30, Ch.0, Th.20.5.7(i)]) and on the other hand, the induced map $(\Omega/M) \otimes_R R_0 \to \Omega^1_{R_0/\mathbb{Z}}$ is an isomorphism ([30, Ch.0, Th.20.5.12(i)]). We deduce a right exact sequence of R-modules

$$N \xrightarrow{j} \Omega^1_{R_0/\mathbb{Z}} \otimes_R k_R \to \Omega^1_{R/S} \otimes_R k_R \to 0.$$

But our choice of the sequence $(g_1, \dots, g_n)$ implies that j is surjective, whence the contention. $\diamondsuit$

Finally, notice that the image of $\overline{\Phi}_{B_{n+1}}$ is the subring $(B_n/pB_n)^p[\beta_n(P^{(n)}), f_1^{1/p^n}, \dots, f_r^{1/p^n}]$. For every $n > 0$, consider the exact sequence

$$\Omega^1_{A/\mathbb{Z}} \otimes_A \kappa(x_n) \xrightarrow{j} \Omega^1_{A_n/\mathbb{Z}} \otimes_{A_n} \kappa(x_n) \to \Omega^1_{A_n/A} \otimes_{A_n} \kappa(x_n) \to 0$$

([30, Ch.0, Th.20.5.7(i)]); the image of j is generated by the image of the generating system $df_\bullet := (df_1, \dots, df_r)$ of the $\kappa(x_0)$-vector space $\Omega^1_{A/\mathbb{Z}} \otimes_A \kappa(x_0)$. However, since f_i admits a p-th root in A_n for every $i = 1, \dots, r$, it is easily seen that the image of $df_\bullet$ vanishes in $\Omega^1_{A_n/\mathbb{Z}}$. Hence $\Omega^1_{A_n/\mathbb{Z}} \otimes_{A_n} \kappa(x_n) = \Omega^1_{A_n/A} \otimes_{A_n} \kappa(x_n)$ is generated by the image of the system $(df_1^{1/p^n}, \dots, df_r^{1/p^n})$. Therefore, to prove the lemma, it suffices to apply claim 9.6.10 to $R := B_n/pB_n$, $I := \beta_n(\mathfrak{m}_{P^{(n)}})R$ and the sequence $(f_1^{1/p^n}, \dots, f_r^{1/p^n})$. $\square$

Theorem 9.6.11. *With the notation of* (9.6.3), *we have :*

(i) *If B_0 is an $\mathbb{F}_p$-algebra, $\overline{\Phi}_{B_1}$ is an isomorphism.*
(ii) *If B_0 is not an $\mathbb{F}_p$-algebra, then there exists $\pi \in B_1$ and $u \in B_1^\times$ such that $\pi^p = pu$, and* $\mathrm{Ker}\,\overline{\Phi}_{B_1} = \pi B_1/pB_1$.

Proof. (i) is immediate from lemma 9.6.8, since in that case Φ_{B_1} is the Frobenius endomorphism of B_1, so it is injective, since the latter is a domain.

(ii): Suppose we have found π as sought, and let $x \in B_1$ whose image in B_1/pB_1 lies in $\mathrm{Ker}\,\overline{\Phi}_{B_1}$; this means that $x^p = py$ holds in B_1 for some $y \in B_1$; hence $(x/\pi)^p \in B_1$, and since B_1 is a normal ring (corollary 6.5.29), we deduce that $x/\pi \in B_1$, *i.e.* $x \in \pi B_1$, which shows the second assertion of (ii). It thus remains only to exhibit π with the required properties.

Assume first that $p \in \mathfrak{m}_A^2$. Then we may write

$$p = \sum_{i=1}^{t} b_i b'_i + \sum_{j=1}^{s} b''_j \cdot \beta_0(x_j)$$

for certain $b_1, b'_1, \dots, b_t, b'_t \in \mathfrak{m}_{B_0}$, $b''_1, \dots, b''_s \in B_0$, and $x_1, \dots, x_s \in \mathfrak{m}_P$. In light of lemma 9.6.8, we may then write

$$b_i = c_i^p + pd_i \quad b'_i = c'^p_i + pd'_i \qquad \text{where } c_i, c'_i, d_i, d'_i \in \mathfrak{m}_{B_1} \text{ for every } i = 1, \dots, t$$

and likewise, $b''_j = c''^p_j + pd''_j$ where $c''_j \in B_1$ for $j = 1, \dots, s$. Moreover, by construction, β_0 extends to a morphism of monoids $\beta_1 : P^{(1)} \to B_1$, and we may write $x_j = y_j^p$ with $y_j \in \mathfrak{m}_{P^{(1)}}$ for $j = 1, \dots, s$. A simple computation then yields

$$p \cdot (1 + e) = \sum_{i=1}^{t} c_i^p c'^p_i + \sum_{j=1}^{s} c''^p_j \cdot \beta_1(y_j)^p \qquad \text{for some } e \in \mathfrak{m}_{B_1}. \tag{9.6.12}$$

However, the right-hand side of (9.6.12) can be written in the form $g^p + ph$ for some $g, h \in \mathfrak{m}_{B_1}$ (details left to the reader); clearly $1 + e - h \in B_1^\times$, whence the contention, in this case.

Next, suppose that $p \notin \mathfrak{m}_A^2$. In this case, recall that

$$\dim_{\kappa(x_0)^{1/p}} \boldsymbol{\Omega}_A = 1 + \dim_{\kappa(x_0)} \Omega^1_{A/\mathbb{Z}} \otimes_A \kappa(x_0).$$

Therefore, after reordering the sequence $(f_1, \dots, f_r)$, we may assume that $df_1, \dots, df_{r-1}$ is a basis of the $\kappa(x_0)$-vector space $\Omega^1_{A/\mathbb{Z}} \otimes_A \kappa(x_0)$. Set

$$B' := (P^{(1)} \otimes_P B_0)[T_1, \dots, T_{r-1}]/(T_1^p - f_1, \dots, T_{r-1}^p - f_{r-1}).$$

Clearly B_1 is a faithfully flat B'-algebra, hence B'/pB' is a B_0/pB_0-subalgebra of B_1/pB_1, so the natural map $B_0/pB_0 \to B'/pB'$ is injective (lemma 9.6.2), and just as in (9.6.3), we deduce that the Frobenius endomorphism of B'/pB' factors through a ring homomorphism $\overline{\Phi}_{B'} : B'/pB' \to B_0/pB_0$, and arguing as in the foregoing, we also see that B' is a local ring. Moreover, by applying claim 9.6.8 with $R := B'/pB'$, $I := \beta_0(\mathfrak{m}_P)R$ and the sequence $(f_1, \dots, f_{r-1})$ we conclude – as in the proof of lemma 9.6.8 – that $\overline{\Phi}_{B'}$ is surjective. Hence, denote by $\mathfrak{m}_{B'}$ the maximal ideal of B'; it follows that there exists $g \in \mathfrak{m}_{B'}$ and $h \in B'$ such that $f_r = g^p + ph$ in B'. Set

$$A' := B'/\mathfrak{m}_{P^{(1)}} B' = A[T_1, \dots, T_{r-1}]/(T_1^p - f_1, \dots, T_{r-1}^p - f_{r-1}).$$

Then A' is a regular local ring, by corollary 4.7.27, applied to the sequence $(\overline{f}_1, \dots, \overline{f}_{r-1})$ consisting of the images of the elements f_i in A. By the same criterion – applied to the similar sequence $(\overline{f}_1, \dots, \overline{f}_r)$ – we see that $A'[T]/(T^p - f_r)$ is regular as well. So once again the same corollary – applied to the element $\overline{f}_r$ of A' – says that the element $\boldsymbol{d}(f_r)$ of $\boldsymbol{\Omega}_{A'}$ does not vanish. However

$$\boldsymbol{d}(f_r) = \boldsymbol{d}(g^p) + \boldsymbol{d}(ph) = pg^{p-1}\boldsymbol{d}(g) + h\boldsymbol{d}(p) + p\boldsymbol{d}(h) = h\boldsymbol{d}(p) \qquad \text{in } \boldsymbol{\Omega}_{A'}$$

(see (4.7.13)). We conclude that $h \in B'^\times$. Lastly, notice that f_r admits a p-th root $f_r^{1/p}$ in B_1, hence $ph = f_r - g^p = (f_r^{1/p} - g)^p + pe$ in B_1 for some $e \in \mathfrak{m}_{B_1}$ (details left to the reader). Since $h - e \in B_1^\times$, we are done. $\square$

Remark 9.6.13. Suppose B_0 is not an $\mathbb{F}_p$-algebra. Then we may construct inductively a sequence of elements $(\pi_n \mid n \in \mathbb{N})$ such that :

(a) $\pi_n \in B_n$ for every $n \in \mathbb{N}$.
(b) $\pi_0 = p$, and $\pi_{n+1}^p/\pi_n \in B_{n+1}^\times$ for every $n \in \mathbb{N}$.
(c) $\pi_n B_n/pB_n = \mathrm{Ker}(\overline{\Phi}_n \circ \cdots \circ \overline{\Phi}_1)$ for every $n > 0$.

Indeed, theorem 9.6.11(ii) provides π_1 as required. Suppose that π_n has already been exhibited for some $n \geq 1$; by lemma 9.6.8, we may then find $\pi_{n+1}, b \in B_{n+1}$ such that $\pi_{n+1}^p = \pi_n + pb$. Hence, $\pi_{n+1}^p = \pi_n u_n$, where $u_n := 1 + \pi_n^{-1} pb \in B_{n+1}^\times$. Condition (c) then follows, as in the proof of theorem 9.6.11 : details left to the reader.

9.6.14. Henceforth we shall restrict to the case where B_0 is not an $\mathbb{F}_p$-algebra, so B_0 is a local integral domain whose field of fractions has characteristic zero, and whose residue field has characteristic p. For every $n \in \mathbb{N}$, set

$$(X_n)_{/p} := \mathrm{Spec}\, B_n/pB_n \qquad V_n := X_n \setminus (X_n)_{/p} \qquad U_n := (X_n, \underline{M})_{\mathrm{tr}} \cap V_n.$$

We consider now a finite morphism $\varphi_0 : Y_0 \to X_0$ with Y_0 a normal scheme, such that φ_0 maps each connected component of Y_0 onto X_0, and such that the restriction $\varphi_0^{-1} U_0 \to U_0$ is a finite étale covering. For every $n \in \mathbb{N}$, we let Y_n be the normalization of X_n in $\varphi^{-1} U_0 \times_{X_0} X_n$, and denote by $\varphi_n : Y_n \to X_n$ the resulting finite morphism ([61, Lemma 1, p.262]), by $\varphi : Y \to X$ the limit of the system $(\varphi_n \mid n \in \mathbb{N})$, and by $W_n \subset X_n$ the *étale locus of* φ_n*, i.e.* the largest open subset such that the restriction $\varphi_n^{-1} W_n \to W_n$ of φ_n is an étale covering (lemma 7.1.7(ii.b) and claim 7.1.8). Let also $(X_{n,\mathfrak{p}} \mid \mathfrak{p} \in \mathrm{Spec}\, P^{(n)})$ be the logarithmic stratification of $(X_n, \underline{M}_n)$ (see (6.5.49)) and set $V_{n,\mathfrak{p}} := X_{n,\mathfrak{p}} \cap V_n$ for every $n \in \mathbb{N}$ and every $\mathfrak{p} \in \mathrm{Spec}\, P^{(n)}$. We call $(V_{n,\mathfrak{p}} \mid \mathfrak{p} \in \mathrm{Spec}\, P^{(n)})$ the *logarithmic stratification* of V_n.

Lemma 9.6.15. *With the notation of* (9.6.14)*, the following holds :*

(i) $V_n \setminus W_n$ *is a union of strata of the logarithmic stratification of* V_n*, for every* $n \in \mathbb{N}$.

(ii) $X_{n+1} \times_{X_n} W_n = W_{n+1}$ *for every sufficiently large* $n \in \mathbb{N}$.

Proof. (i): Since every $X_{n,\mathfrak{p}}$ is irreducible (corollary 6.5.51(iii)), the same holds for each $V_{n,\mathfrak{p}}$. Hence, suppose that $V_n \setminus W_n$ contains a point z of $V_{n,\mathfrak{p}}$, for some $\mathfrak{p} \in \mathrm{Spec}\, P^{(n)}$; we have to show that in this case $V_{n,\mathfrak{p}} \cap W_n = \varnothing$, and since W_n is an open subset of X_n, it suffices to check that the generic point η of $V_{n,\mathfrak{p}}$ does not lie in W_n. However, let $\overline{\eta}$ be any geometric point localized at η; then η lies in W_n if and only if the induced morphism $\varphi_n \times_{X_n} X_n(\overline{\eta})$ is étale (claim 7.1.9). Let also $\overline{z}$ be a geometric point localized at z, and denote by $\underline{M}_{n,\overline{z}}$ (resp. $\underline{M}_{n,\overline{\eta}}$) the stalk at $\overline{z}$ (resp. $\overline{\eta}$) of the logarithmic structure of $(X_n(\overline{z}), \underline{M}_n(\overline{z}))$ (resp. of $(X_n(\overline{\eta}), \underline{M}_n(\overline{\eta}))$). Any choice of a strict specialization morphism $s : X_n(\overline{\eta}) \to X_n(\overline{z})$ induces a strict specialization map $\overline{\sigma} : \underline{M}_{n,\overline{z}} \to \underline{M}_{n,\overline{\eta}}$ (see (2.4.22)) that extends the specialization map $\sigma : \underline{M}_{n,z} \to \underline{M}_{n,\eta}$. Since the natural maps $\underline{M}^\sharp_{n,z} \to \underline{M}^\sharp_{n,\overline{z}}$ and $\underline{M}^\sharp_{n,\eta} \to \underline{M}^\sharp_{n,\overline{\eta}}$ are isomorphisms (see (6.1.8)), and $\sigma^{\sharp\mathrm{gp}}$ is an isomorphism (since z and η lie in the same stratum), the same holds for $\overline{\sigma}^{\sharp\mathrm{gp}}$, and therefore the induced map $\underline{M}^{\mathrm{gp}\vee}_{n,\overline{\eta}} \to \underline{M}^{\mathrm{gp}\vee}_{n,\overline{z}}$ is bijective. Pick any geometric point ξ of $X_n(\overline{z})$ localized at the maximal point, and lift ξ to a geometric point ξ_η of $X_n(\overline{\eta})$; in light of (7.3.52), we conclude that s induces an isomorphism

$$\pi_1((X_n(\overline{\eta}), \underline{M}_n(\overline{\eta}))_{\text{ét}}, \xi_\eta) \xrightarrow{\sim} \pi_1((X_n(\overline{z}), \underline{M}_n(\overline{z}))_{\text{ét}}, \xi).$$

Therefore, $\varphi_n \times_{X_n} X_n(\overline{\eta})$ is étale if and only if it is a trival covering, if and only if the same holds for $\varphi_n \times_{X_n} X_n(\overline{z})$, if and only if $z \in W_n$, whence the assertion.

(ii): For every $n \in \mathbb{N}$, set

$$\mathfrak{Z}_n := \{\mathfrak{p} \in \mathrm{Spec}\, P^{(n)} \mid V_{n,\mathfrak{p}} \neq \varnothing \quad \text{and} \quad V_{n,\mathfrak{p}} \cap W_n = \varnothing\}. \tag{9.6.16}$$

The continuous map $\omega_n : \mathrm{Spec}\, P^{(n+1)} \to \mathrm{Spec}\, P^{(n)}$ induced by the inclusion $P^{(n)} \to P^{(n+1)}$ sends $\mathfrak{Z}_{n+1}$ into $\mathfrak{Z}_n$, for every $n \in \mathbb{N}$. On the other hand, ω_n is also a bijection of finite sets (lemmata 3.1.20(iii) and 3.4.41(i)); we conclude that ω_n restricts to a bijection $\mathfrak{Z}_{n+1} \xrightarrow{\sim} \mathfrak{Z}_n$ for every sufficiently large integer $n \in \mathbb{N}$. For every $n \in \mathbb{N}$, let $g_n : X_{n+1} \to X_n$ be the transition morphism in the maximal tower (9.6.4), in light of (i), it follows that

$$g_n^{-1}(V_n \cap W_n) = V_{n+1} \cap W_{n+1} \qquad \text{for every sufficiently large } n \in \mathbb{N}.$$

On the other hand, the restriction $(X_{n+1})_{/p} \to (X_n)_{/p}$ of g_n is radicial and surjective for every $n \in \mathbb{N}$, so the underlying continuous map is a homeomorphism; since the underlying topological spaces are noetherian, we see as well that

$$g_n^{-1}((X_n)_{/p} \cap W_n) = (X_{n+1})_{/p} \cap W_{n+1} \qquad \text{for every sufficiently large } n \in \mathbb{N}.$$

Summing up, the assertion follows. □

9.6.17. Let $\beta : P^{(\infty)} \to B$ be the chart of the log structure $\underline{M}$ on X from (9.6.3), and notice that the inclusion map $P^{(n)} \to P^{(\infty)}$ induces bijections $\mathrm{Spec}\, P^{(\infty)} \xrightarrow{\sim} \mathrm{Spec}\, P^{(n)}$ for every $n \in \mathbb{N}$ (lemma 3.4.41(i)); especially, $\mathrm{Spec}\, P^{(\infty)}$ is a finite set. Let $I \subset B$ be any ideal; we say that I is a *branch ideal*, if there exist radical ideals $J \subset B$ and $\mathfrak{r} \subset P^{(\infty)}$ such that

$$p \in J \qquad \text{and} \qquad I = J \cap \mathfrak{r}B. \tag{9.6.18}$$

Remark 9.6.19. (i) Let $I \subset B$ be a branch ideal, and pick radical ideals $J \subset B$ and $\mathfrak{r} \subset P^{(\infty)}$ such that (9.6.18) holds. Set $\mathfrak{r}^{(n)} := \mathfrak{r} \cap P^{(n)}$ for every $n \in \mathbb{N}$; from corollary 6.5.36(ii) we know that $\mathfrak{r}^{(n)}B_n$ is a radical ideal of B_n, hence $\mathfrak{r}B$ is a radical ideal of B, and then the same holds for I.

(ii) In the situation of (i), let $\mathfrak{Z} \subset \mathrm{Spec}\, P^{(\infty)}$ be a subset such that $\mathfrak{r} = \bigcap_{\mathfrak{p}\in\mathfrak{Z}} \mathfrak{p}$ (lemma 3.1.15), and set $\mathfrak{p}^{(n)} := \mathfrak{p} \cap B_n$ for every $\mathfrak{p} \in \mathfrak{Z}$ and every $n \in \mathbb{N}$, so $\mathfrak{r}^{(n)} = \bigcap_{\mathfrak{p}\in\mathfrak{Z}} \mathfrak{p}^{(n)}$ for every $n \in \mathbb{N}$. In view of proposition 6.5.32 and lemmata 6.5.17 and 3.1.37, we see that $\mathfrak{r}^{(n)}B = \bigcap_{\mathfrak{p}\in\mathfrak{Z}} \mathfrak{p}^{(n)}B_n$, and therefore

$$\mathfrak{r}B = \bigcap_{\mathfrak{p}\in\mathfrak{Z}} \mathfrak{p}B. \tag{9.6.20}$$

Suppose moreover, that $p \in \mathfrak{p}B$ for some $\mathfrak{p} \in \mathfrak{Z}$; then we may replace J by $J \cap \mathfrak{p}B$ and $\mathfrak{Z}$ by $\mathfrak{Z} \setminus \mathfrak{p}$, after which J is still a radical ideal, and (9.6.18) still holds. Since $\mathfrak{Z}$ is a finite set, we may – after repeating this procedure finitely many times – assume that

$$p \notin \mathfrak{p}B \qquad \text{for every } \mathfrak{p} \in \mathfrak{Z}. \tag{9.6.21}$$

(iii) Furthermore, any branch ideal $I \subset B$ is the radical of I_0B, for some ideal $I_0 \subset B_0$. Indeed, pick J and $\mathfrak{r}$ such that (9.6.18) holds; since the projection $\mathrm{Spec}\, B/pB \to (X_0)_{/p}$ is radicial and surjective, it is clear that J is the radical of J_0B for some ideal $J_0 \subset B_0$, and on the other hand, $\mathfrak{r}B$ is the radical of $\mathfrak{r}^{(0)}B$.

(iv) In the situation of (9.6.14), let $W \subset X$ be the limit of the system $(W_n \mid n \in \mathbb{N})$; then W is an open subset of X, and we may consider the largest ideal $I \subset B$ such that $\mathrm{Spec}\, B/I = X \setminus W$. From lemma 9.6.15 and (9.6.20), it is easily seen that I is a branch ideal of B.

Proposition 9.6.22. *Let $I \subset B$ be any branch ideal, and $J, \mathfrak{r}$ radical ideals such that* (9.6.18) *holds. Then $I^2 = I = \mathfrak{r} \cdot J$, and I fulfills condition* $(\mathbf{B})$ *of* [36, §2.1.6].

Proof. Pick $\mathfrak{Z}$ such that (9.6.21) holds and $\mathfrak{r} = \bigcap_{\mathfrak{p}\in\mathfrak{Z}} \mathfrak{p}$, and for every $n \in \mathbb{N}$ let $\mathfrak{Z}_n \subset \mathrm{Spec}\, P^{(n)}$ be the image of Z; set

$$J_n := J \cap B_n \qquad I_n := I \cap B_n \qquad Q_n := \bigcap_{\mathfrak{p}\in\mathfrak{Z}_n} \mathfrak{p}B_n.$$

Claim 9.6.23. $(\mathfrak{r}B)^2 = \mathfrak{r}B$ and $J^2 = J$, and the ideals $\mathfrak{r}B$ and J fulfill condition $(\mathbf{B})$.

Proof of the claim. Since the p-Frobenius endomorphism of $P^{(\infty)}$ is an automorphism, the first stated identity is clear. Next, set $\overline{J} := J/pB$. By lemma 9.6.8, the Frobenius endomorphism of B/pB is surjective; since $\overline{J}$ is a radical ideal, we deduce that $\overline{J}^2 = \overline{J}$. On the other hand, from remark 9.6.13 we see that $p \in J^2$, from which also the second identity follows easily.

Next, it is clear that $\mathfrak{r}B$ fulfills condition $(\mathbf{B})$, and for J we apply [36, Claim 2.1.9] which reduces checking that J/pJ is generated by the p-th powers of its elements. However, the foregoing already shows that J/pB is generated by the p-th powers of its elements; combining with theorem 9.6.11(ii), the contention follows easily. $\Diamond$

In light of claim 9.6.23, it suffices to show that $J \cap \mathfrak{r}B = \mathfrak{r} \cdot J$, or equivalently, that :

$$\operatorname*{colim}_{n\in\mathbb{N}} (J_n \cap Q_n)/(J_n \cdot Q_n) = 0.$$

Choose a sequence $(\pi_n \mid n \in \mathbb{N})$ as in remark 9.6.13, set $\overline{B}_n := B_n/\pi_n B_n$, let $\overline{J}_n \subset \overline{B}_n$ denote the image of J_n, and likewise define $\overline{Q}_n$ for every $n \in \mathbb{N}$; a standard computation yields a natural isomorphism

$$\mathrm{Tor}_1^{B_n}(B_n/Q_n, B_n/J_n) \xrightarrow{\sim} (J_n \cap Q_n)/(J_n \cdot Q_n)$$

and we remark :

Claim 9.6.24. The natural map $\mathrm{Tor}_1^{B_n}(B_n/Q_n, B_n/J_n) \to \mathrm{Tor}_1^{\overline{B}_n}(\overline{B}_n/\overline{Q}_n, \overline{B}_n/\overline{J}_n)$ is an isomorphism.

Proof of the claim. Taking into account the spectral sequence

$$E^2_{ij} := \mathrm{Tor}_i^{\overline{B}_n}(\mathrm{Tor}_j^{B_n}(B_n/Q_n, \overline{B}_n), B_n/J_n) \Rightarrow \mathrm{Tor}_{i+j}^{B_n}(B_n/Q_n, B_n/J_n)$$

we reduce to checking that $\mathrm{Tor}_1^{B_n}(B_n/Q_n, \overline{B}_n) = 0$. To this aim, consider the short exact sequence

$$\Sigma \quad : \quad 0 \to \pi_n B_n \to B_n \to \overline{B}_n \to 0.$$

From (9.6.21) we see that $\pi_n \notin \mathfrak{p}B_n$, for every $\mathfrak{p} \in \mathfrak{Z}_n$, and Q_n is a radical ideal (corollary 6.5.36(i)) so π_n is a regular element of B_n/Q_n, and then the sought vanishing follows, by inspecting the distinguished triangle $\Sigma \overset{\mathbf{L}}{\otimes}_{B_n} B_n/Q_n$. $\Diamond$

According to remark 9.6.13 and lemma 9.6.8, the Frobenius map $\Phi_{n+1} : \overline{B}_{n+1} \to \overline{B}_{n+1}$ factors as a composition

$$\overline{B}_{n+1} \xrightarrow{\varphi_{n+1}} \overline{B}_n \xrightarrow{j_n} \overline{B}_{n+1}$$

where φ_{n+1} is an isomorphism, and j_n is induced by the inclusion map $B_n \to B_{n+1}$, for every $n \in \mathbb{N}$. We deduce, for every $n, k \in \mathbb{N}$, a commutative diagram

$$\begin{tikzcd}
\overline{B}_{n+k} \arrow[r, "\Phi^k_{n+k}"] \arrow[d, "\varphi_{n,k}"'] & \overline{B}_{n+k} \arrow[d, "\varphi_{n,k}"'] \arrow[dr, equal] & \\
\overline{B}_n \arrow[r, "\Phi^k_n"] & \overline{B}_n \arrow[r, "i_{n,k}"] & \overline{B}_{n+k}
\end{tikzcd}$$

where $i_{n,k}$ is an isomorphism and

$$i_{n,k} \circ \Phi^k_n = j_{n,k} := j_{n+k-1} \circ \cdots \circ j_n \qquad \varphi_{n,k} := \varphi_{n+1} \circ \cdots \circ \varphi_{n+k} \qquad \text{for every } n, k \in \mathbb{N}.$$

Claim 9.6.25. $i_{n,k}\overline{J}_n = \overline{J}_{n+k}$ and $i_{n,k}\overline{Q}_n = \overline{Q}_{n+k}$ for every $n, k \in \mathbb{N}$.

Proof of the claim. The first stated identity holds if and only if $\overline{J}_n = \varphi_{n,k}\overline{J}_{n+k}$, or equivalently $\varphi^{-1}_{n,k}\overline{J}_n = \overline{J}_{n+k}$, for every $n, k \in \mathbb{N}$. However, we have

$$\overline{J}_{n+1} = \overline{\Phi}^{-1}_{n+1}\overline{J}_{n+1} = \varphi^{-1}_{n+1}(j_n^{-1}\overline{J}_{n+1}) = \varphi^{-1}_{n+1}\overline{J}_n$$

whence the contention. For the second identity, it suffices to remark that, by construction, we have $\varphi_{n,k}\overline{Q}_{n+k} = \overline{Q}_n$, for every $n, k \in \mathbb{N}$. $\Diamond$

Claim 9.6.25 implies that $i_{n,k}$ induces isomorphisms

$$(9.6.26) \qquad \overline{B}_n/\overline{J}_n \xrightarrow{\sim} \overline{B}_{n+k}/\overline{J}_{n+k} \qquad \overline{B}_n/\overline{Q}_n \xrightarrow{\sim} \overline{B}_{n+k}/\overline{Q}_{n+k} \qquad \text{for every } n, k \in \mathbb{N}.$$

There follows a commutative diagram

$$\begin{tikzcd}
\mathrm{Tor}_1^{\overline{B}_n}(\overline{B}_n/\overline{J}_n, \overline{B}_n/\overline{Q}_n) \arrow[r, "\alpha_{n,k}"] \arrow[dr, "\beta_{n,k}"'] & \mathrm{Tor}_1^{\overline{B}_n}(\overline{B}_n/\overline{J}_n, \overline{B}_n/\overline{Q}_n) \arrow[d, "\gamma_{n,k}"] \\
& \mathrm{Tor}_1^{\overline{B}_{n+k}}(\overline{B}_{n+k}/\overline{J}_{n+k}, \overline{B}_{n+k}/\overline{Q}_{n+k})
\end{tikzcd}$$

where

- $\beta_{n,k}$ is induced by $j_{n,k}$ and the maps $\overline{B}_n/\overline{J}_n \to \overline{B}_{n+k}/\overline{J}_{n+k}$, $\overline{B}_n/\overline{Q}_n \to \overline{B}_{n+k}/\overline{Q}_{n+k}$ deduced from $j_{n,k}$
- $\gamma_{n,k}$ is the isomorphism induced by $i_{n,k}$ and (9.6.26)
- $\alpha_{n,k}$ is induced by Φ_n^k and the Frobenius maps of $\overline{B}_n/\overline{J}_n$ and $B_n/\overline{Q}_n$.

Now, in view of claim 9.6.24, it suffices to show that, for every $n \in \mathbb{N}$ there exists $k \in \mathbb{N}$ such that $\beta_{n,k} = 0$, or equivalently, such that $\alpha_{n,k} = 0$. However, notice that the Frobenius map of $\overline{B}_n/\overline{J}_n$ factors uniquely through a map $\overline{B}_n/\overline{J}_n \to \overline{B}_n/\overline{J}_n^p$ and the natural projection $\overline{B}_n/\overline{J}_n^p \to \overline{B}_n/\overline{J}_n$; it follows that $\alpha_{n,k}$ factors through the map

$$\mathrm{Tor}_1^{\overline{B}_n}(\overline{B}_n/\overline{J}_n^q, \overline{B}_n/\overline{Q}_n) \to \mathrm{Tor}_1^{\overline{B}_n}(\overline{B}_n/\overline{J}_n, \overline{B}_n/\overline{Q}_n)$$

which is induced by the identity maps of $\overline{B}_n$ and $\overline{B}_n/\overline{Q}_n$, and the natural projection $\overline{B}_n/\overline{J}_n^q \to \overline{B}_n/\overline{J}_n$, with $q := p^k$. Then the sought vanishing follows from corollary 4.4.39(i). $\square$

9.6.27. Let $I \subset B$ be a fixed branch ideal, and consider a pair (X, Z) such that $Z \subset \mathrm{Spec}\, B/I$ and Z is constructible in X. Clearly such a pair is normal (see definition 8.2.25(ii)), and we aim to show that (X, Z) is almost pure for the almost structure given by the basic setup (B, I) supplied by proposition 9.6.22. This will be achieved in several steps.

• To begin with, set $W := X \setminus Z$, and let $\mathscr{A}$ be any étale almost finitely presented $\mathscr{O}_W^a$-algebra. Set also $W_I := X \setminus \mathrm{Spec}\, B/I$; then $\mathscr{A}_{|W_I}$ is a finite étale $\mathscr{O}_{W_I}$-algebra, $W_I = X \times_{X_0} W_{I,0}$ for some open subset $W_{I,0} \subset X_0$ (remark 9.6.19(iii)), and a simple inspection shows that

$$(X, \underline{M})_{\mathrm{tr}} \setminus (X_0)_{/p} \subset W_{I,0}.$$

Let $\psi : W_I \to W_{I,0}$ be the natural projection; by [33, Ch.IV, Prop.17.7.8(ii)] and [32, Ch.IV, Th.8.8.2(ii)], and by virtue of remark 9.6.5, we may assume – after replacing $(X_0, \underline{M})$ by $(X_n, \underline{M}_n)$ for some sufficiently large $n \in \mathbb{N}$ – that there exists a coherent étale $\mathscr{O}_{W_{I,0}}$-algebra $\mathscr{A}_0$ and an isomorphism $\psi^*\mathscr{A}_0 \xrightarrow{\sim} \mathscr{A}_{|W_I}$ of $\mathscr{O}_{W_I}$-algebras. Denote by Y_0 the normalization of X_0 in $\mathrm{Spec}\, \mathscr{A}_0(W_{I,0})$; the resulting morphism $\varphi_0 : Y_0 \to X_0$ is finite (lemma 4.8.4(i)) and induces an isomorphism $(\varphi_{0*}\mathscr{O}_{Y_0})_{|W_{I,0}} \xrightarrow{\sim} \mathscr{A}_0$. Especially, φ_0 is a morphism of the type contemplated in (9.6.14), and the resulting morphism $\varphi : Y \to X$ induces an isomorphism $(\varphi_*\mathscr{O}_Y)_{|W_I} \xrightarrow{\sim} \mathscr{A}_{|W_I}$. However, we have as well $\mathscr{A} = (\mathscr{A}_{|W_I})^\nu$, and since Y is normal, there follows an isomorphism of $\mathscr{O}_W^a$-algebras $(\varphi_*\mathscr{O}_Y^a)_{|W} \xrightarrow{\sim} \mathscr{A}$. Then proposition 8.2.30 and [36, Lemma 8.2.28] reduce to checking that the resulting $\mathscr{O}_X^a$-algebra $\varphi_*\mathscr{O}_Y^a$ is weakly unramified, for every such φ_0.

• Next, in the situation of (9.6.14), suppose additionally that B_0 is strictly henselian; this assumption serves only to ensure that the N-torsion subgroup μ_N of $B_0^\times$ has cardinality equal to N, for every $N > 0$ such that $(N, p) = 1$. Let $k > 0$ be any integer and write $k = p^s \cdot q$, with $s, q \in \mathbb{N}$ and $(q, p) = 1$; set $Q := P$, let $\nu : P \to Q$ be the k-Frobenius map, define

$$C_0' := Q \otimes_P B_0 \qquad C_0 := C_0' \otimes_{B_0} B_s''$$

and endow $X_0' := \mathrm{Spec}\, C_0$ with the log structure $\underline{M}_0'$ deduced from the natural map $Q \to C_0$. Notice that the Frobenius endomorphism of C_0/pC_0 is still a finite map (claim 4.8.37(i)); also, since the induced map $A \to C_0'/\mathfrak{m}_Q C_0'$ is an isomorphism, the image of the sequence $(f_1, \dots, f_r)$ in C_0 is still maximal, in the sense of (9.6.1), and therefore $C_0/\mathfrak{m}_Q C_0$ is still a regular local ring, by corollary 4.7.27. Arguing as in (9.6.3), we deduce that $(X'_0, \underline{M}'_0)$ is a regular log scheme, and we may consider the maximal tower $((X'_n, \underline{M}'_n) \mid n \in \mathbb{N})$ associated to the chart $Q \to C_0$ and the maximal sequence $(f_1^{1/p^s}, \dots, f_r^{1/p^s})$ (see remark 4.7.28(iii)). So, $X'_n = \operatorname{Spec} C_n$ for a finite C_0-algebra C_n, and $\underline{M}'_n$ is given by a chart $Q^{(n)} \to C_n$, where $Q^{(n)}$ is a submonoid of $Q_{\mathbb{Q}}$ containing Q, for every $n \in \mathbb{N}$. For every $n \in \mathbb{N}$, we set $U'_n := U_0 \times_{X_0} X'_n$, we let Y'_n be the normalization of X'_n in $Y \times_{X_0} U'_n$, we denote by

$$\varphi'_n : Y'_n \to X'_n \qquad \text{and} \qquad h_n : (X'_n, \underline{M}'_n) \to (X_n, \underline{M}_n)$$

the resulting finite morphisms, and by $\varphi' : Y' \to X'$ (resp. $h : X' \to X$) the limit of the system of morphisms $(\varphi'_n \mid n \in \mathbb{N})$ (resp. the limit of the system $(h_n \mid n \in \mathbb{N})$). Recall also that the induced morphism $U'_0 \to U_0$ is a torsor for the finite abelian group

$$G := \operatorname{Hom}_{\mathbb{Z}}(Q^{\mathrm{gp}}, \mu_k)$$

(see (7.3.31)). Set

$$C := \bigcup_{n\in\mathbb{N}} C_n \qquad Q^{(\infty)} := \bigcup_{n\in\mathbb{N}} Q^{(n)}.$$

With this notation, we have

$$Q^{(\infty)} = Q \otimes_P P^{(\infty)} \qquad \text{and} \qquad C = Q^{(\infty)} \otimes_P B''.$$

Let also write $Y = \operatorname{Spec} D$ (resp. $Y' = \operatorname{Spec} D'$) for a B-algebra D (resp. for a C-algebra D').

Lemma 9.6.28. *With the notation of* (9.6.27)*, set $W' := X' \setminus h^{-1}Z$. We have :*

(i) *The essentially commutative diagram*

$$\begin{array}{ccc} \mathscr{O}^a_X\text{-}\mathbf{\acute{E}t}_{\mathrm{fr}} & \xrightarrow{\rho} & \mathscr{O}^a_W\text{-}\mathbf{\acute{E}t}_{\mathrm{fr}} \\ \downarrow & & \downarrow \\ \mathscr{O}^a_{X'}\text{-}\mathbf{\acute{E}t}_{\mathrm{fr}} & \xrightarrow{\rho'} & \mathscr{O}^a_{W'}\text{-}\mathbf{\acute{E}t}_{\mathrm{fr}} \end{array}$$

is 2*-cartesian (notation of definition* 8.2.26*).*

(ii) *Especially, if the pair $(X', h^{-1}Z)$ is almost pure, the same holds holds for the pair (X, Z).*

Proof. Obviously, (ii) is an immediate consequence of (i).

(i): A simple inspection shows that h_0 factors through a morphism $(X'_0, \underline{M}'_0) \to (X_s, \underline{M}_s)$, and indeed we have natural isomorphisms of $(X', \underline{M}')$-schemes

$$(X'_n, \underline{M}'_n) \xrightarrow{\sim} (X'_0, \underline{M}'_0) \times_{(X_s, \underline{M}_s)} (X_{n+s}, \underline{M}_{n+s}) \qquad \text{for every } n \in \mathbb{N}$$

that identify the transition morphisms $(X'_{n+1}, \underline{M}'_{n+1}) \to (X'_n, \underline{M}'_n)$ with the base change of the corresponding morphisms for the tower $((X_{n+s}, \underline{M}_{n+s}) \mid n \in \mathbb{N})$. In view of remark 9.6.5, we may then replace $(X_0, \underline{M}_0)$ by $(X_s, \underline{M}_s)$, and P by $P^{(s)}$, and assume that $(k,p) = 1$, so also $(o(G), p) = 1$.

Suppose now that $(\varphi'_* \mathscr{O}^a_{Y'})_{|W'}$ is in the essential image of the restriction functor ρ'; by proposition 8.2.30, this means that D'^a is an étale almost finitely presented $(C, IC)^a$-algebra. Then, taking into account the discussion of (9.6.27), and recalling that ρ and ρ' are fully faithful (lemma 8.2.29(ii)), we are reduced to checking that D^a is an étale almost finitely presented $(B, I)^a$-algebra.

However, the action of G on U'_0 is inherited by C and D', and clearly $C^G = B$ and $D'^G = D$; by corollary 8.6.28(ii), it suffices therefore to show that the action of G on D'^a is horizontal. Denote

$$G \to \operatorname{Aut}(C) \quad : \quad \chi \mapsto \rho_\chi$$

the action of G, let $\chi \in G$ be any element, and $J_\chi \subset C$ the ideal generated by all elements of the form $\rho_\chi(c) - c$, for c ranging over all elements of C. By the discussion in (7.3.31) we get

$$\rho_\chi((q \otimes y) \otimes b) = \chi(q) \cdot (q \otimes y) \otimes b \qquad \text{for every } q \in Q, \, y \in P^{(\infty)} \text{ and } b \in B''.$$

Hence, J_χ is the ideal generated by all elements of the form $(1 - \chi(q)) \cdot q \otimes y \otimes 1$, for all $q \otimes y \in Q^{(\infty)}$. However, since $\chi(q) \in \mu_k$ and $(k, p) = 1$, it is easily seen that $1 - \chi(q)$ either vanishes, or else it is invertible in B_0. Thus, denote by $\mathfrak{q}_\chi \subset Q^{(\infty)}$ the ideal generated by all elements of the form $q \otimes y$, with $\chi(q) \neq 1$; it follows that

$$J_\chi = \mathfrak{q}_\chi \cdot C.$$

Claim 9.6.29. $\mathfrak{q}_\chi$ is a radical ideal.

Proof of the claim. Indeed, say that $(q \otimes y)^n \in \mathfrak{q}_\chi$, so there exist elements $q_1 \otimes y_1$ and x_1 of $Q^{(\infty)}$ such that $(q \otimes y)^n = (q_1 \otimes y_1) \cdot x_1$ and $\chi(q_1) \neq 1$. We may assume that $n = p^t$ for some integer $t \in \mathbb{N}$, and since $Q^{(\infty)}$ is uniquely p-divisible, we may write $q_1 \otimes y_1 = (q_2 \otimes y_2)^n$ and $x_1 = x_2^n$ for some elements $q_2 \otimes y_2$ and x_2 of $Q^{(\infty)}$, and then $q \otimes y = (q_2 \otimes y_2) \cdot x_2$; however, clearly $\chi(q_2) \neq 1$, whence the claim. $\Diamond$

In view of claim 9.6.29, we may write $\mathfrak{q}_\chi = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_s$, for certain $\mathfrak{q}_1, \dots, \mathfrak{q}_s \in \operatorname{Spec} Q^{(\infty)}$ (lemma 3.1.15). Set $\mathfrak{Q} := \{\mathfrak{q}_1, \dots, \mathfrak{q}_s\}$; we also know that $(\mathfrak{q} \cap Q^{(n)}) \cdot C_n$ is a prime ideal of C_n, for every $\mathfrak{q} \in \mathfrak{Q}$ and every $n \in \mathbb{N}$ (corollary 6.5.36(i)), therefore $\mathfrak{q}C$ is a prime ideal of C, for every $\mathfrak{q} \in \mathfrak{Q}$. Especially, the induced map

$$C/J_\chi \to \prod_{\mathfrak{q} \in \mathfrak{Q}} C/\mathfrak{q}C \tag{9.6.30}$$

is injective. Since, by assumption, D'^a is a flat C^a-algebra, it follows that the map of almost modules $D'^a \otimes_C$ (9.6.30) is a monomorphism. Summing up, we are reduced to checking that χ acts trivially on $(D'/\mathfrak{q}D')^a$, for every $\mathfrak{q} \in \mathfrak{Q}$. However, set

$$X'_\mathfrak{q} := \operatorname{Spec} C/\mathfrak{q}C \quad \text{and} \quad W'_\mathfrak{q} := X'_\mathfrak{q} \times_X W_I \qquad \text{for every } \mathfrak{q} \in \mathfrak{Q}.$$

Suppose first that $W'_\mathfrak{q} = \varnothing$; in that case, set $\mathfrak{p} := \mathfrak{q} \cap P^{(\infty)}$, and notice that $\mathfrak{q}$ is the radical of the ideal $\mathfrak{p} \cdot Q^{(\infty)}$, so $\mathfrak{q}C$ is the radical of $\mathfrak{p}C$, and therefore $\operatorname{Spec}(C/\mathfrak{p}C) \times_X W_I = \varnothing$. However, the induced morphism $\operatorname{Spec} C/\mathfrak{p}C \to \operatorname{Spec} B/\mathfrak{p}B$ is surjective, so $\operatorname{Spec}(B/\mathfrak{p}B) \cap W_I = \varnothing$; since $\mathfrak{p}B$ is a prime ideal of B, the latter means that $I \subset \mathfrak{p}B$, whence $(B/\mathfrak{p}B)^a = 0$, so $(D'/\mathfrak{q}D')^a$ vanishes as well, and the assertion is trivial. If $W'_\mathfrak{q} \neq \varnothing$, set $Y'_\mathfrak{q} := Y' \times_{X'} X'_\mathfrak{q}$, let $\varphi'_\mathfrak{q} : Y'_\mathfrak{q} \to X'_\mathfrak{q}$ be the induced morphism, and define

$$\mathscr{D} := \varphi_* \mathscr{O}_Y \qquad \mathscr{D}' := \varphi'_* \mathscr{O}_{Y'} \qquad \mathscr{D}'_\mathfrak{q} := \varphi'_{\mathfrak{q}*} \mathscr{O}_{Y'_\mathfrak{q}}.$$

On the one hand, by assumption $\mathscr{D}'^a_\mathfrak{q}$ is a flat $\mathscr{O}^a_{X'_\mathfrak{q}}$-algebra; on the other hand, $X'_\mathfrak{q}$ is reduced and irreducible, hence the restriction map $\Gamma(X'_\mathfrak{q}, \mathscr{O}_{X'_\mathfrak{q}}) \to \Gamma(W'_\mathfrak{q}, \mathscr{O}_{X'_\mathfrak{q}})$ is injective. Consequently, the restriction map

$$(D'/\mathfrak{q}D')^a = \Gamma(X'_\mathfrak{q}, \mathscr{D}'^a_\mathfrak{q}) \to \Gamma(W'_\mathfrak{q}, \mathscr{D}'^a_\mathfrak{q})$$

is a monomorphism; so, we are reduced to checking that χ acts trivially on $\Gamma(W'_\mathfrak{q}, \mathscr{D}'^a_\mathfrak{q})$. To this aim, we remark that the open subset $W'_I := X' \times_X W_I$ is stable under the action of G, and the restriction $\mathscr{D}'_{|W'_I}$ of $\mathscr{D}'$ to the open subset W'_I is isomorphic to $(h^*\mathscr{D})_{|W'_I}$ ([33, Ch.IV, Prop.17.5.8(iii)]), so the action of G on $\mathscr{D}'_{|W'_I}$ is horizontal, and the assertion follows easily. $\square$

9.6.31. Keep the notation of (9.6.27), and define $\mathfrak{Z}_n$ as in (9.6.16), for every $n \in \mathbb{N}$. As already remarked, in view of lemma 9.6.15(i,ii) we may assume that the map $\operatorname{Spec} P^{(n+1)} \to \operatorname{Spec} P^{(n)}$ sends $\mathfrak{Z}_{n+1}$ bijectively onto $\mathfrak{Z}_n$, for every $n \in \mathbb{N}$. Set $V'_0 := V_0 \times_{X_0} X'_0$, and let $(V'_{0,\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec} Q)$ and $(V_{0,\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec} P)$ be the logarithmic stratifications of V'_0 and respectively V_0, defined as in (9.6.14); notice that the map $\nu^* : \operatorname{Spec} Q \to \operatorname{Spec} P$ induced by ν is bijective (lemma 3.4.41(i)), and clearly

$$V'_{0,\mathfrak{p}} = h_0^{-1} V_{0,\nu^*(\mathfrak{p})} \qquad \text{for every } \mathfrak{p} \in \operatorname{Spec} Q.$$

Let $\mathfrak{Z}'_0 := \nu^{*-1}\mathfrak{Z}_0$, and for every $\mathfrak{p} \in \mathfrak{Z}'_0$, let $\eta'_\mathfrak{p}$ (resp. $\eta_\mathfrak{p}$) denote the generic point of $V'_{0,\mathfrak{p}}$ (resp. of $V_{0,\nu^*(\mathfrak{p})}$), pick geometric points $\overline{\eta}'_\mathfrak{p}$ and $\xi'_\mathfrak{p}$ localized respectively at $\eta'_\mathfrak{p}$ and at a point of $U'_0(\overline{\eta}'_\mathfrak{p}) := U'_0 \times_{X'_0} X'_0(\overline{\eta}'_\mathfrak{p})$. Denote by $\overline{\eta}_\mathfrak{p}$ the image of $\overline{\eta}'_\mathfrak{p}$ in $V_{0,\nu^*(\mathfrak{p})}$, and by $\xi_\mathfrak{p}$ the image of $\xi'_\mathfrak{p}$ in $U_0(\overline{\eta}_\mathfrak{p}) := U_0 \times_{X_0} X_0(\overline{\eta}_\mathfrak{p})$. According to (7.3.30) there follows, for every integer $N > 0$, a commutative diagram of groups

$$\begin{tikzcd}
\pi_1(U'_0(\overline{\eta}'_\mathfrak{p})_{\text{ét}}, \xi'_\mathfrak{p}) \arrow[r] \arrow[d] & \pi_1(U_0(\overline{\eta}_\mathfrak{p})_{\text{ét}}, \xi_\mathfrak{p}) \arrow[d] \\
\underline{M}'^{\text{gp}\vee}_{0,\eta'_\mathfrak{p}} \otimes_\mathbb{Z} \mu_N \arrow[r] & \underline{M}^{\text{gp}\vee}_{0,\eta_\mathfrak{p}} \otimes_\mathbb{Z} \mu_N
\end{tikzcd}$$

(where μ_N is the N-torsion subgroup of $\kappa(\xi_\mathfrak{p})^\times$) whose top arrow is induced by the natural morphism $U'_0(\overline{\eta}'_\mathfrak{p}) \to U_0(\overline{\eta}_\mathfrak{p})$, and whose bottom arrow is induced by $\nu^{\text{gp}\vee} : Q^{\text{gp}\vee} \to P^{\text{gp}\vee}$, *i.e.* by the k-Frobenius map of $P^{\text{gp}\vee}$, for every $\mathfrak{p} \in \mathfrak{Z}'_0$. Now, the restriction

$$\varphi_\mathfrak{p} : Y_0(\overline{\eta}_\mathfrak{p}) := Y_0 \times_{X_0} X_0(\overline{\eta}_\mathfrak{p}) \to X_0(\overline{\eta}_\mathfrak{p})$$

of φ is a tamely ramified covering, hence the action of $\pi_1(U_0(\overline{\eta}_\mathfrak{p})_{\text{ét}}, \xi_\mathfrak{p})$ on $F_\mathfrak{p} := \varphi_\mathfrak{p}^{-1}(\xi_\mathfrak{p})$ factors through a group homomorphism

$$\rho_\mathfrak{p} : \underline{M}^{\text{gp}\vee}_{0,\eta_\mathfrak{p}} \otimes_\mathbb{Z} \mu_N \to \operatorname{Aut}(F_\mathfrak{p})$$

for some sufficiently large $N \in \mathbb{N}$ (theorem 7.3.44). We may then find $k \in \mathbb{N}$ such that the image of $k \cdot P^{\text{gp}\vee}$ in $\underline{M}^{\text{gp}\vee}_{0,\eta_\mathfrak{p}}$ lies in the kernel of $\rho_\mathfrak{p}$, for every $\mathfrak{p} \in \mathfrak{Z}'_0$. Especially, for this choice of k, the image of $\underline{M}'^{\text{gp}\vee}_{0,\eta'_\mathfrak{p}} \otimes_\mathbb{Z} \mu_N$ acts trivially on $\operatorname{Aut}(F_\mathfrak{p})$ via $\rho_\mathfrak{p}$, for every such $\mathfrak{p}$. Consequently, $\pi_1(U'_0(\overline{\eta}'_\mathfrak{p})_{\text{ét}}, \xi'_\mathfrak{p})$ acts trivially on the fibres $\varphi_0'^{-1}(\xi'_\mathfrak{p})$ (by virtue of (1.6.22)); after applying lemma 9.6.15(i) to the morphism φ'_0, we conclude that the étale locus of φ'_0 contains the whole of V'_0.

Proposition 9.6.32. *In the situation of* (9.6.27), *suppose additionally that* $\dim B_0 \leq 2$. *Then the pair* (X, Z) *is almost pure.*

Proof. We consider a finite morphism $\varphi_0 : Y_0 \to X_0$ as in (9.6.14), and by the discussion of (9.6.27), it suffices to show that $\varphi_* \mathscr{O}^a_Y$ is an étale $\mathscr{O}^a_X$-algebra. Moreover, set $V := \operatorname{Spec} B[1/p]$; in view of (9.6.31) and lemma 9.6.28(ii), we may assume that $V_0 \subset W_0$, and therefore $(\varphi_*\mathscr{O}_Y)_{|V}$ is a finite étale $\mathscr{O}_X$-algebra. In this situation, let $I' \subset B$ be the radical of the ideal $I + pB$, and set $Z' := Z \setminus V$; clearly, I' is a branch ideal : indeed, if (9.6.18) and (9.6.21) hold for I, then $I' = J$. It then suffices to show that $\varphi_*\mathscr{O}^a_Y$ is an étale $\mathscr{O}^a_X$-algebra, for the almost structure given by the new setup (B, I'), so we may replace I by I' and Z by Z', and assume that $p \in I$.

The case $\dim B_0 = 0$ does not occur, and if $\dim B_0 = 1$, then B is a valuation ring of rank one, such that the Frobenius endomorphism of B/pB is surjective (lemma 9.6.8), so B is deeply ramified ([36, Prop.6.6.6]); also, clearly I contains the maximal ideal $\mathfrak{m}_B$ of B. If $I = B$, then $Z = \varnothing$, so the morphism φ is étale, and if $I = \mathfrak{m}_B$, the proposition follows from [36, Prop.6.6.2] and proposition 8.2.30.

Thus, we may assume that $\dim B_0 = 2$, and we remark :

Claim 9.6.33. We may assume that P is a free monoid.

Proof of the claim. Clearly $\dim P \leq 2$; if $\dim P = 0$, then $P = \{1\}$, and if $\dim P = 1$, then $P = \mathbb{N}$ (theorem 3.4.16(ii)). If $\dim P = 2$, we may find an integer $k > 0$ such that the k-Frobenius of P factors through injective maps $i : P \to \mathbb{N}^{\oplus 2}$ and $j : \mathbb{N}^{\oplus 2} \to P$ (example 3.4.17(i)). Write $k = p^s \cdot q$ with $s, t \in \mathbb{N}$ and $(q, p) = 1$, set $Q := P$ let $\nu : P \to Q$ be the k-Frobenius map, and define the maximal tower $((X'_n, \underline{M}'_n) \mid n \in \mathbb{N})$ and the morphisms h_n as in (9.6.27), for every $n \in \mathbb{N}$, as well as their limit $h : X' \to X$. Furthermore, set $R := \mathbb{N}^{\oplus 2}$; the map i induces a morphism $\mathrm{Spec}\,(\mathbb{Z}, R) \to \mathrm{Spec}\,(\mathbb{Z}, P)$ of log schemes, and the fibre product

$$(X''_0, \underline{M}''_0) := \mathrm{Spec}\,(\mathbb{Z}, R) \times_{\mathrm{Spec}\,(\mathbb{Z},P)} (X_0, \underline{M}_0)$$

is a regular log scheme (claim 7.3.36). Notice that $X''_0 = \mathrm{Spec}\, B_R$, with $B_R := R \otimes_P B_0$, and the induced map $A \to A_R := B_R/\mathfrak{m}_R B_R$ is an isomorphism, so the image of the sequence $(f_1, \dots, f_r)$ in A_R is again maximal. Therefore, we may form the maximal tower $((X''_n, \underline{M}''_n) \mid n \in \mathbb{N})$ associated to the chart $R \to B_R$ and the maximal sequence $(f_1, \dots, f_r)$, and a simple inspection shows that h_n factors as a composition

$$(X'_n, \underline{M}'_n) \xrightarrow{h'_n} (X''_n, \underline{M}''_n) \xrightarrow{h''_n} (X_n, \underline{M}_n) \qquad \text{for every } n \in \mathbb{N}$$

where h'_n is induced by the maps $j_n : R^{(n)} \to Q^{(n)}$ extending j, and h''_n is likewise induced by the corresponding maps $i_n : P^{(n)} \to R^{(n)}$. Let $h' : X' \to X''$ (resp. $h'' : X'' \to X$) be the limit of the system of morphisms $(h'_n \mid n \in \mathbb{N})$ (resp. $(h''_n \mid n \in \mathbb{N})$), set $W'' := h''^{-1}W$, $W' := h^{-1}W$, and denote by $g'' : W'' \to W$ (resp. $g : W' \to W$) the restriction of h'' (resp. of h). Suppose that the pair $(X'', h''^{-1}Z)$ is almost pure (for the almost structure of the basic setup (B, I)). Let $\mathscr{A}$ be a given étale $\mathscr{O}^a_W$-algebra of finite rank; by assumption, $g''^*\mathscr{A}$ extends to an étale $\mathscr{O}^a_{X''}$-algebra of finite rank $\mathscr{B}$, and then $h'^*\mathscr{B}$ is an étale $\mathscr{O}^a_{X'}$-algebra of finite rank whose restriction to W' is isomorphic to $g^*\mathscr{A}$. By lemma 9.6.28(i), it follows that $\mathscr{A}$ extends to an étale $\mathscr{O}^a_X$-algebra of finite rank, and by virtue of proposition 8.2.30, this shows that (X, Z) is almost pure. Lastly, say that $X'' = \mathrm{Spec}\, B''$, denote by $I'' \subset B''$ the radical of IB'', and notice that I'' is a branch ideal of B''; clearly if the pair $(X'', h''^{-1}Z)$ is almost pure for the almost structure given by the setup (B'', I''), then the same pair shall be almost pure also relative to the basic setup (B, I). The claim follows. $\Diamond$

In view of claim 9.6.33 and corollary 6.5.35, we may assume that B_n is a regular local ring of dimension 2, for every $n \in \mathbb{N}$. On the other hand, since Y_n is normal, we have

$$\mathrm{depth}\, \mathscr{O}_{Y_n,y} \geq \min(2, \dim \mathscr{O}_{Y_n,y}) \qquad \text{for every } y \in Y_n$$

from which it follows that $\mathscr{O}_{Y_n,y}$ is a flat $\mathscr{O}_{X_n,\varphi_n(y)}$-module, for every $y \in \mathbb{N}$ ([75, Th.4.4.15]), and consequently $\mathscr{D} := \varphi_*\mathscr{O}_Y$ is a flat $\mathscr{O}_X$-algebra. Let $x \in X$ be the closed point, and denote

$$j_n : X_n \backslash \{x_n\} \to X_n \quad \text{for every } n \in \mathbb{N}, \text{ and } \quad j : X \backslash \{x\} \to X$$

the open immersions. Notice that $\mathrm{depth}\, B_n = 2$, hence $j_{n*}j_n^*\mathscr{O}_{X_n} = \mathscr{O}_{X_n}$ for every $n \in \mathbb{N}$, so that $j_*j^*\mathscr{O}_X = \mathscr{O}_X$ (proposition 5.1.15(ii)). Therefore $j_*j^*\mathscr{D} = \mathscr{D}$ (claim 8.2.12), and then lemma 8.2.11 reduces to checking that $j^*\mathscr{D}^a$ is an étale almost finitely presented $j^*\mathscr{O}^a_X$-algebra (for the almost structure given by the basic setup (B, I)). To this aim, it suffices to prove that the pair $(X \backslash \{x\}, Z \backslash \{x\})$ is almost pure; by corollary 8.2.31, we are further reduced to showing that the pair $(X(x'), Z(x'))$ is almost pure for every $x' \in X \backslash \{x\}$. However, since $p \in I$, the assertion is obvious for every $x' \in \mathrm{Spec}\, B[1/p]$; if $x' \in \mathrm{Spec}\, B/pB$, let $x'_n \in X_n$ be the image of x', for every $n \in \mathbb{N}$. Then $X(x')$ is the limit of the system of schemes $(X_n(x'_n) \mid n \in \mathbb{N})$, and the discussion of 9.6.6 says that the corresponding tower $(X_n(x'_n), \underline{M}_n(x'_n) \mid n \in \mathbb{N})$ is still maximal; notice moreover, that $I_{x'} := I \cdot \mathscr{O}_{X,x'}$ is a branch ideal of $\mathscr{O}_{X,x'}$, so we can replace the basic setup (B, I) by the basic setup $(\mathscr{O}_{X,x'}, I_{x'})$. After these preparations, the assertion for (X, Z) is reduced to the same assertion for $(X(x'), Z(x'))$, and notice that $\dim X(x') < 2$. If

$\dim X(x') = 0$, the point x' is the generic point of X, and the assertion is obvious. Lastly, the case where $\dim X(x') = 1$ is already known, by the foregoing. □

The next step consists in introducing suitable normalized lengths for B-modules, and the classes of presentable and almost presentable modules.

Theorem 9.6.34. *In the situation of* (9.6.3)*, the ring B is ind-measurable.*

Proof. (See remark 8.3.56(i) for the definition of ind-measurable ring.) One argues as in the proof of theorem 9.3.48, with some simplifications. We have to exhibit a sequence $(d_n \mid n \in \mathbb{N})$ of normalizing factors fulfilling conditions (a) and (b) of definition 8.3.51, where the λ_n occurring in *loc.cit.* is meant to be the usual length function for finitely generated λ_n-modules supported at the closed point x_n of X_n. Now, fix $n \in \mathbb{N}$, set

$$T_n := P^{\mathrm{gp}}_{\mathbb{R}} / P^{(n)\mathrm{gp}}$$

and endow T_n with its invariant measure $d\mu_n$ of total volume equal to 1. For every $\gamma \in P^{\mathrm{gp}}_{\mathbb{R}}$, let $[\gamma] \in T_n$ be the equivalence class of γ; notice that the $P^{(n)}$-module

(9.6.35) $$S_{[\gamma]} := \gamma P^{(n)\mathrm{gp}} \cap P_{\mathbb{Q}}$$

is finitely generated (proposition 3.3.22(ii)) and depends only on the class $[\gamma]$, and for any given finitely generated B_n-module M supported at x_n, consider the function

$$l_M : T_n \to \mathbb{N} \qquad [\gamma] \mapsto \lambda_n(S_{[\gamma]} \otimes_{P^{(n)}} M).$$

Let $e_1, \dots, e_r$ be a basis of the free $\mathbb{Z}$-module P^{gp}, and define $\Omega_n \subset P^{\mathrm{gp}}_{\mathbb{R}}$ as in the proof of theorem 9.3.48, so that Ω_n is a fundamental domain for the lattice $P^{(n)\mathrm{gp}}$, and 0 lies in the interior of Ω_n. Denote also by $\Sigma \subset T_n$ the image of $P_{\mathbb{R}} \cap \Omega_n$.

Claim 9.6.36. There is a partition of T_n into finitely many measurable subsets $\Theta_1, \dots, \Theta_t$, independent of M, such that :

(i) l_M restricts to a constant function on each Θ_i.
(ii) Let $\Theta \in \{\Theta_1, \dots, \Theta_t\}$ be the subset containing $[0] \in T_n$; then $\Theta \cap \Sigma$ has measure > 0.

Proof of the claim. According to proposition 3.3.35(i,iii), the set $\mathscr{S} := \{\gamma^{-1}S_{[\gamma]} \mid \gamma \in \Omega_n\}$ is finite, and for every non-empty $S \in \mathscr{S}$, the set $\{\gamma \in \Omega_n \mid \gamma^{-1}S_{[\gamma]} = S\}$ is the intersection of Ω_n with a $\mathbb{Q}$-linearly constructible subset. It follows that the same must hold also in case $S = \varnothing$. The image in T_n of any such $\mathbb{Q}$-linearly constructible subset is obviously measurable, whence (i). Moreover, in view of our choice of Ω, assertion (ii) follows easily from claim 3.3.40. ◊

Let $m \geq n$ be any integer; since B''_m is a free B''_n-module of rank $p^{r(m-n)}$ (notation of (9.6.1)), we may compute :

$$\lambda_m(B_m \otimes_{B_n} M) = \frac{p^{r(m-n)}}{[\kappa(x_m) : \kappa(x_n)]} \cdot \sum_{[\gamma] \in P^{(m)\mathrm{gp}}/P^{(n)\mathrm{gp}}} l_M([\gamma]).$$

However, lemma 9.6.8 easily implies that $\kappa(x_{n+1})^p = \kappa(x_n)$ for every $n \in \mathbb{N}$, whence

$$[\kappa(x_{n+1}) : \kappa(x_n)] = p^{e_n} \qquad \text{where } e_n := \Omega^1_{\kappa(x_n)/\mathbb{Z}}$$

by virtue of [30, Ch.IV, Th.21.4.5]. But by the same token, the field $\kappa(x_m)$ is isomorphic to $\kappa(x_n)$ for every $m \geq n$, so e_n is actually independent of n, and we get

$$[\kappa(x_m) : \kappa(x_n)] = p^{e_0(m-n)} \qquad \text{for every } m \geq n.$$

Therefore, set

$$d_n := p^{n \cdot \dim B_0} \qquad \text{for every } n \in \mathbb{N}.$$

We claim that $(d_n \mid n \in \mathbb{N})$ is a suitable sequence of normalizing factors for B. Indeed, claim 9.6.36(i) says that l_M is a measurable function on T_n, and the foregoing, together with the discussion of (9.6.6) implies that :

$$\lambda(B \otimes_{B_n} M) := \lim_{m \to +\infty} \lambda_m(B_m \otimes_{B_n} M) = d_n^{-1} \int_{T_n} l_M d\mu_n$$

(recall that $\dim P = \mathrm{rk}_{\mathbb{Z}} P^{\mathrm{gp}}$, by corollary 3.4.10(i)), so condition (a) holds for this choice of factors. Next, fix $\varepsilon > 0$, let $N \to N'$ be a surjection of finitely generated B_n-modules supported at x_n, and suppose that $d_n^{-1}(\lambda_n(N) - \lambda_n(N')) \geq \varepsilon$. Since $\lambda_n(N) = l_N(0)$ (and likewise for N'), we deduce that

$$\lambda(B \otimes_{B_n} N) - \lambda(B \otimes_{B_n} N') \geq \varepsilon \cdot \int_{\Theta} d\mu_n$$

where $\Theta \in \{\Theta_1, \dots, \Theta_t\}$ is the unique subset of T_n such that $[0] \in \Theta$, so the volume of Θ is > 0, by claim 9.6.36(ii). This shows that condition (b) holds as well, and concludes the proof of the theorem. $\square$

Proposition 9.6.37. *Let $I \subset B$ be any branch ideal, M any B-module supported at the closed point $x \in X$, and such that $M^a = 0$, for the almost structure given by (B, I). Then $\lambda(M) = 0$.*

Proof. By theorem 8.3.62(i) (and remark 8.3.56(i)), we may assume that M is finitely generated. Then, by an easy induction, we further reduce to the case where $M = Bt$ for some $t \in M$, and the annihilator of t contains $\mathfrak{q} := \mathfrak{m}_{B_0}^k$, for some $k \in \mathbb{N}$. By theorem 8.3.62(ii), we may then further assume that $M = B/(\mathfrak{q}B + I)$. Pick $J \subset B$ and $\mathfrak{Z} \subset \mathrm{Spec}\, P^{(\infty)}$ such that (9.6.18) holds with $\mathfrak{r} = \bigcap_{\mathfrak{p} \in \mathfrak{Z}} \mathfrak{p}$, and set

$$J_n := B_n \cap J \qquad \overline{B}_n := B_n/\mathfrak{q}B_n \qquad \overline{J}_n := J_n\overline{B}_n \qquad \text{for every } n \in \mathbb{N}$$

as well as $\mathfrak{p}^{(n)} := P^{(n)} \cap \mathfrak{p}$ for every $n \in \mathbb{N}$ and $\mathfrak{p} \in \mathfrak{Z}$. There follow B_n-linear maps

$$M^{(n)} := \overline{B}_n / \prod_{\mathfrak{p} \in \mathfrak{Z}} \mathfrak{p}^{(n)} \cdot \overline{J}_n \to M$$

such that M is the increasing union of the images of these maps, and by lemma 8.3.57(i), it suffices to check that

$$\lim_{n \to \infty} d_n^{-1} \cdot \lambda_n(M^{(n)}) = 0.$$

We remark :

Claim 9.6.38. There exists an integer $N \in \mathbb{N}$ such that J_n and $\mathfrak{p}^{(n)}$ admit systems of generators of cardinality $\leq N$, for every $n \in \mathbb{N}$, and every $\mathfrak{p} \in \mathfrak{Z}$.

Proof of the claim. Recall that the p-Frobenius of $P^{(n+1)}$ induces an isomorphism $P^{(n+1)} \xrightarrow{\sim} P^{(n)}$, and clearly the latter restricts to an isomorphism $\mathfrak{p}^{(n+1)} \xrightarrow{\sim} \mathfrak{p}^{(n)}$ of $P^{(n+1)}$-modules, for every $n \in \mathbb{N}$ and every $\mathfrak{p} \in \mathfrak{Z}$. Since $\mathfrak{Z}$ is a finite set, the sought bound on the number of generators for all $\mathfrak{p}^{(n)}$ is an immediate consequence. Next, let $(\pi_n \mid n \in \mathbb{N})$ be a system of elements of B as in remark 9.6.13; especially $\pi_n \in J_n$ for every $n \in \mathbb{N}$, and the Frobenius endomorphism of B_{n+1} induces an isomorphism $\varphi_n : B_{n+1}/\pi_{n+1}B_{n+1} \xrightarrow{\sim} B_n/\pi_n B_n$. Since J_{n+1} is the radical of $J_n B_{n+1}$ for every $n \in \mathbb{N}$, it follows easily that φ_n maps $J_{n+1}/\pi_{n+1}B_{n+1}$ isomorphically onto $J_n/\pi_n B_n$. The sought bound on the number of generators for J_n is an immediate consequence. $\diamondsuit$

Now, say that $\mathfrak{Z} = \{\mathfrak{p}_1, \dots \mathfrak{p}_k\}$, and set

$$M_i^{(n)} := \prod_{j=1}^{i-1} \mathfrak{p}_i^{(n)} \cdot \overline{J}_n \qquad \text{for } i = 1, \dots, k, \text{ and every } n \in \mathbb{N}.$$

If N is as in claim 9.6.38, then it is easily seen that the B_n-module $M_i^{(n)}/M_{i+1}^{(n)}$ admits a system of generators of cardinality $\leq N^i$, for every $i = 1, \dots, k$ and every $n \in \mathbb{N}$, and therefore

$$\lambda_n(M^{(n)}) \leq \lambda_n(\overline{B}_n/\overline{J}_n) + \sum_{i=1}^{k} N^i \cdot \lambda_n(\overline{B}_n/\mathfrak{p}_i^{(n)}\overline{B}_n) \qquad \text{for every } n \in \mathbb{N}.$$

We may thus consider separately the cases where $M = B/(J + \mathfrak{q}B)$ and $M = B/(\mathfrak{p}B + \mathfrak{q}B)$, for any $\mathfrak{p} \in \mathfrak{Z}$.

- Suppose first, that $M = B/(\mathfrak{p}B + \mathfrak{q}B)$. Set $Q^{(n)} := P^{(n)}/\mathfrak{p}^{[n]}$, and let

$$C_0 := \overline{B}_0/\mathfrak{p}^{(0)}\overline{B}_0 \qquad C'_n := Q^{(n)} \otimes_{Q^{(0)}} C_0 \qquad C_n := C'_n \otimes_{B_0} B''_n \qquad \text{for every } n \in \mathbb{N}$$

where B''_n is defined as in (9.6.1), so that C_n is both a cyclic B_n-module, and a free C'_n-module of rank p^{rn}, for every $n \in \mathbb{N}$, and it is easily seen that the natural map $\bigcup_{n\in\mathbb{N}} C_n \to M$ is an isomorphism of B-modules. Set $d'_n := p^{n\cdot\dim P} \cdot [\kappa(x_n) : \kappa(x_0)]^{-1}$ for every $n \in \mathbb{N}$. With this notation, and taking into account lemma 8.3.57(i), it then suffices to show :

Claim 9.6.39. $\lim_{n\to\infty} d'^{-1}_n \cdot \lambda_n(C'_n) = 0$.

Proof of the claim. Denote by $F_\mathfrak{p} \subset P_\mathbb{R}$ the unique face such that $\mathfrak{p} = P^{(\infty)} \setminus F_\mathfrak{p}$ (proposition 3.4.7(ii)), let $T' := F^{\mathrm{gp}}_\mathbb{R}/(P^{\mathrm{gp}} \cap F^{\mathrm{gp}}_\mathbb{R})$, and for every $[\gamma] \in T'$ set $S'_{[\gamma]} := \gamma P^{\mathrm{gp}} \cap F_\mathfrak{p}$. We consider the function

$$l' : T' \to \mathbb{R} \qquad [\gamma] \mapsto \lambda_0(S'_\gamma \otimes_{B_0} C'_n) \qquad \text{for every } [\gamma] \in T'$$

where λ_0 is the standard length function for B_0-modules. Arguing as in the proof of theorem 9.6.34, we see that l' is measurable, for the invariant measure $d\mu'$ of T' of total volume equal to 1, and indeed

$$\int_{T'} l' d\mu' = \lim_{n\to\infty} d''^{-1}_n \cdot \lambda_n(C'_n)$$

where $d''_n := p^{n\cdot\dim F_\mathfrak{p}} \cdot [\kappa(x_n) : \kappa(x_0)]^{-1}$ for every $n \in \mathbb{N}$. Since $\dim F_\mathfrak{p} < \dim P$, the claim follows. ◊

- Next, suppose that $M = B/(J + \mathfrak{q}B)$. In this case, set $C_n := B_n/\pi_n B_n$ for every $n \in \mathbb{N}$ and $\overline{\mathfrak{q}} := \mathfrak{q}C_0$, where $(\pi_n \mid n \in \mathbb{N})$ is the system of elements supplied by remark 9.6.13; as already observed, the Frobenius endomorphism of C_{n+1} factors through an isomorphism

$$C_{n+1} \xrightarrow{\sim} C_n \qquad \text{for every } n \in \mathbb{N}. \tag{9.6.40}$$

Then, under the identification $C_n \xrightarrow{\sim} C_0$ induced by the isomorphisms (9.6.40), the C_n-module $C_n/\mathfrak{q}C_n$ is identified with $C_0/\Phi^n_{C_0}(\overline{\mathfrak{q}})C_0$, and lemma 8.3.57(i) reduces to showing that

Claim 9.6.41. $\lim_{n\to\infty} d_n^{-1} \cdot \lambda_0(C_0/\Phi^n_{C_0}(\overline{\mathfrak{q}})C_0) = 0$.

Proof of the claim. Say that $\overline{\mathfrak{q}}$ is generated by N elements; then $\overline{\mathfrak{q}}^{Np^n} \subset \Phi^n_{C_0}(\overline{\mathfrak{q}})C_0$ for every $n \in \mathbb{N}$. On the other hand, by dimension theory (see [61, Th.13.4]), the function $k \mapsto \lambda_0(C_0/\overline{\mathfrak{q}}^k)$ is a polynomial of degree $e := \dim C_0$, hence we may find a constant $C > 0$ such that

$$\lambda_0(C_0/\Phi^n_{C_0}(\overline{\mathfrak{q}})C_0) < C \cdot k^e \qquad \text{for every sufficiently large } k \in \mathbb{N}.$$

Since $e = \dim B_0 - 1$, the claim follows. □

9.6.42. Let M and N be any two B-modules supported at the closed point $x \in X$, and such that M^a and N^a are isomorphic (for the almost structure given by some branch ideal I); as already for the case of measurable K^+-algebras, proposition 9.6.37 easily implies that $\lambda(M) = \lambda(N)$ (cp. proposition 8.3.66), and, as in (8.3.66), we let $(B, I)^a\text{-}\mathbf{Mod}_{\{s\}}$ be the full subcategory of $(B, I)^a\text{-}\mathbf{Mod}$ whose objects are the B^a-modules such that $M_!$ is supported at x, and we set

$$\lambda(M) := \lambda(M_!) \qquad \text{for every } M \in \mathrm{Ob}((B, I)^a\text{-}\mathbf{Mod}_{\{s\}}).$$

And again, with this definition, it is clear that theorem 8.3.62(i,ii) extends *mutatis mutandis* to almost modules.

9.6.43. Now, by construction, B_m is a finitely presented B_n-module for every $n, m \in \mathbb{N}$ with $m \geq n$; hence in the situation of (9.4.1), we may take $R_\bullet := (B_n \mid n \in \mathbb{N})$. Fix $n \in \mathbb{N}$, set $\Gamma := P^{(\infty)}/P^{(n)}$, and recall that B is a Γ-graded B_n-algebra, (see (9.6.3)). We let $\mathscr{C}_n$ be the smallest full subcategory of the category of finitely generated B_n-modules such that the following holds :

- $B_m \in \mathrm{Ob}(\mathscr{C}_n)$ for every $m \geq n$
- For every finitely presented B_n-modules M and N, the direct sum $M \oplus N$ is an object of $\mathscr{C}_n$ if and only if both M and N are objects of $\mathscr{C}_n$.

Following definition 9.4.6, we thus obtain a well defined class of presentable B-modules; likewise, for any branch ideal $I \subset B$ we also get well defined classes of presentable and almost presentable $(B, I)^a$-modules.

Proposition 9.6.44. *Let M be an almost presentable B^a-module, $N \subset M$ a submodule supported at the closed point $x \in X$, and suppose that $\lambda(N) = 0$. Then $N = 0$.*

Proof. Arguing as in the proof of proposition 9.4.10, we reduce easily to the case where $M = (TB)^a$ for some sequence of functors $(T_n \mid n \in \mathbb{N})$ as in definition 9.4.6, and $N = (B\tau)^a$ for some $\tau \in T_n B_n$, with $\lambda(B\tau) = 0$. The latter identity, together with lemma 8.3.57(i) and remark 8.3.56(i), implies that

$$\lim_{k \to \infty} d_{n+k}^{-1} \cdot \lambda_{n+k}(B_{n+k}\tau) = 0$$

where $(d_n \mid n \in \mathbb{N})$ is the sequence of normalizing factors found in the proof of theorem 9.6.34. Let Ω_n be a fundamental domain of the lattice $P^{(n)\mathrm{gp}}$ in $P^{\mathrm{gp}}_{\mathbb{R}}$ as in the proof of theorem 9.6.34; also, for each $[\gamma] \in T_n := P^{\mathrm{gp}}_{\mathbb{R}}/P^{(n)\mathrm{gp}}$, define $S_{[\gamma]}$ as in (9.6.35), and let $\{\Theta_1, \dots \Theta_t\}$ be the partition of T_n provided by claim 9.6.36, so that, for any $\gamma, \lambda \in \Omega_n$, the classes $[\gamma], [\lambda] \in T_n$ lie in the same Θ_i if and only if $\gamma^{-1}S_{[\gamma]} = \lambda^{-1}S_{[\lambda]}$. Let $\Theta := \Theta_i$ be the unique subset in the partition, such that $[0] \in \Theta$, and $\Sigma \subset T_n$ the image of $P_{\mathbb{R}} \cap \Omega_n$, so that $\Theta \cap \Sigma$ has measure > 0, relative to the invariant measure $d\mu_n$ on T_n, of total volume equal to 1. Now, let $B_{[\gamma],m} := S_{[\gamma]} \otimes_P B''_m$ for every $[\gamma] \in \Gamma$ and every integer $m \geq n$, where $S_{[\gamma]}$ is defined as in (9.6.35), and B''_m is as in (9.6.1); say that $\gamma \in \Theta \cap P^{(m)}$ for some $m \geq n$, and denote

$$B_{[0],m} \xrightarrow{\mu_\gamma} B_{[\gamma],m} \xrightarrow{j_\gamma} B_m$$

respectively the scalar multiplication, given by the rule : $b \mapsto \gamma \cdot b$ for every $b \in B_{[0],m}$, and the inclusion map. Since γ lies in Θ, the map μ_γ is an isomorphism, and we have a commutative diagram

$$\begin{tikzcd}
B_{[0],m} \otimes_{B_n} T_n B_n \arrow[r, "\mu_\gamma \otimes T_n B_n"] \arrow[d, "\psi_{B_{[0],m}}"'] & B_{[\gamma],m} \otimes_{B_n} T_n B_n \arrow[r, "j_\gamma \otimes T_n B_n"] \arrow[d, "\psi_{B_{[\gamma],m}}"'] & B_m \otimes_{B_n} T_n B_n \arrow[d, "\psi_{B_m}"] \\
T_n B_{[0],m} \arrow[r, "T_n \mu_\gamma"] & T_n B_{[\gamma],m} \arrow[r, "T_n j_\gamma"] & T_n B_m .
\end{tikzcd}$$

From this and from (9.4.5), a simple inspection shows that $T_n\mu_\gamma$ restricts to an isomorphism

$$B_{[0],m}\tau \xrightarrow{\sim} B_{[\gamma],m}\tau \qquad \text{for every } \gamma\in\Theta\cap P^{(m)}$$

whence a lower bound :

$$(9.6.45)\quad d_{n+k}^{-1}\cdot\lambda_{n+k}(B_{n+k}\tau)\geq d_n^{-1}\cdot p^{-(r+\dim P)k}\cdot c_{n+k}\cdot\lambda_n(B_{[0],n+k}\tau) \qquad \text{for every } k\in\mathbb{N}$$

where c_{n+k} is the cardinality of $\Theta\cap P^{(n+k)}$. However, notice that $B_{[0],n+k}=B_n\otimes_{B''_n}B''_{n+k}$ is a free B_n-module of rank p^{rk}, and pick a basis $(e_i \mid i=1,\dots,p^{kr})$ of this B_n-module. Thus, the B_n-linear map $\mu_i : B_n\to B_ne_i$ is an isomorphism for every $i\leq p^{kr}$, so we may argue as in the foregoing, to deduce that $T_n\mu_i$ restricts to an isomorphism $B_n\tau\xrightarrow{\sim}B_ne_i\tau$, for every $i=1,\dots,p^{kr}$. Combining with (9.6.45), we obtain the lower bound

$$\lim_{k\to\infty}d_{n+k}^{-1}\cdot\lambda_{n+k}(B_{n+k}\tau)\geq\lim_{k\to\infty}d_n^{-1}\cdot p^{-k\cdot\dim P}\cdot c_{n+k}\cdot\lambda_n(B_n\tau)=d_n^{-1}\cdot\lambda_n(B_n\tau)\cdot\int_{\Theta\cap\Sigma}d\mu_n$$

whence $\lambda_n(B_n\tau)=0$, and finally $\tau=0$, as stated. □

9.6.46. Let (B,I) be as in (9.6.27), and pick radical ideals $J\subset B$ and $\mathfrak{r}\subset P^{(\infty)}$ such that (9.6.18) holds. For the next step, we construct ideals $J_{\mathbf{A}},I_{\mathbf{A}}\subset\mathbf{A}(B)^+$, as follows. Define $\overline{u}_B:\mathbf{E}(B)^+\to B/pB$ as in (4.6.25), and set

$$J_{\mathbf{E}}:=\overline{u}_B^{-1}(J/pB)\subset\mathbf{E}(B)^+.$$

Since J is a radical ideal, J/pB is a radical ideal of B/pB, and therefore $J_{\mathbf{E}}$ is a radical ideal of $\mathbf{E}(B)^+$; especially

$$(9.6.47)\qquad \Phi_{\mathbf{E}(B)^+}(J_{\mathbf{E}})=J_{\mathbf{E}}$$

(notation of (4.6.25)). We let $J_{\mathbf{A}}\subset\mathbf{A}(B)^+$ be the ideal generated by $(\tau_B(x)\mid x\in J_{\mathbf{E}})$ (notation of (4.6.29)). Since the Teichmüller mapping is multiplicative, (9.6.47) implies that

$$(9.6.48)\qquad J_{\mathbf{A}}^2=J_{\mathbf{A}}.$$

Denote by $\overline{\beta}:P^{(\infty)}\to B/pB$ the composition of the chart β (notation of (9.6.17)) and the projection $B\to B/pB$. It is easily seen that the mapping

$$\beta_{\mathbf{E}}:P^{(\infty)}\to\mathbf{E}(B)^+ \qquad \gamma\mapsto(\overline{\beta}(\gamma^{1/p^n})\mid n\in\mathbb{N})$$

is a morphism of monoids (for the multiplication law of $\mathbf{E}(B)^+$). Since the Teichmüller mapping is multiplicative, we deduce a morphism of monoids

$$\beta_{\mathbf{A}}:=\tau_B\circ\beta_{\mathbf{E}}:P^{(\infty)}\to\mathbf{A}(B)^+$$

and taking into account (4.6.32), we see that the resulting diagram

$$\begin{tikzcd} P^{(\infty)} \arrow[r, "\beta"] \arrow[d, "\beta_{\mathbf{A}}"'] & B \arrow[d] \\ \mathbf{A}(B)^+ \arrow[r, "u_B"] & B^\wedge \end{tikzcd}$$

commutes (where $B^\wedge$ is the p-adic completion of B, the right vertical arrow is the natural completion map, and $u_B:\mathbf{A}(B)^+\to B^\wedge$ as in (4.6.29)). We define

$$I_{\mathbf{A}}:=\beta_{\mathbf{A}}(\mathfrak{r})\cdot J_{\mathbf{A}}.$$

Lemma 9.6.49. *With the notation of* (9.6.46)*, we have :*

(i) $u_B(J_{\mathbf{A}})=JB^\wedge$ *and* $u_B(I_{\mathbf{A}})=IB^\wedge$.
(ii) $I_{\mathbf{A}}^2=I_{\mathbf{A}}$ *and* $I_{\mathbf{A}}$ *satisfies condition* $(\mathbf{B})$ *of* [36, §2.1.6].

Proof. (i): To begin with, let $(\pi_n \mid n \in \mathbb{N})$ be a sequence of elements of B, as in remark 9.6.13; hence $\pi_0 = p$ and

$$\pi_{n+1}^p = (1 + x_n) \cdot \pi_n \quad \text{with} \quad x_n \in \pi_1^{p-1} B \qquad \text{for every } n \in \mathbb{N}.$$

The sequence $(\overline{\pi}_n \mid n \in \mathbb{N})$ consisting of the images of the π_n in B/pB, defines an element $\overline{\pi} \in \mathbf{E}(B)^+$ such that $\overline{u}_B(\overline{\pi}) = 0$, so $\overline{\pi} \in J_{\mathbf{E}}$. According to (4.6.32), we may compute

$$\widehat{\pi} := u_B \circ \tau_B(\overline{\pi}) = \lim_{n\to\infty} \pi_n^{p^n} = p \cdot \prod_{n=0}^{\infty} (1 + x_n)^{p^n}$$

where the convergence is relative to the p-adic topology; so $\widehat{\pi} B^\wedge = pB^\wedge$, and therefore $pB^\wedge \subset u_B(J_{\mathbb{A}})$. The assertion for $J_{\mathbf{A}}$ is then reduced to the identity $\overline{u}_B(J_{\mathbf{E}}) = J/pB$, which is clear, since $\overline{u}_B$ is surjective. Next, it is clear that $u_B(\beta_{\mathbf{A}}(\mathfrak{r}) \cdot \mathbf{A}(B)^+) = \mathfrak{r}B^\wedge$; on the other hand, proposition 9.6.23 implies that $IB^\wedge = \mathfrak{r} \cdot JB^\wedge$, so the assertion for $I_{\mathbf{A}}$ follows as well.

(ii): Since $P^{(\infty)}$ is p-divisible, we have $\beta_{\mathbf{A}}(\mathfrak{r})^2 \cdot \mathbf{A}(B)^+ = \beta_{\mathbf{A}}(\mathfrak{r}) \cdot \mathbf{A}(B)^+$; combining with (9.6.48), we deduce that $I_{\mathbf{A}} = I_{\mathbf{A}}^2$. Likewise, it is clear that $\beta_{\mathbf{A}}(\mathfrak{r}) \cdot \mathbf{A}(B)^+$ fulfills condition $(\mathbf{B})$, so it suffices to check that the same holds for $J_{\mathbf{A}}$; in view of [36, Claim 2.1.9], the latter follows from (9.6.47). $\square$

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